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SADDLEPOINT APPROXIMATIONS TO OPTION PRICES[1]

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The use of saddlepoint approximations in statistics is a well-established technique for computing the distribution of a random variable whose moment generating function is known. In this paper, we apply the methodology to computing the prices of various European-style options, whose returns processes are not the Brownian motion with drift assumed in the Black–Scholes paradigm. Through a number of examples, we show that the methodology is generally accurate and fast.

1. Introduction. We are going to be concerned with the pricing of a European put option on a share whose price at time t is denoted $\exp(X_t)$. According to arbitrage-pricing theory, the time-0 price of the option is

$$\text{Price} = \mathbb{E}\, e^{-rT}(e^{\alpha} - e^{X_T})^{+}, \tag{1.1}$$

where $\mathbb{P}$ is the (risk-neutral) pricing measure[2], the expiry of the option is T, and the strike is e^{α}. For the time being, we assume a constant interest rate r. In the case where X is a Brownian motion with constant drift, the price is given by the Black–Scholes formula, but the assumptions underlying the Black–Scholes analysis are often questioned, and various other models for the returns have been considered; see [7] for a selection of the models considered. Without attempting to pick out any "good" alternatives from the vast array already on offer, what we shall do here is show how classical statistical techniques for computing (approximations to) the tails of distributions may often be applied to such pricing problems.

The first step is to rewrite the price (1.1) as

$$\text{Price} = e^{\alpha - rT}\mathbb{P}(X_T < \alpha) - e^{-rT}\mathbb{E}[e^{X_T}\colon X_T < \alpha], \tag{1.2}$$

the difference of two terms. It will be our standing assumption that the cumulant-generating function K of X_T, defined by

$$\mathbb{E}\exp(zX_T) \equiv \exp(K(z)), \tag{1.3}$$

is finite in some open strip $\{z\colon a_- < \mathscr{R}(z) < a_+\}$ containing the imaginary axis, where $\mathscr{R}(z)$ denotes the real part of complex z, and a_- and $a_+ > 1$ may

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[2]Of course, the examples we shall be discussing will be examples of incomplete markets, so there is no unique equivalent martingale measure. We shall cut through the soul-searching and assume we have reached an equivalent martingale measure we are happy to work with.
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be infinite. With this assumption, we can rewrite (1.2) in such a way that the two parts of the expression appear quite similar, namely,

$$\text{Price} = e^{\alpha - rT}\mathbb{P}(X_T < \alpha) - e^{-rT+K(1)}\mathbb{P}_1(X_T < \alpha), \tag{1.4}$$

where we define the probabilities $\mathbb{P}_y$ by

$$\mathbb{E}_y \exp(zX_T) \equiv \mathbb{E}[\exp((z+y)X_T)]\, e^{-K(y)} \tag{1.5}$$

for any $y \in (a_-, a_+)$. Clearly, the cumulant-generating function (CGF) of the law $\mathbb{P}_y$ is given as

$$K_y(z) = K(y+z) - K(y), \tag{1.6}$$

so if we can find an (approximate) expression for $\mathbb{P}(X_T < \alpha)$ using the CGF K, we are in a position to find an approximate price for the put option. Computing such approximations is the business of the classical *saddlepoint method* of statistics; the main ideas of the method are explained with examplary clarity by Daniels [2] and Wood, Booth and Butler [12], and we could not hope to better these. In the Appendix, we summarize the method (without proof) and refer to [2] or [12] for more details. For an extremely thorough presentation of the entire method, see [8]. If we know the CGF of X_T, we can in principle compute the price of the option by inverting the Fourier transform, and with the fast Fourier transform this can be done reasonably rapidly. Indeed, the saddlepoint method starts from the Fourier inversion formula, but by considering a well-chosen contour of integration and approximating the principal contribution of the integrand, it turns out that no numerical integration is needed to come up with an approximation which is usually extremely accurate. The other virtue of the saddlepoint method is that the approximation to the price is actually an analytic expression, so it is possible to discover (for example) the local behavior of the price as some parameter is varied.

