The Annals of Applied Probability
1999, Vol. 9, No. 3, 871–903

OPTIMAL LONG TERM GROWTH RATE OF EXPECTED UTILITY OF WEALTH

BY WENDELL H. FLEMING[1] AND SHUENN-JYI SHEU[2]

Brown University and Academia Sinica

An optimal investment policy model for the long term growth of expected utility of wealth is considered. The utility function is HARA with exponent $-\infty < \gamma < 1$. The problem can be reformulated as an infinite time horizon, risk sensitive control problem. Then the dynamic programming equations for different HARA exponents and different policy constraints are studied. We obtain some estimates for the solution of each equation. This can be used to derive an optimal policy with some interesting properties.

1. Introduction. In this paper we consider an optimal investment policy model, in which the goal is to maximize the long term growth rate of expected utility of wealth. For simplicity, only one risky and one riskless asset are considered, and transactions costs are ignored. In the traditional Merton model, the stock price P_t is a logarithmic Brownian motion with drift. However, we consider a model introduced by Platen and Rebolledo (1996) in which $L_t = \log P_t$ is subject to Ornstein–Uhlenbeck type random fluctuations about a deterministic trend. See (2.3). We consider a HARA utility function of wealth, with exponent $-\infty < \gamma < 1$. The case $\gamma = 0$ corresponds to log utility function. In Section 2, when $\gamma \neq 0$, we reformulate the problem as an infinite time horizon, risk sensitive stochastic control problem of the kind considered in Fleming and McEneaney (1995). The control u_t at time t is the fraction of wealth invested in the risky asset. We assume that the deterministic log stock price trend $\bar{L}_t$ is linear in t and that stock price volatility σ is constant. The state y_t is $L_t - \bar{L}_t$ plus a suitable constant; it satisfies the linear stochastic differential equation (2.9). The problem is then to choose a control which maximizes or minimizes, according to $\gamma > 0$ or $\gamma < 0$, the long term growth rate of the expectation of an exponential-of-integral cost criterion

$$E \exp \int_0^T l(y_t, u_t)\, dt,$$

where $l(y, u)$ is as in (2.10). Dynamic programming leads to a differential equation (2.13) for Λ and $W(y)$, where Λ is the optimal long term growth rate and $W(y)$ has the role of a cost potential function.

In Sections 3, 4 and 5, we study the case with $0 < \gamma < 1$ in detail. In Section 3, we consider the case of no control constraints, that is, $-\infty < u_t < \infty$.

Received February 1998; revised October 1998.
[1]Supported in part by NSF Grant DMS-95-31276.
[2]Supported in part by NSC Grant 86-2115-001-008.
AMS 1991 *subject classifications*. Primary 90A09, 93E20; secondary 60H30, 90A19.
Key words and phrases. Long term growth rate, Ornstein–Uhlenbeck process, risk sensitive control, dynamical programming equation, optimal policy.

In this case the problem has an explicit solution, with $W(y)$ a quadratic. The corresponding optimal investment policy is a linear, decreasing function of y. See (3.4). In Sections 4 and 5 we consider the problem with the control constraint $u_t \in U$, where U is either a finite interval or $U = [0, \infty)$. Section 4 is concerned with bounds on the derivatives $W_y(y)$ and $W_{yy}(y)$ which do not depend on U. Then in Section 5 we consider $U = [0, \infty)$, corresponding to a no short-selling constraint. It is shown that a solution Λ, $W(y)$ to the dynamic programming equation (5.1) exists and satisfies the required bounds (5.2). The corresponding optimal control policy $u^*(y)$ in (5.3) is no longer explicit, but is expressed in terms of $W_y(y)$. Theorem 5.1 verifies that Λ is indeed the optimal growth rate and $u^*(y)$ an optimal policy.

In Section 6, we consider the case $\gamma < 0$. The dynamic programming equation is interpreted as the dynamic programming equation of a differential game. The analysis developed in the previous sections can no longer be applied. For the case with no control constraints, the equation again has an explicit solution, with $W(y)$ a quadratic. We also have the result that Λ is the minimal long term growth rate. However, it is worth mentioning the following. When $0 > \gamma > -3$, $u^*(y)$ defined by (3.4) is an optimal investment policy. However, if $\gamma \leq -3$, $u^*(\cdot)$ is no longer optimal. In fact, if $\gamma < -3$, the investment policy $u^*(\cdot)$ gives infinite HARA expected utility of wealth in finite time T, if T is sufficiently large. However, if $\gamma \leq -3$, suitable truncation of $u^*(\cdot)$ gives a nearly optimal investment policy. For the case with no short-selling constraint, the dynamic programming equation can be suitably transformed to an equation of the type in Sections 3, 4 and 5. From this a solution with proper estimates can be obtained. This analysis can also be applied to the cases with other type of constraints.

In Section 7, we discuss the asymptotics of $\Lambda = \tilde{\Lambda}^{(\gamma)}$, $W(y) = \tilde{W}^{(\gamma)}(y)$ as $\gamma \to 0$. We show that after suitable normalization they converge to the limit which relates to the investment problem with log utility function.

Similar long term growth rate problems with transactions costs were considered in Taksar, Klass and Assaf (1988), Fleming, Grossman, Vila and Zariphopoulou (1990). In those papers log Brownian motion price fluctuations were considered, and the problem was reduced to a one-dimensional singular stochastic control problem. It would be interesting to include transactions costs in the model which we consider. Other models for optimizing long term growth rates of expected utility of wealth were considered in Bielecki and Pliska (1997), Cvitanic and Karatzas (1995), Konno, Pliska and Suzuki (1993) and references cited there. Further perspective on optimal investment models in the context of risk sensitive stochastic control is given in Fleming (1995).

2. Problem formulation. We consider an infinite time horizon optimal investment model, with one risky and one riskless asset. Let x_t denote the investor's wealth at time $t \geq 0$ and u_t the *fraction* of wealth in the risky asset. Then $u_t x_t$ is the amount in the risky asset and $(1 - u_t)x_t$ the amount in the riskless asset. We require that $u_t \in U$, where U is some given interval. Different U are considered in Sections 3, 4 and 5 for $0 < \gamma < 1$. In Section 3, we take $U = (-\infty, \infty)$, corresponding to no investment control constraints. Later we will take for U a finite interval, or $U = [0, \infty)$ corresponding to a no short-selling constraint. Let P_t denote the price per share of the risky asset and r the riskless interest rate. Then

$$dx_t = x_t[r(1-u_t)\,dt + u_t P_t^{-1} dP_t], \tag{2.1}$$

with given initial wealth $x_0 > 0$. The model for the dynamics of P_t will be given below.

We shall first consider a HARA parameter γ, with $0 < \gamma < 1$. At the end of this section and in Sections 6, 7, we will consider the necessary changes for other HARA parameters. We wish to maximize the long term exponential growth rate of the expectation of $\gamma^{-1} x_T^\gamma$ as $T \to \infty$. In the classical Merton model,

$$P_t^{-1}\,dP_t = R\,dt + \sigma\,dw_t, \tag{2.2}$$

where $R > r$ and w_t is a Brownian motion. For the Merton model, an explicit calculation using dynamic programming gives the optimal growth rate,

$$\Lambda_m = \frac{\gamma(R-r)^2}{2\sigma^2(1-\gamma)} + \gamma r.$$

The optimal fraction u_m^* invested in the risky asset is

$$u_m^* = \frac{R-r}{\sigma^2(1-\gamma)}.$$

Instead of (2.2) we consider the following model, which belongs to a class considered by Platen and Rebolledo (1996). Let

$$L_t = \log P_t$$

and assume that

$$dL_t = c(\bar{L}_t - L_t)\,dt + \sigma_t\,d\tilde{w}_t, \qquad c > 0, \tag{2.3}$$

where $\bar{L}_t$, σ_t are deterministic and $\tilde{w}_t$ is a Brownian motion under some probability measure $\tilde{\mathbb{P}}$. Then

$$P_t^{-1}\,dP_t = dL_t + \tfrac{1}{2}\sigma_t^2\,dt.$$

Let us rewrite the expectation $\tilde{E}x_T^\gamma$ in terms of an expected exponential-of-integral criterion, which involves only L_t and u_t. We apply the Itô differential rule to $\log x_t^\gamma = \gamma \log x_t$ and obtain, after a routine calculation,

$$\tilde{E}x_T^\gamma = x_0^\gamma \tilde{E}\exp\left\{\gamma\int_0^T \left[r(1-u_t) + \tfrac{1}{2}(u_t - u_t^2)\sigma_t^2 + cu_t(\bar{L}_t - L_t)\right]dt + \gamma\int_0^T \sigma_t u_t\,d\tilde{w}_t\right\}.$$

We get rid of the stochastic integral term by writing

$$\gamma \int_0^T \sigma_t u_t \, d\tilde{w}_t = \int_0^T \left[\gamma \sigma_t u_t \, d\tilde{w}_t - \tfrac{1}{2}\gamma^2 \sigma_t^2 u_t^2 \, dt\right] + \tfrac{1}{2} \int_0^T \gamma^2 \sigma_t^2 u_t^2 \, dt$$

and using a Girsanov transformation. This changes the stochastic differential equation (2.3) to

$$dL_t = c(\bar{L}_t - L_t)\, dt + \gamma \sigma_t^2 u_t \, dt + \sigma_t \, dw_t, \tag{2.4}$$

where w_t is a Brownian motion under the transformed probability measure $\mathbb{P}$ and

$$\tilde{E} x_T^\gamma = x_0^\gamma E \exp \int_0^T l_t \, dt, \tag{2.5}$$

$$l_t = \tfrac{1}{2}\sigma_t^2 \gamma(\gamma - 1) u_t^2 + \gamma\{r(1 - u_t) + [c(\bar{L}_t - L_t) + \tfrac{1}{2}\sigma_t^2] u_t\}. \tag{2.6}$$

Here E is expectation under $\mathbb{P}$. This change of probability measure argument is valid provided σ_t is bounded and

$$\tilde{E} \exp \theta u_t^2 \le C \tag{2.7}$$

for some positive constants θ, C. See Liptser and Shiryayev [(1977), page 220].

We interpret the stochastic differential equation (2.4) as the dynamics of a stochastic control problem, in which L_t is the state and u_t the control at time t. We require that u_t is $\mathscr{F}_t$ progressively measurable, for some w_t-adapted increasing family of σ-algebras $\mathscr{F}_t$ [see, e.g., Fleming and Soner (1992), Chapter 4], and that (2.7) holds. In particular, u_t may be obtained from a Lipschitz-continuous control policy $\underline{u}$,

$$u_t = \underline{u}(t, L_t).$$

For fixed finite T, one can consider the problem of choosing u_t on $0 \le t \le T$ to maximize the expectation on the right side of (2.5). Let

$$\mathscr{V}(L, T) = \log \sup_{u_\bullet} E_L \exp \int_0^T l_t \, dt,$$

where the subscript L indicates the initial data $L_0 = L$.

We anticipate that, under suitable assumptions on $\bar{L}_t$ and σ_t, $T^{-1}\mathscr{V}(L, T)$ tends to a limit Λ as $T \to \infty$, and that Λ can be interpreted as the optimal long term growth rate of expected utility of wealth. In this paper, the analysis is carried out only under the assumptions

$$\begin{aligned} &\sigma_t = \sigma \text{ is constant,} \\ &\bar{L}_t = \mu t + \bar{L}_0, \ \ \mu, \bar{L}_0 \text{ constant.} \end{aligned} \tag{2.8}$$

[It would also be interesting to allow σ_t and $\bar{L}_t$ to vary periodically, as in Platen (1996).]

