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LARGE DEVIATIONS AND YOUNG MEASURES FOR A POISSONIAN MODEL OF BIPHASED MATERIAL

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We prove a large deviations principle for the Young measures of a stochastic homogenization model of Poissonian biphased material.

0. Introduction.

0.1. *Stochastic homogenization.* Assume that a nonhomogeneous material occupies a domain D of $\mathbf{R}^d$ and that its properties at point x are described by a parameter $a(x)$. The complete determination of the function a: $D \to \mathbf{R}$ is equivalent to a microscopic description of the material. However, if the scale of the inhomogeneities is small, the function a is highly irregular and such a description is impossible to get. It is at the same time irrelevant if one is interested only in the macroscopic properties of the material. To get round this difficulty, one replaces a by a well-chosen family of random functions,

$$a_\varepsilon\colon \Omega \times D \to \mathbf{R}, \qquad (\omega, x) \mapsto a_\varepsilon(\omega, x),$$

where Ω is a probability space endowed with the probability measure P. Here, $\omega \in \Omega$ represents the randomness of a_ε, that is, of the material, and ε is the typical scale of the irregularities of this material. When ε goes to zero, one hopes that the behavior of the random material $a_\varepsilon(\omega, \cdot)$ converges, in a sense which has to be precised, to the behavior of the actual material. This "mean" description, called stochastic homogenization, is the subject of an extensive physical and mathematical literature; see Kozlov (1980), Yurinskiĭ (1991) and Jikov, Kozlov and Oleinik (1994), for instance. In the model studied below, the convergence is easy to establish and we describe the speed of this convergence.

0.2. *Young measures.* One is often led to describe nonlinear functionals of the functions a_ε and of their limit. To this end, an efficient approach is to study, instead of the functions a_ε, their Young measures ν_ε. Following Young (1942) and others, this formalism is developped by Michel and Robert (1994) in the context of the thermodynamical limit of infinite dimensional dynamical systems. The Young measure ν associated to a measurable function a: $D \to \mathbf{R}$ is defined by

$$\nu(B) = \int_D \mathbf{1}_B(x, a(x))\, dx$$

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for every Borel set B of $D \times \mathbf{R}$, where $\mathbf{1}_B$ is the characteristic function of B. Two functions a which yield the same measure ν must coïncide, up to null sets. More importantly, a mixture of different functions a can still be described by a similar measure [see Section 0.4 and Michel and Robert (1994) for the physical meaning of Young measures]. The main asset of the random Young measures ν_ε associated to a_ε is that ν_ε can converge, even though a_ε does not converge in a usual sense. In Theorem 1 of this paper, we characterize the asymptotic behavior of

$$P(\nu_\varepsilon \in A)$$

for ε going to zero, when A is an asymptotically rare measurable set, that is, when A does not contain the limit of the ν_ε.

0.3. *Poissonian biphased material.* This model describes a biphased material whose irregularities have random scale and shape. Another random model, studied in dimension 1 by Baldi (1988), and subsequently in any dimension by Michel and Robert (1994) and by Michel and Piau (1998), assumes that the scale of the irregularities of the function a_ε, for a given value of ε, is constant through the material. On the other hand, in Baldi's model, the values of the parameter $a(x)$ are random, whereas in the present one, they are fixed. This Poissonian biphased model is called Poisson blob model or Swiss cheese model in Meester and Roy (1996). It is extensively studied by Sznitman in a series of papers. See, for example, Sznitman (1995), which gives, among other results, the behavior of a standard Brownian motion in the nonoccupied part of the material, that is, in one of its two phases (the part D_ε, in our notations below).

We choose two values of the parameter $a(x)$, for example 0 and 1, a shape S, that is, a bounded domain of $\mathbf{R}^d$, and a bounded domain D of $\mathbf{R}^d$ of volume $|D| = 1$, for example, $D := [0,1)^d$. For given $\kappa > 0$ and $\varepsilon > 0$, $\mathscr{P}_\varepsilon := \mathscr{P}(\kappa\,\varepsilon^{-d})$ is the support of a random Poisson measure on $\mathbf{R}^d$ with constant intensity $\kappa\,\varepsilon^{-d}$. The set $O_\varepsilon(S)$ of the Poissonian obstacles is then

$$O_\varepsilon(S) := \mathscr{P}_\varepsilon + \varepsilon S = \bigcup_{x \in \mathscr{P}_\varepsilon} x + \varepsilon S$$

and $D_\varepsilon := D \setminus O_\varepsilon(S)$ is the part of D which is not covered by the obstacles. Set $a_\varepsilon(x) := 1$ if $x \in D_\varepsilon$ and $a_\varepsilon(x) := 0$ if $x \in D \setminus D_\varepsilon$. Hence,

$$a_\varepsilon(\omega, x) = \mathbf{1}(x \in D_\varepsilon(\omega)).$$

Lastly, ν_ε is the random Young measure associated to a_ε.