In Section 2, we explore a number of examples where the log-price process X is a process with stationary independent increments, or *Lévy* process (see [10], Chapter VI or [1] for more background on Lévy processes). As a simple first example, we take the situation where X is a drifting Brownian motion plus a compound Poisson process. We compute the prices of the option, using numerical integration (FFT), and compare with the saddlepoint approximation. Our next example takes X to be a gamma process, and computes the price by FFT and by saddlepoint approximation, and our final example uses the hyperbolic distribution of returns advocated by Barndorff-Nielsen, and Eberlein and Keller [3]. Once again, we compute the price by exact means, and compare with the saddlepoint approximation. Further examples of this kind are left to the reader; [7] lists a number that have been studied in the past. Gerber and Shiu [4] consider pricing of options on a share whose log price is a Lévy process. They arrive at an expression ((2.15) in [4]) for the price of a European call which is equivalent to (1.2) above and study a number of examples. They argue also that one can find a similar expression for the price of an exchange option (Corollary 1 in [4]), and it is clear that the saddlepoint method can as well be used for computing the approximate value of such an expression.

As a further application of the saddlepoint method, we remark that the prices of options in various stochastic volatility–stochastic interest rate models (as in [5] or [11], for example) can be computed, since all that is needed for the saddlepoint method is a simple expression for the characteristic function.

2. Lévy returns.

2.1. *Jump-diffusion processes.* The first application of the method is to the case in which the prices are modelled by a jump-diffusion process, specifically X is a drifting Brownian motion plus a compound Poisson process in which the size of the jumps is normal with mean a and variance γ^2. The function K is then

$$K(z) = T\left(c\, z + \frac{\sigma^2}{2}\, z^2 + \lambda\left(\exp\left(a\, z + \frac{\gamma^2}{2}\, z^2\right) - 1\right)\right), \tag{2.1}$$

where

$$c = r - \frac{\sigma^2}{2} - \lambda\left(\exp\left(a + \frac{\gamma^2}{2}\right) - 1\right). \tag{2.2}$$

Let us fix the values of the parameters as in the following table:

σ	r	S_0	λ	a	γ
0.1	0.05	1	5	−0.001	0.1

and let $T \in \{0.1 + 0.05\, k\}_{k=1}^{10}$ and $\alpha \in \{-0.11 + 0.01\, k\}_{k=1}^{21}$. Figure 1 displays the price surface obtained using the Lugannani and Rice saddlepoint approximation.

In Figure 2 we can see the difference between the prices computed using the saddlepoint approximation and the prices computed by numerical integration.

We then compute the volatilities that are implied by the LR prices. Recall that the Black–Scholes option pricing formula for a put option with strike price K, maturity T, volatility σ, interest rate r and initial price of the underlying asset S_0 is

$$P_{\mathrm{BS}}(r, \sigma, T, S_0, K) = K \exp(-r\, T)\, \Phi(-d_2) - S_0\, \Phi(-d_1), \tag{2.3}$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)\, T}{\sigma\sqrt{T}} \tag{2.4}$$