It is convenient to replace L_t by an equivalent state variable y_t, obtained by subtracting the linear trend $\bar{L}_t$ in the log price dynamics, plus a suitable constant,

$$y_t = L_t - \bar{L}_t + c^{-1}\mu.$$

Then by (2.4),

$$dy_t = -cy_t\,dt + \gamma\sigma^2 u_t\,dt + \sigma\,dw_t. \tag{2.9}$$

If $u_t \equiv 0$, then y_t is an Ornstein–Uhlenbeck process. Moreover, by (2.6), $l_t = l(y_t, u_t)$ where

$$l(y, u) = -\frac{a}{2}u^2 + \gamma(b - cy)u + \gamma r, \tag{2.10}$$

$$\begin{aligned} a &= \sigma^2\gamma(1-\gamma),\\ b &= \tfrac{1}{2}\sigma^2 + \mu - r. \end{aligned} \tag{2.11}$$

Let

$$V(y, T) = \log \sup_{u_\cdot} E_y \exp \int_0^T l(y_t, u_t)\,dt, \tag{2.12}$$

which equals $\mathscr{V}(L, T)$ above. We use as in Fleming and McEneaney (1995) the heuristic

$$V(y, T) \sim \Lambda T + W(y) \quad \text{as } T \to \infty,$$

where Λ, $W(y)$ satisfy the dynamic programming equation

$$\begin{aligned} \Lambda = {} & \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r\\ & + \max_{u\in U}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y)\right]. \end{aligned} \tag{2.13}$$

If U is a finite interval, then results in Fleming and McEneaney [(1995), Section 7] ensure that (2.13) has a solution and the Λ is the optimal long term growth rate. Therefore, we shall focus on the case $U = (-\infty, \infty)$, for which Λ and $W(y)$ are found explicitly in Section 3, and the case $U = [0, \infty]$ considered in Section 5.

The function $W(y)$ has a role similar to a cost potential function for ergodic stochastic control. Once W is known, an optimal investment control policy $u^*(y)$ can be obtained by taking arg max over U in (2.13).

For the HARA parameter γ in $\gamma < 0$, we change sup to inf in (2.12). Then we change max to min in (2.13) for the dynamic programming equation.

For $\gamma = 0$, this is the case of the logarithmic utility (Kelly) criterion; we consider

$$V(y, T) = \sup_u E_y\left[\int_0^T \left(-\frac{\sigma^2}{2}u_t^2 + (b - cy_t) + r\right)dt\right].$$

Here y_t satisfies

$$dy_t = -cy_t\,dt + \sigma\,dw_t.$$

The dynamic programming equation is

$$\Lambda = \frac{\sigma^2}{2}W_{yy} - cyW_y + \sup_{u\in U}\left[-\frac{\sigma^2}{2}u^2 + (b - cy)u + r\right]. \tag{2.14}$$

Note that when $\gamma = 0$, the optimal u^* in (2.14) can be found directly without knowing W. See Section 7.

3. Unconstrained case. In this section we assume that $0 < \gamma < 1$ and $U = (-\infty, \infty)$. Thus both short selling ($u < 0$) and borrowing at rate $r(u > 1)$ are allowed. An explicit solution to the dynamic programming solution (2.13) is readily obtained as follows. With no investment control constraints, (2.13) becomes

$$\Lambda = \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r + \frac{\gamma^2(b - cy + \sigma^2 W_y)^2}{2a} \tag{3.1}$$

with a, b as in (2.11). We look for a quadratic solution

$$W(y) = \tfrac{1}{2}Ay^2 + By. \tag{3.2}$$

After a routine calculation we find that (3.1) holds for

$$A = \frac{c}{\sigma^2}[1 - (1-\gamma)^{1/2}], \tag{3.3)(a}$$

$$B = -\frac{\gamma b}{\sigma^2}, \tag{3.3)(b}$$

$$\Lambda = \frac{c}{2}[1 - (1-\gamma)^{1/2}] + \gamma r + \frac{\gamma b^2}{2\sigma^2}. \tag{3.3)(c}$$

By taking arg max over $U = (-\infty, \infty)$ in (2.13), we obtain the following candidate $u^*(y)$ for an optimal investment policy:

$$u^*(y) = \frac{b - cy + \sigma^2 W_y(y)}{\sigma^2(1-\gamma)}.$$

Since $W_y(y) = Ay + B$,

$$u^*(y) = -\frac{c(1-\gamma)^{-1/2}}{\sigma^2}y + \frac{b}{\sigma^2}. \tag{3.4}$$

The corresponding solution y_t^* to (2.9), with $u_t = u_t^* = u^*(y_t^*)$ is

$$dy_t^* = -c(1 + \gamma(1-\gamma)^{-1/2})y_t^*\,dt + \gamma b\,dt + \sigma\,dw_t. \tag{3.5}$$

For any initial state $y_0^* = y$, the process y_t^* is an ergodic, Gaussian, Markov process.

Theorem 3.1 justifies calling Λ the optimal long term growth rate and $u^*(y)$ an optimal investment policy. At the end of the section we will also verify that $T^{-1}V(y,T) \to \Lambda$ as $T \to \infty$, where $V(y,T)$ solves the finite horizon dynamic programming equation. There is another quadratic solution to (3.1), with $A = c\sigma^{-2}[1+(1-\gamma)^{1/2}]$. However, the corresponding solution to (3.5) is not ergodic, and this solution to (3.1) is irrelevant.

THEOREM 3.1 (Verification theorem). *Let* $U = (-\infty, \infty)$.

(a) *For every admissible control process* u_t,

$$\Lambda \geq \limsup_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t, u_t)\,dt.$$

(b) *If* $u_t^* = u^*(y_t^*)$ *as in* (3.4), (3.5), *then*

$$\Lambda = \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^*, u_t^*)\,dt.$$

PROOF. We rewrite (2.13) in exponentiated form. Let $\psi = e^W$. Then

$$\Lambda\psi = \max_{u\in U}[\mathscr{L}^u\psi + l(y,u)\psi], \tag{3.6}$$

where $l(y,u)$ is as in (2.10) and

$$\mathscr{L}^u\psi = \frac{\sigma^2}{2}\psi_{yy} + (-cy + \gamma\sigma^2 u)\psi_y \tag{3.7}$$

is the generator of the controlled process y_t in (2.9) in case of constant control $(u_t \equiv u)$.

To prove (a), let $\tilde{l} = l - \Lambda$. The Itô differential rule gives

$$\begin{aligned} d&\left[\psi(y_t)\exp\int_0^t \tilde{l}(y_s,u_s)\,ds\right] \\ &= \left[\mathscr{L}^{u_t}\psi(y_t) + \tilde{l}(y_t,u_t)\psi(y_t)\right]\exp\int_0^t \tilde{l}(y_s,u_s)\,ds + dM_t, \end{aligned}$$

where M_t is a local martingale. By (3.6), $\mathscr{L}^u\psi(y) + \tilde{l}(y,u)\psi(y) \leq 0$ for all y, u.

Let τ_R denote the exit time of y_t from the ball $\{|y| \leq R\}$. Then for every $T < \infty$,

$$E_y\psi(y_{T\wedge\tau_R})\exp\int_0^{T\wedge\tau_R} \tilde{l}(y_t,u_t)\,dt \leq \psi(y).$$

Let $R \to \infty$ and use Fatou's lemma to get

$$E_y\psi(y_T)\exp\int_0^T \tilde{l}(y_t,u_t)\,dt \leq \psi(y).$$

Since $A > 0$ in (3.2), $\psi(y) \geq \exp(-K)$ for some K. Hence

$$\exp(-K)\exp(-\Lambda T)E_y\exp\int_0^T l(y_t,u_t)\,dt \leq \exp(W(y)).$$

This implies (a).

To prove (b), we argue as in the proof of Fleming and McEneaney [(1995), Theorem 3.5] (see also the proof of Theorem 5.1 below). We make a Girsanov change of probability measure from $\mathbb{P}$ to $\mathbb{P}^0$ corresponding to adding a term $\sigma^2 W_y(y_t^*)\,dt$ on the right side of (3.5):

$$\begin{aligned} dy_t^* = -c(1+\gamma(1-\gamma)^{-1/2})y_t^*\,dt + \gamma b\,dt \\ + \sigma^2 W_y(y_t^*)\,dt + \sigma\,dw_t^0, \end{aligned}$$

with w_t^0 a $\mathbb{P}^0$-Brownian motion. Since $W_y(y) = Ay + B$, with A, B as in (3.3), we obtain

$$dy_t^* = -c(1-\gamma)^{-1/2} y_t^*\,dt + \sigma\,dw_t^0. \tag{3.8}$$

Thus, y_t^* is a Gaussian, Markov ergodic process under $\mathbb{P}^0$. Since $E\exp\theta(y_t^*)^2$ is bounded for $\theta > 0$ small enough, the Girsanov transformation is justified. From (3.6),

$$\Lambda\psi(y) = \mathscr{L}^{u^*(y)}\psi(y) + l(y, u^*(y))\psi(y).$$

We then have [Fleming and McEneaney (1995), page 1889]

$$E_y \exp\int_0^T l(y_t^*, u_t^*)\,dt = \exp(\Lambda T + W(y))E_y^0 \exp[-W(y_T^*)]. \tag{3.9}$$

Moreover, as $T\to\infty$, $E_y^0\exp[-W(y_T^*)]$ tends to the expectation of $\exp(-W)$ under the equilibrium probability measure. This implies (b). □

REMARK 3.1. From (3.4), $u^*(y)$ is a decreasing function of y. This property remains true if investment control constraints are imposed, although one no longer has such an explicit formula for $u^*(y)$. See Sections 4, 5. Since $y_t = L_t - \bar{L}_t + c^{-1}\mu$ and $L_t = \log P_t$, with this model the optimal fraction of wealth in the risky asset decreases as the price P_t increases.

FINITE TIME HORIZON. For $T < \infty$, the function $V(y, T)$ in (2.12) can be shown by a similar analysis to be quadratic in y:

$$V(y,T) = \tfrac{1}{2}A(T)y^2 + B(T)y + C(T). \tag{3.10}$$

The coefficients satisfy differential equations obtained from the time-dependent version of (3.1),

$$V_T = \frac{\sigma^2}{2}V_{yy} + \frac{\sigma^2}{2}V_y^2 - cyV_y + \gamma r + \frac{\gamma^2(b - cy + \sigma^2 V_y)^2}{2a},$$

with initial data $V(y,0) = 0$. An elementary analysis shows that $A(T)\to A$, $B(T)\to B$ and $(dC/dT)(T)\to\Lambda$ as $T\to\infty$, with A, B, Λ as in (3.3). This implies that, for every y,

$$\Lambda = \lim_{T\to\infty}\frac{V(y,T)}{T}. \tag{3.11}$$

4. Finite control interval. In this section we assume that $0 < \gamma < 1$ and U is a closed finite interval with $0 \in U$. In particular, we may consider $U = U_M = [0, M]$, corresponding to no short-selling and borrowing constraints. By Fleming and McEneaney [(1995), Theorem 7.1], there exists a solution Λ, $W(y)$ to (2.13), such that $W_y(y)$ is bounded. Moreover, by Fleming and McEneaney (1995), Theorem 7.2, Λ is the optimal growth rate in the sense described in Theorem 3.1.

The main purpose of this section is to derive bounds for Λ and $W_y(y)$ which do not depend on U. We also find that $W(y)$ is convex and $W(y) - (2\sigma^2)^{-1}cy^2$ is concave. First,

$$\gamma r \leq \Lambda \leq \Lambda_0, \tag{4.1}$$

where Λ_0 is the unconstrained optimal growth rate when $U = (-\infty, \infty)$. The left inequality is immediate from considering $u_t \equiv 0$, and the right inequality follows, since imposing control constraints cannot increase the optimal growth rate.

