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LARGE DEVIATIONS OF COMBINATORIAL DISTRIBUTIONS II. LOCAL LIMIT THEOREMS

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We derive a general local limit theorem for probabilities of large deviations for a sequence of random variables by means of the saddlepoint method on Laplace-type integrals. This result is applicable to parameters in a number of combinatorial structures and the distribution of additive arithmetical functions.

1. Main result. This paper is a sequel to our paper [18] where we derived a general central limit theorem for probabilities of large deviations applicable to many classes of combinatorial structures and arithmetic functions. The ranges of large deviations treated here are usually referred to as "moderate deviations" in probability literature (cf., e.g., [23]); we follow the same terms as in Part I [18] to keep consistency. In this paper we consider corresponding local limit theorems. More precisely, given a sequence of integral random variables $\{\Omega_n\}_{n\geq 1}$ each of maximal span 1 (see below for definition), we are interested in the asymptotic behavior of the probabilities

$$\Pr\{\Omega_n = m\}, \qquad (m \in \mathbb{N},\ m = \mu_n \pm x_n\sigma_n,\ \mu_n := \mathbf{E}\,\Omega_n,\ \sigma_n^2 := \operatorname{Var}\Omega_n),$$

as $n \to \infty$, where x_n can tend to ∞ with n at a rate that is restricted to $O(\sigma_n)$. Our interest here is not to derive asymptotic expression for $\Pr\{\Omega_n = m\}$ valid for the widest possible range of m, but to show that for m lying in the interval $\mu_n \pm O(\sigma_n^2)$, very precise asymptotic formulas can be obtained. These formulas are in close connection with our results in [18]. Although local limit theorems receive a constant research interest [2, 3, 7, 14, 13, 24], our approach and results, especially Theorem 1, seem rarely discussed in a systematic manner.

Recall that a lattice random variable X is said to be of *maximal span* h if X takes only values of the form $b + hk$, $k \in \mathbb{Z}$, for some constants b and $h > 0$, and there does not exist b' and $h' > h$ such that X takes only values of the form $b' + h'k$.

Let us now state our main result. Let $\{\Omega_n\}_{n\geq 1}$ be a sequence of random variables taking only integral values. Suppose that Ω_n is of maximal span 1 for $n \geq n_0$ ($n_0 \geq 1$). Assume further that the moment generating functions $M_n(s) := \mathbf{E}\,e^{\Omega_n s} = \sum_{m\in\mathbb{Z}} \Pr\{\Omega_n = m\}e^{ms}$ satisfy

$$M_n(s) = e^{\phi(n)u(s)+v(s)}(1 + O(\kappa_n^{-1})), \qquad n \to \infty, \tag{1}$$

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uniformly for $|s| \le \rho$, $s \in \mathbb{C}$, $\rho > 0$, where the following hold:

1. $\{\phi(n)\}$ is a sequence of n such that $\phi(n) \to \infty$ as $n \to \infty$;
2. $u(s)$ and $v(s)$ are functions of s independent of n and are analytic for $|s| \le \rho$; furthermore $u''(0) \ne 0$;
3. $\kappa_n \to \infty$;
4. the moment generating functions $M_n(s)$ satisfy condition (A): there exist constants $0 < \varepsilon \le \rho$ and $c = c(\varepsilon, r) > 0$, where $\varepsilon > 0$ may be taken arbitrarily small but fixed, such that

$$\left|\frac{M_n(r+it)}{M_n(r)}\right| = O\big(e^{-c\phi(n)}\big), \tag{A}$$

uniformly for $-\rho \le r \le \rho$ and $\varepsilon \le |t| \le \pi$, as $n \to \infty$.

As in [18], let us introduce the following notation:

$$u_m := u^{(m)}(0), \qquad v_m := v^{(m)}(0), \qquad m = 1, 2, 3, \dots,$$
$$\mu_n := u_1\phi(n), \qquad \sigma_n^2 := u_2\phi(n).$$

The symbol $[z^n]f(z)$ denotes the coefficient of z^n in $f(z)$.

THEOREM 1. *If $m = \mu_n + x\sigma_n$, $x = o(\sqrt{\phi(n)})$, then the probabilities $\Pr\{\Omega_n = m\}$ satisfy asymptotically*

$$\begin{aligned}\Pr\{\Omega_n = m\} = \frac{e^{-x^2/2+\phi(n)Q(\xi)}}{\sqrt{2\pi u_2\phi(n)}}\Bigg(1 + \sum_{1\le k\le \nu} \frac{\Pi_k(x)}{(u_2\phi(n))^{k/2}} \\ + O\bigg(\frac{|x|^{\nu+1}+1}{\phi^{(\nu+1)/2}(n)} + \frac{|x|+1}{\kappa_n\sqrt{\phi(n)}}\bigg)\Bigg),\end{aligned} \tag{2}$$

where $\xi = x/\sigma_n$, $Q(\xi) = Q(u;\xi) = \sum_{k\ge 3} q_k\xi^k$ is analytic at 0 with the coefficients q_k defined by

$$q_k = \frac{-1}{k}[w^{k-2}]u''(w)\left(\frac{u'(w)-u_1}{u_2 w}\right)^{-k}, \qquad k = 3, 4, 5, \dots, \tag{3}$$

ν is a nonnegative integer (depending upon the error term κ_n^{-1}) and the $\Pi_k(x)$ are polynomials of degree k such that $\Pi_{2j}(x)$ has only even powers of x and $\Pi_{2j-1}(x)$ has only odd powers of x, for $j = 1, 2, \dots$.

Note that the Cramér-type power series $Q(y)$ is exactly the same as in Theorem 1 of [18].

This theorem generalizes a result of Richter [28] on local limit theorem for large deviations for sums of independent and identically distributed random variables. Richter's result has been generalized by many authors; see [4, 21] and the references therein.

In a combinatorial context, general (multidimensional) local limit theorems have been derived by Gao and Richmond [13] under different settings. Their methods, including the Fourier inversion formula and the technique of "shifting the mean," although apparently different, have the same analytic character. On the other hand, our results are comparatively more precise.

The asymptotic relation (1) is a priori a local one and need not be uniformly valid throughout the region $-\rho \le \Re s \le \rho$ and $|\Im s| \le \pi$. In fact, the functions u and v may present discontinuities there (usually at $\Im s = \pm\pi$). When (1) holds in the above extended rectangle, condition (A) can be replaced by the following one: there exist constants $0 < \varepsilon < \pi$ and $\delta = \delta(\varepsilon) > 0$, such that

$$\Re\, u(r + it) - u(r) < -\delta, \qquad -\rho \le r \le \rho,\ \ \varepsilon \le |t| < \pi.$$

Such a condition is used in [13]. Note that the remaining cases $t = \pm\pi$ will not cause further difficulty due to the uniform continuity of characteristic functions.