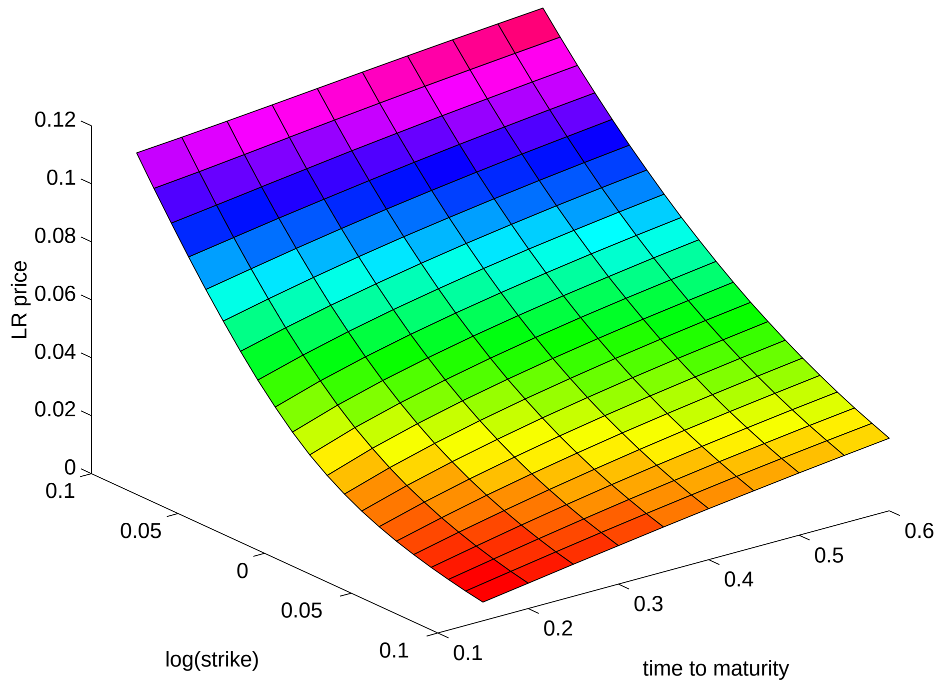

FIG. 1. *Saddlepoint approximation option prices.*

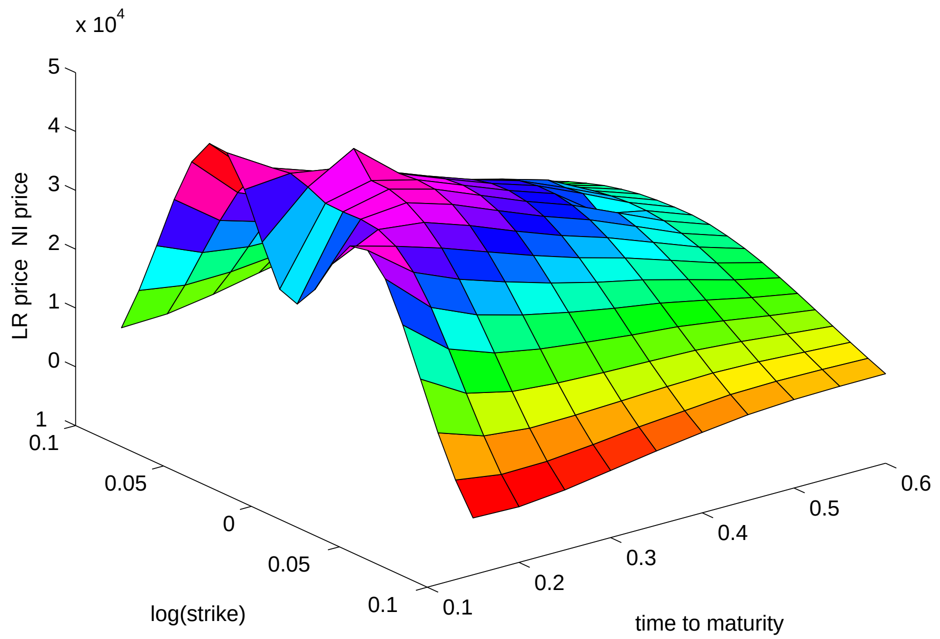

FIG. 2. *Difference between LR prices and numerical integration prices.*

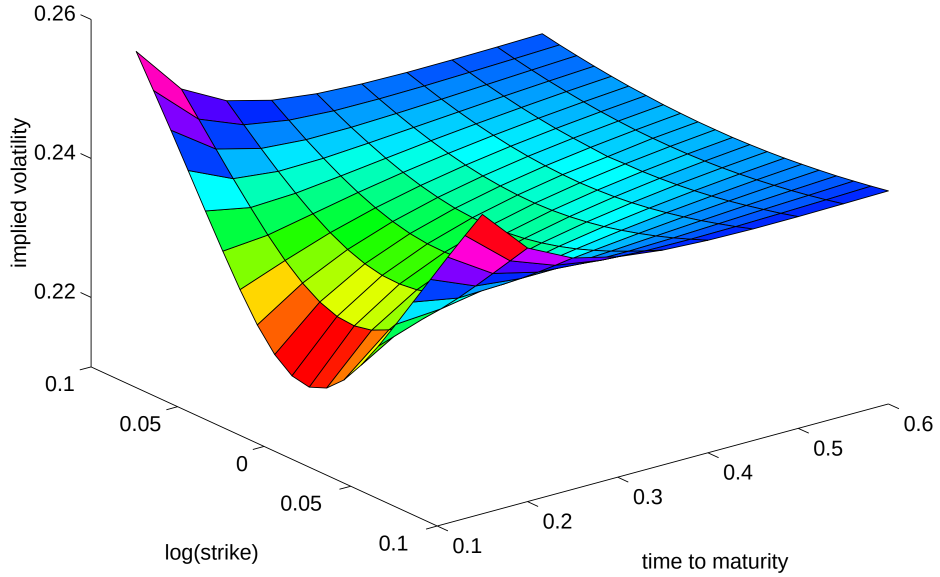

FIG. 3. *Implied volatility surface.*

and

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)\,T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{2.5}$$

Figure 3 displays the volatility surface obtained by computing the value of the volatility parameter that is needed to obtain the LR price using the BS formula when $r, T, S_0, K \equiv \exp(\alpha)$ assume the same values in both cases.

Finally, we match the variance of the log price in the standard BS model and the BS model with jumps, by taking the BS volatility $\hat{\sigma} = \sqrt{\sigma^2 + \lambda(a^2 + \gamma^2)}$ and compute the put option prices $P_{\mathrm{BS}}(r, \hat{\sigma}, T, S_0, \exp(\alpha))$ using the Black and Scholes formula. The results are displayed in Figure 4. As we can see, the errors are ten times bigger if one tries to use the Black and Scholes formula with the "volatility" obtained from the second moment of the jump diffusion model.

We display some of the results in Table 1. [Note that the prices that are reported in all tables are rounded if the fifth digit is greater than or equal to 5 and truncated otherwise; the relative error $100\,(|\mathrm{price}_{\mathrm{NI}} - \mathrm{price}_{\mathrm{LR}}|/\mathrm{price}_{\mathrm{NI}})$ is computed before such operation takes place.]

In Table 2 we give the volatility implied by the prices of Table 1.

2.2. *Gamma processes.* As a second example we consider the case in which the returns of the stock are modeled by a subordinated process given by a