Let us rewrite (2.13) as the dynamic programming equation of an ergodic stochastic control problem, with two controls $u \in U$, $v \in \mathbb{R}^1$. For this purpose, note that

$$\frac{\sigma^2}{2} W_y^2 = \max_{v \in \mathbb{R}^1} \left[\sigma v W_y - \tfrac{1}{2} v^2\right].$$

Since W_y is bounded, it suffices to replace the max over $\mathbb{R}^1$ by the max over $\{|v| \leq B\}$ for some B. As in Fleming and McEneaney [(1995), Section 7], consider the discounted cost criterion

$$J_\rho(y; u_\bullet, v_\bullet) = E_y \int_0^\infty e^{-\rho t}\left[l(y_t, u_t) - \tfrac{1}{2} v_t^2\right] dt,$$

where y_t satisfies

$$dy_t = (-cy_t + \gamma\sigma^2 u_t + \sigma v_t)\, dt + \sigma\, dw_t \tag{4.2}$$

with $y_0 = y$, and w_t a $\mathscr{F}_t$-adapted Brownian motion on some fixed probability space $(\Omega, \{\mathscr{F}_t\}, P)$. By differentiating with respect to the initial data y in(4.2) and using the fact that $l_y = -c\gamma u$ in (2.10), we have that

$$J_{\rho y} = E_y \int_0^\infty \exp(-(\rho + c)t) l_y(y_t, u_t)\, dt$$

does not in fact depend on y. Thus, $J_\rho(y; u_\bullet, v_\bullet)$ is linear in y for each pair $u_\bullet v_\bullet$ of $\mathscr{F}_t$-progressively measurable processes ($u_t \in U$, $|v_t| \leq B$.)

Since the supremum of any family of linear functions is convex,

$$W_\rho(y) = \sup_{u_\bullet, v_\bullet} J_\rho(y; u_\bullet, v_\bullet)$$

is convex. As $\rho \to 0$ (through a sequence),

$$W(y) = \lim_{\rho \to 0}[W_\rho(y) - W_\rho(0)].$$

We have proved the following lemma.

LEMMA 4.1. *$W(y)$ is convex.*

Let us next obtain bounds for W_y which do not depend on U.

LEMMA 4.2. (a) *$|W_y(y)| \le c_1|y| + c_2$, where c_1, c_2 do not depend on U.*
(b) *If $U = [0, M]$, then $W_y(y) \le 0$ for all y. Moreover, for $y \ge 0$, $-c_3 \le W_y(y)$ where c_3 does not depend on M.*

PROOF. By taking $u = 0$ in (2.13) and using Lemma 4.1, we have

$$\Lambda \ge \frac{\sigma^2}{2} W_y^2 - cyW_y + \gamma r. \tag{4.3}$$

By (4.1), $\Lambda \le \Lambda_0$. This implies (a). To obtain (b), note that $l_y = -c\gamma u \le 0$ for $u \ge 0$. Then $J_{\rho y} \le 0$ for all $u_\bullet, v_\bullet$, which implies that $W(y)$ is a nonincreasing function of y. Then (b) follows from this and (4.3). □

LEMMA 4.3. *$W(y) - c(2\sigma^2)^{-1}y^2$ is concave.*

PROOF. In (4.2) we make the change of variable

$$\tilde{v}_t = -\frac{c}{\sigma} y_t + v_t.$$

Then

$$dy_t = (\gamma\sigma^2 u_t + \sigma\tilde{v}_t)\,dt + \sigma\,dw_t, \tag{4.4}$$

and

$$\begin{aligned} l(y, u) - \frac{1}{2}v^2 &= l(y, u) - \frac{1}{2}\left(\tilde{v} + \frac{c}{\sigma}y\right)^2 \\ &= -\frac{a}{2}u^2 + \gamma bu - \frac{1}{2}\tilde{v}^2 - \frac{c^2}{2\sigma^2}y^2 - \frac{c}{\sigma^2}(\gamma\sigma^2 u + \sigma\tilde{v})y. \end{aligned}$$

From (4.4) and Itô's rule,

$$d\frac{c}{2\sigma^2}y_t^2 = \left[\frac{c}{\sigma^2}(\gamma\sigma^2 u_t + \sigma\tilde{v}_t)y_t + \frac{1}{2}c\right]dt + dM_t,$$

where M_t is a martingale (recall that u_t and v_t are bounded in this section). Then

$$d\left[\frac{c}{2\sigma^2}y_t^2 e^{-\rho t}\right] = \left[\frac{c}{\sigma^2}(\gamma\sigma^2 u_t + \sigma\tilde{v}_t)y_t + \frac{1}{2}c - \frac{c}{2\sigma^2}\rho y_t^2\right]e^{-\rho t}dt + e^{-\rho t}dM_t,$$

$$\begin{aligned} &E_y \int_0^T e^{-\rho t}\frac{c}{\sigma^2}(\gamma\sigma^2 u_t + \sigma\tilde{v}_t)y_t\,dt \\ &\quad = E_y \int_0^T e^{-\rho t}\left(\frac{c}{2\sigma^2}\rho y_t^2 - \frac{1}{2}c\right)dt + E_y\left[\frac{c}{2\sigma^2}y_T^2 e^{-\rho T}\right] - \frac{c}{2\sigma^2}y^2. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&E_y \int_0^T e^{-\rho t}\left(l(y_t, u_t) - \frac{1}{2}v_t^2\right) dt \\
&\qquad = E_y \int_0^T e^{-\rho t}(\tilde{l}(y_t, u_t) - \frac{1}{2}\tilde{v}_t^2)dt - E_y\left[\frac{c}{2\sigma^2} y_T^2 e^{-\rho T}\right] + \frac{c}{2\sigma^2} y^2, \\
&\tilde{l}(y_t, u_t) = -\frac{a}{2}u^2 + \gamma b u - \frac{c}{2\sigma^2}(c+\rho)y^2 + \gamma r.
\end{aligned}
\tag{4.5}
$$

Since u_t and v_t are bounded and (4.2) holds,

$$
E_y \int_0^\infty e^{-\rho t}(u_t^2 + \tilde{v}_t^2)\, dt < \infty,
$$

$$
\lim_{T\to\infty} e^{-\rho T} E_y y_T^2 = 0.
$$

Therefore, (4.5) implies (with J_ρ as above)

$$
\begin{aligned}
J_\rho(y; u_\bullet, v_\bullet) &= \tilde{J}_\rho(y; u_\bullet, \tilde{v}_\bullet) + \frac{c}{2\sigma^2}y^2, \\
\tilde{J}_\rho(y; u_\bullet, \tilde{v}_\bullet) &= E_y \int_0^\infty e^{-\rho t}\left(\tilde{l}(y_t, u_t) - \frac{1}{2}\tilde{v}_t^2\right) dt.
\end{aligned}
\tag{4.6}
$$

Since $\tilde{l}(y, u)$ is concave, (4.4) is linear, $\tilde{J}_\rho$ is a concave function of $(y, u_\bullet, \tilde{v}_\bullet)$. By (4.6) and Fleming and Rishel [(1975), page 196], $W_\rho(y) - c(2\sigma^2)^{-1}y^2$ is concave. Since $W(y)$ is the limit of $W_\rho(y) - W_\rho(0)$ as $\rho \to 0$ through a subsequence, we obtain Lemma 4.3. □

From Lemma 4.1 and Lemma 4.3 we have

$$
0 \le W_{yy}(y) \le \frac{c}{\sigma^2}. \tag{4.7}
$$

5. No short-selling constraint. Let us now assume that $U = [0, \infty)$. Again we consider $0 < \gamma < 1$. The first step of the analysis is to find Λ and $W(y)$ satisfying (2.13) with suitable growth behavior of $W_y(y)$ as $y \to \pm\infty$. Let $U_M = [0, M]$ and let $\Lambda_M, W_M(y)$ be the solution to (2.13) considered in Section 4 when U is replaced by U_M. Then Λ_M is nondecreasing as M increases, and satisfies the uniform bounds (4.1). Moreover, W_{My} and W_{Myy} satisfy the uniform bounds in Lemma 4.2 and (4.7).

We normalize $W_M(y)$ by taking $W_M(0) = 0$. Then we let $M \to \infty$ (through some sequence) to obtain Λ and $W(y)$ satisfying

$$
\begin{aligned}
\Lambda &= \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r \\
&\quad + \max_{u\ge 0}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y)\right],
\end{aligned}
\tag{5.1}
$$

$$W_y(y) \leq 0, \tag{5.2)(a}$$

$$W_y(y) \geq -C \quad \text{for } y \geq 0, \tag{5.2)(b}$$

$$0 \leq W_{yy}(y) \leq \frac{c}{\sigma^2}. \tag{5.2)(c}$$

In the Appendix we add some remarks about the behavior of Λ and $W(\cdot)$. The maximum in (5.1) is attained at $u = u^*(y)$, where $u^*(y) = 0$ if and only if $b - cy + \sigma^2 W_y(y) \leq 0$. By (4.7), $b - cy + \sigma^2 W_y(y)$ is nonincreasing. Hence

$$I = \{y: b - cy + \sigma^2 W_y(y) \leq 0\}$$

is an interval. Let us show that $I = [y^*, \infty)$ for some y^*. Since $W_y(y)$ is bounded for $y \geq 0$, $y \in I$ for all y sufficiently large. Suppose that $I = (-\infty, \infty)$. Then

$$\frac{b}{y} - c + \sigma^2 \frac{W_y(y)}{y} \geq 0 \quad \text{for } y < 0,$$

$$\liminf_{y \to -\infty} \frac{W_y(y)}{y} \geq \frac{c}{\sigma^2}.$$

Together with (5.2)(c), $y^{-1} W_y(y) \to \sigma^{-2} c$ as $y \to -\infty$. If $I = (-\infty, \infty)$, the last term in (5.1) is always 0. We divide (5.1) by y^2 and let $y \to -\infty$ to obtain

$$0 = \frac{\sigma^2}{2} \left(\frac{c}{\sigma^2} \right)^2 - \frac{c^2}{\sigma^2} = -\frac{c^2}{2\sigma^2} \neq 0,$$

a contradiction. Thus $I = [y^*, \infty)$.

Since $a = \sigma^2 \gamma (1 - \gamma)$, we then have the following candidate for the optimal investment control policy:

$$u^*(y) = \begin{cases} \dfrac{b - cy + \sigma^2 W_y(y)}{\sigma^2 (1 - \gamma)}, & \text{if } y < y^*, \\ 0, & \text{if } y \geq y^*. \end{cases} \tag{5.3}$$

Note that u^* is a nonincreasing function of y since $b - cy + \sigma^2 W_y$ is nonincreasing by (5.2)(c). It remains to verify that $u^*(y)$ is indeed optimal. Let y_t^* be the solution to

$$dy_t^* = [-cy_t^* + \gamma \sigma^2 u^*(y_t^*)]\, dt + \sigma\, dw_t \tag{5.4}$$

with $y_0^* = y$. For $y < y^*$,

$$-cy + \gamma \sigma^2 u^*(y) \geq -cy,$$

and equality holds for $y \geq y^*$. This implies that y_t^* is an ergodic Markov process. See the results in Khasminskii [(1980), Chapter IV, Section 4]. As in Section 3, we shall need to make a change of probability measure from $\mathbb{P}$ to

$\mathbb{P}^0$ such that

$$dy_t^* = [-cy_t^* + \gamma\sigma^2 u^*(y_t^*) + \sigma^2 W_y(y_t^*)]\,dt + \sigma\,dw_t^0, \tag{5.5}$$

where w_t^0 is a $\mathbb{P}^0$-Brownian motion. Then

$$-cy + \gamma\sigma^2 u^*(y) + \sigma^2 W_y(y) \le -cy \quad \text{if } y \ge y^*.$$

For $y < y^*$, (5.3) implies that

$$-cy + \gamma\sigma^2 u^*(y) + \sigma^2 W_y(y) = \frac{\sigma^2 W_y(y) - cy}{1-\gamma} + \frac{\gamma b}{1-\gamma}.$$

For $y < y^*$, (3.1) holds. We divide by y^2 and let $y \to -\infty$ (through a sequence). Then by (5.2)(c)

$$\frac{W_y(y)}{y} \to \bar{A} \le \frac{c}{\sigma^2},$$

$$0 = \frac{\sigma^2}{2}\bar{A}^2 - c\bar{A} + \frac{\gamma^2}{2a}(-c + \sigma^2\bar{A})^2.$$

This implies

$$0 = \frac{\sigma^2}{2}\bar{A}^2 - c\bar{A} + \frac{\gamma c^2}{2\sigma^2},$$

$$\bar{A} = \frac{c}{\sigma^2}(1 - \sqrt{1-\gamma}).$$

[Note that $\bar{A} = A$ in (3.3)(a).] Then as $y \to -\infty$,

$$-cy + \gamma\sigma^2 u^*(y) + \sigma^2 W_y(y) \sim -c(1-\gamma)^{1/2} y.$$

This implies that, under $\mathbb{P}^0$, y_t^* is an ergodic Markov process and $E_y^0|y_T^*|^\beta$ is bounded independent of T for any $\beta > 0$.