In general, the availability of condition (A), usually a result of aperiodicity of coefficients, separates, on the one hand, the central limit theorem from the local limit theorem for lattice random variables. It also has, on the other hand, a straightforward effect on estimates of probabilities of large deviations, as may be simply seen by the following rough arguments which can be justified using similar (and simpler) proof techniques of Theorem 1. By the integral formula (5) below, we have the upper bound

$$\Pr\{\Omega_n = m\} \le \frac{e^{-mr}}{2\pi} \int_{-\pi}^{\pi} |M_n(r + it)|\, dt.$$

If condition (A) is not available, we have

$$\Pr\{\Omega_n = m\} \le e^{-mr} M_n(r) = O\big(e^{-mr+u(r)\phi(n)}\big);$$

on the other hand, assuming condition (A) [cf. (7) below],

$$\begin{aligned} \Pr\{\Omega_n = m\} &= O\bigg(e^{-mr+u(r)\phi(n)} \int_{-\varepsilon}^{\varepsilon} e^{\phi(n)\Re(u(r+it)-u(r))}\, dt + e^{-mr-c\phi(n)}\bigg) \\ &= O\big(\phi(n)^{-1/2} e^{-mr+u(r)\phi(n)}\big), \end{aligned} \tag{4}$$

for all m and a suitable choice of $r > 0$ [in a way to minimize the value of $e^{-mr+u(r)\phi(n)}$]. If $m = a\phi(n)$ and the real solution r_0 to the equation $u'(s) = a$ satisfies $-\rho \le r_0 \le \rho$, then we take $r = r_0$; otherwise, we take $r = \rho$ so that the right-hand side of (4) is uniformly small. Therefore, the smooth (or regularity) condition (A), which reflects the concentration of moment generating functions around real axis, yields the correct order of tail probabilities. Estimates for $\Pr\{\Omega_n \ge m\}$ for $|m/\phi(n) - u_1| > \varepsilon > 0$ are easily derived by summation of the right-hand side of (4).

We prove Theorem 1 in the next section; the method of proof is based on the saddlepoint method on Laplace-type integrals. Many immediate consequences of this result will be given in Section 3. These results will then be applied to the combinatorial schemes of Flajolet and Soria [11, 12] in Section 4. Finally, we discuss some examples in Section 5. We conclude this paper with an extension of our results.

Throughout this paper, all generating functions (ordinary, exponential, bivariate, etc.) denote functions *analytic at* 0 with *nonnegative coefficients*. Following a number-theoretic convention, the symbols O and $\ll$ are equivalent and will be used interchangeably as is convenient. All limits, (including O, $\ll$, o and $\sim$), whenever unspecified, will be taken as $n \to \infty$. The symbols ε, δ always represent sufficiently small (but fixed, namely, independent of the major asymptotic parameter), positive numbers whose values may differ from one occurrence to another.

2. Proof of Theorem 1. Write $m = \mu_n + x\sigma_n$, $x = O(\sqrt{\phi(n)})$. We start from the integral formula

$$\Pr\{\Omega_n = m\} = \frac{1}{2\pi i}\int_{r-i\pi}^{r+i\pi} M_n(z)e^{-mz}\,dz =: I_{n,m}, \qquad -\rho \le r \le \rho. \tag{5}$$

Divide the integral $I_{n,m}$ into two parts: $I_{n,m} = I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon} M_n(r+it)e^{-mr-mit}\,dt; \qquad I_2 = \frac{1}{2\pi}\int_{\varepsilon<|t|\le\pi} M_n(r+it)e^{-mr-mit}\,dt.$$

Consider first I_2. By condition (A) and (1), we obtain

$$|I_2| \ll M_n(r)e^{-mr-c\phi(n)} \ll e^{u(r)\phi(n)-mr-c\phi(n)}. \tag{6}$$

For I_1, we write (1) in the form

$$M_n(s) = T_n(s)(1+R_n(s)),$$

where $R_n(s)$ satisfies $R_n(0) = 0$ and $R_n(s) \ll \kappa_n^{-1}$ for $|s| \le \rho$. For small ε, we have, by Cauchy's integral formula, the better estimate

$$R_n(s) \ll R_n'(0)|s| \ll \kappa_n^{-1}|s|,$$

for $|s| \le \varepsilon$. Thus, we can write

$$I_1 = \frac{e^{u(r)\phi(n)+v(r)}}{2\pi e^{mr}}(I_3 + I_4),$$

where

$$I_3 = \int_{-\varepsilon}^{\varepsilon} \exp((u(r+it)-u(r))\phi(n) + v(r+it) - v(r) - mit)\,dt,$$

$$I_4 \ll \kappa_n^{-1}\int_{-\varepsilon}^{\varepsilon} |r+it| \exp(\Re(u(r+it)-u(r))\phi(n))\,dt.$$

For sufficiently small ε, there exists a constant $\gamma > 0$ such that $\Re(u(r+it) - u(r)) < -\gamma t^2$, since $u''(r) > 0$ ($\log M_n(r)$ being convex; cf. [18]). Thus

$$I_4 \ll \kappa_n^{-1}\int_{-\varepsilon}^{\varepsilon} (|r| + |t|)e^{-\gamma\phi(n)t^2}\,dt \ll \kappa_n^{-1}\big(|r|\phi(n)^{-1/2} + \phi(n)^{-1}\big). \tag{7}$$

We now take, as in [18], r to be the (approximate) saddlepoint of the integrand of I_3; namely, r satisfies the equation

$$u'(r)\phi(n) = m = u_1\phi(n) + x\sqrt{u_2\phi(n)}. \tag{8}$$

The solution always exists (cf. [18]) whenever $x = o(\sqrt{\phi(n)})$ and satisfies

$$r \ll \frac{|x|}{\sqrt{\phi(n)}} \le \rho,$$

for n large enough. This implies that

$$I_4 \ll \kappa_n^{-1}\big(|r|\phi(n)^{-1/2} + \phi(n)^{-1}\big) \ll \frac{|x|+1}{\kappa_n \phi(n)}. \tag{9}$$

On the other hand, with this r, we have

$$I_3 = \int_{-\varepsilon}^{\varepsilon} \exp\bigg(\phi(n) \sum_{k\ge 2} \frac{u^{(k)}(r)}{k!}(it)^k + \sum_{k\ge 1} \frac{v^{(k)}(r)}{k!}(it)^k\bigg)\,dt.$$

Since $u''(r) > 0$, we can carry out the change of variable $y = \lambda_n t = \sqrt{u''(r)\phi(n)}\,t$ and obtain

$$I_3 = \lambda_n^{-1} \int_{-\varepsilon\lambda_n}^{\varepsilon\lambda_n} e^{-y^2/2} \exp\bigg(\sum_{k\ge 1} \frac{1}{k!}\bigg(\frac{iy}{\lambda_n}\bigg)^k \bigg(v^{(k)}(r) + \frac{u^{(k+2)}(r)(iy)^2}{(k+1)(k+2)u''(r)}\bigg)\bigg)\,dy.$$

