Algebra & Number Theory

Volume 13
2019
No. 8

Multiplicity one for wildly ramified representations

Daniel Le

msp

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1807

Multiplicity one for wildly ramified representations

Daniel Le

Let F be a totally real field in which p is unramified. Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a modular Galois representation which satisfies the Taylor–Wiles hypotheses and is generic at a place v above p. Let $\mathfrak{m}$ be the corresponding Hecke eigensystem. We show that the $\mathfrak{m}$-torsion in the mod p cohomology of Shimura curves with full congruence level at v coincides with the $\mathrm{GL}_2(k_v)$-representation $D_0(\bar{r}|_{G_{F_v}})$ constructed by Breuil and Paškūnas. In particular, it depends only on the local representation $\bar{r}|_{G_{F_v}}$, and its Jordan–Hölder factors appear with multiplicity one. This builds on and extends work of the author with Morra and Schraen and, independently, Hu–Wang, which proved these results when $\bar{r}|_{G_{F_v}}$ was additionally assumed to be tamely ramified. The main new tool is a method for computing Taylor–Wiles patched modules of integral projective envelopes using multitype tamely potentially Barsotti–Tate deformation rings and their intersection theory.

1. Introduction

Let $F/\mathbb{Q}$ be a totally real field which is unramified at a rational prime p. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ is a Galois representation occurring in the $\mathbb{F}$-cohomology of a Shimura curve $X_{/F}$ with corresponding Hecke eigensystem $\mathfrak{m}$ (see Section 5). Suppose that the corresponding quaternion algebra splits at p. Let v be a place of F dividing p, let K^v be a compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$ and $K_v(n)$ the n-th principal congruence subgroup at v. One expects that the analogues of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and mod p local-global compatibility for $\mathrm{GL}_2(\mathbb{Q})$ describe the $\mathrm{GL}_2(F_v)$-representation

$$\pi' = \mathrm{Hom}_{G_F}(\bar{r}, \varinjlim_n H^1(X(K^v K_v(n)), \mathbb{F})[\mathfrak{m}_{\bar{r}}])$$

in the completed cohomology of X, at least up to multiplicities, in terms of $\bar{\rho} \overset{\text{def}}{=} \bar{r}|_{G_{F_v}}$. In fact, we study a related representation $\pi = (M^{\min})^*$ (see Section 5), which is minimal with respect to multiplicities. Such analogues are unknown at present, although [Breuil 2014; Emerton et al. 2015] show that if $\bar{r}$ satisfies the usual Taylor–Wiles hypotheses and $\bar{\rho}$ is generic, then π contains one of infinitely many $\mathrm{GL}_2(F_v)$-representations constructed by [Breuil and Paškūnas 2012]. The idea, as explained in [Breuil 2014], behind the constructions of [Breuil and Paškūnas 2012] is that if one can show that the restriction of π to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_{F_v})$ satisfies certain multiplicity one properties, then π

MSC2010: 11S37.
Keywords: Galois deformations, mod p Langlands program.

must contain a Diamond diagram of the form $D(\bar{\rho}, \iota)$. These multiplicity one properties, which one might view as minimalist conjectures for multiplicities, were established in [Emerton et al. 2015].

That the family of representations containing a diagram $D(\bar{\rho}, \iota)$ is infinite is unfortunate and warrants further investigation of π. One part of a Diamond diagram $D(\bar{\rho}, \iota)$ is a $\mathrm{GL}_2(k_v)$-representation denoted $D_0(\bar{\rho})$, which is a subrepresentation of $\pi|_{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ (see [Breuil 2014, Proposition 9.3]), and thus a subrepresentation of the invariants of π under the first principal congruence subgroup $K_v(1)$ of $\mathrm{GL}_2(\mathcal{O}_{F_v})$. Our main result is the following:

Theorem 1.1 (Corollary 5.2). *If $\bar{r}$ satisfies the Taylor–Wiles hypotheses and $\bar{\rho}$ is generic, see Definition 4.1, then the $\mathrm{GL}_2(k_v)$-representation $\pi^{K_v(1)}$ is isomorphic to $D_0(\bar{\rho})$. In particular, it only depends on $\bar{\rho}$ and is multiplicity free.*

One can view this result as showing that π satisfies a minimality property: $\pi^{K_v(1)}$ is as small as possible. A similar result has been announced by Hu–Wang.

The main tool in the proof of Theorem 1.1 is the Taylor–Wiles patching method. Diamond [1997] and Fujiwara [2006] discovered that the Cohen–Macaulay property of patched modules could be combined with local algebra results of Auslander, Buchsbaum, and Serre to rederive and generalize mod p multiplicity one results of Mazur for modular forms with level away from p. Emerton et al. [2015] proved similar results for modular forms with level at p by introducing two gluing methods to calculate patched modules from smaller ones to which the Diamond–Fujiwara trick applied. The first method is a version of Nakayama's lemma and uses the submodule structure of mod p reductions of Deligne–Lusztig representations. The second method combines the submodule structure above with the intersection theory of special fibers of tamely potentially Barsotti–Tate deformation rings.

When $\bar{\rho}$ is tamely ramified, [Hu and Wang 2018; Le et al. 2016b] show that the patched modules of projective envelopes of irreducible $\mathbb{F}[\mathrm{GL}_2(k_v)]$-modules are cyclic modules by describing the submodule structure of these projective envelopes and using the Nakayama method of [Emerton et al. 2015] (see Proposition 4.6). However, the gluing methods of [loc. cit.] are insufficient when $\bar{\rho}$ is wildly ramified. Indeed, these methods only glue together characteristic p patched modules, but when $\bar{\rho}$ is wildly ramified there is more than one isomorphism class of $\mathbb{F}[\mathrm{GL}_2(k_v)]$-modules satisfying the multiplicity one properties for $\pi^{K_v(1)}$ established in [loc. cit.].

We introduce a variant of the intersection theory method of [loc. cit.], which uses the intersection theory of integral tamely potentially Barsotti–Tate deformation rings. Let $W(\mathbb{F})$ denote the Witt vectors of $\mathbb{F}$. The first step (Proposition 4.6) is to show that the methods of [loc. cit.] still apply to certain quotients of generic $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-projective envelopes (which are projective envelopes in the abelian category of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-modules generated by lattices in some fixed set of Deligne–Lusztig representations). If such a quotient is reducible rationally, then it can be written as a submodule of the direct sum of two smaller quotients with p-torsion cokernel (see Proposition 2.4). This reflects a kind of transversality: while these subcategories do not give a direct product decomposition of the category of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-modules, if two subquotients of lattices in two distinct Deligne–Lusztig representations are isomorphic, they must be p-torsion. By exactness of patching and this exact sequence, it turns out that the patched modules of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-projective envelopes are then determined by the patched modules of these quotients (this depends crucially on the fact that all such patched modules turn out to be cyclic).

It remains to actually compute these patched modules using intersection theory in a multitype Barsotti–Tate framed deformation space, which we define to be the Zariski closure in the unrestricted framed deformation space of $\bar{\rho}$ of potentially Barsotti–Tate Galois representations with tame inertial type in some fixed set. That the resulting patched module is cyclic comes from the fact that the multitype Barsotti–Tate deformation rings exhibit a similar kind of transversality: two lattices in potentially Barsotti–Tate Galois representations of two distinct generic tame inertial types can be congruent modulo p, but never modulo p^2.