It remains to verify that Λ is the optimal growth rate and that $u^*(y)$ is an optimal control policy. Unlike the unconstrained case in Section 3, we do not know that $W(y)$ is bounded below. Hence the proof of Theorem 3.1(a) cannot be used when $U = (0, \infty)$. However, (5.1) and (5.2) imply $|yW_y(y)| \le c_1$ if $y \ge 0$, for some c_1. This implies a logarithmically growing upper bound for $-W(y)$, which will be used in the proof of Theorem 5.1(a). Unlike the situation in Section 4, $U = [0, \infty)$ is not compact. In defining admissible control processes u_t, we must ensure that the Girsanov transformations used in the proof of Theorem 5.1 are valid. To avoid technicalities in this regard, let us admit only those u_t which arise via some control policy,

$$u_t = u(t, y_t),$$

where $u(t, y)$ is locally Lipschitz and satisfies for suitable $\alpha_1(T), \alpha_2(T)$,

$$|u(t, y)| \le \alpha_1(T)|y| + \sigma_2(T) \quad \text{for } 0 \le t \le T.$$

THEOREM 5.1 (Verification theorem). *Let* $U = [0, \infty)$ *and let* Λ, $W(y)$ *satisfy* (5.1) *and* (5.2).

(a) *For every admissible control process* u_t,

$$\Lambda \geq \limsup_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t, u_t)\, dt.$$

(b) *If* $u_t^* = u^*(y_t^*)$ *as in* (5.3), (5.4), *then*

$$\Lambda = \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^*, u_t^*)\, dt.$$

PROOF. Let

$$H^*(y) = \gamma r + \max_{u\geq 0}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y(y))\right],$$

$$h^*(y, u) = \gamma r - \frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y(y)) - H^*(y).$$

As in (2.9),

$$dy_t = (-cy_t + \gamma\sigma^2 u_t)\, dt + \sigma\, dw_t, \tag{5.6}$$

with $y_0 = y$ and w_t a $\mathbb{P}$-Brownian motion. From the Itô differential rule applied to $W(y_t)$ and (5.1),

$$\begin{aligned}\int_0^T l(y_t, u_t)\, dt = \Lambda T &+ \int_0^T \left[l(y_t, u_t) - H^*(y_t) - \frac{\sigma^2}{2} W_y^2(y_t) + \gamma\sigma^2 u_t W_y(y_t)\right] dt \\ &+ \int_0^T \sigma W_y(y_t)\, dw_t + W(y) - W(y_T).\end{aligned}$$

We change probability measure, from $\mathbb{P}$ to $\hat{\mathbb{P}}$, corresponding to adding $\sigma^2 W_y(y_t)$ to the drift in (5.6),

$$dy_t = [-cy_t + \gamma\sigma^2 u_t + \sigma^2 W_y(y_t)]\, dt + \sigma\, d\hat{w}_t, \tag{5.7}$$

with $\hat{w}_t$ a $\hat{\mathbb{P}}$-Brownian motion. Then

$$E_y \exp \int_0^T l(y_t, u_t)\, dt = \exp(\Lambda T)\psi(y)\hat{E}_y\left[\exp \int_0^T h^*(y_t, u_t)\, dt \psi^{-1}(y_T)\right], \tag{5.8}$$

where $\psi = \exp W$. We make another change in probability measure, from $\hat{\mathbb{P}}$ to $\tilde{\mathbb{P}}$, corresponding to adding $\gamma\sigma^2(u^*(y_t) - u_t)$ to the drift in (5.7),

$$dy_t = [-cy_t + \gamma\sigma^2 u^*(y_t) + \sigma W_y(y_t)]\, dt + \sigma\, d\tilde{w}_t, \tag{5.9}$$

with $\tilde{w}_t$ a $\tilde{\mathbb{P}}$-Brownian motion. Note that under $\tilde{\mathbb{P}}$, y_t is identical in probability law to the solution y_t^* to (5.5) under $\mathbb{P}^0$. Moreover, by (5.8),

$$\begin{aligned}&E_y \exp \int_0^T l(y_t, u_t)\, dt \\ &\quad = \exp(\Lambda T)\psi(y)\tilde{E}_y\left[\exp \zeta_T \cdot \exp \int_0^T h^*(y_t, u_t)\, dt \psi^{-1}(y_T)\right],\end{aligned} \tag{5.10}$$

$$\zeta_T = \int_0^T \left[\gamma\sigma(u_t - u_t^*)\,d\tilde{w}_t - \tfrac{1}{2}\gamma^2\sigma^2(u_t - u_t^*)^2\,dt\right].$$

We use Hölder's inequality, with $p^{-1}+q^{-1} = 1$ and q sufficiently large, together with

$$\tilde{E}_y\left[\exp\int_0^T p\gamma\sigma(u_t - u_t^*)\,d\tilde{w}_t - \tfrac{1}{2}p^2\gamma^2\sigma^2(u_t - u_t^*)^2\,dt\right] \le 1$$

to obtain

$$\begin{aligned} &E_y \exp\int_0^T l(y_t, u_t)\,dt \\ &\quad\le \exp(\Lambda T)\psi(y)\left\{\tilde{E}_y\left[\exp\int_0^T qh^*(y_t, u_t)\right.\right. \\ &\qquad\qquad \left.\left. + \frac{p}{2}\gamma^2\sigma^2(u_t - u_t^*)^2\,dt\psi^{-q}(y_T)\right]\right\}^{1/q}. \end{aligned} \tag{5.11}$$

From (5.3) and the fact that $b - cy + \sigma^2 W_y(y) \le 0$ for $y \ge y^*$, we have

$$\begin{aligned} h^*(y,u) &\le -\frac{a}{2}u^2 = -\frac{a}{2}(u - u^*(y))^2 \quad \text{if } y \ge y^*, \\ h^*(y,u) &= -\frac{a}{2}(u - u^*(y))^2 \quad \text{if } y < y^*. \end{aligned}$$

Therefore, for q large enough,

$$qh^*(y_t, u_t) + \frac{p}{2}\gamma^2\sigma^2(u_t - u_t^*)^2 \le 0,$$

and by (5.11),

$$E_y \exp\int_0^T l(y_t, u_t)\,dt \le \exp(\Lambda T)\psi(y)\{\tilde{E}_y\psi^{-q}(y_T)\}^{1/q}. \tag{5.12}$$

However, (5.1) and (5.2) imply that $\psi^{-q}(y_T) \le K(1 + |y_T|)^\beta$ for suitable constants K, β. As noted in the remarks following (5.5), $\tilde{E}|y_T|^\beta$ is bounded independent of T. This implies (a). When $u_t = u_t^*$, $h^*(y_t, u_t) = 0$. Moreover, $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$ and $\tilde{E}_y\psi^{-1}(y_T)$ tends to its equilibrium value (positive) as $T \to \infty$. This implies (b). $\square$

6. Results for $\gamma < 0$. In this section we wish to consider the maximal long term exponential growth rate of $\gamma^{-1}Ex_T^\gamma$ for $\gamma < 0$. This is equivalent to minimize Ex_T^γ. In this case, $a < 0$ [see (2.11)]. We change sup to inf in (2.12) and max to min in (2.13). In particular, we have the dynamic programming equation

$$\begin{aligned} \Lambda &= \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r \\ &\quad + \min_{u\in U}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y)\right]. \end{aligned} \tag{6.1}$$

If there is no investment constraint, the same calculation as in Section 3 gives a solution Λ, $W(y)$ to (6.1) with $W(y)$ quadratic as in (3.2) and (3.3). Note that $A < 0$ in the case $\gamma < 0$. We may conjecture that the investment policy $u^*(y)$ in (3.4) is optimal. It is surprising that this is no longer true unless $0 > \gamma > -3$. To see this, by (3.9),

$$E_y\left[\exp\left(\int_0^T l(y_t, u_t)\,dt\right)\right] = \exp(\Lambda T + W(y))E_y^0[\exp(-W(y_T^*))].$$

For $0 > \gamma > -3$,

$$E_y^0[\exp(-W(y_t^*))] \to \int \exp(-W(y))\,d\mu^*(y), \qquad T \to \infty,$$

where μ^* is the equilibrium distribution of y_t^* satisfying (3.8),

$$d\mu^*(y) = \frac{1}{\sqrt{2\pi\sigma_\infty^2}} \exp\left(-\frac{1}{2\sigma_\infty^2} y^2\right),$$

$$\sigma_\infty^2 = \frac{\sigma^2}{2c}\sqrt{1-\gamma}.$$

For $\gamma = -3$,

$$\frac{1}{T}\log E_y^0[\exp(-W(y_T^*))] \to \infty, \qquad T \to \infty.$$

For $\gamma < -3$, $E_y^0[\exp(-W(y_T^*))] = \infty$ when T is large enough ($T \geq T_1$). By (2.5) this is equivalent to $\tilde{E}x_T^{*\gamma} = \infty$ for $T \geq T_1$. Here x_t^* is the solution to (2.1) when $u_t = u_t^* = u^*(y_t)$ where

$$dy_t = -cy_t\,dt + \sigma\,d\tilde{w}_t.$$

However, we will show that Λ is still the minimal cost for the control problem. The verification theorem stated in the following indicates that the optimal policy may not exist for $\gamma < -3$. The results also show that $u^*(y)$ can be used to construct a nearly optimal policy if this happens.

THEOREM 6.1 (Verification theorem). *Let* $U = (-\infty, \infty)$, $\gamma < 0; \Lambda, W(y)$ *and* $u^*(y)$ *be defined by* (3.3) *and* (3.4); $u_t^* = u^*(y_t^*)$ *with* y_t^* *defined by* (3.5).

(a) *For every admissible control process* u_t,

$$\Lambda \leq \liminf_{T\to\infty} \frac{1}{T}\log E_y \exp\int_0^T l(y_t, u_t)\,dt.$$

(b) *If* $0 > \gamma > -3$, *then*

$$\Lambda = \lim_{T\to\infty} \frac{1}{T}\log E_y \exp\int_0^T l(y_t^*, u_t^*)\,dt.$$

(c) *If* $\gamma \le -3$, *there exist* $u^{(n)}(y)$, $n = 1, 2, 3, \dots$ *such that*

$$\Lambda = \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^{(n)}, u_t^{(n)})\, dt.$$

We may take $u^{(n)}(y) = \varphi^{(n)}(y)u^*(y)$ *with* $\varphi^{(n)}(y) = \varphi((1/n)y)$, $\varphi(y)$ *is smooth and has compact support*, $0 \le \varphi \le 1$, $\varphi(y) = 1$ *in a neighborhood of* 0. *Alternatively, we may take* $u^{(n)}(y) = \min\{n, \max\{-n, u^*(y)\}\}$.

Note that in (c), the order of limits as $n \to \infty$ and $T \to \infty$ cannot be reversed. Indeed, if $\gamma < -3$, for fixed $T \ge T_1$ the limit as $n \to \infty$ is $+\infty$.

PROOF. For (a), the proof is similar to that of Theorem 3.1(a); just change max to min and reverse inequalities to get

$$E_y\left[\psi(y_T) \exp \int_0^T \tilde{l}(y_t, u_t)\, dt\right] \ge \psi(y).$$

Since $\psi(y) \le \exp(K)$ for some constant K,

$$\exp(K) \exp(-\Lambda T) E_y \exp \int_0^T l(y_t, u_t)\, dt \ge \psi(y).$$

Then (a) follows from this.