0.4. *Notations and result.* We give some conventions that will be in force below, before stating our result. The relation

$$P(\nu_\varepsilon \in A) \sim \exp(-\varepsilon^{-d}\, i(A))$$

is a shorthand for the following assertion. The sign $\sim$ means that the lower and the upper bounds of a large deviations principle (LDP) “à la Varadhan” hold: there exists a function i with values in $[0, +\infty]$, called the rate function of the LDP, such that, for every open set G and every closed set F, one has

$$\liminf_{\varepsilon \to \infty} \varepsilon^d \log P(\nu_\varepsilon \in G) \geq -i(G),$$

$$\limsup_{\varepsilon \to \infty} \varepsilon^d \log P(\nu_\varepsilon \in F) \leq -i(F).$$

For every set A, one writes $i(A) = \inf\{i(\nu);\ \nu \in A\}$. The level sets $i^{-1}([0, t])$ of the rate function i are supposed to be compact. (This implies that i is lower semicontinuous.)

NOTATION 1. Call $\mathscr{Y}$ the set of the Young measures ν^p for any measurable function p: $D \to [0, 1]$, where ν^p is defined by

$$\nu^p(B) = \int_D [p(x)\,\mathbf{1}_B(x, 1) + (1 - p(x))\,\mathbf{1}_B(x, 0))]\,dx.$$

Call $\mathscr{Z}$ the subset of $\mathscr{Y}$ which corresponds to the $\{0, 1\}$ valued functions p.

The Young measures ν^0 and ν^1 describe a homogeneous material of constant characteristic. The Young measures of $\mathscr{Z}$ describe biphased materials. For instance, setting $p = \mathbf{1}(D_\varepsilon)$, one recovers the Young measures ν_ε introduced in Section 0.3. Any value of $p(x)$ between 0 and 1 should be viewed as the representation of a mixture at point x of the 1-material and of the 0-material in the proportions $p(x)$ and $1 - p(x)$.

Recall that the weak topology on the space of Borel bounded measures on a Hausdorff topological space X is defined as follows: μ_n converges weakly to μ iff $\mu_n(f)$ converges to $\mu(f)$ for every f bounded continuous on X; see Dembo and Zeitouni (1992) for instance. The vague topology is defined in a similar way with the continuous and compactly supported functions. As the weak topology contains the vague topology, a LDP has more content if it is valid for the weak topology than for the vague one. About the weak and vague topologies, see the discussion of Remark 1 in Section 1.1.

THEOREM 1. (i) *When ε goes to zero, the volume $|D_\varepsilon|$ converges in probability to a constant $s := \exp(-\kappa|S|)$. Furthermore, ν_ε converges weakly in probability to the measure $\nu^s = s\nu^1 + (1 - s)\nu^0$.*

(ii) *The volume $|D_\varepsilon|$ satisfies a LDP of rate function j; that is,*

$$P(|D_\varepsilon| \in A) \sim \exp(-\varepsilon^{-d}\, j(A)).$$

The function j is convex, null at s, finite on $[0, 1]$ and infinite on $\mathbf{R} \setminus [0, 1]$.

(iii) *The measures $(\nu_\varepsilon)_\varepsilon$ satisfy a LDP in the space of bounded Borel measures, endowed with the topology of the weak convergence, of the form*

$$P(\nu_\varepsilon \in A) \sim \exp(-\varepsilon^{-d}\, i(A)).$$

The rate function i is given by

$$i(\nu^p) = \int_D j(p(x))\,dx$$

for any $\nu^p \in \mathscr{Y}$, and by $i(\nu) = +\infty$ if $\nu \notin \mathscr{Y}$.

0.5. *Remarks.* The only Young measures of $\mathscr{Y}$ associated to a function a are the elements of $\mathscr{Q}$. Hence, the weak limit $\nu^s = s\nu^1 + (1-s)\nu^0$ of ν_ε is not the Young measure of the constant function $a(x) = s$ (nor of any other function), although $a(\cdot) = s$ is the weak limit of a_ε. The same phenomenon occurs for the models of Baldi and of Facchinetti and Russo, which are studied in Michel and Piau (1998).

Any $\nu^p \in \mathscr{Y}$ is a weak limit of elements of $\mathscr{Q}$, that is, of Young measures ν^{p_n} which describe a real material. To see this in dimension 1, assume first that p is continuous and define p_n as follows: if $x \in [0, 1)$ is such that $nx \in [k, k + p(k/n))$ for an integer k, then $p_n(x) = 1$; else $p_n(x) = 0$. If p is not continuous, replace $p(k/n)$ by the mean value of p over $[k/n, (k+1)/n)$ in this construction.