Define polynomials $P_k(z)$ by the formal expansion (cf. [22], page 149; [16], Section 2.4)

$$\exp\bigg(\sum_{k\ge 1} \frac{1}{k!}\bigg(\frac{z}{\lambda_n}\bigg)^k \bigg(v^{(k)}(r) + \frac{u^{(k+2)}(r)z^2}{(k+1)(k+2)u''(r)}\bigg)\bigg) = \sum_{k\ge 0} \frac{P_k(z)}{\lambda_n^k},$$

the degree of $P_k(z)$ being equal to $3k$ ($k \ge 0$). The same argument as that used in [16], Lemma 2 (cf. also [6], Lemma 2, Chapter VII; [22], page 151) leads to

$$I_3 = \sum_{0\le k\le \nu} \frac{1}{\lambda_n^{k+1}} \int_{-\infty}^{\infty} \exp\bigg(-\frac{y^2}{2}\bigg) P_k(iy)\,dy + O\bigg(\frac{1}{\lambda_n^{\nu+2}}\bigg),$$

for any nonnegative integer ν. Now from the explicit expression of $P_k(z)$ (cf. [16], page 67)

$$P_k(z) = z^k \sum_{\substack{j_1+2j_2+\cdots+kj_k=k\\ j_1,j_2,\dots,j_k\ge 0}} \prod_{1\le l\le k} \frac{1}{l!\,j_l!}\bigg(v^{(l)}(r) + \frac{u^{(l+2)}(r)z^2}{(l+1)(l+2)u''(r)}\bigg)^{j_l},$$

$$k = 1, 2, 3, \dots,$$

we observe that the $P_{2k}(z)$ contain only even powers of z (from z^{2k}, $z^{2k+2}, \dots,$ z^{6k}) and that the $P_{2k+1}(z)$ contain only odd powers of z (from z^{2k+1}, $z^{2k+3}, \dots,$ z^{6k+3}) for $k \ge 0$. It follows that $\int_{-\infty}^{\infty} e^{-y^2/2} P_{2k+1}(iy)\,dy = 0$ for all nonnegative integer k. Thus we obtain the asymptotic expansion

$$I_3 \sim \frac{\sqrt{2\pi}}{\sqrt{u''(r)\phi(n)}}\bigg(1 + \sum_{k\ge 1} \frac{d_{2k}(r)}{(u''(r)\phi(n))^k}\bigg), \tag{10}$$

where $(c_{kj}(r) := [z^{2j}]P_{2k}(z))$,

$$d_{2k}(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} P_{2k}(iy)\, dy$$
$$= \sum_{k \le j \le 3k} (-1)^j \frac{(2j)!}{2^j j!} c_{kj}(r), \qquad k = 1, 2, 3, \dots .$$

Returning to $\Pr\{\Omega_n = m\}$, we have, by (5)–(10),

$$\Pr\{\Omega_n = m\} = \frac{e^{u(r)\phi(n)+v(r)-mr}}{\sqrt{2\pi u''(r)\phi(n)}} \Bigg(1 + \sum_{1 \le k \le \nu} \frac{d_{2k}(r)}{(u''(r)\phi(n))^k} + O\bigg(\frac{1}{\phi^{\nu+1}(n)} + \frac{|r|}{\kappa_n \sqrt{\phi(n)}} + \frac{1}{\kappa_n \phi(n)}\bigg)\Bigg), \tag{11}$$

for some nonnegative integer ν. In particular,

$$d_2(r) = \frac{1}{8} - \frac{v''(r)}{2} - \frac{v'(r)^2}{2} + \frac{v'(r)u'''(r)}{2u^{(4)}(r)} - \frac{5u'''(r)^2}{24u^{(4)}(r)^2}.$$

Note that when $x = 0$ (this implies that the μ_n are integers), we obtain $(r = 0)$

$$\Pr\{\Omega_n = \mu_n\} = \frac{1}{\sqrt{2\pi u_2 \phi(n)}} \Bigg(1 + \sum_{1 \le k \le \nu} \frac{d_{2k}(0)}{(u_2\phi(n))^k} + O\bigg(\frac{1}{\phi^{\nu+1}(n)} + \frac{1}{\kappa_n \phi(n)}\bigg)\Bigg),$$

(probability of the mode). Let us now prove (2). Set $\xi = x/\sigma_n$. In [18], we showed that

$$\phi(n)(u(r) - ru'(r)) = -\frac{x^2}{2} + \phi(n)Q(\xi),$$

and similarly, we can write

$$\frac{e^{v(r)} d_{2k}(r)}{u''(r)^{k+1/2}} = \frac{1}{u_2^{k+1/2}} \bigg(\sum_{j \ge 0} b_{kj}\, \xi^j\bigg), \qquad k = 0, 1, 2, \dots,$$

for some coefficients b_{kj}. Theorem 1 follows from collecting terms according to the powers of ξ (or x):

$$\Pi_{2k}(x) = \sum_{0 \le j \le k} b_{k-j,\,2j} x^{2j}, \quad \text{and} \quad \Pi_{2k+1}(x) = \sum_{0 \le j \le k} b_{k-j,\,2j+1} x^{2j+1},$$

for $k = 0, 1, 2, \dots$. In particular,

$$\Pi_0(x) = 1, \qquad \Pi_1(x) = \bigg(v_1 - \frac{u_3}{2u_2}\bigg)x,$$
$$\Pi_2(x) = \bigg(\frac{v_2}{2} + \frac{v_1^2}{2} - \frac{v_1 u_3}{u_2} + \frac{5u_3^2}{8u_2^2} - \frac{u_4}{4u_2}\bigg)x^2 + \frac{1}{8} - \frac{v_2}{2} - \frac{v_1^2}{2} + \frac{v_1 u_3}{2u_4} - \frac{5u_3^2}{24u_4^2}.$$

This completes the proof. □

3. Some corollaries of Theorem 1. The following two corollaries are immediate consequences of Theorem 1.

COROLLARY 1. *If* $m = \mu_n + x\sigma_n$, $x = o(\phi(n)^{1/2})$, *then*

$$\Pr\{\Omega_n = m\} = \frac{e^{-x^2/2+\phi(n)Q(\xi)}}{\sqrt{2\pi u_2 \phi(n)}}\left(1 + O\left(\frac{|x|+1}{\sqrt{\phi(n)}}\right)\right).$$

COROLLARY 2. *If* $m = \mu_n + x\sigma_n$, $x = o(\phi(n)^{1/6})$, *then*

$$\Pr\{\Omega_n = m\} = \frac{e^{-x^2/2}}{\sqrt{2\pi u_2 \phi(n)}}\left(1 + O\left(\frac{|x|^3+1}{\sqrt{\phi(n)}}\right)\right).$$

Other corollaries like those in Section 3 of [18] can be derived.