For (b), no change in the proof is needed.

Now to prove (c). We take $u^{(n)}(y) = \varphi^{(n)}(y)u^*(y)$ with $\varphi^{(n)}(y)$ given above. The proof works for other cases as well. We know that Λ, $W(y)$ defined in (3.2) and (3.3) satisfy, by (6.1),

$$\Lambda = \frac{\sigma^2}{2} W_{yy}(y) + \frac{\sigma^2}{2} W_y^2(y) + (-cy + \gamma\sigma^2 u^*(y)) W_y(y) + l^{u^*}(y),$$

$$l^u(y) = -\frac{a}{2} u^2 + \gamma u(b - cy) + \gamma r.$$

Let

$$\varphi^*(y) = \exp(W(y)).$$

Then φ^* is the function constructed as follows. Denote

$$L^*\varphi(y) = \frac{\sigma^2}{2} \varphi_{yy}(y) + (-cy + \gamma\sigma^2 u^*(y))\varphi_y(y) + l^{u^*}(y)\varphi(y).$$

Then $L^*\varphi^* = \Lambda\varphi^*$. Let

$$G(y) = \frac{1}{\sigma^2} \int_0^y (-cv + \gamma\sigma^2 u^*(v))\, dv.$$

The transformation

$$\varphi \to f = \varphi e^G$$

induces $\hat{L}^* f$ by the rule

$$\hat{L}^* f = L^* \varphi e^G.$$

By an easy calculation,

$$\hat{L}^* f(y) = \frac{\sigma^2}{2} f_{yy}(y) + \hat{l}^*(y) f(y),$$
$$\hat{l}^*(y) = -\frac{c^2}{2\sigma^2} y^2 - \frac{\sigma^2}{2}\gamma u^*(y)^2 + \gamma b u^*(y) + \gamma r - \frac{\sigma^2}{2} G_{yy}(y).$$

After simplification,

$$\hat{l}^*(y) = -\frac{c^2}{2\sigma^2}\frac{1}{1-\gamma} y^2 - \frac{bc}{2\sigma^2}\frac{\gamma}{\sqrt{1-\gamma}} y + \frac{b^2}{2\sigma^2}\gamma + \frac{c}{2}\left(1 + \frac{\gamma}{\sqrt{1-\gamma}}\right) + \gamma r.$$

Then $\hat{L}^*$ can be regarded as a self-adjoint operator on $L^2(R, dy)$ such that it has a compact resolvent. See Reed and Simon [(1978), Theorem XIII.67]. We see that Λ is precisely the principal eigenvalue of this operator and $\varphi^* e^G$ is the corresponding normalized eigenfunction.

Similarly, we consider the operator

$$L^{(n)}\varphi(y) = \frac{\sigma^2}{2}\varphi_{yy}(y) + (-cy + \gamma\sigma^2 u^{(n)}(y))\varphi_y(y) + l^{u^{(n)}}(y)\varphi(y),$$
$$G^{(n)}(y) = \frac{1}{\sigma^2}\int_0^y (-cv + \gamma\sigma^2 u^{(n)}(v))\, dv,$$

and the transformation

$$\varphi \to f = \varphi \exp(G^{(n)}),$$

which induces $\hat{L}^{(n)} f$ by the rule

$$\hat{L}^{(n)} f = L^{(n)}\varphi \exp(G^{(n)}).$$

Again, we have

$$\hat{L}^{(n)} f(y) = \frac{\sigma^2}{2} f_{yy}(y) + \hat{l}^{(n)}(y) f(y),$$
$$\hat{l}^{(n)}(y) = -\frac{c^2}{2\sigma^2} y^2 - \frac{\sigma^2}{2}\gamma u^{(n)}(y)^2 + \gamma b u^{(n)}(y) + \gamma r - \frac{\sigma^2}{2} G^{(n)}_{yy}(y).$$

We note that there are $a_1 > 0, b_1$ such that for all n,

$$\hat{l}^{(n)}(y) \le -a_1 y^2 + b_1.$$

The operator $\hat{L}^{(n)}$ is self-adjoint on $L^2(R, dy)$ and has compact resolvent. Denote $\Lambda^{(n)}$, $f^{(n)}$ the principal eigenvalue and the corresponding eigenfunction with $f^{(n)}(0) = 1$. Then it is easy to see that $f^{(n)}$, $n = 1, 2, 3, \dots$, is a compact family of functions in $L^2(R, dy)$. From this we can show that

$$\Lambda^{(n)} \to \Lambda, \qquad f^{(n)} \to f^* = \varphi^* e^G \quad \text{in } L^2(R, dy)$$

as $n \to \infty$. Since

$$\Lambda^{(n)} = \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^{(n)}, u_t^{(n)})\, dt$$

with $u_t^{(n)} = u^{(n)}(y_t^{(n)})$, (c) follows from this. □

If investment control constraints are imposed, we no longer have an explicit solution for (6.1). However, by using the relation

$$\frac{\sigma^2}{2}W_y^2 = \max_{v\in\mathbb{R}^1}\left[\sigma v W_y - \frac{1}{2}v^2\right],$$

the equation now can be interpreted as the dynamic programming equation of a stochastic differential game of the kind considered in Fleming and McEneaney (1995). Assume that $U = U_M = [0, M]$ as considered in Section 4. By Fleming and McEneaney [(1995), Theorem 7.1], there exist Λ, $W(y)$ satisfying (6.1) with $W_y(y)$ bounded. But the proof for the uniform estimates in Lemma 4.2 cannot be applied here. We should not expect Lemmas 4.1 and 4.3 to be correct. Therefore, for $\gamma < 0$, the results in Section 5 are not immediately valid. In the rest we shall show that similar results hold here by using a different argument. The analysis is given only for $U = [0, \infty)$. A similar argument can be applied to the cases $U = U_M$, $0 < M < \infty$. We leave the details of the latter cases to the interested reader. This analysis is based on carefully examining (6.1), which will provide a solution Λ, $W(y)$ with properties similar to those in Section 5. The verification theorem, as Theorem 5.1, shows that Λ is the maximal exponential growth rate for $\gamma^{-1}Ex_T^{\gamma}$. We note that for $U = U_M$, $0 < M < \infty$; this argument also gives some uniform estimates for $W = W_M$ similar to those in Lemma 4.2.

In the rest we assume $\gamma < 0$, $\quad U = [0, \infty)$.

Equation (6.1) can be written as

$$\begin{aligned}\Lambda &= \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r\\ &\quad + \min_{u\geq 0}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y)\right]\\ &= \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r\\ &\quad + \gamma(1-\gamma)F\left(\frac{1}{1-\gamma}(b - cy + \sigma^2 W_y)\right).\end{aligned}\tag{6.2}$$

Here

$$\begin{aligned}F(v) &= \max_{u\geq 0}\left[-\frac{\sigma^2}{2}u^2 + uv\right]\\ &= \begin{cases}\dfrac{1}{2\sigma^2}v^2, & \text{if } v \geq 0,\\ 0, & \text{if } v < 0.\end{cases}\end{aligned}$$

Therefore, if $b - cy + \sigma^2 W_y > 0$, then

$$\begin{aligned}\Lambda &= \frac{\sigma^2}{2}W_{yy} + \frac{\sigma^2}{2}W_y^2 - cyW_y + \gamma r\\ &\quad + \frac{1}{2\sigma^2}\frac{\gamma}{1-\gamma}(b - cy + \sigma^2 W_y)^2\end{aligned}$$

$$= \frac{\sigma^2}{2} W_{yy} + \frac{\sigma^2}{2} \frac{1}{1-\gamma} W_y^2 - \left(c \frac{1}{1-\gamma} y - \frac{\gamma}{1-\gamma} b \right) W_y$$
$$+ \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b - cy)^2 + \gamma r.$$

If $b - cy + \sigma^2 W_y \le 0$, then

$$\begin{aligned}
\Lambda &= \frac{\sigma^2}{2} W_{yy} + \frac{\sigma^2}{2} W_y^2 - cyW_y + \gamma r \\
&= \frac{\sigma^2}{2} W_{yy} + \frac{\sigma^2}{2} \frac{1}{1-\gamma} W_y^2 - \left(c \frac{1}{1-\gamma} y - \frac{\gamma}{1-\gamma} b \right) W_y + \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b-cy)^2 \\
&\quad + \gamma r - \frac{\sigma^2}{2} \frac{\gamma}{1-\gamma} W_y^2 - \frac{\gamma}{1-\gamma} (b-cy) W_y - \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b-cy)^2 \\
&= \frac{\sigma^2}{2} W_{yy} + \frac{\sigma^2}{2} \frac{1}{1-\gamma} W_y^2 - \left(c \frac{1}{1-\gamma} y - \frac{\gamma}{1-\gamma} b \right) W_y + \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b-cy)^2 \\
&\quad + \gamma r - \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b - cy + \sigma^2 W_y)^2.
\end{aligned}$$

That is,

$$\begin{aligned}
\Lambda &= \frac{\sigma^2}{2} W_{yy} + \frac{\sigma^2}{2} \frac{1}{1-\gamma} W_y^2 - \left(c \frac{1}{1-\gamma} y - \frac{\gamma}{1-\gamma} b \right) W_y + \frac{1}{2\sigma^2} \frac{\gamma}{1-\gamma} (b-cy)^2 \\
&\quad + \gamma r + \max_{u \ge 0} \left[\frac{\sigma^2}{2} \gamma(1-\gamma) u^2 + \gamma u (b - cy + \sigma^2 W_y) \right].
\end{aligned}$$

Denote

$$\tilde{W}(y) = \frac{1}{1-\gamma} W(y), \qquad \tilde{\Lambda} = \frac{1}{1-\gamma} \Lambda.$$

Then

$$\begin{aligned}
\tilde{\Lambda} &= \frac{\sigma^2}{2} \tilde{W}_{yy} + \frac{\sigma^2}{2} \tilde{W}_y^2 - \left(c \frac{1}{1-\gamma} y - \frac{\gamma}{1-\gamma} b \right) \tilde{W}_y + \frac{1}{2\sigma^2} \frac{\gamma}{(1-\gamma)^2} (b-cy)^2 \\
&\quad + \frac{\gamma r}{1-\gamma} + \max_{u \ge 0} \left[\frac{\sigma^2}{2} \gamma u^2 + \frac{\gamma}{1-\gamma} u (b - cy + \sigma^2 (1-\gamma) \tilde{W}_y) \right].
\end{aligned} \tag{6.3}$$

We further define

$$\tilde{\tilde{W}}_y = \tilde{W}_y - \alpha y - \beta, \tag{6.4}$$

with

$$\alpha = \frac{c}{\sigma^2} \frac{1}{1-\gamma} - \frac{c}{\sigma^2} \frac{1}{\sqrt{1-\gamma}}, \qquad \beta = -\frac{1}{\sigma^2} \frac{\gamma}{1-\gamma} b.$$

Note that $(1-\gamma)\alpha = A$ with A as in (3.3). Then

$$
\begin{aligned}
\tilde{\Lambda} &= \frac{\sigma^2}{2}\tilde{\tilde{W}}_{yy} + \frac{\sigma^2}{2}\tilde{\tilde{W}}_y^2 - \frac{c}{\sqrt{1-\gamma}} y\tilde{\tilde{W}}_y + \left(\frac{\sigma^2}{2}\alpha + \frac{\gamma r}{1-\gamma} + \frac{1}{2\sigma^2}b^2\frac{\gamma}{1-\gamma}\right) \\
&\quad + \max_{u\geq 0}\left[\frac{\sigma^2}{2}\gamma u^2 + \gamma u\left(b - c\frac{1}{\sqrt{1-\gamma}}y + \sigma^2\tilde{\tilde{W}}_y\right)\right].
\end{aligned}
\tag{6.5}
$$

Comparing with (5.1), we can apply the same argument as in Section 5 to obtain a solution $\tilde{\Lambda}$, $\tilde{\tilde{W}}$ for (6.5) [hence a solution Λ, W for (6.2) through relations (6.3) and (6.4)] with the following estimates:

1. $\tilde{\tilde{W}}$ is convex;
2. $\tilde{\tilde{W}} - (1/2\sigma^2)(c/\sqrt{1-\gamma})y^2$ is concave;
3. $\tilde{\tilde{W}}_y \geq 0$ for all y;
4. $\tilde{\tilde{W}}_y(y) \leq c_1$ for all $y \leq 0$ for some c_1.