We now give a brief overview of the following sections. In Section 2, we generalize some of the results of [Le et al. 2016b] and prove the key result (Proposition 2.4) gluing integral projective envelopes from their quotients. In Section 3, we define and calculate multitype Barsotti–Tate deformation rings — this is the other key technical input. To compare Kisin modules for varying tame types, it is much more convenient to choose eigenbases for Kisin modules which are not always gauge bases in the sense of [Emerton et al. 2015, Section 7.3]. This requires generalizing [Le et al. 2018, Theorem 4.1]. The main result, Theorem 3.6, of this section computes some multitype Barsotti–Tate framed deformation spaces. In Section 4, we calculate the abstract patched modules of projective envelopes using the Nakayama method and our integral intersection theory method. In Section 5, we apply the results of Section 4 to the cohomology of Shimura curves using the Taylor–Wiles method.

1A. *Notation.* If F is any field, we write $\bar{F}$ for a separable closure of F and $G_F := \mathrm{Gal}(\bar{F}/F)$ for the absolute Galois group of F.

Let $f \in \mathbb{N}$ and $q = p^f$. Let $\mathcal{O}_K$ be the Witt vectors $W(\mathbb{F}_q)$ of $\mathbb{F}_q$. Let $K = \mathcal{O}_K[p^{-1}]$ be the unramified extension of $\mathbb{Q}_p$ of degree f. Let E be an extension of K with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$. This induces embeddings $\mathcal{O}_K \hookrightarrow \mathcal{O}$ and $\iota_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$. For $i \in \mathbb{Z}/f$, let $\iota_i = \iota_0 \circ \varphi^i$ be the i-th Frobenius twist of ι_0. We fix an embedding $\mathbb{F} \hookrightarrow \bar{\mathbb{F}}_q$. We will denote by $(\cdot)^*$ the $\mathbb{F}$-linear dual, and by $(\cdot)^\vee$ the contragredient of a representation.

Let G (resp. G^{der}) be the algebraic group $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathrm{GL}_2$ (resp. $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathrm{SL}_2$), and let $T \subset G$ (resp. $T^{\mathrm{der}} \subset G^{\mathrm{der}}$) be the diagonal torus. Let $X^*(T)$ (resp. $X^*(T^{\mathrm{der}})$) denote the group of characters of T (resp. T^{der}). Let $X_*(T)$ and $X_*(T^{\mathrm{der}})$ similarly denote groups of cocharacters. By the embeddings ι_i, $X^*(T)$ is identified with $X^*(T \times_{\mathbb{F}_p} \mathbb{F}) \cong X^*(\prod_{i \in \mathbb{Z}/f} \mathbb{G}_m^2)$, which is identified with $(\mathbb{Z}^2)^{\mathbb{Z}/f}$ in the usual way. A similar identification for $X_*(T)$ is made. For a character $\mu \in X^*(T)$, we write μ_i as the i-th factor of μ so that $\mu = \sum_{i \in \mathbb{Z}/f} \mu_i$.

Let $\eta^{(i)} \in X^*(T)$ (resp. $\alpha^{(i)} \in X_*(T)$) be the dominant fundamental character (resp. the positive coroot) represented by $(1, 0)$ (resp. $(1, -1)$) in the i-th factor and 0 elsewhere. Let $\eta = \sum_{i \in \mathbb{Z}/f} \eta^{(i)}$. Let $\omega^{(i)}$ be the restriction of $\eta^{(i)}$ to T^{der}.

Let W be the Weyl group of G and G^{der}, which is similarly identified with $S_2^{\mathbb{Z}/f}$. Here, S_2 denotes the permutation group on two elements. We denote the trivial element of S_2 by id. Then W acts naturally on $X^*(T)$ and $X^*(T^{\text{der}})$. Let π be the automorphism of $X^*(T)$ and W which acts by a shift so that $\pi(x)_i = x_{i-1}$. Then the action on $X^*(T)$ induced by the relative Frobenius morphism on T is given by $p\pi^{-1}$, while the action of the relative Frobenius on W is given by π.

For a dominant character $\mu \in X^*(T)$ we write $V(\mu)$ for the Weyl module for G defined in [Jantzen 1987, II.2.13(1)]. It has a unique simple G-quotient $L(\mu)$. If $\mu = \sum_i \mu_i$ is p-restricted (i.e., $0 \le \langle \mu, \alpha^{(i)} \rangle \le p$ for all i), then $L(\mu) = \otimes_i L(\mu_i)$ by the Steinberg tensor product theorem as in [Herzig 2009, Theorem 3.9]. Let $F(\mu)$ be the restriction of $L(\mu)$ to $\mathrm{GL}_2(\mathbb{F}_q)$, which remains irreducible by [Herzig 2009, A.1.3]. Every irreducible $\mathrm{GL}_2(\mathbb{F}_q)$-representation is of this form, and we call such a representation a *Serre weight*. Note that $F(\mu) \cong F(\lambda)$ if and only if $\mu \cong \lambda \mod (p-\pi)X^0(T)$, where $X^0(T)$ is the kernel of the restriction map $X^*(T) \to X^*(T^{\text{der}})$.

Recall that to a pair $(s, \lambda) \in W \times X^*(T)$, [Herzig 2009, Lemma 4.2] attaches a (virtual) representation of $\mathrm{GL}_2(\mathbb{F}_q)$, which we denote $R_s(\lambda)$. In each use below, $R_s(\lambda)$ will in fact denote a true representation.

An *inertial type* for a local field L is a continuous E-representation τ of the inertial subgroup I_L, whose action factors through a finite quotient and can be extended to G_L. For our purposes, all inertial types will be two-dimensional. In this case, Henniart [2002, Annexe A] attaches to τ a smooth irreducible finite-dimensional $\mathrm{GL}_2(\mathcal{O}_L)$-representation $\sigma(\tau)$ over E (see also [Emerton et al. 2015, Section 1.9]). We call the association of τ and $\sigma(\tau)$ the inertial local Langlands correspondence. An inertial type τ is called *tame* if τ factors through the tame quotient of I_L. The tame inertial types are exactly those τ such that $\sigma(\tau)$ factors through $\mathrm{GL}_2(k_L)$ where k_L is the residue field of L.

For any characteristic 0 field F, let $\varepsilon : G_F \to \mathbb{Z}_p^\times \subset \mathcal{O}^\times$ denote the p-adic cyclotomic character and $\bar{\varepsilon}$ denote its reduction modulo ϖ. We now let F be K. Let $\mathbb{C}_p(i)$ denote $\varepsilon^i \otimes_E \mathbb{C}_p$, where the tensor product is over any embedding $E \hookrightarrow \mathbb{C}_p$. Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation over E. For each embedding $\kappa : E \hookrightarrow \mathbb{C}_p$, let $\mathrm{HT}_\kappa(V)$ be the multiset of integers such that $-i$ appears with multiplicity $\dim_{\mathbb{C}_p}(V \otimes_\kappa \mathbb{C}_p(i))^{G_K}$. Then in particular $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$ for all embeddings κ. We say that a two-dimensional representation V is (*potentially*) *Barsotti–Tate* if V is (potentially) crystalline with $\mathrm{HT}_\kappa(V) = \{0, 1\}$ for all embeddings κ. If τ is an inertial type, we say that V is potentially Barsotti–Tate of type τ if the action of I_K on the potentially crystalline Dieudonné module of V is isomorphic to τ.