In fact, the convex set $\mathscr{Y}$ is the weak closure of $\mathscr{Q}$. (Furthermore, the measures of $\mathscr{Q}$ are the extremal points of $\mathscr{Y}$.) Hence, it is not a surprise that the domain of the rate function of a LDP for the measures ν_ε in the weak topology contains $\mathscr{Y}$ (and in fact, is equal to $\mathscr{Y}$). (We do not use this remark in our proofs.)

The rate function j is the solution of an optimization problem but the explicit form of j is unknown. For computations in dimension 1, see section 2.4. In higher dimensions, an open problem is to know whether j depends of the shape of the obstacle S, or only of its volume $|S|$, that is, only of the point s where j is null.

Theorem 1 can be generalized to the case where D is a bounded domain of $\mathbf{R}^d$ of volume $|D| = 1$ with rectifiable boundary. One can also replace $\mathscr{P}_\varepsilon$ by a Poisson process of intensity $\varepsilon^{-d}\kappa(x)\,dx$ on $\mathbf{R}^d$ for any locally integrable function $\kappa \geq 0$. The regularity of D ensures, for every k, the existence of an "almost k-uniform" partition of D, that is a partition of D such that the following properties hold:

(i) The partition has at most $c\,k$ cells.
(ii) The volume of each cell is at most k^{-1}.
(iii) The diameter of the cells goes uniformly to zero.
(iv) The total volume of the cells whose volume is different of k^{-1} goes to zero [see condition (P1) in Michel and Piau (1997)].

The existence of these partitions allows working with D as with $[0, 1)^d$ and we refer to Michel and Piau (1998) for a discussion. Denote by $j(\kappa, \cdot)$ the rate function of the LDP for $|D_\varepsilon|$ when $\mathscr{P}_\varepsilon$ is of constant intensity $\varepsilon^{-d}\kappa$ and by j_κ this rate function when the intensity of the Poisson process is $\varepsilon^{-d}\kappa(x)\,dx$.

Then, one has

$$j_\kappa(v) = \inf\left\{\int_D j(\kappa(x), h(x))\,dx;\ \int_D h(x)\,dx = v\right\}.$$

The above infimum may be restricted to the functions h with values in $[0, 1]$. The Young measures ν_ε then satisfy a LDP of rate function

$$i_\kappa(\nu^p) = \int_D j(\kappa(x), p(x))\,dx$$

for any $\nu^p \in \mathscr{Y}$, and $i_\kappa(\nu) = +\infty$ if $\nu \notin \mathscr{Y}$.

The rest of the paper is devoted to the proof of Theorem 1. Note that part (i) of the theorem is a consequence of parts (ii) and (iii). Section 1 contains the proof that a LDP for $|D_\varepsilon|$ implies a LDP for ν_ε [(ii) implies (iii)] and Section 2 contains the proof of a LDP for $|D_\varepsilon|$ [(ii) holds].

1. Large deviations for the Young measures.

1.1. *Legendre transforms.* In order to prove that (iii) holds when (ii) holds, we show, as in Michel and Piau (1998), that (ii) implies that the family $(\nu_\varepsilon)_\varepsilon$ satisfies the hypotheses of a known theorem. Recall that the space of bounded Borel measures on $D \times \{0, 1\}$ endowed with the topology of the weak convergence is a locally convex topological space. Its dual can be identified with the space C_b of bounded continuous functions on $D \times \{0, 1\}$; see Section 6.2 of Dembo and Zeitouni (1992). Assume first that the family $(\nu_\varepsilon)_\varepsilon$ is exponentially tight. Assume also that the functional

$$\tau_\varepsilon(f) := \varepsilon^d \log E[\exp(\varepsilon^{-d}\nu_\varepsilon(f))]$$

converges as ε goes to zero to a limit $\tau(f)$, for every $f \in C_b$. Assume lastly that τ is finite and Gateaux differentiable on C_b. Then, by Corollary 4.6.14 of Dembo and Zeitouni (1992), the family $(\nu_\varepsilon)_\varepsilon$ satisfies a LDP whose rate function i is the Legendre transform of τ; that is,

$$i(\nu) := \sup\{\nu(f) - \tau(f);\ f \in C_b\}.$$

In our situation, the law of ν_ε charges only the measures whose first marginal is the Lebesgue measure on D (or, in the usual terminology, the Young measures of base the Lebesgue measure). As $D \times \{0, 1\}$ is a compact set, this set of measures is compact, hence $(\nu_\varepsilon)_\varepsilon$ is exponentially tight.

REMARK 1. When the measures are defined on a noncompact space, the set of Young measures is a closed set, noncompact, for the weak topology. But the closure of this set for the vague topology is compact. Hence, the argument of exponential tightness is valid with the vague topology and one has to deduce the LDP for the weak topology from the LDP for the vague topology. This detour is needed when the set of values of a or D itself are not compact. Then, a LDP for the weak topology has more content than a LDP for the vague one. For the details of this discussion, see Michel and Piau (1998).