As in [18], define the real function Λ of y, $y > -1$, by

$$\Lambda(y) := (1+y)\log(1+y) - y - \frac{y^2}{2} = \sum_{k\geq 3} \frac{(-1)^k}{k(k-1)}\, y^k,$$

the series being convergent for $|y| \leq 1$.

THEOREM 2. *Let* $u(s) = e^s - 1$ *in Theorem* 1 *and assume that* $\kappa_n^{-1} = o(\phi(n)^{-M})$ *for any* $M > 0$. *Then, for* $m = \phi(n) + x\sqrt{\phi(n)}$, $x = o(\sqrt{\phi(n)})$, *we have*

$$\Pr\{\Omega_n = m\} = \frac{\exp(-x^2/2 - \phi(n)\Lambda(x/\sqrt{\phi(n)}))}{\sqrt{2\pi\phi(n)}} \times \left(1 + \sum_{1\leq k\leq \nu} \frac{\Pi_k(x)}{\phi(n)^{k/2}} + O\left(\frac{|x|^{\nu+1}+1}{\phi(n)^{(\nu+1)/2}}\right)\right), \tag{12}$$

or, equivalently, the Poisson approximation formula

$$\Pr\{\Omega_n = m\} = \frac{\phi(n)^m}{m!} e^{-\phi(n)}\left(1 + \sum_{1\leq k\leq \nu} \frac{\varpi_k(x)}{\phi(n)^{k/2}} + O\left(\frac{|x|^{\nu+1}+1}{\phi(n)^{(\nu+1)/2}}\right)\right), \tag{13}$$

for some polynomials $\varpi_k(x)$ *and some nonnegative integer* ν.

PROOF. For (12), the saddle point r in (8) satisfies $e^r = 1 + \xi = m(\phi(n))^{-1}$. For (13), we have, by (11),

$$\begin{aligned}\Pr\{\Omega_n = m\} &= \frac{e^{-\phi(n)+m+v(r)}\phi(n)^m}{\sqrt{2\pi m}\, m^m}\left(1 + \sum_{1\leq k\leq \nu} \frac{d_{2k}(r)}{m^k} + O\left(\frac{1}{m^{\nu+1}}\right)\right) \\ &= \frac{e^{-\phi(n)}\phi(n)^m}{m!}\, e^{v(r)}\left(1 + \sum_{1\leq k\leq \nu} \frac{\widetilde{d}_{2k}(r)}{m^k} + O\left(\frac{1}{m^{\nu+1}}\right)\right),\end{aligned}$$

by applying Stirling's formula backwards. The required result follows from expanding each factor $e^{v(r)}\widetilde{d}_{2k}(r)/m^k$ in powers of ξ. □

In particular,

$$\varpi_1(x) = v_1 x, \qquad \varpi_2(x) = \tfrac{1}{2}(v_1^2 + v_2 - v_1)(x^2 - 1)$$

and in general the $\varpi_k(x)$ have the same degree property as $\Pi_k(x)$.

From the Poisson approximation formula (13), we can easily derive precise asymptotics for, say, the total variation distance of Ω_n and a Poisson distribution with mean $\phi(n)$. Also our Theorem 1 is useful for further asymptotics of different probability metrics; compare, for example, [1] and the references cited there.

4. Combinatorial schemes of Flajolet and Soria. In this section, we apply the theorems derived in previous sections to the combinatorial distributions studied by Flajolet and Soria [11, 12]. These distributions are classified according to the type of singularity of their bivariate generating functions.

4.1. *The* exp–log *class.* We state a definition for logarithmic function slightly stronger than that in [12] and [18].

A generating function $C(z)$ is called *logarithmic* (cf. [11]) if (i) $C(z)$ is analytic for $z \in \Delta$:

$$\Delta := \{z\colon |z| \le \zeta + \varepsilon \quad \text{and} \quad |\arg(z-\zeta)| \ge \delta\}\backslash\{\zeta\}, \qquad \varepsilon > 0,\ 0 < \delta < \pi/2,$$

$\zeta > 0$ being the radius of convergence and the sole singularity of C for $z \in \Delta$, and (ii) there exists a constant $a > 0$, such that for $z \sim \zeta$, $z \in \Delta$,

$$C(z) = a \log \frac{1}{1 - z/\zeta} + H((1 - z/\zeta)^{1/b}),$$

where $b = 1, 2, 3, \ldots$, $H(u)$ is analytic at 0 and satisfies the expansion

$$H(u) = K + \sum_{k\ge 1} h_k u^k,$$

convergent for $|u| \le \varepsilon$, K being some constant.

For brevity, we say that C is logarithmic with parameters (ζ, a, K, b).

Now consider generating functions of the form

$$P(w, z) = \sum_{n,\, m\ge 0} P_n(w) z^n = e^{wC(z)} Q(w, z),$$

where $C(z)$ is logarithmic with parameters (ζ, a, K, b) and $Q(w, z)$ satisfies the following two conditions:

1. As a function of z, $Q(w, z)$ is analytic for $|z| \le \zeta$, namely, it has a larger radius of convergence than C.
2. As a function of w, $Q(w, \zeta)$ is analytic for $|w| \le \eta$, where $\eta > 1$.

Roughly, these assumptions imply that for any fixed w, $|w| \le \eta$, $P(w, z)$ satisfies

$$P(w, z) = e^{Kw} Q(w, \zeta)\left(1 - \frac{z}{\zeta}\right)^{-aw} \left(1 + O\big((1 - z/\zeta)^{1/b}\big)\right), \qquad z \sim \zeta,\ z \notin [\zeta, \infty),$$

and $P(w, z)$ is analytically continuable to a Δ-region. We can then apply the singularity analysis of Flajolet and Odlyzko [10] to deduce the asymptotic formula

$$P_n(w) := [z^n]P(w, z) = \frac{\zeta^{-n} n^{aw-1}}{\Gamma(aw)} e^{Kw} Q(w, \zeta)\big(1 + O(n^{-1/b})\big), \tag{14}$$

the O-term being uniform with respect to w, $|w| \le \eta$. We note that although an asymptotic expansion in terms of the ascending powers of $n^{-1/b}$ can be derived under our assumptions, the expression (14) suffices for our purposes.

Since $\eta > 1$, $P_n(1)$ is well defined. Thus for the moment generating functions $M_n(s)$ of the random variables Ω_n defined by $M_n(s) := \mathbf{E}\, e^{\Omega_n s} = P_n(e^s)/P_n(1)$, we have

$$M_n(s) = e^{(e^s-1)a \log n} \frac{e^{K(e^s-1)}\Gamma(a)Q(e^s, \zeta)}{\Gamma(ae^s)Q(1, \zeta)} \big(1 + O(n^{-1/b})\big),$$

uniformly for $-\log \eta \le \Re s \le \log \eta$ and $|\Im s| \le \pi$. Note that $\log \eta > 0$. The application of Theorem 2 is straightforward since from (14) the $P_n(w)$ are aperiodic for sufficiently large n and condition (A) is easily checked.