The results for $W(y)$ are summarized in the following theorem.

THEOREM 6.2. *Equation* (6.2) *has a solution* Λ, W *with* $W(0) = 0$. *This solution satisfies the following properties*:

(i) $W(y) - \frac{1}{2}Ay^2$ *is convex*;
(ii) $W(y) - (c/2\sigma^2)y^2$ *is concave*;
(iii) $W_y(y) - Ay \geq B$ *for all* y [*see* (3.3) *for* B];
(iv) $W_y(y) - Ay \leq c_1$ *for all* $y \leq 0$ *for some* c_1.

Given a solution Λ, W of (6.2) in Theorem 6.2, we define $u^*(y)$ to be the $\arg\min[\cdots]$ in the right-hand side of (6.2). That is,

$$
u^*(y) = \begin{cases} 0, & \text{if } b - cy + \sigma^2 W_y(y) < 0, \\ \dfrac{1}{\sigma^2}\dfrac{1}{1-\gamma}(b - cy + \sigma^2 W_y(y)), & \text{if } b - cy + \sigma^2 W_y(y) \geq 0. \end{cases}
$$

Theorem 6.2(ii) implies that $\sigma^2 W_y(y) - cy + b$ is a nonincreasing function. The argument after (5.2) can be applied to

$$
\sigma^2\tilde{\tilde{W}}_y - \frac{c}{\sqrt{1-\gamma}}y + b = \frac{1}{1-\gamma}[\sigma^2 W_y(y) - cy + b]
$$

and shows that there is y^*, $-\infty < y^* < \infty$ such that

$$
\{y\colon \sigma^2 W_y(y) - cy + b \geq 0\} = (-\infty, y^*].
$$

Therefore,

$$
u^*(y) = \begin{cases} 0, & \text{if } y > y^*, \\ \dfrac{1}{\sigma^2}\dfrac{1}{1-\gamma}(b - cy + \sigma^2 W_y(y)), & \text{if } y \leq y^*. \end{cases}
\tag{6.6}
$$

We shall prove that Λ is the optimal growth rate. However, $u^*(\cdot)$ is an optimal control policy only when $0 > \gamma > -3$. As in Section 5, we admit only those u_t as a control policy which arise via

$$u_t = u(t, y_t)$$

with $u(t, y_t)$ being locally Lipschitz and satisfying, for suitable α_1, α_2,

$$0 \le u(t, y) < \alpha_1 |y| + \alpha_2$$

for all t and y.

THEOREM 6.3 (Verification theorem). *Let Λ, W be the solution given in Theorem 6.2 and u^* be defined in (6.6). Then the following hold.*

(a) *For every admissible control process u_t,*

$$\Lambda \le \liminf_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t, u_t)\, dt.$$

(b) *Let $0 > \gamma > -3$. If $u_t^* = u^*(y_t^*)$, then*

$$\Lambda = \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^*, u_t^*)\, dt.$$

(c) *Let $\gamma \le -3$. Then there are $u^{(n)}(y)$ such that*

$$\Lambda = \lim_{n\to\infty} \lim_{T\to\infty} \frac{1}{T} \log E_y \exp \int_0^T l(y_t^{(n)}, u^{(n)}(y_t^{(n)})\, dt.$$

Here $u^{(n)}(y)$ can be chosen as in Theorem 6.1(c) by using $u^*(y)$.

PROOF. The argument is similar to that in Theorem 5.1. We give only a sketch.

Let

$$H_*(y) = \gamma r + \min_{u\ge 0}\left[-\frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y(y))\right],$$

$$h_*(y, u) = \gamma r - \frac{a}{2}u^2 + \gamma u(b - cy + \sigma^2 W_y(y)) - H_*(y).$$

As in (5.8), we have

$$\begin{aligned} &E_y \exp \int_0^T l(y_t, u_t)\, dt \\ &\quad = \exp(\Lambda T)\psi(y)\tilde{E}_y\left[\exp \zeta_T \exp \int_0^T h_*(y_t, u_t)\, dt \psi^{-1}(y_T)\right], \end{aligned} \tag{6.7}$$

See the notation there.

By (6.1) with $U = [0, \infty)$ and the fact that the min is 0 when $y \ge y^*$, then using the argument after (5.5) we can show

$$\frac{W_y(y)}{y} \to 0 \quad \text{as } y \to \infty. \tag{6.8}$$

On the other hand, Theorem 6.2(iii) and 6.2(iv) imply

$$\frac{W_y(y)}{y} \to A \quad \text{as } y \to -\infty. \tag{6.9}$$

Equation (6.8) implies that for any $\delta > 0$ there is b such that

$$|W(y)| \le \delta y^2 + b, \qquad y \ge 0.$$

Then (6.9) implies that for any $a_1, a_2 > 0$ with $-2a_1 < (1-\gamma)\alpha < -2a_2$ there are b_1, b_2 such that

$$-a_1 y^2 + b_1 \le W(y) \le -a_2 y^2 + b_2 \quad \text{for all } y < 0.$$

Therefore,

$$\psi^{-1}(y) = \exp(-W(y)) \ge c_1 \exp(-\delta |y|^2) \quad \text{for all } y$$

for some $c_1 > 0$.

We use Hölder's inequality with $p^{-1} + q^{-1} = 1$ and p sufficiently large,

$$\tilde{E}_y \psi^{-1/p}(y_T) \le \left(\tilde{E}_y\left[\exp \zeta_T \exp \int_0^T h_*(y_t, u_t)\, dt \psi^{-1}(y_T)\right]\right)^{1/p}$$
$$\times \left(\tilde{E}_y\left[\exp -\frac{q}{p}\zeta_T \ \exp -\frac{q}{p}\int_0^T h_*(y_t, u_t)\, dt\right]\right)^{1/q}.$$

Similar to $h^*(y,u)$ in the proof of Theorem 5.1, we have

$$h_*(y,u) \ge -\frac{a}{2}(u - u^*(y))^2 = -\frac{\sigma^2\gamma(1-\gamma)}{2}(u-u^*)^2.$$

Then

$$\tilde{E}_y\left[\exp -\frac{q}{p}\zeta_T \ \exp -\frac{q}{p}\int_0^T h_*(y_t, u_t)\, dt\right] \le 1$$

by using

$$\tilde{E}_y\left[\exp \int_0^T -\frac{q}{p}\gamma\sigma(u_t - u_t^*)\, d\tilde{w}_t - \frac{1}{2}\frac{q^2}{p^2}\gamma^2\sigma^2(u_t - u_t^*)^2\, dt\right] \le 1$$

and

$$\frac{1}{2}\gamma^2\sigma^2\left(\frac{q}{p} + \frac{q^2}{p^2}\right) - \frac{\sigma^2\gamma(1-\gamma)}{2}\frac{q}{p} \le 0$$

when qp^{-1} is small enough. On the other hand, by the argument at the end of the proof, $\tilde{E}_y \psi^{-1/p}(y_T)$ converges as T tends to ∞. Then (a) follows from these.

To prove (b), using the relations (6.7) for $u_t = u_t^* = u^*(y_t^*)$, $y_t = y_t^*$, we have

$$E_y \exp \int_0^T l(y_t^*, u_t^*)\, dt = \exp(\Lambda T)\psi(y) E_y \psi^{-1}(y_T^*). \tag{6.10}$$

We will show that y_t^* has the invariant measure $\mu^*(\cdot)$ and $E_y\psi^{-1}(y_T)$ converges to $\int \psi^{-1}(y)\,d\mu^*(y)$. Thus (b) follows.

The drift for the diffusion y_t^* is given by

$$b^*(y) = -cy + \gamma\sigma^2 u^*(y) + \sigma^2 W_y(y).$$

If $y \geq y^*$, then

$$\begin{aligned} b^*(y) &= -cy + \sigma^2 W_y(y), \\ \frac{b^*(y)}{y} &\to -c \quad \text{as } y \to \infty \end{aligned} \tag{6.11}$$

by (6.8). If $y < y^*$, then

$$\begin{aligned} b^*(y) &= \frac{1}{1-\gamma}(\sigma^2 W_y(y) - cy) + \frac{\gamma}{1-\gamma} b, \\ \frac{b^*(y)}{y} &\to -\frac{c}{\sqrt{1-\gamma}} \quad \text{as } y \to -\infty \end{aligned} \tag{6.12}$$

by (6.9).

From (6.11) and (6.12), the process y_t^* is ergodic and has unique invariant measure $\mu^*(\cdot)$. Let θ be a number satisfying

$$\theta < \frac{c}{\sqrt{1-\gamma}} \frac{1}{\sigma^2}.$$

Then by Itô's rule applied to the function $f(y) = \exp\theta y^2$ and using the properties (6.11) and (6.12), we can show that $E_y f(y_T^*)$ is bounded in T and converges to $\int f(y) d\mu^*(y)$. The same holds for $E_y\psi^{-1}(y_T^*)$.

The statement (c) can be proved as Theorem 6.1(c) by using the properties (6.8) and (6.9) and the results in Theorem 6.2. We omit the details. □

REMARK 6.4. As $\gamma \to -\infty$, one is concerned with totally risk-averse limits. Let us consider the unconstrained case $U = (-\infty, \infty)$ and write $\Lambda^{(\gamma)} = \gamma^{-1}\Lambda$, $u^* = u^{*(\gamma)}$. By (3.3)(c) and (3.4), as $\gamma \to -\infty$,

$$\Lambda^{(\gamma)} \to r + \frac{b^2}{2\sigma^2},\ u^{*(\gamma)} \to \frac{b}{\sigma^2}.$$

A direct calculation shows that the constant control $\bar{u} = \sigma^{-2}b$ gives the growth rate $\gamma(r + (2\sigma^2)^{-1}b^2)$ for the expected utility of wealth in (2.5). Thus, $\bar{u}$ is approximately optimal for $\gamma < 0$, $|\gamma|$ large. This is a different result from the Merton model, for which the optimal fraction $u_m^{*(\gamma)}$ of wealth in the risky asset tends to 0 as $\gamma \to -\infty$.

7. Limit as $\gamma \to 0$. Let Λ, $W(y)$, $u^*(y)$ be defined either in Sections 3, 4 or 5. It is convenient to introduce the following notations. To indicate dependence on γ, we write $\Lambda = \tilde{\Lambda}^{(\gamma)}$, $W = \tilde{W}^{(\gamma)}$, $u^* = u^{*(\gamma)}$ and $l = \tilde{l}^{(\gamma)}$. Moreover, let

$$\tilde{\Lambda}^{(\gamma)} = \gamma \Lambda^{(\gamma)},$$
$$\tilde{W}^{(\gamma)}(y) = \gamma W^{(\gamma)}(y),$$
$$\tilde{l}^{(\gamma)}(y, u) = \gamma l^{(\gamma)}(y, u).$$

We consider the limiting behavior of $\Lambda^{(\gamma)}$, $W^{(\gamma)}$ and $u^{*(\gamma)}$ as $\gamma \to 0$.