2. Quotients of generic $\mathrm{GL}_2(\mathbb{F}_q)$-projective envelopes

Suppose that $\mu \in X^*(T)$ and that $1 \le \langle \mu - \eta, \alpha^{(i)} \rangle < p-2$ for all $i \in \mathbb{Z}/f$. Let σ be $F(\mu - \eta)$. Let $\tilde{R}_\mu$ (resp. R_μ) be the projective $\mathcal{O}_K[\mathrm{GL}_2(\mathbb{F}_q)]$-envelope (resp. the projective $\mathbb{F}_q[\mathrm{GL}_2(\mathbb{F}_q)]$-envelope) of σ. Let S be the set $\{\pm\omega^{(i)}\}_i$ and let I be a subset of S. Recall from [Le et al. 2016b, Definition 3.5] that (with respect to μ) we attach to a subset $J \subset S$ a Serre weight σ_J. Let $R_{\mu,I}$ be the universal object among quotients of R_μ that do not contain $\sigma_{\{\omega\}}$ as a Jordan–Hölder factor for all ω in I. Recall from [Le et al. 2016b, Section 3] that there is a filtration $\mathrm{Fil}^{\boldsymbol{k}}$ on R_μ which induces a filtration $\mathrm{Fil}^{\boldsymbol{k}}$ on $R_{\mu,I}$. Similarly, we can construct a filtration $\mathrm{Fil}^k_\otimes = \sum_{|\boldsymbol{k}|=k} \mathrm{Fil}^{\boldsymbol{k}}$ on R_μ and $R_{\mu,I}$. Let $W_{\boldsymbol{k},I}$ be $\mathrm{gr}^{\boldsymbol{k}} R_{\mu,I}$.

Proposition 2.1. *We have an isomorphism* $W_{\boldsymbol{k},I} \cong \bigoplus_{J \subset S, \boldsymbol{k}(J)=\boldsymbol{k}, J \cap I = \varnothing} \sigma_J$.

Proof. This follows from [Le et al. 2016b, Proposition 3.6 and Theorem 3.14]. □

If I is a subset of S such that $I \cap \{\pm\omega^{(i)}\}$ has size at most one for all i, let $T_{\sigma,I}$ be the set of Deligne–Lusztig representations over K of the form $R_w(\mu - w\eta)$ where $w_i = \mathrm{id}$ (resp. $w_i \neq \mathrm{id}$) if $\omega^{(i)} \in I$ (resp. $-\omega^{(i)} \in I$). Fix an embedding $\tilde{R}_\mu \hookrightarrow \bigoplus_{\sigma(\tau)\in T_{\sigma,\varnothing}} \sigma(\tau)$. Let $\tilde{R}_{\mu,I}$ be the quotient of $\tilde{R}_\mu$ isotypic for the set $T_{\sigma,I}$ (which does not depend on the above embedding). Note that $\tilde{R}_{\mu,\varnothing}$ is equal to $\tilde{R}_\mu$.

Proposition 2.2. *The reduction of $\tilde{R}_{\mu,I}$ modulo p is $R_{\mu,I}$.*

Proof. For each $\omega \in I$, $\sigma_{\{\omega\}} \notin \mathrm{JH}(\bar{\sigma}(\tau))$ for all $\sigma(\tau) \in T_{\sigma,I}$. Thus, there is a canonical quotient map $R_{\mu,I} \to \bar{R}_{\mu,I}$, where $\bar{R}_{\mu,I}$ is the reduction of $\tilde{R}_{\mu,I}$. By Proposition 2.1, $R_{\mu,I}$ has length $2^{2f-\#I}$. Since $\bar{R}_{\mu,I}$ is the reduction of a lattice in the direct sum of $2^{f-\#I}$ types, each of whose reduction has length 2^f [Diamond 2007], it also has length $2^{2f-\#I}$. Since both objects have the same length, this surjection must be an isomorphism. □

Again, let $I \subset S$. Let $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ be $\mathrm{Fil}^{\boldsymbol{k}} R_{\mu,I}/(\mathrm{Fil}^{k+2}_{\otimes} R_{\mu,I} \cap \mathrm{Fil}^{\boldsymbol{k}} R_{\mu,I})$. Note that $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ is multiplicity free since $W_{\boldsymbol{k},\boldsymbol{k}+1,\varnothing}$ (which is $W_{\boldsymbol{k},\boldsymbol{k}+1}$ in [Le et al. 2016b, Section 3]) is by [loc. cit, Proposition 3.6 and Lemma 3.7].

Proposition 2.3. *Suppose that $J \subset J'$, $\#J' \setminus J = 1$, and $J' \cap I = \varnothing$. Let $\boldsymbol{k}$ and $\boldsymbol{k}'$ be $\boldsymbol{k}(J)$ and $\boldsymbol{k}(J')$, respectively. Then there is a subquotient of $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ which is the unique up to isomorphism nontrivial extension of σ_J by $\sigma_{J'}$.*

Proof. This follows immediately from Proposition 2.1 and [Le et al. 2016b, Proposition 3.8]. □

Proposition 2.4. *Suppose that the size of $I \cap \{\pm\omega^{(i)}\}$ is at most one for all i and that $I \cap \{\pm\omega^{(j)}\} = \varnothing$ for some j. Then there is an exact sequence*

$$0 \to \tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \to R_{\mu,I\cup\{\pm\omega^{(j)}\}} \to 0, \tag{2-1}$$

where the second (resp. third) map is the sum (resp. difference) of the natural projections.

Proof. The second map of (2-1) is clearly injective since it is after inverting p and $\tilde{R}_{\mu,I}$ is $\mathcal{O}_K$-flat. We claim that the cokernel of this map is p-torsion. Let $\sigma_{\{\omega^{(j)}\}} = F(\mu' - \eta)$ and consider a map $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ such that the composition with the projection

$$\tilde{R}_{\mu,I} \twoheadrightarrow R_{\mu,I} \twoheadrightarrow R_{\mu,I}/\,\mathrm{Fil}^2_{\otimes} R_{\mu,I}$$

is nonzero. The composition of $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with the natural surjection $\tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}}$ is zero since $\sigma_{\{\omega^{(j)}\}} \notin \mathrm{JH}(R_{\mu,I\cup\{\omega^{(j)}\}})$.

Lemma 2.5. *The image of the composition $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with the natural surjection $\tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ contains $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$.*

With Lemma 2.5 and its analogue for $\tilde{R}_{\mu,I\cup\{\omega^{(j)}\}}$, we would see that the image of

$$\tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$$

contains $p\tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$, establishing our claim.