THEOREM 3. *Let Ω_n be defined as above. If $m = a \log n + x\sqrt{a \log n}$, $x = o(\sqrt{\log n})$, then we have*

$$\begin{aligned}\Pr\{\Omega_n = m\} = {} & \frac{\exp(-x^2/2 - a \log n \cdot \Lambda(x/\sqrt{a \log n}))}{\sqrt{2\pi a \log n}} \\ & \times \left(1 + \sum_{1\le k\le \nu} \frac{\Pi_k(x)}{(a \log n)^{k/2}} + O\left(\frac{|x|^{\nu+1} + 1}{(\log n)^{(\nu+1)/2}}\right)\right).\end{aligned}$$

In other words, Ω_n obeys asymptotically a Poisson distribution of parameter $a \log n$:

$$\Pr\{\Omega_n = m\} = \frac{(a \log n)^m}{m!} e^{-a \log n} \left(1 + \sum_{1\le k\le \nu} \frac{\varpi_k(x)}{(a \log n)^{k/2}} + O\left(\frac{|x|^{\nu+1} + 1}{(\log n)^{(\nu+1)/2}}\right)\right).$$

PROOF. Take $\phi(n) = a \log n$ in Theorem 2. $\square$

4.2. *The algebraic–logarithmic class.* Next, let us consider generating functions of the form

$$\begin{aligned} P(w, z) &= \sum_{n,\, m\ge 0} P_n(w) z^n \\ &= \frac{1}{(1 - wC(z))^{\alpha}} \left(\log \frac{1}{1 - wC(z)}\right)^{\beta}, \qquad C(0) = 0, \end{aligned} \tag{15}$$

where $\beta \in \mathbb{N}$, $\alpha \geq 0$, and $\alpha + \beta > 0$. Define the moment generating functions $M_n(s)$ of the random variables Ω_n by the coefficients of $P_n(w)$:

$$M_n(s) = \sum_{m\geq 0} \Pr\{\Omega_n = m\}\, e^{ms} = \frac{[z^n]P(e^s, z)}{[z^n]P(1, z)}. \tag{16}$$

DEFINITION (1-regular function [12]). A generating function $C(z) \not\equiv z^q$ $(q = 0, 1, 2, \ldots)$ analytic at $z = 0$ is called *1-regular* if there exists a positive number $\rho < \zeta$, ζ being the radius of convergence of $C(z)$, such that $C(\rho) = 1$. Assume, without loss of generality, that $C(z)$ is aperiodic, namely, $C(z) \not\equiv z^e \sum_{n\geq 0} c_n z^{nd}$ for some integers $e \geq 0$ and $d \geq 2$. (This aperiodic condition is slightly stronger than the one in [12], page 166.)

We need the following simple lemma.

LEMMA 1. *Let $P_n(w) = [z^n]P(w, z)$, where $P(w, z)$ has the form* (15). *$C(z)$ is aperiodic iff $P_n(w)$ is aperiodic for $n \geq n_0$, n_0 depending only upon β and the period of C.*

PROOF. If $C(z) = z^q D(z^p)$, where $q \geq 0$, $p \geq 1$ are integers and D is a power series, then

$$P_n(w) = \sum_{\substack{qk+pj=n\\ k,\, j\geq 0}} A_k B_{kj} w^k, \qquad n \geq 1,$$

where

$$A_k = [z^k]\left(\frac{1}{1-z}\right)^{\alpha}\left(\log\frac{1}{1-z}\right)^{\beta} \quad \text{and} \quad B_{kj} = [z^{pj}]D(z^p)^k.$$

Since $A_k > 0$ for $k \geq \beta$, the result follows from considering the cases $p = 1$ (C being aperiodic) and $p \geq 2$ (C being periodic). $\square$

Assume that $C(z)$ is 1-regular and $\rho(w)$ satisfies $\rho(1) = \rho$ and $C(\rho(w)) = w^{-1}$ for $|w - 1| \leq \varepsilon$. By our periodicity condition, implicit function theorem and singularity analysis, we obtain (cf. [10, 12, 19]),

$$P_n(w) = \frac{\rho(w)^{-n} n^{\alpha-1}(\log n)^{\beta}}{(w\rho(w)C'(\rho(w)))^{\alpha}}\big(1 + \varepsilon_{\alpha,\beta}(n)\big), \tag{17}$$

uniformly for $|w - 1| \leq \varepsilon$, where

$$\varepsilon_{\alpha,\beta}(n) \ll \begin{cases} n^{-1}, & \text{if } (\alpha > 0 \text{ and } \beta = 0) \text{ or } (\alpha = 0 \text{ and } \beta = 1),\\ (\log n)^{-1}, & \text{if } (\alpha = 0 \text{ and } \beta \geq 2) \text{ or } (\alpha > 0 \text{ and } \beta > 0). \end{cases}$$

This result gives the local behavior of $P_n(w)$ for $w \sim 1$. To apply Theorem 1, we need an estimate for $|P_n(re^{it})|$ for r near 1 and $\varepsilon \leq |t| \leq \pi$.

A uniform estimate for $P_n(w)$. Observe first that $\rho(w)$ satisfies $|\rho(re^{it})| \geq \rho(r)$ for r and t in the region of validity of (17), in particular, for $|re^{it}-1| \leq \varepsilon$. For, otherwise, using the fact that the coefficients of $C(z)$ are nonnegative,

$$r^{-1} = |C(\rho(re^{it}))| \leq C(|\rho(re^{it})|) < C(\rho(r)) = r^{-1},$$

a contradiction. We deduce, by 1-regularity of $C(z)$, that

$$\left|\frac{P_n(re^{it})}{P_n(r)}\right| \ll \exp(-q(r)nt^2), \tag{18}$$

holds uniformly for $|r-1| \leq \delta$ and $|t| \leq \varepsilon$, where $q(r) > 0$ is sufficiently small.

We now show that such an estimate subsists for $\varepsilon \leq |t| \leq \pi$, namely, $M_n(s)$ satisfies condition (A).

LEMMA 2. *If $C(z)$ is 1-regular and $\alpha \geq 0$, $\beta \in \mathbb{N}$ satisfies $\alpha+\beta > 0$, then there exists an absolute constant $q = q(r) > 0$, independent of n, t, α, β, such that, for n sufficiently large,*

$$P_n(r\exp(it)) \ll P_n(r)\exp(-q(r)nt^2), \tag{19}$$

holds uniformly for $1-\delta \leq r \leq 1+\delta$ and $-\pi \leq t \leq \pi$.

PROOF. The case $|t| \leq \varepsilon$ having been proved in (18), we consider the remaining range $\varepsilon \leq |t| \leq \pi$.