1.2. *Computation of* τ. We compute the limit of $\tau_\varepsilon(f)$ for every bounded continuous function f. As in Michel and Piau (1998), the main idea is to use the case where f is piecewise constant. The restrictions of the Poisson process are then independent and their laws are still Poisson.

NOTATION 2. For any function $f\colon D \times \{0,1\} \to \mathbf{R}$, we use the following shorthands: $f_0 := f(\cdot, 0)$, $f_1 := f(\cdot, 1)$ and $\widetilde{f} = f_1 - f_0$.

For any continuous bounded f, since D is compact, f_0, f_1 and $\widetilde{f}$ are bounded and uniformly continuous on D. For $k \geq 1$, there exists a function $g_k\colon D \to \mathbf{R}$ which is constant on each cell

$$D_n^k := k^{-1}(n + D), \qquad n \in \mathbf{Z}^d,$$

such that the uniform norm ε_k of $g_k - \widetilde{f}$ goes to zero when k goes to infinity. One has

$$\nu_\varepsilon(f) = \nu^s(f) + \int_D [\mathbf{1}(D_\varepsilon)(\widetilde{f}\,) - s\,\widetilde{f}\,].$$

Replacing $\widetilde{f}$ by g_k in the last term above yields an error on $\nu_\varepsilon(f)$ which is between $-\varepsilon_k$ and ε_k. Denote by g_n^k the value of g_k on D_n^k and introduce

$$\begin{aligned}\sigma_\varepsilon(g_k) &:= E\left[\exp\left(\varepsilon^{-d} \int_D (\mathbf{1}(D_\varepsilon) - s)\, g_k\right)\right]\\ &= E\left[\prod_n \exp\left(\varepsilon^{-d}\, g_n^k\, (|D_n^k \cap D_\varepsilon| - s\,|D_n^k|)\right)\right].\end{aligned}$$

The random variables $|D_n^k \cap D_\varepsilon|$ are "almost" independent from each other. To see this, denote by V_n^k the volume of the part of D_n^k which is not covered by the obstacles attached to points of the Poisson process $\mathscr{P}_\varepsilon$ *which belong themselves to the cell* D_n^k. Denote by V_ε the volume of the part of D which is not covered by the obstacles attached to points of the Poisson process $\mathscr{P}(\kappa\,(k\varepsilon)^{-d})$ which belong to D. Then, the random variables $(V_n^k)_n$ are i.i.d. and the homogeneity and scaling properties of the Poisson process imply that V_n^k has the law of $k^{-d}\, V_\varepsilon$. Furthermore,

$$|D_n^k \cap D_\varepsilon| \leq V_n^k \leq |D_n^k \cap D_\varepsilon| + O(k^{-(d-1)}\varepsilon).$$

To see this, notice that the difference between the two volumes can only be caused by points $x \in D_n^k$ which are covered by an obstacle εS attached to a point $y \notin D_n^k$. Hence, the point x must be at distance $O(\varepsilon)$ from the boundary of D_n^k. The total volume of such points is $O(k^{-(d-1)}\varepsilon)$, hence the above inequalities hold. For a given k, the error due to the replacement of each $|D_n^k \cap D_\varepsilon|$ by V_n^k in $\sigma_\varepsilon(g_k)$ is at most

$$\exp(\varepsilon^{-d}\, k\, O(\varepsilon)) = \exp(\varepsilon^{-d}\, O(\varepsilon)).$$

This shows that

$$\sigma_\varepsilon(g_k) = \exp(\varepsilon^{-d}\, O(\varepsilon)) \prod_n E[\exp(\varepsilon^{-d}\, k^{-d}\, g_n^k\, (V_\varepsilon - s))]. \tag{1}$$

The difference between V_ε and $|D_{k\varepsilon}|$ is seen to be $O(\varepsilon)$ by the same method, hence one can replace V_ε by $|D_{k\varepsilon}|$ in each factor of (1), causing an error on the exponent of each exponential of at most $O(\varepsilon)$. One goes back to $\tau_\varepsilon(f)$ by summing these errors and this yields

$$\tau_\varepsilon(f) = \nu^s(f) + \varepsilon(k) + O(\varepsilon) + \sum_n \varepsilon^d \log E[\exp((k\varepsilon)^{-d}\, g_n^k\, (|D_{k\varepsilon}| - s))],$$

where $\varepsilon(k)$ is a number between $-\varepsilon_k$ and ε_k. We use now the LDP satisfied by $|D_\varepsilon|$. By Varadhan's lemma, for any value of t, one has