Proof of Lemma 2.5. Fix a map $\tilde{R}_\mu \to \tilde{R}_{\mu'}$ such that the composition with the projection to $R_{\mu'}/\operatorname{Fil}^2_\otimes R_{\mu'}$ is nonzero. It suffices to show that the image, denoted Q, of the composition of $\tilde{R}_\mu \to \tilde{R}_{\mu'}$ with the above $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ is $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$. On the one hand, we see that Q is in $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ by reducing modulo p and using Propositions 2.2 and 2.3. Let $\sigma(\tau)$ be a Jordan–Hölder factor of $\tilde{R}_{\mu,I}[p^{-1}]$ and let $\sigma^\circ(\tau) \subset \sigma(\tau)$ be the unique lattice up to homothety with cosocle isomorphic to σ [Emerton et al. 2015, Lemma 4.1.1]. Fix a surjection from $\tilde{R}_{\mu,I}$ to $\sigma^\circ(\tau)$. By reducing $\bmod p$, we see that the image of the composition of $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with this surjection is a saturated lattice $\sigma^{\circ\circ}(\tau)$ with cosocle $\sigma_{\{\omega^{(j)}\}}$. Similarly, the image of Q under this surjection is a saturated lattice in $\sigma^{\circ\circ}(\tau)$ with cosocle isomorphic to σ. This lattice is $p\sigma^\circ(\tau)$ by [Emerton et al. 2015, Theorem 5.1.1]. Thus, the composition $Q \subset p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \twoheadrightarrow p\sigma^\circ(\tau)$ is an isomorphism upon taking cosocles. We see that Q must be equal to $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$. □

Let R be the cokernel of the second map in (2-1), which is p-torsion by our first claim. Then the exact sequence

$$0 \to \tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \to R \to 0 \tag{2-2}$$

induces an exact sequence

$$R_{\mu,I} \to R_{\mu,I\cup\{\omega^{(j)}\}} \oplus R_{\mu,I\cup\{-\omega^{(j)}\}} \to R \to 0 \tag{2-3}$$

by Proposition 2.2. By taking cosocles, (2-3) induces an exact sequence

$$\operatorname{cosoc} R_{\mu,I} \to \operatorname{cosoc} R_{\mu,I\cup\{\omega^{(j)}\}} \oplus \operatorname{cosoc} R_{\mu,I\cup\{-\omega^{(j)}\}} \to \operatorname{cosoc} R \to 0 \tag{2-4}$$

Note that $\operatorname{cosoc} R_{\mu,I}$, $\operatorname{cosoc} R_{\mu,I\cup\{\omega^{(j)}\}}$, and $\operatorname{cosoc} R_{\mu,I\cup\{-\omega^{(j)}\}}$ are all isomorphic to σ and that the composition of first map of (2-4) with either projection is nonzero. Thus $\operatorname{cosoc} R$ is isomorphic to σ and the restriction of the second map of (2-4) to either summand is nonzero. We conclude that the restriction of the second map in (2-3) to either summand is surjective. By definition, the maximal representation which is a quotient of both $R_{\mu,I\cup\{\omega^{(j)}\}}$ and $R_{\mu,I\cup\{-\omega^{(j)}\}}$ is $R_{\mu,I\cup\{\pm\omega^{(j)}\}}$. Thus, there is a surjection $R_{\mu,I\cup\{\pm\omega^{(j)}\}} \twoheadrightarrow R$. On the other hand, it is easy to see that the composition $R_{\mu,I} \to R_{\mu,I\cup\{\omega^{(j)}\}} \oplus R_{\mu,I\cup\{-\omega^{(j)}\}} \to R_{\mu,I\cup\{\pm\omega^{(j)}\}}$ is zero, where the second map is the difference of the natural projections. Thus, there is a surjection $R \twoheadrightarrow R_{\mu,I\cup\{\pm\omega^{(j)}\}}$. Since R and $R_{\mu,I\cup\{\pm\omega^{(j)}\}}$ are finite length objects, they must be isomorphic. □

3. Multitype Barsotti–Tate deformation rings

3A. ***Étale φ-modules.*** Let K_∞ be the infinite extension obtained by adjoining compatible p-power roots of $-p$ to K. Let $\mathcal{O}_{\mathcal{E},K}$ denote the p-adic completion of $\mathcal{O}_K((v))$, and let $\mathcal{O}_{\mathcal{E}^{\mathrm{un}},K}$ denote the p-adic completion of a maximal connected étale extension of $\mathcal{O}_{\mathcal{E},K}$. For R a complete local Noetherian $\mathcal{O}$-algebra, let $\Phi\text{-}\operatorname{Mod}^{\mathrm{et}}(R)$ be the category of étale φ-modules over $\mathcal{O}_{\mathcal{E},K} \otimes_{\mathbb{Z}_p} R$, and let $\operatorname{Rep}_{G_{K_\infty}}(R)$ be the category of (continuous) representations of G_{K_∞} over R. Fontaine defined an exact antiequivalence of tensor categories

$$\mathbb{V}^* : \Phi\text{-}\operatorname{Mod}^{\mathrm{et}}(R) \to \operatorname{Rep}_{G_{K_\infty}}(R)$$

by $\mathbb{V}^*(\mathcal{M}) = ((\mathcal{M} \otimes \mathcal{O}_{\mathcal{E}^{\mathrm{un}},K})^{\varphi=1})^\vee$.

For a natural number d, let $\varpi_d \in E$ be a root of $u^{p^{df}-1} + p$. Let K_d be the degree d unramified extension of K. We define the fundamental character

$$\begin{aligned} \omega_{df} : G_{K_d} &\to \mathcal{O}^\times \\ g &\mapsto \frac{g(\varpi_d)}{\varpi_d}, \end{aligned}$$

which does not depend on the choice of ϖ_d. For $\alpha \in \mathbb{F}^\times$, denote by nr_α the unramified character of G_K taking a geometric Frobenius element to α.

Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation. If $\bar{\rho}$ is reducible, then it is an extension of

$$\mathrm{nr}_{\alpha'}\, \omega_f^{\sum_{i=0}^{f-1} \mu_{2,i} p^i} \quad \text{by} \quad \mathrm{nr}_\alpha\, \omega_f^{\sum_{i=0}^{f-1} \mu_{1,i} p^i}$$

for some dominant p-restricted character $\mu_{\bar{\rho}} = (\mu_{1,i}, \mu_{2,i})_i \in X^*(T)$ and some α and $\alpha' \in \mathbb{F}^\times$. If $\bar{\rho}$ is irreducible, then $\bar{\rho}$ is

$$\mathrm{Ind}_{G_{K_2}}^{G_K} \mathrm{nr}_{-\alpha}\, \omega_{2f}^{\sum_{i=0}^{f-1} \mu_{1,i} p^i + p^f \sum_{i=0}^{f-1} \mu_{2,i} p^i}$$

where $\mu_{\bar{\rho}}$ again is a dominant p-restricted element of $X^*(T)$ and $\alpha \in \mathbb{F}^\times$. We note that the main result of this paper in the case when $\bar{\rho}$ is irreducible already appears in [Le et al. 2016b; Hu and Wang 2018], and so this case can be ignored if the reader desires. [Buzzard et al. 2010] attaches to $\bar{\rho}$ a set $W(\bar{\rho})$ of Serre weights (see also [Breuil 2014, Section 4, Proposition A.3] with the notation $\mathcal{D}(\bar{\rho})$).