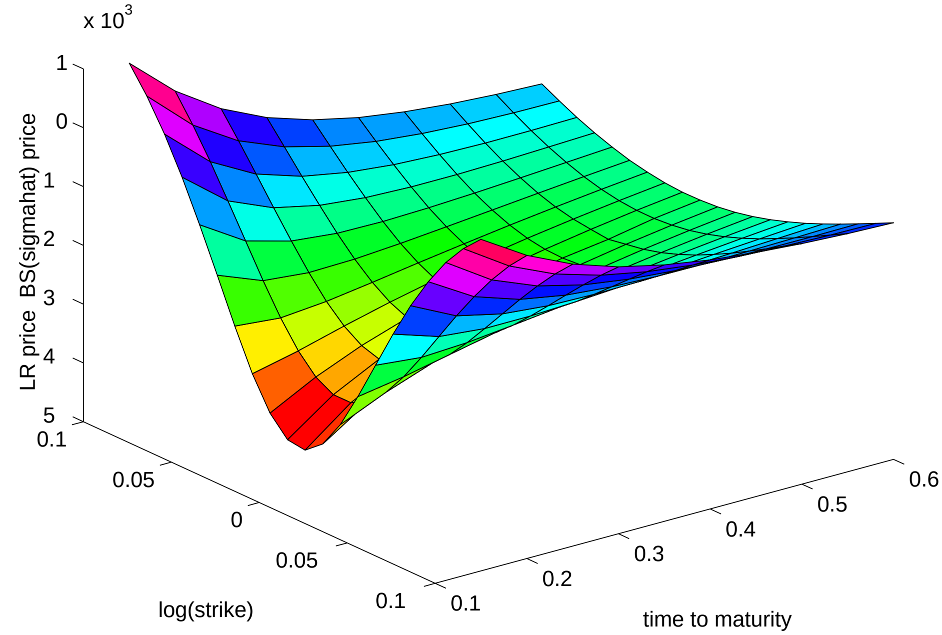

FIG. 4. *Difference between LR prices and prices computed using BS with volatility $\hat{\sigma}$.*

gamma process subordinated to Wiener process ($X(t) := \sigma\, W(G(t))$). In this case the cumulant generating function is given by

$$K(z) = T\left(c\, z + \log\left(\frac{\beta}{\beta - (\sigma^2/2)z^2}\right)\right), \tag{2.6}$$

where

$$c = r - \log\left(\frac{\beta}{\beta - \sigma^2/2}\right).$$

The use of these kinds of processes has been suggested in [6] and [4].

TABLE 1

Time to maturity	$\alpha = -0.05$			$\alpha = 0$			$\alpha = 0.05$		
	LR	**NI**	**%**	**LR**	**NI**	**%**	**LR**	**NI**	**%**
0.25	0.0210	0.0208	1.21	0.0393	0.0388	1.06	0.0688	0.0684	0.51
0.5	0.0347	0.0346	0.44	0.0542	0.0540	0.44	0.0812	0.0809	0.27
1	0.0515	0.0514	0.13	0.0711	0.0710	0.12	0.0959	0.0958	0.09
2	0.0691	0.0690	0.03	0.0877	0.0877	0.03	0.1101	0.1101	0.02
5	0.0844	0.0844	0.00	0.0999	0.0999	0.00	0.1177	0.1177	0.00

TABLE 2

Maturity	**$\alpha = -0.05$**	**$\alpha = 0$**	**$\alpha = 0.05$**
0.25	0.2314	0.2259	0.2306
0.5	0.2372	0.2358	0.2366
1	0.2408	0.2406	0.2408
2	0.2430	0.2429	0.2430
5	0.2443	0.2443	0.2443

The values for the parameters that have been used are

σ	r	S_0	β
0.1	0.05	1	0.25

.

Table 3 gives the results obtained using the Lugannani–Rice and the numerical integration methods and shows the relative errors.

The values of the parameters (there is only one degree of freedom in the choice) have been chosen in such a way that the implied volatilities are between 0.1 and 0.2, as Table 4 illustrates.

2.3. *Hyperbolic returns.* As a last example we consider the case in which the return of the share is modelled, at any time t, by a random variable with hyperbolic distribution. This choice has been suggested by Barndorff-Nielsen and has been analyzed in [3].

The function K is given by

$$K(z) = T\left(r\,z + \frac{a - \sqrt{a^2 + \sigma^2\,\sigma_0^2\,z(1-z)}}{\sigma_0^2} \right). \tag{2.7}$$

We consider the following parameter values:

σ	r	S_0	σ_0	a
0.25	0.05	1	0.7	1

.

TABLE 3

Time to maturity	**$\alpha = -0.05$**			**$\alpha = 0$**			**$\alpha = 0.05$**		
	LR	**NI**	%	**LR**	**NI**	%	**LR**	**NI**	%
0.25	0.0084	0.0114	26.39	0.0145	0.0218	33.58	0.0519	0.0565	8.15
0.5	0.0179	0.0199	10.04	0.0309	0.0337	8.32	0.0592	0.0620	4.61
1	0.0310	0.0319	2.85	0.0468	0.0477	1.98	0.0704	0.0714	1.32
2	0.0466	0.0448	0.65	0.0604	0.0606	0.47	0.808	0.0811	0.34
5	0.0546	0.0547	0.09	0.0675	0.0675	0.07	0.0828	0.0828	0.06