Consider first the case $\alpha > 0$ and $\beta = 0$. We use induction on n. By Lemma 1, there exists a constant $n_0 > 1$ independent of α such that $P_n(w)$ is aperiodic for $n \geq n_0$. It is easily seen, by distinguishing two cases: $|t| \leq \varepsilon$ and $\varepsilon \leq |t| \leq \pi$, that (19) is satisfied [$q(r)$ can be chosen small enough so that it depends only upon C but not upon α].

Now suppose that (19) holds for $n_0 \leq n \leq N-1$. Differentiating the defining equation (15) of $P_n(w)$ and multiplying both sides by the factor $z(1-wC(z))$ yields the recurrence relation

$$NP_N(w) = w\sum_{1\leq k\leq N} c_k P_{N-k}(w)(\alpha k + N - k), \qquad N \geq 1, \tag{20}$$

with $P_0(w) = 1$, where $c_k := [z^k]C(z)$. Set $w = re^{it}$, where $1-\delta \leq r \leq 1+\delta$ and $\varepsilon \leq |t| \leq \pi$. We divide the estimate of the sum in (20) into two parts separated at $k = \lfloor N/2 \rfloor$. For $\lfloor N/2 \rfloor < k \leq N$, we use (17) in the form

$$P_N(r) \ll \rho(r)^{-N}N^{\alpha-1}, \qquad N \geq 1, \tag{21}$$

and obtain

$$re^{it}\sum_{\lfloor N/2 \rfloor < k \leq N} c_k P_{N-k}(re^{it})(\alpha k + N - k) \ll (\alpha+1)\rho(r)^{-N}N^{\alpha}\sum_{\lfloor N/2 \rfloor < k \leq N} c_k \rho(r)^k.$$

When δ is chosen sufficiently small, say, $\rho(1+\delta) < \zeta$, the series $C(z) = \sum_{n\geq 1} c_n z^n$ is convergent for $|z| \leq \rho(r)+\eta$, $0 < \eta < \zeta - \rho(r)$. Thus, by Cauchy's inequality, $c_k \ll (\rho(r)+\eta)^{-k}$, as $k \to \infty$. We obtain

$$(\alpha+1)\rho(r)^{-N}N^{\alpha}\sum_{\lfloor N/2\rfloor<k\le N} c_k\rho(r)^k \ll (\alpha+1)N^{\alpha}\rho(r)^{-N}\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}$$
$$\ll (\alpha+1)NP_N(r)\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}.$$

Returning to the recurrence (20), we obtain

$$NP_N(re^{it}) = re^{it}\sum_{1\le k\le N/2} c_kP_{N-k}(re^{it})(\alpha k+N-k)$$
$$+\,O\left((\alpha+1)NP_N(r)\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}\right).$$

Now we apply the induction hypothesis and the estimate (21) to the right-hand side:

$$P_N(re^{it}) \ll (\alpha+1)\sum_{1\le k\le N/2} c_kP_{N-k}(r)\exp(-q(r)(N-k)t^2)$$
$$+(\alpha+1)P_N(r)\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}$$
$$\ll (\alpha+1)\rho(r)^{-N}N^{\alpha-1}\exp(-q(r)Nt^2)\sum_{1\le k\le N/2} c_k\bigl(\rho(r)\exp(q(r)t^2)\bigr)^k$$
$$+(\alpha+1)P_N(r)\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}$$
$$\ll (\alpha+1)P_N(r)\left(\exp(-q(r)Nt^2)+\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2}\right).$$

Once η (depending only upon C) is fixed, $q(r)$ can be chosen in a way that it is consistent throughout the analyses and that

$$\left(1+\frac{\eta}{\rho(r)}\right)^{-N/2} \le \exp(-q(r)Nt^2), \qquad \varepsilon \le |t| \le \pi.$$

Thus the estimate (19) is established in the case $\alpha > 0$.

For the general case when $\beta > 0$, since β is a positive integer, we can differentiate the recurrence (20) β-times with respect to α (writing $P_n^{[\alpha,\beta]}(w)$ in this case):

$$NP_N^{[\alpha,\beta]}(w) = w\sum_{1\le k\le N} c_k\bigl((\alpha k+N-k)P_{N-k}^{[\alpha,\beta]}(w)+k\beta P_{N-k}^{[\alpha,\beta-1]}(w)\bigr), \qquad N\ge\beta,$$

and the estimate (19) follows the same line of argument and an induction on β. □

Returning to $M_n(s)$, from (17), we obtain

$$M_n(s) = \left(\frac{e^s\rho(e^s)C'(\rho(e^s))}{\rho(1)C'(\rho(1))}\right)^{-\alpha} \frac{\rho(e^s)^{-n}}{\rho(1)^{-n}}\big(1+\varepsilon_{\alpha,\beta}(n)\big), \tag{22}$$

uniformly for s in the disk $|s| \le \varepsilon$. The mean and the variance of Ω_n satisfy (cf. [12, 19])

$$\mathbf{E}\,\Omega_n = \alpha_1 n + O(1) \quad \text{and} \quad \operatorname{Var}\Omega_n = \alpha_2 n + O(1),$$

where the two constants α_1 and α_2 are defined by ($\rho := \rho(1)$):

$$\alpha_1 = \frac{1}{\rho C'(\rho)} \quad \text{and} \quad \alpha_2 = \frac{1}{\rho^2 C'(\rho)^2} + \frac{C''(\rho)}{\rho C'(\rho)^3} - \frac{1}{\rho C'(\rho)}.$$

THEOREM 4. *Let the random variables Ω_n be defined by* (16), *where C is* 1*-regular. Then for $m = \alpha_1 n + x\sqrt{\alpha_2 n}$, $x = o(\sqrt{n})$, we have*

$$\begin{aligned}\Pr\{\Omega_n = m\} &= \frac{\exp(-x^2/2 + nQ(\xi))}{\sqrt{2\pi\alpha_2 n}}\\ &\quad\times\left(1 + \frac{Bx}{\sqrt{\alpha_2 n}} + O\left(\frac{x^2+1}{n} + \frac{(|x|+1)\varepsilon_{\alpha,\beta}(n)}{\sqrt{n}}\right)\right),\end{aligned}$$

where $\xi = x/\sqrt{\alpha_2 n}$, $Q(\xi) = Q(u;\xi)$ with $u(s) = -\log(\rho(e^s)/\rho(1))$, and the coefficient B is given by ($C_j := C^{(j)}(\rho)$):

$$B = \frac{\alpha}{\rho C_1} + \frac{\alpha C_2}{C_1^2} - \frac{\rho^2 C_1^4 - \rho^2 C_3 C_1 - 3\rho^2 C_2 C_1^2 + 3\rho^2 C_2^2 - 3\rho C_1^3 + 3\rho C_1 C_2 + 2C_1^2}{2\rho C_1^2(C_1 - \rho C_1^2 + \rho C_2)}.$$

PROOF. By Theorem 1, (22) and Lemma 2. □

An alternative approach to obtaining the above local limit theorem is as follows. By the defining equation (15) of $P_n(w)$, we have the relation

$$[w^m z^n]P(w,z) = A_m\,[z^n]C^m(z) \quad \text{where } A_k := [z^k]\left(\frac{1}{1-z}\right)^{\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}.$$

When $m \asymp n$, namely, there exist two constants $0 < \nu_1 < \nu_2 < 1$ such that $\nu_1 n \le m \le \nu_2 n$, the first term A_k is easily treated by singularity analysis (cf. [10]) and the second term by the saddlepoint method (cf. [7, 14, 20, 21]). From there a local limit theorem as above can be obtained.