In both the reducible and irreducible cases, we now assume that $\mu_{\bar{\rho}} \in X^*(T)$ with $\mu_i = (\mu_{1,i}, \mu_{2,i}) = (c_i, 1)$ with $3 < c_i < p-2$ for all $i \in \mathbb{Z}/f$. For $i \in \mathbb{Z}/f$, let a_i be an element of $\mathbb{F}$. Let $\mathcal{M} = \prod_i \mathbb{F}((v))\mathfrak{e}^i \oplus \mathbb{F}((v))\mathfrak{f}^i$ be the φ-module defined by

$$\begin{aligned} i \neq 0 &: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}} \mathfrak{e}^i + a_{i-1} v^{c_{f-i}} \mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = v\mathfrak{f}^i, \end{cases} \\ i = 0,\ \bar{\rho} \text{ reducible} &: \begin{cases} \varphi(\mathfrak{e}^{f-1}) = \alpha v^{c_0} \mathfrak{e}^0 + \alpha a_{f-1} v^{c_0} \mathfrak{f}^0, \\ \varphi(\mathfrak{f}^{f-1}) = \alpha' v \mathfrak{f}^0, \end{cases} \\ i = 0,\ \bar{\rho} \text{ irreducible} &: \begin{cases} \varphi(\mathfrak{e}^{f-1}) = \alpha v^{c_0} \mathfrak{f}^0, \\ \varphi(\mathfrak{f}^{f-1}) = -v \mathfrak{e}^0, \end{cases} \end{aligned}$$

(here the i-th factor corresponds to the embedding ι_{-i}).

Proposition 3.1. *There are unique values $a_i \in \mathbb{F}$ for $i \in \mathbb{Z}/f$ such that $\mathbb{V}^*(\mathcal{M})$ is isomorphic to the restriction $\bar{\rho}|_{G_{K_\infty}}$.*

Proof. Note that $\bar{\rho}$ is Fontaine–Laffaille by the genericity condition. We use Fontaine–Laffaille theory as in [Breuil 2014, Appendix A]. We address the case when $\bar{\rho}$ is reducible and leave the irreducible case to the reader. Let $M = \bigoplus_{i\in\mathbb{Z}/f} M^{(i)}$ with $M^{(i)} = k_E e^{(i)} \oplus k_E f^{(i)}$ be the Fontaine–Laffaille module with

$$\begin{gathered}\mathrm{Fil}^1 M^{(i)} = M^{(i)}, \quad \mathrm{Fil}^2 M^{(i)} = \mathrm{Fil}^{c_{f-i}} M^{(i)} = k_E f^{(i)}, \quad \mathrm{Fil}^{c_{f-i}+1} M^{(i)} = 0,\\ \varphi(e^{(i)}) = e^{(i+1)}, \quad \varphi_{c_{f-i}}(f^{(i)}) = f^{(i+1)} + a_{i-1}e^{(i+1)} \quad \text{for } i \neq 1,\\ \varphi(e^{(1)}) = \alpha' e^{(2)}, \quad \varphi_{c_{f-1}}(f^{(1)}) = \alpha f^{(2)} + \alpha' a_0 e^{(2)},\end{gathered}$$

for $a_i \in k_E$ such that $\bar\rho \cong \mathrm{Hom}_{\mathrm{Fil}^\bullet,\varphi_\cdot}(M, A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ (see e.g., [Breuil 2014, (16)]).

Let $\mathfrak{M}$ be the $\mathbb{F}_q[\![v]\!] \otimes_{\mathbb{Z}_p} \mathbb{F}$-submodule of $\mathcal{M}$ generated by $(\mathfrak{e}_i)_{i\in\mathbb{Z}/f}$ and $(\mathfrak{f}_i)_{i\in\mathbb{Z}/f}$. Note that φ maps $\mathfrak{M}$ to itself. Then a calculation (see [Emerton et al. 2015, Section 7.4] with $J = \varnothing$) shows that $\Theta_{p-1}(\mathfrak{M}) \cong \mathcal{F}_{p-1}(M)$, where the functors Θ_{p-1} and $\mathcal{F}_{p-1}$ are introduced in [loc. cit, Appendix A]. The result now follows from [loc. cit, Propositions A.3.2 and A.3.3]. □

For the rest of this section, we fix, for each $i \in \mathbb{Z}/f$, $a_i \in \mathbb{F}$, the unique element as in Proposition 3.1. In doing so, we thus fix $\mathcal{M}$. If $\bar\rho$ is irreducible, let $S_{\bar\rho}$ be the set $\{-\omega^{(0)}, \omega^{(1)}, \dots, \omega^{(f-1)}\}$. Otherwise, let $S_{\bar\rho}$ be the set $\{\omega^{(i)} \mid a_{f-1-i} = 0\}$.

Proposition 3.2. *The set $W(\bar\rho)$ equals $\{\sigma_J \mid J \subset S_{\bar\rho}\}$ where σ_J is defined with respect to $\mu_{\bar\rho}$.*

Proof. This follows from a direct calculation using [Breuil 2014, Section 4]. □

3B. ***Kisin modules and deformation rings.*** To describe tamely potentially Barsotti–Tate deformation rings, we will use the theory of Kisin modules with descent datum. Let τ be the tame principal series type $\eta_1 \oplus \eta_2 : I_K \to \mathrm{GL}_2(\mathbb{F}_q)$ where $\eta_k = \omega_f^{-\boldsymbol{a}_k^{(0)}}$ for $k = 1$ and 2 and

$$\boldsymbol{a}_k^{(j)} = \sum_{i=0}^{f-1} a_{k,-j+i}\, p^i,$$

where $a_{k,i} \in \mathbb{Z}$. We will suppose throughout that $2 \le |a_{1,i} - a_{2,i}| \le p-3$ for all $i \in \mathbb{Z}/f$ and call such a tame principal series type *generic*. We will say a tame inertial type τ' is generic if its restriction to the quadratic unramified extension of K is a generic principal series type.

The *orientation* of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is the element $s \in W$ such that $\boldsymbol{a}_{s_j(1)}^{(j)} > \boldsymbol{a}_{s_j(2)}^{(j)}$. By an abuse of notation, we say that the orientation of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is an orientation for τ if τ can be expressed in terms of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ as above.

Let R be an $\mathcal{O}$-algebra. For a principal series type τ, we will consider Kisin modules over R with descent datum of type τ (see [Le et al. 2018, Definition 2.4]). We will say that such a Kisin module $\mathfrak{M}_R$ is in $Y^{(0,1),\tau}(R)$ if the cokernels of $\phi_{\mathfrak{M}_R} : \varphi^*(\mathfrak{M}_R) \to \mathfrak{M}_R$ and $\phi_{\det\mathfrak{M}_R} : \varphi^*(\det\mathfrak{M}_R) \to \det\mathfrak{M}_R$ are annihilated by $E(u) = u^{q-1} + p$. Let v be u^{q-1}.

Let s be an orientation for a generic tame principal series type τ and $\mathfrak{M}_R$ be an element of $Y^{(0,1),\tau}(R)$. Then $\mathfrak{M}_R$ can be described by the matrices $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M}_R\otimes_R\mathbb{F},s_{i+1}(2)})$ after choosing an eigenbasis β (see [loc. cit., Definition 2.11]). The following is a generalization of [loc. cit., Theorem 4.1] in the case of GL_2, where β is allowed to have a slightly more general form than a gauge basis.