$$\lim_{\varepsilon\to 0} \varepsilon^d \log E[\exp(k^{-d}\, \varepsilon^{-d}\, t\, |D_{k\varepsilon}|)] = k^{-d}\, j^*(t),$$

where j^* is the Legendre transform of the rate function j, also called its convex conjugate. For a given k, we apply the estimate of Varadhan's lemma for the k^d values $t = g_n^k$. One gets

$$\begin{aligned}\tau_\varepsilon(f) &= \nu^s(f) + O(\varepsilon) + \varepsilon(k) + \sum_n k^{-d}[j^*(g_n^k) - s\, g_n^k + c(g_n^k, k\varepsilon)]\\ &= \nu^s(f) + O(\varepsilon) + \varepsilon(k) + \int_D [j^*(g_k) - s\, g_k + c(g_k, k\varepsilon)].\end{aligned}$$

Here, the functions $c(t, k\varepsilon)$ go to zero when ε goes to zero, for any given t and k. Hence, the limit points of $\tau_\varepsilon(f)$ are of the form

$$\nu^s(f) + \varepsilon(k) + \int_D [j^*(g_k) - s\, g_k]. \tag{2}$$

Since $|D_\varepsilon| \in [0, 1]$ and $[0, 1]$ is closed, the rate function j for $|D_\varepsilon|$ is infinite outside of $[0, 1]$. Hence, its Legendre transform j^* is 1-Lipschitz continuous.

The proof of this basic fact is as follows: assume that a rate function m is infinite outside of $[-v_0, v_0]$. Then, its Legendre transform m^* is given a priori by

$$m^*(x) = \sup\{m_x(v);\, v \in \mathbf{R}\}, \qquad m_x(v) = xv - m(v),$$

but the assumption on m implies that the supremum defining m^* can be restricted to $v \in [-v_0, v_0]$. Hence, the difference $|m^*(x) - m^*(y)|$ is bounded by the uniform norm of $m_x - m_y$ on $[-v_0, v_0]$, which is $|x - y|\, v_0$.

Similarly, $|D_\varepsilon| - s \in [-1, +1]$, the rate function for $|D_\varepsilon| - s$ is the translate $j(\cdot + s)$ of j and the Legendre transform of $j(\cdot + s)$ is

$$x \mapsto j^*(x) - sx.$$

Hence, this last function is 1-Lipschitz continuous. The replacement of g_k by $\widetilde{f}$ in (2) yields an error on the result of at most $\varepsilon(k)$. When k goes to infinity, $\varepsilon(k)$ goes to zero, so that we proved the following:

$$\lim_{\varepsilon\to 0} \tau_\varepsilon(f) = \nu^s(f) + \int_D [j^*(\widetilde{f}) - s\, \widetilde{f}\,] =: \tau(f).$$

The function $f \mapsto \tau(f)$ is finite on C_b. We now show that τ is Gateaux differentiable at any $f \in C_b$. For any $h \in C_b$ and $t > 0$,

$$t^{-1}[\tau(f+th) - \tau(f)] = \nu^s(h) - s\int_D \widetilde{h} + \int_D t^{-1}[j^*(\widetilde{f} + t\widetilde{h}) - j^*(\widetilde{f})].$$

We split the last integral into an integral where $\widetilde{h} > 0$ and another integral where $\widetilde{h} < 0$. Because j^* is convex, these two integrals converge monotonically when t goes to 0^+ to an integral of the right derivative j^*_r of j^*, respectively, of its left derivative j^*_l. Hence the Gateaux differential of τ at f computed on h exists and is

$$\nu^s(h) + \int_D [j^*_r(\widetilde{f})\mathbf{1}(\widetilde{h} > 0) + j^*_l(\widetilde{f})\mathbf{1}(\widetilde{h} < 0) - s\widetilde{h}].$$

1.3. *Evaluation of* i. The Legendre transform of τ is

$$i(\nu) := \sup\{\nu(f) - \tau(f);\ f \in C_b\}.$$

We first show that $i(\nu)$ is infinite if $\nu \notin \mathscr{Y}$. Assume that $i(\nu)$ is finite. Since we know that $|j^*(x) - sx| \le |x|$, one has, for any $f \in C_b$,

$$t(\nu(f) - \nu^s(f)) - |t|\int_D |\widetilde{f}| \le \nu(tf) - \tau(tf) \le i(\nu)$$

for any value of t. (Recall that $\widetilde{f} := f_1 - f_0$.) Letting t go to $\pm\infty$, one sees that $|\nu(f) - \nu^s(f)|$ is bounded by the integral of $|\widetilde{f}|$. The same relation holds for measurable functions.