5. Examples.

Let us consider some typical examples. More examples can be found in [16, 18] and the references cited there.

EXAMPLE 1 (Connected components in random mappings). By random mapping (cf. [21]), we mean a random single-valued mapping of the set $\{1, 2, \ldots, n\}$ into itself. Structurally, any such mapping can be viewed as a set (partitional complex) of connected components each of which is a cycle of rooted labeled (or Cayley) trees. The bivariate generating function for random mappings is given by

$$\exp\left(w\log\frac{1}{1-C(z)}\right), \qquad |z|<e^{-1},$$

where $C(z)=z\,e^{C(z)}$ enumerates Cayley trees. From the singular expansion (cf. [9]),

$$\begin{aligned}C(z)&=1-\sqrt{2\,(1-ez)}-\tfrac{1}{3}(1-ez)\\&\quad+\sum_{k\ge 3}c_k(1-ez)^{k/2}, \qquad z\sim e^{-1},\ z\notin(e^{-1},\infty),\end{aligned}\tag{23}$$

we get

$$\log\frac{1}{1-C(z)}=\frac{1}{2}\log\frac{1}{1-ez}+H(\sqrt{1-ez}), \qquad z\sim e^{-1},\ z\notin(e^{-1},\infty),$$

where the logarithm takes its principal value and $H(t)$ is analytic at $t=0$ with $H(0)=0$. Theorem 3 applies to ξ_n, the number of connected components in a random mapping of size n, and we obtain, for example,

$$\begin{aligned}\Pr\{\xi_n=m\}&=\frac{\exp(-x^2/2-\frac{1}{2}\log n\cdot\Lambda(x/\sqrt{\frac{1}{2}\log n}))}{\sqrt{\pi\log n}}\\&\quad\times\left(1+\sum_{1\le k\le\nu}\frac{\Pi_k(x)}{(\frac{1}{2}\log n)^{k/2}}+O\left(\frac{|x|^{\nu+1}+1}{(\log n)^{(\nu+1)/2}}\right)\right),\end{aligned}$$

uniformly for $m=\frac{1}{2}\log n+x\sqrt{\frac{1}{2}\log n}$, $x=o(\sqrt{\log n})$. In particular, $\Pi_1(x)=\frac{1}{2}(\log 2+\gamma)x$, γ being Euler's constant. This improves a result of Pavlov [27].

EXAMPLE 2 (Components in ordered random mappings). In a random mapping, the order of its components is not taken into account. We can consider its ordered counterpart with generating function

$$\frac{1}{1-w\log(1/(1-C(z)))},\tag{24}$$

where as above $C(z)$ is the generating function for Cayley trees. For $\varepsilon\le\Re w\le\varepsilon^{-1}$, the solution of the denominator (in z) of (24) is seen to be $\rho(w)=(1-e^{-1/w})e^{-(1-e^{-1/w})}$. Let Ξ_n denote the number of connected components in an ordered random mapping. Applying Theorem 4 yields

$$\Pr\{\Xi_n=m\}=\frac{\exp(-x^2/2+nQ(y))}{\sqrt{2\pi\alpha_2 n}}\left(1-\frac{\alpha_3 x}{\sqrt{\alpha_2 n}}+O\left(\frac{x^2+1}{n}\right)\right), \qquad y:=\frac{x}{\sqrt{\alpha_2 n}},$$

where

$$\alpha_1 = e^{-1}(e-1)^{-1}, \qquad \alpha_2 = (2e^4 - 5e^3 + 2e^2 + 3e - 1)e^{-3}(e-1)^{-2},$$
$$\alpha_3 = (e^3 - 3e^2 + 1 + e^{2+e^{-1}} - e^{1+e^{-1}})e^{-1}(e-1)^{-1}$$

and $Q(y)$ can be calculated by using formula (3) with

$$u(s) = -\log \frac{\exp(1-e^{-e^{-s}})(1-\exp(-e^{-s}))}{\exp(1-e^{-1})(1-e^{-1})}.$$

EXAMPLE 3 (Very large deviations). The interest of considering ordered random mappings in the last example is to derive the following formula for ξ_n (using the same notation as in Examples 1 and 2):

$$\Pr\{\xi_n = m\} = \frac{\exp(-2 - e^{-1}n - x^2/2 + nQ(y))}{\sqrt{\alpha_2}\, m!\,(1-\exp(-1))^{n+1}} \times \left(1 - \frac{\alpha_3 x}{\sqrt{\alpha_2 n}} + O\left(\frac{x^2+1}{n}\right)\right), \tag{25}$$

for $m = \alpha_1 n + x\sqrt{\alpha_2 n}$, $x = o(\sqrt{n})$, $Q(y)$ being as in Example 2. To prove (25), we start from the observation

$$[w^m z^n]\exp(wL(z)) = \frac{1}{m!}[w^m z^n]\frac{1}{1-wL(z)},$$

where $L(z) = -\log(1 - C(z))$. By the definitions of ξ_n and Ξ_n,

$$\Pr\{\xi_n = m\} = \frac{[w^m z^n]\exp(wL(z))}{[z^n]e^{L(z)}}, \qquad \Pr\{\Xi_n = m\} = \frac{[w^m z^n](1-wL(z))^{-1}}{[z^n](1-L(z))^{-1}},$$

for $n, m = 0, 1, 2, \ldots$, we have

$$\Pr\{\xi_n = m\} = \frac{[z^n](1-L(z))^{-1}}{m!\,[z^n]e^{L(z)}}\Pr\{\Xi_n = m\}. \tag{26}$$

The result (25) follows from this relation and the asymptotics of $[z^n]e^{L(z)}$ and $[z^n](1 - L(z))^{-1}$ which are easily obtained by singularity analysis (cf. [10]) using (23).

Similarly, let ξ_n denote the number of cycles in a random permutation of size n. We obtain

$$\Pr\{\xi_n = m\} = \frac{\exp(-x^2/2 + nQ(y))}{m!\sqrt{2\pi n}\,(1-e^{-1})^n}\left(1 + \frac{e^2 - e - 1}{\sqrt{n}}\,x + O\left(\frac{x^2+1}{n}\right)\right),$$

for $m = n/(e-1) + x\sqrt{n}/(e-1)$, $x = o(\sqrt{n})$, where $y = (e-1)x/\sqrt{n}$, $Q(y) = Q(u; y)$ with $u(s) = -\log((1-e^{-e^{-s}})/(1-e^{-1}))$.