For the unconstrained case, by (3.3)(c),

$$\Lambda^{(\gamma)} \to \Lambda^{(0)} = \frac{c}{4} + r + \frac{b^2}{2\sigma^2},$$
$$W^{(\gamma)}(y) \to W(y)^{(0)} = \frac{c}{4\sigma^2} y^2 - \frac{b}{\sigma^2} y,$$
$$u^{*(\gamma)}(y) \to u^{*(0)}(y) = \frac{b - cy}{\sigma^2}$$

as $\gamma \to 0$ with $\gamma > 0$. Similar result holds for $\gamma \to 0$, $\gamma < 0$. We see that $\Lambda^{(0)}$, $u^{*(0)}(y)$ are the optimal growth rate and optimal investment policy for the investment problem with log utility function. Moreover,

$$\Lambda^{(0)} = \frac{\sigma^2}{2} W^{(0)}_{yy} - cyW^{(0)}_y + r + \frac{1}{2\sigma^2}(b - cy)^2.$$

We remark that it is easy to get the optimal policy directly for the investment problem with log utility function. In this case, the control u_t disappears from (2.9) and in (2.10) $l(y, u)$ is replaced by

$$l^{(0)}(y, u) = -\frac{\sigma^2}{2} u^2 + (b - cy)u + r,$$

optimal policy is to take $\arg\max_u$ of $l^{(0)}(y, u)$, which gives $u^{*(0)}(y)$ (if there are investment control constraints, we take $\arg\max_{u \in U}$).

This result also holds for the cases with other constraints. However, the proof will require more argument. For example, (5.2)(c) or Theorem 6.1 says that in general $|W^{(\gamma)}_y(y)|$ is bounded by $|\gamma|^{-1}$. This is not good when $\gamma \to 0$. We expect a bound independent of γ. See (7.7) below which provides an interesting bound for $W^{(\gamma)}_y(y)$. In the following, we shall consider only the case with no short-selling constraint studied in Section 5 and Section 6 [$U = [0, \infty)$].

Denote

$$F(v) = \max_{u \geq 0} \left[-\frac{\sigma^2}{2} u^2 + uv \right].$$

Then

$$F(v) = \begin{cases} \dfrac{1}{2\sigma^2} v^2, & \text{if } v > 0, \\ 0, & \text{if } v \leq 0. \end{cases}$$

By (5.1) and (6.2), the following equation holds for all γ:

$$\begin{aligned}\Lambda^{(\gamma)} &= \frac{\sigma^2}{2} W_{yy}^{(\gamma)} + \frac{\sigma^2}{2}\gamma (W_y^{(\gamma)})^2 - cyW_y^{(\gamma)} \\ &\quad + r + (1-\gamma)F\left(\frac{1}{1-\gamma}(b - cy + \sigma^2\gamma W_y^{(\gamma)})\right).\end{aligned} \tag{7.1}$$

In the rest we shall assume $0 < \gamma < 1$. The argument for $\gamma < 0$ is similar. To see that $\Lambda^{(\gamma)} \to \Lambda^{(0)}$, first by Jensen's inequality,

$$\frac{1}{\gamma}\log E_y \exp \int_0^T l(y_t, u_t)\, dt \geq \frac{1}{\gamma} E_y \int_0^T l(y_t, u_t)\, dt = E_y \int_0^T l^{(\gamma)}(y_t, u_t)\, dt$$

holds for any T and *admissible* control u, process y_t in (2.9). Since

$$l^{(\gamma)}(y,u) \geq l^{(0)}(y,u),$$

we have $\Lambda^{(\gamma)} \geq \Lambda^{(0)}$. On the other hand, with $u_t = u_t^* = u^*(y_t^*)$, u^* as in (5.3), we have the expansion

$$\begin{aligned}&\frac{1}{\gamma}\log E_y \exp \int_0^T l(y_t^*, u_t^*)\, dt \\ &\qquad = \int_0^T l^{(\gamma)}(y_t^*, u_t^*)\, dt + \gamma \operatorname{var}_y\left(\int_0^T l^{(\gamma)}(y_t^*, u_t^*)\, dt\right) + \cdots.\end{aligned}$$

Here $\operatorname{var}_y(\cdots)$ denote the variance of a random variable with respect to P_y. The last term should be negligible if a bound for $W_y^{(\gamma)}$ such as (7.7) is known. Since $\Lambda^{(\gamma)}$ is the limit of the left-hand side divided by T as $T \to \infty$, this would imply

$$\limsup_{\gamma \to 0} \Lambda^{(\gamma)} \leq \Lambda^{(0)}.$$

Then

$$\lim_{\gamma \to 0} \Lambda^{(\gamma)} = \Lambda^{(0)}.$$

In the following, we shall give more details of the analysis. We can obtain the finer asymptotics for $\Lambda^{(\gamma)}$ and also prove the convergence of $W^{(\gamma)}$.

By dividing (4.1) by γ and using (3.3)(c), we have

$$r \leq \Lambda^{(\gamma)} \leq \frac{c}{2\gamma}(1 - (1-\gamma)^{1/2}) + r + \frac{b^2}{2\sigma^2}. \tag{7.2}$$

Then use (5.2), $F \geq 0$ and (7.1) to get

$$|W_y^{(\gamma)}(y)| \leq c_1 \frac{1}{y}, \qquad y > 0 \tag{7.3}$$

for some $c_1 > 0$.

To estimate $W_y^{(\gamma)}$ for $y < 0$, we use $W_{yy}^{(\gamma)} \geq 0$ [see (5.2)(c)], $F \geq 0$ and (7.1) to get

$$\frac{\sigma^2}{2} W_{yy}^{(\gamma)} - cyW_y^{(\gamma)} \leq c_2,$$

$c_2 = (c/2\gamma)(1-(1-\gamma)^{1/2}) + r + (b^2/2\sigma^2)$. We integrate this equation from y to ∞,

$$-W_y^{(\gamma)}(y)\exp\left(-\frac{c}{\sigma^2}y^2\right) \leq c_2 \int_y^\infty \exp\left(-\frac{c}{\sigma^2}x^2\right)dx \leq c_3,$$

with $c_3 = c_2\sigma\sqrt{\pi/c}$. Together with (5.2)(a), we have

$$|W_y^{(\gamma)}(y)| \leq c_3 \exp\left(\frac{c}{\sigma^2}y^2\right) \tag{7.4}$$

for all y.

We need to improve the estimate in (7.4). We consider $\tilde{\Lambda}^{(\gamma)}$, $\tilde{W}^{(\gamma)}(y)$ instead of $\Lambda^{(\gamma)}$, $W^{(\gamma)}(y)$. Denote

$$V^{(\gamma)}(y) = (1-\gamma)F\left(\frac{1}{1-\gamma}(b - cy + \sigma^2 \tilde{W}_y^{(\gamma)})\right) + r - \Lambda^{(\gamma)}.$$

Then

$$\begin{aligned} &\frac{\sigma^2}{2}\tilde{W}_{yy}^{(\gamma)} + \frac{\sigma^2}{2}(\tilde{W}_y^{(\gamma)})^2 - cy\tilde{W}_y^{(\gamma)} + \gamma V^{(\gamma)}(y) = 0, \\ &\frac{\sigma^2}{2}\left(\tilde{W}_y^{(\gamma)}(y) - \frac{c}{\sigma^2}y\right)^2 = \frac{c^2}{2\sigma^2}y^2 - \frac{\sigma^2}{2}\tilde{W}_{yy}^{(\gamma)} - \gamma V^{(\gamma)}(y). \end{aligned} \tag{7.5}$$

By (5.2),

$$|V^{(\gamma)}(y)| \leq c_1(1+|y|^2), \qquad 0 \leq \tilde{W}_{yy}^{(\gamma)} \leq \frac{c}{\sigma^2}$$

for some c_1. Let γ be small and $y < 0$, $|y|$ be large enough. Here (7.5) implies

$$\begin{aligned} &\frac{\sigma}{\sqrt{2}}\left(\tilde{W}_y^{(\gamma)}(y) - \frac{c}{\sigma^2}y\right) \\ &\quad = \frac{\pm 1}{\sqrt{2}}\frac{c}{\sigma}y\left(1 + \frac{2\sigma^2}{c^2y^2}\left(-\frac{\sigma^2}{2}\tilde{W}_{yy}^{(\gamma)}(y) - \gamma V^{(\gamma)}(y)\right)\right)^{1/2}. \end{aligned} \tag{7.6}$$

We cannot have a + sign in (7.6). Otherwise,

$$\frac{\sigma}{\sqrt{2}}\tilde{W}_y^{(\gamma)}(y) = \sqrt{2}\frac{c}{\sigma}y + \gamma\ O(|y|) + O\left(\frac{1}{|y|}\right).$$

Then $W_y^{(\gamma)}(y) = \gamma^{-1}\tilde{W}_y^{(\gamma)}(y)$ cannot be bounded in γ as $\gamma \to 0$. See (7.4). Therefore,

$$\begin{aligned}
&\frac{\sigma}{\sqrt{2}}\left(\tilde{W}_y^{(\gamma)}(y) - \frac{c}{\sigma^2}y\right)\\
&\quad = \frac{-1}{\sqrt{2}}\frac{c}{\sigma}y\left(1 - \frac{2\sigma^2}{c^2y^2}\left(-\frac{\sigma^2}{2}\tilde{W}_{yy}^{(\gamma)}(y) - \gamma V^{(\gamma)}(y)\right)\right)^{1/2}\\
&\quad = \frac{-1}{\sqrt{2}}\frac{c}{\sigma}y + \tilde{W}_{yy}^{(\gamma)}\left[\frac{\sigma^3}{2\sqrt{2}c}\frac{1}{y} + O\left(\frac{1}{|y|^3}\right) + \gamma\, O\left(\frac{1}{|y|}\right)\right] + O(\gamma|y|).
\end{aligned}$$

The last step is by expansion,

$$(1+\alpha)^{1/2} = 1 + \tfrac{1}{2}\alpha + O(\alpha^2)$$

with

$$\alpha = -\frac{2\sigma^2}{c^2y^2}\left(-\frac{\sigma^2}{2}\tilde{W}_{yy}^{(\gamma)}(y) - \gamma V^{(\gamma)}(y)\right).$$

Then we can write

$$\tilde{W}_{yy}^{(\gamma)}(y) - f^{(\gamma)}(y)\tilde{W}_y^{(\gamma)}(y) = g^{(\gamma)}(y),$$

where

$$f^{(\gamma)}(y) = 2\frac{c}{\sigma^2}y\left[1 + O\left(\frac{1}{y^2}\right) + O(\gamma)\right], \qquad g^{(\gamma)}(y) = O(\gamma y^2).$$

By integrating this equation from $-\infty$ to $y < 0$, we have

$$\tilde{W}_y^{(\gamma)}(y)\exp(-h^{(\gamma)}(y)) = \gamma\, O\left(\int_{-\infty}^{y} x^2 \exp(-h^{(\gamma)}(x))\,dx\right).$$

Here $(d/dy)h^{(\gamma)}(y) = f^{(\gamma)}(y)$,

$$h^{(\gamma)}(y) = \frac{c}{\sigma^2}y^2\left(1 + O\left(\frac{1}{|y|}\right) + O(\gamma)\right).$$

From this, (7.3) and (7.4), we can deduce

$$|\tilde{W}_y^{(\gamma)}(y)| \le c_1\gamma(1+|y|)$$

for some $c_1 > 0$. This is equivalent to

$$|W_y^{(\gamma)}(y)| \le c_1(1+|y|) \quad \text{for all } y. \tag{7.7}$$

This is an improvement of the estimate in (7.4) that we need. These estimates imply the following result.