The choice of $f = \mathbf{1}_A \otimes \mathbf{1}$ gives $\nu(A \times \{0,1\}) = \nu^s(A \times \{0,1\}) = |A|$, hence the first marginal of ν is the Lebesgue measure. In particular, ν is a probability measure which may be decomposed along its first marginal into probability measures over $\{0,1\}$. Since D is a Borel space, a measurable version of this conditioning exists. Denoting by $p(x)\delta_1 + (1-p(x))\delta_0$ the conditioned measure at x, one sees that there exists a measurable function $p\colon D \to [0,1]$ such that $\nu = \nu^p$. We now compute $i(\nu^p)$. One has

$$\begin{aligned}\nu^p(f) - \tau(f) &= \nu^p(f) - \nu^s(f) - \int_D [j^*(\widetilde{f}) - s\widetilde{f}]\\ &= \int_D [p(x)\widetilde{f}(x) - j^*(\widetilde{f}(x))]\,dx.\end{aligned}$$

From the definition of the Legendre transform and from the convexity of j, one has for every y the following inequality:

$$p(x)\,y - j^*(y) \le (j^*)^*(p(x)) = j(p(x)).$$

Hence, $i(\nu^p)$ is at most the integral of $j(p)$. We now show that this is its exact value.

The application $f \mapsto \nu^p(f) - \tau(f)$ is continuous for the L^1 norm and the continuous bounded functions are a dense subset of L^1. Hence, $i(\nu^p)$ is also the supremum of the integrals of $p\,g - j^*(g)$ on D for all the functions $g \in L^1$. Assume for the moment that j is finite on $]0,1[$ (we prove in Section 2.3 that

j is finite on $[0, 1]$). By convexity, for every $x \in]0, 1[$, there exists a supporting line of the graph of j at point x, that is, a slope $y(x)$ such that

$$j(x') \geq j(x) + y(x)(x' - x)$$

for every x'. One sees then that $x'\, y(x) - j(x')$ is maximum at $x' = x$, that is, that $j^*(y(x)) = x\, y(x) - j(x)$. For every $a > 0$ such that $a \leq s \leq 1 - a$, denote by p_a the truncation of p at levels a and $1 - a$, that is,

$$p_a := (1 - a) \wedge (p \vee a).$$

Then, $g_a := y \circ p_a$ is bounded, hence integrable, and

$$p_a\, g_a - j^*(g_a) = j(p_a).$$

Furthermore, still from the convexity of j, $y(a) \leq 0 \leq y(1 - a)$. One gets

$$\begin{aligned} p\, g_a - j^*(g_a) &= j(p_a) - y(a)(p - a)^- + y(1 - a)(p - (1 - a))^+ \\ &\geq j(p_a). \end{aligned}$$

Finally, $i(\nu^p)$ is at least the integral of $j(p_a)$ on D. When a goes to zero, p_a goes to p. The rate function j is lower semicontinuous hence the lim inf of the integral of $j(p_a)$ is greater than the integral of $j(p)$, and this ends the proof that (ii) implies (iii).

2. Large deviations for the volumes. Section 2.1 is not logically necessary, but it can help the reader to understand in a simple setting the proofs of the following sections.

2.1. *Dimension* 1. The Poisson process $\mathscr{P}_\varepsilon$ of intensity κ/ε is invariant by translations and the scaled process $(t\,\mathscr{P}_\varepsilon)$ is a Poisson process of intensity $\kappa/(t\varepsilon)$. Hence, it suffices to show that, for a Poisson process $\mathscr{P}(a)$ of given intensity $a := \kappa\,|S|$ and for $k := (\varepsilon\,|S|)^{-1}$ going to infinity, the measure of the part of $[0, k]$ which is not covered by the obstacles $\mathscr{P}(a) + [0, 1]$ satisfies a LDP. Denote by W_n the measure of the uncovered part of $[n, n + 1]$ and

$$\begin{aligned} X_n &:= \sup\{x \in [0, 1];\ [n, n + x] \cap \mathscr{P}(a) = \varnothing\}, \\ Y_n &:= \sup\{y \in [0, 1];\ [n + 1 - y, n + 1] \cap \mathscr{P}(a) = \varnothing\}. \end{aligned}$$

The measure V_k of the uncovered part of $[0, k]$ is the sum of W_n from $n = 0$ to $k - 1$. The random variables W_n are not independent but

$$W_n = f(X_n, Y_{n-1}) := (X_n + Y_{n-1} - 1)^+.$$

The random variables $Z_n := (X_n, Y_n)$ are i.i.d. and their law is known (see below). Hence, the process $(W_n)_n$ is stationary, with values in the compact set $[0, 1]$, and mixing since W_n and W_m are independent as soon as $|n - m| \geq 2$. The LDP for stationary sequences [see Theorem 6.4.4 of Dembo and Zeitouni (1992)] ensures the existence of the limit

$$\lambda(t) := \lim_{k\to\infty} k^{-1} \log E[\exp(t\, V_k)],$$

as well as a LDP for V_k/k. Its rate function j is convex, since it is the Legendre transform of λ; that is,

$$j(v) := \sup\{t\,v - \lambda(t);\ t \in \mathbf{R}\}.$$

Since $\lambda(\cdot)$ is convex, λ is in fact the function j^* of Section 1. Notice that $\exp j^*(t)$ is the largest eigenvalue of a Perron–Frobenius operator (see below for an explicit formula in dimension 1) and that $j(v)$ is also the solution of a problem of entropy minimization, a problem whose effective solution is impossible without the value of $\exp j^*(t)$.