In general, since the mean and the variance of Ξ_n and ξ_n [defined by (26)] are different (due to the large factor $m!$), an asymptotic formula for one provides large deviations (from the mean) for the other. This observation has formerly been applied in [3] for polynomials of binomial type.

EXAMPLE 4 (Random mapping patterns). Random mapping patterns are equivalence classes of random mappings and, structurally, they are multisets of cycles of rooted unlabeled trees. Namely,

$$P(w, z) = \exp\biggl(\sum_{k\geq 1} \frac{w^k}{k} S(z^k)\biggr),$$

where $S(z)$ is the generating function for cycles of unlabeled Cayley trees:

$$S(z) = \sum_{k\geq 1} \frac{\varphi(k)}{k} \log \frac{1}{1-T(z^k)}, \qquad T(z) = z \exp\biggl(\sum_{k\geq 1} \frac{T(z^k)}{k}\biggr).$$

By the implicit function theorem, Otter [26] proved that $T(z)$ has the singular expansion

$$T(z) = 1 - \tau_1\sqrt{1-z/\rho} + \tau_2(1-z/\rho) + \cdots, \qquad 0 < \rho < 1,\ \tau_1 > 0,$$

for $z \sim \rho$ and $z \notin (\rho, \infty)$. This yields

$$S(z) = \log \frac{1}{1-T(z)} + R(z),$$

where $R(z)$ is analytic for $|z| \leq \rho$. It follows that

$$P(w, z) = \exp\biggl(\frac{w}{2} \log \frac{1}{1-z/\rho} + wH\bigl((1-z/\rho)^{1/2}\bigr) + wR(z) + U(w, z)\biggr),$$

where $U(w, z)$ is analytic for $|w| < \rho^{-1}$ and $|z| \leq \rho$,

$$H(u) = -\log c_1 + \sum_{k\geq 1} h_k u^k.$$

Thus Theorem 3 applies with $a = \frac{1}{2}$.

Similarly, our results apply to the cases where we count the number of distinct components and the number of components in square-free random mapping patterns, namely, random mapping patterns in which no two components are of the same size. We can also consider the ordered random mapping patterns in Examples 2 and 3.

EXAMPLE 5 (Prime factors of integers). Let $\omega(k)$ denote the number of distinct prime factors of k. We can then consider the sequence of random variables ξ_n, which takes the values $\omega(k)$, $1 \leq k \leq n$, with probability n^{-1}. Then the probability generating function $P_n(z) = \mathbf{E}\, z^{\xi_n} = n^{-1} \sum_{1\leq k\leq n} z^{\omega(k)}$ satisfies the asymptotic expression due to Selberg [29] (cf. also [22], Lemma 9.2 and [30], Chapter II.6):

$$P_n(z) = V(z)(\log n)^{z-1}\biggl(1 + O\biggl(\frac{1}{\log n}\biggr)\biggr), \qquad n \to \infty,$$

uniformly for $|z| \le M$, for any $M > 1$, where

$$V(z) = \frac{1}{\Gamma(z)} \prod_{p:\,\mathrm{prime}} \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{z}{p-1}\right),$$

the product being extended over all prime numbers p. Applying Theorem 2, we obtain, for $m = \log\log n + x\sqrt{\log\log n}$, $x = o(\sqrt{\log\log n})$,

$$\Pr\{\xi_n = m\} = \frac{\exp(-x^2/2 - \log\log n \cdot \Lambda(x/\sqrt{\log\log n}))}{\sqrt{2\pi \log\log n}} \times \left(1 + \sum_{1\le k\le \nu} \frac{\Pi_k(x)}{(\log\log n)^{k/2}} + O\left(\frac{|x|^{\nu+1}+1}{(\log\log n)^{(\nu+1)/2}}\right)\right),$$

a formula that can be deduced by a result of Selberg [29] (cf. also [8], [25] and [30], Theorem II.6.4) but does not seem to have been stated in this form. The same results hold for $\Omega(n)$, the total number of prime divisors of n (with multiplicities), and for many other arithmetic functions; see [18].

6. Extension. From the proof of Theorem 1 [see formula (11)], it is obvious that we have in fact proved more, namely, the proof of Theorem 1 extends to the case $x = O(\sigma_n)$. In such a large deviation range, it is more convenient to consider $\Pr\{\Omega_n = a\phi(n)\}$ with $a > 0$ (cf. [20]). The saddlepoint equation (8) $u'(r)\phi(n) = m$ then simplifies to $u'(r) = a$.

THEOREM 5. *Let $m = a\phi(n) > 0$. Suppose that the solution r of the equation $u'(s) = a$ exists and satisfies $-\rho \le r \le \rho$. Then $\Pr\{\Omega_n = m\}$ satisfies (11) uniformly in m.*

In particular, if $u(s) = e^s - 1$, then the result (11) can be stated differently:

$$\Pr\{\Omega_n = m\} = \frac{\phi^m(n)}{m!} e^{-\phi(n)+v(r)} \left(1 + \sum_{1\le k\le\nu} \frac{e_{2k}(r)}{m^k} + O\left(\frac{1}{m^{\nu+1}} + \frac{|r|}{\kappa_n\sqrt{\phi(n)}} + \frac{1}{\kappa_n\phi(n)}\right)\right),$$

for some polynomials $e_{2k}(r)$ of $r = \log a$. In this case, it is more convenient to work with the probability generating function of Ω_n and apply Selberg's method (cf. [29], [30], Chapter II.6), which is a variant of the usual saddlepoint method; see also [16, 17] for details.

7. Conclusion. The model that we developed in [16, 18, 19] and in this paper may be termed an "analytic scheme for moment generating functions" by which the similarity of the statistical properties of many apparently different structures (like the number of cycles in permutations and the number of prime factors in integers) is well explained by the analytic properties of their moment generating functions. At first glance, these properties seem difficult to establish. But for concrete combinatorial and arithmetic problems, we have demonstrated, by using analytic methods, that analytic properties of moment generating functions are well reflected by the singularity type of the associated bivariate generating functions. Thus a classification according to the latter and then the use of suitable analytic methods, such as singularity analysis (cf. [10]), allow us to derive the required properties for moment generating functions in a rather systematic and general way. This explains roughly why there are so many similarities between the number of cycles in permutations and the number of connected components in 2-regular graphs [5], because the dominant singularity of the corresponding bivariate generating functions are both of type exp–log. Such an approach is also rather robust under structural perturbations when one considers, for example, structures without components of prescribed sizes or with some components appearing at most a specified number of times, and so on. A detailed study in this direction can be found in [16]. The uniformity afforded by the singularity analysis is also useful for other probabilistic properties of combinatorial parameters; see, for example, [1] and [15].

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