TABLE 4

Maturity	$\alpha = -0.05$	$\alpha = 0$	$\alpha = 0.05$
0.25	0.1722	0.1391	0.1676
0.5	0.1751	0.1616	0.1692
1	0.1830	0.1786	0.1793
2	0.1901	0.1890	0.1890
5	0.1962	0.1959	0.1959

TABLE 5

Time to maturity	$\alpha = -0.05$			$\alpha = 0$			$\alpha = 0.05$		
	LR	**NI**	%	**LR**	**NI**	%	**LR**	**NI**	%
0.25	0.0150	0.0198	24.35	0.0316	0.0368	14.29	0.0609	0.0662	8.01
0.5	0.0312	0.0336	6.99	0.0496	0.0521	4.86	0.0762	0.0788	3.30
1	0.0500	0.0510	2.01	0.0689	0.0700	1.58	0.0933	0.0945	1.22
2	0.0692	0.0696	0.58	0.0876	0.0880	0.49	0.1097	0.1102	0.41
5	0.0865	0.0866	0.11	0.1020	0.1021	0.10	0.1197	0.1199	0.09

TABLE 6

Maturity	$\alpha = -0.05$	$\alpha = 0$	$\alpha = 0.05$
0.25	0.2254	0.2157	0.2191
0.5	0.2331	0.2289	0.2291
1	0.2397	0.2379	0.2375
2	0.2442	0.2435	0.2432
5	0.2478	0.2475	0.2474

Table 5 gives the results obtained using the Lugannani–Rice and the numerical integration methods and shows the relative errors.

Once more, the values of the parameters have been chosen in such a way that the implied volatilities are around 20% as Table 6 illustrates.

3. Conclusions. We have shown how the saddlepoint method can be used to price European puts on assets whose return process is more general than the standard Gaussian model. The key feature is that the moment generating function of returns must be sufficiently explicit that we can analyze it. Various examples with Lévy returns have been shown to be amenable to this approach, which also embraces many stochastic volatility–stochastic interest rate models discussed in the literature. The accuracy of the approximation improves as the expiry increases; this is not surprising, since for longer expiry, the return distribution will be a better approximation to the Gaussian base used in the saddlepoint approximation. For expiry one year or more, we get accuracy of the order of 2%, comparable to the accuracy of parameter estimates (or even a lot better!). Thus the approximation is good enough to be useful and is able to compute thousands of options a second, so it is *very* fast. There is scope for improving the accuracy considerably, by more cunning choice of the comparison distribution, but this choice would depend on just what return distribution one wished to work with, and this is more an econometric issue, for which there are no clear answers.

APPENDIX

We give the briefest explanation of the saddlepoint method, without any attempt at proof. Jensen [8] gives a careful account.

As explained above, our goal is to approximate the tail probabilities $\mathbb{P}(X > a)$, where X is a random variable whose distribution is known through its cumulant generating function (CGF) K:

$$\mathbb{E}\exp(z\,X) = \exp K(z).$$

The CGF K is assumed analytic in some strip containing the imaginary axis. Typically, K will be reasonably tractable, but the distribution F of X will not be.

By Fourier inversion,

$$\begin{aligned}\mathbb{P}(X > a) &= \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\exp(-\varepsilon(x-a))\,I_{\{x>a\}}\,F(dx)\\ &= \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{\exp(-i\,t\,a)}{\varepsilon+i\,t}\exp\{K(i\,t)\}\frac{dt}{2\,\pi}\end{aligned} \tag{A.1}$$

since the Fourier transform of $x \to \exp(\varepsilon\,a - \varepsilon\,x)\,I_{\{x>a\}}$ is $t \to \exp(i\,t\,a)(\varepsilon - i\,t)^{-1}$.