Theorem 3.3. *Let τ be a tame generic principal series type and let $s = (s_i)_i \in W$ be an orientation for τ. Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Let $\mathfrak{M}_R \in Y^{(0,1),\tau}(R)$ with $\mathrm{Mat}_{\bar{\beta}}(\phi^{(i)}_{\mathfrak{M}_R\otimes_R\mathbb{F},s_{i+1}(2)})$ given by*

$$\bar{A}_1 = \begin{pmatrix} v & \\ a_i v & 1 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} & 1 \\ v & \end{pmatrix}, \quad \bar{A}_3 = \begin{pmatrix} & 1 \\ v & a_i \end{pmatrix}, \quad \text{or} \quad \bar{A}_4 = \begin{pmatrix} 1 & \\ & v \end{pmatrix}$$

for $i \neq 0$ and $\bar{A}_j \begin{pmatrix} \alpha & \\ & \alpha' \end{pmatrix}$ for $i = 0$, where $\bar{\beta}$ is an eigenbasis for $\mathfrak{M}_R \otimes_R \mathbb{F}$. Then there is a unique eigenbasis β of $\mathfrak{M}_R$ up to scaling lifting $\bar{\beta}$ such that $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M}_R,s_{i+1}(2)})$ is given by

$$A_1 = \begin{pmatrix} v+p & \\ (X_i+[a_i])v & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -Y_i & 1 \\ v & X_i \end{pmatrix}, A_3 = \begin{pmatrix} -p(X_i+[a_i])^{-1} & 1 \\ v & X_i+[a_i] \end{pmatrix}, \text{ or } A_4 = \begin{pmatrix} 1 & -Y_i \\ & v+p \end{pmatrix},$$

respectively, for $i \neq 0$ and $A_j D(\alpha, \alpha')$ with A_j as above for $i = 0$. Here $[\cdot]$ denotes the Teichmüller lift, $X_i Y_i = p$ for A_2, and

$$D(\alpha, \alpha') = \begin{pmatrix} [\alpha] + X_\alpha & \\ & [\alpha'] + X_{\alpha'} \end{pmatrix}.$$

Proof. The proof is similar to the proofs of [loc. cit., Theorems 4.1 and 4.16] which prove existence and uniqueness of β, respectively. We describe some of the key points. We modify [loc. cit., Definition 4.2], defining $d_R(P) = \min_k 2v_R(r_k) + k$ if $P = \sum_k r_k v^k \in R[\![v]\!]$. Then the analogue of [loc. cit., Proposition 4.3] holds (see [loc. cit., Remark 4.4]). The entry in the middle column of [loc. cit., Table 5] becomes

$$\begin{pmatrix} & 1^* \\ v(\leq 0) & 0^* \end{pmatrix}, \quad \begin{pmatrix} \leq 0 & 0^* \\ 1^* & \leq 0 \end{pmatrix}, \quad \begin{pmatrix} \leq 0 & 0^* \\ 1^* & \leq 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0^* & \leq 0 \\ & 1^* \end{pmatrix},$$

respectively, and we modify [loc. cit., Definition 4.5] for $E^{(i)}$ appropriately. For $1 \leq m, k \leq 2$, we define $\delta(A^{(i)}_{mk})$ to be $d_R(E^{(i)}_{mk})$ if $\bar{A}^{(i)} \neq \bar{A}_3$. If $\bar{A}^{(i)} = \bar{A}_3$, we define $\delta(A^{(i)}_{mk})$ to be $d_R(E^{(i)}_{mk})$ (resp. $d_R(E^{(i)}_{mk}) + 1$) if $k = 1$ (resp. if $k = 2$). Finally, we let

$$\delta(A^{(i)}) = \min_{1 \leq m,k \leq 2} \{\delta(A^{(i)}_{mk})\}.$$

The analogue of [loc. cit., Proposition 4.6] holds, replacing $3 + d_R(x^{(j)})$ with $2 + d_R(x^{(j)})$. We define the notion of pivots for $\bar{A}^{(i)} \neq \bar{A}_3$ as in the [loc. cit., Definition 4.8], and define the pivots in the case of $\bar{A}^{(i)} = \bar{A}_3$ to be the same as the pivots in the case of $\bar{A}_2$. The analogue of [loc. cit., Lemma 4.10] holds except that the second equation of [loc. cit.] is changed to $A^{(i)}_{22} = vP_{22} + [a_i] + Q_{22}$ when $\bar{A}^{(i)} = \bar{A}_3$. Then the analogues of [loc. cit., Proposition 4.11, Proposition 4.13, and Lemma 4.14] give the eigenbasis β.

We give more details for the algorithm in the case $\bar{A}^{(i)} = \bar{A}_3$. We let $\delta > 1$ be an integer. Suppose that $\delta(A^{(i)})$, which is necessarily greater than one, is δ. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A^{\prime,(i)} \stackrel{\text{def}}{=} D_{22}(x)A^{(i)}$ satisfies $\delta(A^{\prime,(i)}) \geq \delta$ and $\delta(A^{\prime,(i)}_{21}) > \delta$. Note the crucial role played by the definition of $\delta(A^{\prime,(i)}_{22})$ as $d_R(E^{\prime,(i)}_{22}) + 1$ in this case. Moreover, these inequalities still hold after right multiplication by a conjugate of $D_{22}(x)^\varphi$ by a permutation matrix. This is the analogue of [loc. cit., Proposition 4.6], where the notation I^φ is defined.

Suppose next that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{21}) > \delta$. Then there exists an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A'^{,(i)} \overset{\text{def}}{=} U_{12}(x)A^{(i)}$ satisfies $\delta(A'^{,(i)}) \geq \delta$ and $\delta(A'^{,(i)}_{11})$, $\delta(A'^{,(i)}_{21}) > \delta$ (note that $\delta(A'^{,(i)}_{21}) = \delta(A^{(i)}_{21})$). Again, we use that $\delta(A'^{,(i)}_{12}) = d_R(E'^{,(i)}_{12})+1$. Moreover, these inequalities still hold after right multiplication by a conjugate of $U_{12}(x)^{\varphi}$ by a permutation matrix by the genericity assumption.

Suppose next that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{11})$, $\delta(A^{(i)}_{21}) > \delta$. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A'^{,(i)} \overset{\text{def}}{=} D_{11}(x)A^{(i)}$ satisfies $\delta(A'^{,(i)}) \geq \delta$ and $\delta(A'^{,(i)}_{11})$, $\delta(A'^{,(i)}_{21})$, $\delta(A'^{,(i)}_{12}) > \delta$ using that $A^{(i)}_{11} \in m_R \cdot R[\![v]\!]$. Moreover, these inequalities still hold after right multiplication by a conjugate of $D_{11}(x)^{\varphi}$ by a permutation matrix.

Suppose finally that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{11})$, $\delta(A^{(i)}_{21})$, $\delta(A^{(i)}_{12}) > \delta$. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A'^{,(i)} \overset{\text{def}}{=} L_{21}(x)A^{(i)}$ satisfies $\delta(A'^{,(i)}) \geq \delta + 1$ using again that $A^{(i)}_{11} \in m_R \cdot R[\![v]\!]$. Moreover, these inequalities still hold after right multiplication by a conjugate of $L_{21}(x)^{\varphi}$ by a permutation matrix by the genericity assumption. Repeating these four steps repeatedly gives the analogue of [loc. cit., Proposition 4.13] in this case.