2.2. *Higher dimensions.* In dimension $d \geq 2$, few things change. After a translation and a scaling, we can and will assume that $S \subset [0,1]^d$. Call $\pi(n)$ the part of $\mathscr{P}(a)$ which belongs to the elementary cube $c(n) := n + [0,1)^d$ for $n \in \mathbf{Z}^d$. More precisely,

$$\pi(n) := [\mathscr{P}(a) \cap c(n)] - n.$$

The field $(\pi(n),\ n \in \mathbf{Z}^d)$ is i.i.d. and each $\pi(n)$ takes values in the space of the finite subsets of $[0,1)^d$, endowed with the Hausdorff distance between compact sets. We claim that the volume W_n of the uncovered part of $c(n)$ is a continuous functional of the family $(\pi(n-e),\ e \in \{0,1\}^d)$. To see this, notice that, if the Hausdorff distance between two samples $(\pi(n-e))_e$ and $(\pi'(n-e))_e$ is less than ε, the uncovered points for π which are at distance more than ε of the covered part for π are uncovered by π'. Hence, W_n is Lipschitz continuous with respect to $(\pi(n-e),\ e \in \{0,1\}^d)$.

Finally, we may apply a contraction principle to the LDP for lattice systems with finite range interactions of Deuschel, Stroock and Zessin (1991) (see their Theorem 1.3). It proves that $k^{-d} V_k$ satisfies a LDP of speed k^d and rate function j, where

$$V_k := \sum_{n \in N(k)} W_n, \qquad N(k) := \{n \in \mathbf{Z}^d;\ c(n) \subset [0,k)^d\}$$

is the volume of the uncovered part of $[0,k)^d$. Once again, the rate function j is convex and may be seen, either as the Cramér transform of the largest positive eigenvalue of a Perron–Frobenius operator or as the solution of a problem of minimization of an entropy.

2.3. *Properties of j.* In any dimension, V_k/k^d belongs to $[0,1]$ and converges almost surely to s. To see this, consider the translation operator

$$\theta_m\colon (W_n,\ n \in \mathbf{Z}^d) \mapsto (W_{n+m},\ n \in \mathbf{Z}^d).$$

For any $m \in \mathbf{Z}^d$, θ_m is ergodic on $[0,1]^{\mathbf{Z}^d}$ for the law of $(W_n)_n$. To see this, notice that such an operator is equivalent to a translation of all the Poisson process by $-m$, which does not change its law. The ergodicity follows from the fact that, due to the i.i.d. property of the process $(\pi(n),\ n \in \mathbf{Z}^d)$, any possible local configuration of the Poisson process is a.s. approximately realized around some point of $m\mathbf{Z}$; this is equivalent to saying that, for any neighborhood of any given local configuration, there exists a translation which is a power of θ_m and which sends $(W_n)_n$ in this neighborhood. Hence, the ergodic theorem yields the a.s. convergence of V_k/k^d to

$$E[W_1] = E[|D_\varepsilon|] = \int_D P(x \notin \mathscr{P}_\varepsilon + \varepsilon S)\, dx$$
$$= P(\mathscr{P}_\varepsilon \cap (-\varepsilon S) = \varnothing) = \exp(-\kappa\, |S|).$$

The function j is null at point s and infinite outside of $[0, 1]$. Let us show that $j(0)$ and $j(1)$ are finite; this will imply that j is finite on $[0, 1]$. First, $D_\varepsilon = D$ if and only if no point of $\mathscr{P}_\varepsilon$ is in $D - \varepsilon S$. This shows that

$$P(|D_\varepsilon| = 1) = P(\mathscr{P}_\varepsilon \cap (D - \varepsilon S) = \varnothing) = \exp[-\kappa \varepsilon^{-d}(1 + o(1))],$$

hence $j(1) = \kappa$. On the other hand, since S is an open set, there exists a finite number c of cells of volumes less than v such that $A + S$ completely covers D as soon as A contains at least one point in each cell. Hence, the existence of at least one point of $\mathscr{P}_\varepsilon$ in $c\varepsilon^{-d}$ cells of volumes at most $v\varepsilon^d$ ensures that D is completely covered by $O_\varepsilon(S)$. In other words,

$$P(|D_\varepsilon| = 0) \geq (1 - \exp(-\kappa \varepsilon^{-d} v \varepsilon^d))^{c\varepsilon^{-d}},$$

hence $j(0) \leq -c \log(1 - \exp(-\kappa v))$. This ends the proof of (ii).