Now letting $\varepsilon \downarrow 0$ in (A.1) may be problematic because we have a pole at zero, but by Cauchy's theorem we have for *any* $c > 0$ in the strip of analyticity of K,

$$\begin{aligned}\int_{-i\infty}^{i\infty}\frac{\exp\{-z\,a+K(z)\}}{\varepsilon+z}\,\frac{dz}{2\,\pi\,i} &= \int_{c-i\infty}^{c+i\infty}\frac{\exp\{-z\,a+K(z)\}}{\varepsilon+z}\,\frac{dz}{2\,\pi\,i}\\ &\to \int_{c-i\infty}^{c+i\infty}\frac{\exp\{-z\,a+K(z)\}}{z}\,\frac{dz}{2\,\pi\,i}, \qquad \varepsilon\downarrow 0\end{aligned} \tag{A.2}$$

and the key to the saddlepoint method is a cunning choice of c. In fact, we choose c so that the function $K(x) - a\,x$ is minimized:

$$K'(c) = a.$$

This value of c will be strictly positive if and only if $a > K'(0) \equiv \mathbb{E}\,X$, which we assume from now on. If $a < \mathbb{E}\,X$, we estimate $\mathbb{P}(X < a)$ mutatis mutandis. [Incidentally, the name "saddlepoint" comes from the fact that the function $z \to K(c+z) - a\,(c+z)$ looks like $\frac{1}{2}z^2\,K''(c)$ for small z, and the real part of this is the saddle-shaped function $(x, y) \to \frac{1}{2}K''(c)\,(x^2 - y^2)$.]

The saddlepoint approximation is achieved by comparison with some "base" distribution with CGF K_0. Classically, this is the Gaussian distribution, for which $K_0(z) = \frac{1}{2}z^2$, but it is important to realize that one may use other base distributions. The base distribution is assumed nice enough that we can find the distribution F_0 quite explicitly. By shifting and scaling F_0, we may transform K_0 to $z \to -\xi z + K_0(\lambda z) \equiv \tilde{K}_0(z)$, say, for real constants ξ, λ, which we may choose to make the minimum of $\tilde{K}_0$ at c and to match the second derivatives of $\tilde{K}_0$ and K there. This turns out not to be the right thing though, because, although the behavior at $z = c$ is well approximated, the behavior at $z = 0$ is not. Instead we pick ξ so that

$$\min_x K_0(x) - \xi x = \min_x K(x) - a x$$

and then suppose that we have an analytic map $z \to w(z)$ such that

$$K_0(w) - \xi w = K(z) - a z. \tag{A.3}$$

We have in particular that $\hat{w} \equiv w(c)$ is the place where $K_0(x) - \xi x$ is minimized, and $w(0) = 0$. Hence by change of variable in (A.2), with Γ the image of $c + i\,\mathbb{R}$ under w,

$$\begin{aligned}
\mathbb{P}(X > a) &= \int_\Gamma \exp\{K_0(w) - \xi w\} \frac{1}{z} \frac{dz}{dw} \frac{dw}{2\pi i} \\
&= \int_\Gamma \exp\{K_0(w) - \xi w\} \frac{1}{w} \frac{dw}{2\pi i} \\
&\quad + \int_\Gamma \exp\{K_0(w) - \xi w\} \left(\frac{1}{z} \frac{dz}{dw} - \frac{1}{w} \right) \frac{dw}{2\pi i}.
\end{aligned}$$

The first term is nothing other than $1 - F_0(\xi)$, and for the second term, we note that there is no singularity of the integrand at $w = 0$; since $w(0) = 0$, we have that $z = w\,(dz/dw)(0) + O(w^2)$ for small w. So this allows us to expand the term $(1/z)(dz/dw) - 1/w$ about $w = \hat{w}$ and collect terms; the power-series expansion for $z = z(w)$ about $w = \hat{w}$ can be evaluated to any desired order using (A.3), since the power-series expansions of K_0 and K are assumed known.

The resulting integrals of the form $\int_\Gamma w^n \exp\{K_0(w) - \xi w\}\, dw$ can be written down in terms of the (known) density of F_0, and its derivatives. Rather surprisingly, for many practical applications, one term is enough; in this case, the expansion gives the celebrated Lugannani–Rice formula (see [9]).

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