THEOREM 7.1. *Consider the case with no short-selling constraint studied in Section* 5. *Then*

$$\Lambda^{(\gamma)} \to \Lambda^{(0)}, \qquad W^{(\gamma)}(y) \to W^{(0)}(y)$$

uniformly for y *in compact sets as* $\gamma \to 0$. *We have the equation*

$$\Lambda^{(0)} = \frac{\sigma^2}{2} W^{(0)}_{yy} - cyW^{(0)}_y + r + F(b - cy). \tag{7.8}$$

$\Lambda^{(0)}$ *is the long term optimal growth rate for the investment problem with* log *utility function*. *Moreover*,

$$\text{(7.9)(a)} \qquad 0 \le W^{(0)}_{yy},$$

$$\text{(7.9)(b)} \qquad 0 \le -W^{(0)}_y(y) \le c_1(1 + |y|) \quad \textit{for all } y,$$

$$0 \le -W^{(0)}_y(y) \le c_1 \frac{1}{y} \quad \textit{if } y > 0.$$

As $\gamma \to 0$, *we have the expansion*

$$\begin{aligned} \Lambda^{(\gamma)} &= r + \sqrt{\frac{c}{\pi}} \frac{1}{\sigma} \int_{-\infty}^{\infty} F(b - cy) \exp\left(-\frac{c}{\sigma^2} y^2\right) dy \\ &\quad + \gamma\left(\frac{\sigma^2}{2} \sqrt{\frac{c}{\pi}} \frac{1}{\sigma} \int_{-\infty}^{\infty} (W^{(0)}_y(y))^2 \exp\left(-\frac{c}{\sigma^2} y^2\right) dy \right.\\ &\quad + \sqrt{\frac{c}{\pi}} \frac{1}{\sigma} \int_{-\infty}^{\infty} (-F(b - cy) + F'(b - cy) \\ &\quad \left. \times (b - cy + \sigma^2 W^{(0)}_y(y))) \exp\left(-\frac{c}{\sigma^2} y^2\right) dy \right) + o(\gamma) \end{aligned} \tag{7.10}$$

PROOF. By (7.2), (7.3) and (7.7), $\Lambda^{(\gamma)}, W^{(\gamma)}$ converges through a subsequence to $\Lambda^{(0)}, W^{(0)}$. It is clear that (7.8) holds.

Integrating (7.8) from $-\infty$ to ∞,

$$\Lambda^{(0)} = r + \frac{\sqrt{c}}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} F(b - cy) \exp\left(-\frac{c}{\sigma^2} y^2\right) dy.$$

Then integrate (7.8) again to get

$$\begin{aligned} -\frac{\sigma^2}{2} W^{(0)}_y(y) \exp\left(-\frac{c}{\sigma^2} y^2\right) &= (\Lambda^{(0)} - r) \int_y^{\infty} \exp\left(-\frac{c}{\sigma^2} x^2\right) dx \\ &\quad - \int_y^{\infty} F(b - cx) \exp\left(-\frac{c}{\sigma^2} x^2\right) dx. \end{aligned}$$

Therefore, $\Lambda^{(0)}, W^{(0)}$ are uniquely determined from (7.8) and (7.9). This also shows the convergence of $\Lambda^{(\gamma)}, W^{(\gamma)}$ as $\gamma \to 0$.

We know (7.8) is the same as (2.14) for $U = [0, \infty)$, which is the dynamic programming equation for the investment problem with log utility function.

A verification theorem (similar to Theorem 3.1) shows that Λ^0 is the optimal growth rate for this investment problem.

Properties in (7.9) are consequences of (5.2), (7.3) and (7.7).

We now prove (7.10). By integrating (7.1) from $-\infty$ to ∞,

$$\Lambda^{(\gamma)} = r + \frac{\sqrt{c}}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} \frac{\sigma^2}{2}\gamma(W_y^{(\gamma)}(y))^2 \exp\left(-\frac{c}{\sigma^2}y^2\right) dy$$

$$+ \frac{\sqrt{c}}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} (1-\gamma)F\left(\frac{1}{1-\gamma}(b - cy + \sigma^2\gamma W_y^{(\gamma)}(y))\right) \exp\left(-\frac{c}{\sigma^2}y^2\right) dy.$$

Equation (7.10) follows from the expansion of this at $\gamma = 0$. This completes the proof. □

APPENDIX

In this Appendix, we add a few remarks about the behavior of W and Λ for the case considered in Section 5. These are stated in the following.

THEOREM A.1. *Assume* $0 < \gamma < 1$, $U = [0, \infty)$. *Then we have* $\Lambda \geq \gamma(r + (2\sigma^2)^{-1}b^2)$ *if* $b > 0$. *In general, we have* $\Lambda > \gamma r$. *Moreover, we have the following upper bound for* Λ:

(a) *If* $b \leq 0$, *then* $\Lambda \leq \gamma r + c/2$.
(b) *If* $b > 0$, *then* $\Lambda \leq \gamma r + (c/2) + (\gamma b^2/2\sigma^2)$.

THEOREM A.2. *Assume* $0 < \gamma < 1$, $U = [0, \infty)$. *When* $y \to \infty$,

$$W(y) = -\frac{\Lambda - \gamma r}{c}\ln y + O(1).$$

PROOF OF THEOREM A.1. We first derive the upper bound for Λ in (a) and (b). Note that (b) follows from the fact that (3.3)(c) gives an upper bound for Λ for any b, in particular for $b > 0$. We now assume that $b \leq 0$ and prove (a).

Let $u_{\bullet}$ be an admissible control process in Theorem 5.1, y_t be the process defined by (2.9). Then by Itô's differential rule,

$$dy_t^2 = (-2cy_t^2 + 2\gamma\sigma^2 u_t y_t + \sigma^2)\,dt + 2\sigma y_t\,dw_t.$$

Then

$$-\gamma c\int_0^T u_t y_t\,dt = \frac{c}{2\sigma^2}\left[-y_T^2 + y_0^2 + \sigma^2 T - 2c\int_0^T y_t^2\,dt + 2\sigma\int_0^T y_t\,dw_t\right],$$

$$\int_0^T l(y_t, u_t)\,dt = \frac{c}{2\sigma^2}[-y_T^2 + y_0^2] + \left(\gamma r + \frac{c}{2}\right)T$$

$$+ \int_0^T \left(-\frac{a}{2}u_t^2 - \frac{c^2}{2\sigma^2}y_t^2 + \gamma b u_t\right) dt$$

$$+ \frac{c}{\sigma}\int_0^T y_t\,dw_t - \frac{c^2}{2\sigma^2}\int_0^T y_t^2\,dt,$$

$$E \exp \int_0^T l(y_t, u_t)\, dt = \hat{E}\biggl[\exp\biggl(\frac{c}{2\sigma^2}[-y_T^2 + y_0^2] + \Bigl(\gamma r + \frac{c}{2}\Bigr)T + \int_0^T \Bigl(-\frac{a}{2}u_t^2 - \frac{c^2}{2\sigma^2}y_t^2 + \gamma b u_t\Bigr)\, dt\biggr)\biggr].$$

Here $\hat{E}[\cdots]$ is the expectation with respect to the probability measure $\hat{P}$ under which

$$dy_t = \gamma\sigma^2 u_t\, dt + \sigma\, d\hat{w}_t,$$

$\hat{w}_t$ is a Brownian motion. We change the probability measure by applying the Girsanov theorem. Since $b \le 0$, the term $\int_0^T (\cdots)\, dt$ on the right-hand side is negative, the last expectation has an upper bound

$$\exp\Bigl(\gamma r + \frac{c}{2}\Bigr)T \exp \frac{c}{2\sigma^2} y_0^2.$$

From this,

$$\Lambda \le \gamma r + \frac{c}{2}$$

as asserted.

If $b > 0$, then $\Lambda \ge \gamma(r + (2\sigma^2)^{-1}b^2)$ by Remark 6.4. In the rest, we prove $\Lambda > \gamma r$ holds in general. For this, we choose a particular $u = u(y)$ as follows. Let

$$u(y) = \begin{cases} \dfrac{\gamma}{a}(b - cy), & \text{if } b - cy \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Denote

$$\hat{b}(y) = -cy + \gamma\sigma^2 u(y), \qquad \hat{l}(y) = l(y, u(y)).$$

That is,

$$\hat{b}(y) = \begin{cases} -\dfrac{c}{1-\gamma}y + \dfrac{\gamma}{1-\gamma}b, & \text{if } b - cy \ge 0, \\ -cy, & \text{if } b - cy < 0, \end{cases}$$

$$\hat{l}(y) = \begin{cases} \gamma r + \dfrac{\gamma^2}{2a}(b - cy)^2, & \text{if } b - cy \ge 0, \\ \gamma r, & \text{if } b - cy < 0. \end{cases}$$

We denote $\hat{y}_t$ the diffusion satisfying the equation

$$d\hat{y}_t = \hat{b}(\hat{y}_t)\, dt + \sigma\, dw_t.$$

Then $\hat{y}_t$ is ergodic with the invariant density $\hat{p}(\cdot)$,

$$\hat{p}(y) = \begin{cases} \alpha_1 \exp\left(-\dfrac{c}{\sigma^2}\dfrac{1}{1-\gamma}y^2 + \dfrac{2b}{\sigma^2}\dfrac{\gamma}{1-\gamma}y\right), & \text{if } b - cy \geq 0, \\ \alpha_2 \exp\left(-\dfrac{c}{\sigma^2}y^2\right), & \text{if } b - cy < 0, \end{cases}$$

where α_1, α_2 are chosen such that $\hat{p}(\cdot)$ is a continuous function and satisfies $\int_{-\infty}^{\infty} \hat{p}(y)\,dy = 1$. That is, α_1, α_2 satisfy the relation

$$\alpha_2 = \alpha_1 \exp \frac{b^2}{c\sigma^2}\frac{\gamma}{1-\gamma},$$

$$1 = \alpha_1 \int_{-\infty}^{b/c} \exp\left(-\frac{c}{\sigma^2}\frac{1}{1-\gamma}y^2 + \frac{2b}{\sigma^2}\frac{\gamma}{1-\gamma}y\right) dy + \alpha_2 \int_{b/c}^{\infty} \exp\left(-\frac{c}{\sigma^2}y^2\right) dy,$$

which determine α_1, α_2 uniquely.

By Jensen's inequality,

$$E \exp \int_0^T \hat{l}(\hat{y}_t)\,dt \geq \exp\left(E \int_0^T \hat{l}(\hat{y}_t)\,dt\right).$$

By the ergodic theorem,

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} E \int_0^T \hat{l}(\hat{y}_t)\,dt &= \int_{-\infty}^{\infty} \hat{l}(y)\hat{p}(y)\,dy \\ &= \gamma r + \int_{-\infty}^{b/c} \frac{\gamma^2}{2a}(b-cy)^2 \hat{p}(y)\,dy. \end{aligned}$$

From Theorem 5.1(a),

$$\Lambda \geq \limsup_{T\to\infty} \frac{1}{T} E \int_0^T \hat{l}(\hat{y}_t)\,dt = \gamma r + \int_{-\infty}^{b/c} \frac{\gamma^2}{2a}(b-cy)^2 \hat{p}(y)\,dy. \tag{A.1}$$

This implies $\Lambda > \gamma r$. $\square$

We remark that, although (A.1) provides a lower bound for Λ, this is smaller compared with the upper bound in Theorem A.1(a) and (b). We are not sure which gives a better estimate for Λ.

PROOF OF THEOREM A.2. By (5.1), for some $c_1 > 0$, we have

$$y|W_y(y)| \leq c_1 \quad \text{if } y \geq 0.$$

Therefore,

$$\begin{aligned} \Lambda - \gamma r &= \frac{\sigma^2}{2}W_{yy} - cyW_y + \frac{\sigma^2}{2}W_y^2 \\ &= \frac{\sigma^2}{2}W_{yy} - cyW_y + O\left(\frac{1}{y^2}\right). \end{aligned}$$

We integrate this relation to get

$$-W_y(y)\exp\left(-\frac{c}{\sigma^2}y^2\right) = 2\frac{\Lambda-\gamma r}{\sigma^2}\int_y^\infty x\exp\left(-\frac{c}{\sigma^2}x^2\right)dx + O\left(\int_y^\infty \frac{1}{x^2}\exp\left(-\frac{c}{\sigma^2}x^2\right)dx\right)$$

for $y > 0$. From this,

$$W_y(y) = -\frac{1}{c}(\Lambda-\gamma r)\frac{1}{y} + O\left(\frac{1}{y^2}\right),$$

then

$$W(y) = -\frac{1}{c}(\Lambda-\gamma r)\ln y + O(1)$$

as $y \to \infty$. This completes the proof. □

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DIVISION OF APPLIED MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND 02912

INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
NANKANG, TAIPEI
TAIWAN, ROC