2.4. *Explicit formulas in dimension* 1. In dimension 1, one can estimate $j^*(t)$ explicitly. The common law of the random variables $Z_n = (X_n, Y_n)$ (see above) is

$$P(Z_0 \in (dx, dy)) = \exp(-a)\, \delta_1(dx)\, \delta_1(dy) + a \exp(-a)\, \delta_{1-x}(dy)\, \mathbf{1}_{[0,1]}(x)\, dx$$
$$+ a^2 \exp(-a(x + y))\, \mathbf{1}_{[0,1]}(x)\, \mathbf{1}_{[0,1]}(y)\, \mathbf{1}_{[0,1]}(x + y)\, dx\, dy.$$

The random variables X_n and Y_n follow the law of $\mathscr{E} \wedge 1$ where $\mathscr{E}$ is an exponential random variable of mean a^{-1}. Hence,

$$P(X_0 \in dx) = P(Y_0 \in dx) = e^{-a}\, \delta_1(dx) + a\, e^{-ax}\, \mathbf{1}_{[0,1]}(x)\, dx.$$

Recall that $f(x, y) := (x + y - 1)^+$ and define Q_t on $L^1([0, 1])$ by

$$Q_t h(x) := E[h(X_0) \exp(t\, f(x, Y_0))].$$

Setting $h_0 := 1$ and $h_{k+1} := Q_t h_k$, one gets

$$h_k(x) = E[\exp(t\, V_k) \,|\, X_k = x],$$
$$E[\exp(t\, V_k)] = E[h_k(X_0)] = E[Q_t^k(\mathbf{1})(X_0)].$$

Hence, $k^{-1} \log E[\exp(t\, V_k)]$ is equivalent, when k goes to infinity, to $\log \Lambda(t)$, where $\Lambda(t)$ is the largest positive eigenvalue of the Perron–Frobenius operator Q_t. One sees that $\Lambda(t) = \exp j^*(t)$.

Although an explicit formula of $j(v)$ seems difficult to get, one can write down the operator Q_t, for any $t \neq a$, as the following:

$$Q_t h(x) = \exp(-a)\exp(tx)\,h(1) + \int_0^1 a\exp(-ay)\,h(y)\,dy + \int_0^x a\exp(a)\,\frac{t}{t-a}\,(\exp(t(x-y)) - \exp(a(x-y)))\,h(y)\,dy.$$

The expression of Q_a can be deduced by continuity. Furthermore, the very definition of Q_t and the fact that $W_n \in [0,1]$ show that $j^*(t)$ has the sign of t and is between 0 and t.

Notice that Q_t operates by duality on the finite measures μ over $[0,1]$ by $Q_t^*(\mu)(h) = \mu(Q_t(h))$. The usual method of computation of $\Lambda(t)$ is to find $\beta_t > 0$ and a positive measure μ_t such that $Q_t^*(\mu_t) = \beta_t \mu_t$. Then, $\beta_t = \Lambda(t)$. In the present case, such a measure can be explicitly written down as

$$\mu_t(dx) = \delta_1(dx) + (r-t)\exp(r(1-x))\mathbf{1}_{[0,1]}(x)\,dx,$$

provided $\beta_t = e^r$ and for a suitable value of r. We skip the details of this computation and give the implicit equation that it yields for $j^*(t) = \log\Lambda(t)$,

$$2j^*(t) = (t-a) + \left((t-a)^2 + 4ate^{-a}\exp(-j^*(t))\right)^{1/2}. \tag{3}$$

Equation (3) has a unique solution $j^*(t)$ in $(0,t)$ for $t > 0$ and in $(-t, 0)$ for $t < 0$. For instance,

$$j^*(a)^2 \exp(j^*(a)) = a^2\exp(-a).$$

The fact that $j^*(t)$ is the solution of (3) seems to imply that no closed formula for $j^*(t)$ or $j(v)$ holds.

Finally, we mention that, for $t \geq 0$, one can get an upper bound of $j^*(t)$, which is better than the obvious one, $j^*(t) \leq t$, by writing

$$1 \leq \exp(2j^*(t)) \leq E[\exp(2t\,f(X_1, Y_0))] = E[\exp(2t\,W_1)].$$

This last inequality is a consequence of the Cauchy–Schwarz inequality, applied to the random variables

$$\exp\left(t\sum_n W_{2n}\right) \quad \text{and} \quad \exp\left(t\sum_n W_{2n+1}\right),$$

and of the fact that the random variables W_n that are in one of these two sums are independent.

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