We deduce the forms of A_i from the condition that $v + p$ must divide the determinant. Finally, the analogue of [loc. cit., Theorem 4.16] proves the uniqueness of β up to scaling. In the notation of [loc. cit.], we obtain the equation

$$\tilde{A}^{(i)}_2 + v^2 \tilde{A}^{(i)}_2 M^{(i)} = \tilde{A}^{(i)}_1 + I^{(i+1)} \tilde{A}^{(i)}_1 \tag{3-1}$$

(see [loc. cit., (4.2)]). Suppose that $d_R(I^{(j)}) \geq \delta \geq 1$ for all j. Then one can show that $d_R(I^{(j)}) \geq \delta + 1$ for all j. This implies that $I^{(j)} = 0$ for all j. We again give more details in the case $\bar{A}^{(i)} = \bar{A}_2$ or $\bar{A}_3$. The other cases are treated similarly. Let k be 1 or 2. We first compare the $(k, 1)$-entries of (3-1) to see that $d_R(I^{(i+1)}_{k2}) \geq \delta + 1$. Using this and the $(k, 2)$-entries of (3-1), we see that $d_R(I^{(i+1)}_{k1}) \geq \delta + 1$. □

For the rest of the section, let $\bar{\rho}$ be as in Section 3A and let $\mathcal{M}$ be as in Proposition 3.1 so that $\bar{\rho}|_{G_{K_\infty}}$ is isomorphic to $\mathbb{V}^*(\mathcal{M})$. Moreover, for simplicity, assume that $\bar{\rho}$ is reducible. Recall the definition of $S_{\bar{\rho}}$ from Section 3A.

Let s and s' be in W such that one of the following holds for each $i \in \mathbb{Z}/f$:

(1) s_i and s'_i are both id.

(2) s_i and s'_i are both not id.

(3) s_i is id, but s'_i is not, and $i \in S_{\bar{\rho}}$.

We say that $i \in \mathbb{Z}/f$ is case (1), (2), or (3) if the above relevant condition holds.

Proposition 3.4. *Let s and s' be in W as above. Let τ be the tame generic inertial type with $\sigma(\tau) \cong R_s(\mu_{\bar{\rho}} - s'\eta)$. Let R be the ring $\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}, X_\alpha, X_{\alpha'}]\!]/(h_i)$ where for each $i \in \mathbb{Z}/f$, h_i is Y_i, $X_iY_i - p$, $Y_i - p$, or X_i if $f - 1 - i$ is case (1), (2) with $\omega^{(f-i)} \in S_{\bar{\rho}}$, (2) with $\omega^{(f-i)} \notin S_{\bar{\rho}}$, or (3), respectively.*

Let $\mathcal{M}_R = \prod_i R((v))\mathfrak{e}^i \oplus R((v))\mathfrak{f}^i$ be the φ-module defined by

$$f-i \text{ is case (1)}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}-1}(v+p)\mathfrak{e}^i + (X_{i-1}+[a_{i-1}])v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = v\mathfrak{f}^i, \end{cases}$$

$$f-i \text{ is case (2)},\ \omega^{(f-i)} \in S_{\bar\rho}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f-i \text{ is case (2)},\ \omega^{(f-i)} \notin S_{\bar\rho}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + (X_{i-1}+[a_{i-1}])v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -p(X_{i-1}+[a_{i-1}])^{-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f-i \text{ is case (3)}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + (v+p)\mathfrak{f}^i, \end{cases}$$

with the usual modification for $i=0$. Then $\mathbb{V}^*(\mathcal{M}_R)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of type τ.

Proof. Define $w^* \in W$ and $s_\tau \in S_2$ to be the unique elements such that

$$w^*_{f-1} = \mathrm{id} \quad \text{and} \quad (w^*)^{-1} s\pi(w^*) = (s_\tau, \mathrm{id}, \dots, \mathrm{id}).$$

Then the Deligne–Lusztig representations $R_s(\mu_{\bar\rho} - s'\eta)$ and $R_{(s_\tau,\mathrm{id},\dots,\mathrm{id})}((w^*)^{-1}(\mu_{\bar\rho} - s'\eta))$ are isomorphic by [Herzig 2009, Lemma 4.2]. Moreover, (the quadratic base change of) $R_{(s_\tau,\mathrm{id},\dots,\mathrm{id})}((w^*)^{-1}(\mu_{\bar\rho} - s'\eta))$ is a generic principal series. Define $w = (w_i)_i$ by $w_i = (w^*_{f-1-i})^{-1}$ for $i \in \mathbb{Z}/f$. Then one easily checks that w is an orientation for $(w^*)^{-1}(\mu_{\bar\rho} - s'\eta)$. Let $\mathfrak{M}_R$ be the Kisin module (with quadratic unramified descent) of tame inertial type (the quadratic unramified base change of) $\tau(s_\tau, -(w^*)^{-1}(\mu_{\bar\rho} - s'\eta))$ with $A^{(i-1)} = \mathrm{Mat}_\beta(\phi^{(i-1)}_{\mathfrak{M}_R, w_i(2)})$ given by A_1, A_2, A_3, or A_4 if $f-i$ is case (1), $f-i$ is case (2) and $f-i \in S_{\bar\rho}$, $f-i$ is case (2) and $f-i \notin S_{\bar\rho}$, or $f-i$ is case (3), respectively. We claim that $T^*_{dd}(\mathfrak{M}_R \otimes_{\mathcal{O}} \mathbb{F})$ is isomorphic to the restriction to G_{K_∞} of $\bar\rho$. Assuming this, by Theorem 3.3 and the analogue of [Le et al. 2018, Sections 5.2 and 6], $T^*_{dd}(\mathfrak{M}_R)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of type τ.

Let L be $K((-p)^{1/e})$ with $e = q-1$ if $s_\tau = \mathrm{id}$ and $K_2((-p)^{1/e})$ with $e = q^2-1$ otherwise. Let Δ be the Galois group $\mathrm{Gal}(L/K)$. We claim that

$$(\mathfrak{M}_R \otimes_{\mathcal{O}_{\mathcal{E},K}} \mathcal{O}_{\mathcal{E},L})^\Delta \cong \mathcal{M}_R.$$

This would finish the proof including the claim in the previous paragraph since the restriction to G_{K_∞} of $\bar\rho$ is isomorphic to $\mathcal{M}$ by Proposition 3.1, and clearly $\mathcal{M}_R \otimes_{\mathcal{O}} \mathbb{F}$ is isomorphic to $\mathcal{M}$.

Let $\mu_{\bar\rho}$ be $(\mu_i)_i$. Let v^λ denote the torus element obtained by applying the coweight λ to $v \stackrel{\text{def}}{=} u^e$. By [Le et al. 2016a, Proposition 3.1.2], we see that a Kisin module (with quadratic unramified descent) of tame inertial type (the quadratic unramified base change of) τ with $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M}, w_{i+1}(2)})$ given by $A^{(i)}$ (resp. $A^{(i)} s_0^{-1} D(\alpha, \alpha') s_0$) for $i < f-1$ (resp. for $i = f-1$) gives a φ-module $\mathcal{M} = \prod_i \mathbb{F}((v))\mathfrak{e}'^i \oplus \mathbb{F}((v))\mathfrak{f}'^i$

with $\varphi(\mathfrak{e}'^{i-1}, \mathfrak{f}'^{i-1}) = M'_{i-1}(\mathfrak{e}'^{i}, \mathfrak{f}'^{i})$ where

$$\begin{aligned} M'_i &= w_{i+1} A^{(i)} v^{w_{i+1}^{-1}(w^*_{f-1-i})^{-1}(\mu_{f-1-i} - s'_{f-1-i}\eta)} (w_{i+1})^{-1} \\ &= (w^*_{f-2-i})^{-1} A^{(i)} v^{w^*_{f-2-i}(w^*_{f-1-i})^{-1}(\mu_{f-1-i} - s'_{f-1-i}\eta)} w^*_{f-2-i} \\ &= (w^*_{f-2-i})^{-1} A^{(i)} v^{s_{f-1-i}^{-1}(\mu_{f-1-i} - s'_{f-1-i}\eta)} w^*_{f-2-i} \end{aligned}$$

for $i < f-1$ and $M'_{f-1} = A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} v^{(w_0^*)^{-1}(\mu_0 - s_0\eta)}$. Changing to the bases $(\mathfrak{e}^i, \mathfrak{f}^i) = (\mathfrak{e}'^i, \mathfrak{f}'^i)(w^*_{f-2-i})^{-1}$, we see that $\mathcal{M}$ is given by $(M_i)_i$ where

$$\begin{aligned} M_i &= A^{(i)} v^{s_{f-1-i}^{-1}(\mu_{f-1-i} - s'_{f-1-i}\eta)} w^*_{f-2-i} (w^*_{f-1-i})^{-1} \\ &= A^{(i)} v^{s_{f-1-i}^{-1}(\mu_{f-1-i} - s'_{f-1-i}\eta)} s_{f-1-i}^{-1} \\ &= A^{(i)} s_{f-1-i}^{-1} v^{\mu_{f-1-i} - s'_{f-1-i}\eta} \end{aligned}$$

for $i < f-1$ and

$$\begin{aligned} M'_{f-1} &= A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} v^{(w_0^*)^{-1}(\mu_0 - s'_0\eta)} (w_0^*)^{-1} \\ &= A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} (w_0^*)^{-1} v^{\mu_0 - s'_0\eta} \\ &= A^{(f-1)} s_0^{-1} v^{\mu_0 - s'_0\eta} D(\alpha, \alpha'). \end{aligned}$$

The proposition is now deduced by substituting for $A^{(i)}$, s, and $\mu_{\bar\rho}$. □

If τ is an inertial type, let R^τ parametrize potentially Barsotti–Tate (framed) liftings of $\bar\rho$ of type τ. If T is a set of inertial types for K, then we let $\operatorname{Spec} R^T$ be the Zariski closure of $\bigcup_{\tau \in T} \operatorname{Spec} R^\tau[p^{-1}]$ in the universal (framed) lifting space $\operatorname{Spec} R^\square_{\bar\rho}$ of $\bar\rho$.

For applications to Shimura curves and algebraic modular forms on definite quaternion algebras, it is convenient to consider fixed determinant deformation rings. If $\psi : G_K \to \mathcal{O}^\times$ is a continuous character, let $R^{\psi,\square}_{\bar\rho}$ be the quotient of $R^\square_{\bar\rho}$ parametrizing (framed) liftings of $\bar\rho$ with determinant $\psi\varepsilon$. Let $R^{\psi,\tau}$ be the simultaneous quotient of $R^{\psi,\square}_{\bar\rho}$ and R^τ parametrizing potentially Barsotti–Tate (framed) liftings of $\bar\rho$ of type τ and determinant $\psi\varepsilon$. We can similarly define the quotient $R^{\psi,T}$ of R^T. If $R^{\psi,\tau}$ is nonzero, then R^τ must be nonzero, ψ must lift $\bar\varepsilon^{-1}\det\bar\rho$, and $\psi|_{I_K}$ must be $\det\tau$. For all sets of types T considered below, the determinants of all elements of T coincide.

Now fix a Serre weight σ in $W(\bar\rho)$. Suppose that $\sigma = \sigma_J$ for $J \subset S_{\bar\rho}$ where σ_J is defined with respect to $\mu_{\bar\rho}$. Let I be a subset of S such that $I \cap \{\pm\omega^{(i)}\}$ has size at most one for all $i \in \mathbb{Z}/f$. Let $T_{J,I}$ be the set of inertial types τ such that $\sigma(\tau)$ is of the form $R_s(\mu_{\bar\rho} - s'\eta)$ where s and s' have the restrictions given by the following table:

s_i, s'_i	$i \notin J$	$i \in J$
$\{\pm\omega^{(i)}\} \cap I = \varnothing$	$s_i = s'_i$	$s' \neq \mathrm{id}$
$\omega^{(i)} \in I$	$s_i = s'_i = \mathrm{id}$	$s_i = s'_i \neq \mathrm{id}$
$-\omega^{(i)} \in I$	$s_i = s'_i \neq \mathrm{id}$	$s_i = \mathrm{id}, s'_i \neq \mathrm{id}$

Lemma 3.5. *Define $w_J \in W$ by $w_{J,i-1} = \mathrm{id}$ if and only if $i \notin J$ for all $i \in \mathbb{Z}/f$. Then the set of tame inertial types $T_{J,I}$ corresponds by inertial local Langlands to the set $T_{\sigma, w_J(I)}$ of Deligne–Lusztig representations defined in Section 2.*

Proof. This is a computation using the definitions and [Herzig 2009, Theorem 5.2]. Note that in the notation of [loc. cit.], $\gamma'_{\sigma,\tau}$ in this case is equal to the Kronecker symbol for σ and τ. Another method of proof is to use [Le et al. 2016b, Proposition 2.10] and verify that if $V_\phi(\tau) \cong R_s(\mu)$, then $W^?(\tau) = \mathrm{JH}(\bar{R}_{sw_0}(\mu - sw_0\eta))$. □

Theorem 3.6. *There is an isomorphism to a formal power series ring over $\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}]\!]/(g_i(J, I))_i$ from $R^{T_{J,I}}$, where $g_i(J, I)$ is given by the following table*:

$g_i(J,I)$	$\omega^{(f-1-i)} \notin S_{\bar\rho}$	$\omega^{(f-1-i)} \in S_{\bar\rho} \setminus J$	$\omega^{(f-1-i)} \in J$
$\{\pm\omega^{(f-1-i)}\} \cap I = \varnothing$	$Y_i(Y_i - p)$	$Y_i(X_iY_i - p)$	$X_i(X_iY_i - p)$
$\omega^{(f-1-i)} \in I$	Y_i	Y_i	$X_iY_i - p$
$-\omega^{(f-1-i)} \in I$	$Y_i - p$	$X_iY_i - p$	X_i

If $I \subset I'$, then $g_i(J, I') \mid g_i(J, I)$ for all $i \in \mathbb{Z}/f$ and $R^{T_{J,I'}}$ is the quotient of $R^{T_{J,I}}$ by the ideal $(g_i(J, I'))_i$. Analogous results hold for $R^{\psi, T_{J,I}}$ provided that ψ is chosen so that $R^{\psi,T_{J,I}}$ is nonzero for any, or equivalently all, choices of I as above.

Remark 3.7. Since twisting by the universal unramified deformation of the trivial character gives an isomorphism $R^T \cong R^{\psi,T}[\![X]\!]$ (assuming $R^{\psi,T}$ is nonzero), the fixed determinant case follows from the first part of Theorem 3.6, and we ignore it below (see [Emerton et al. 2015, Remark 7.2.2]).

Proof. Since $R^{T_{J,I}}$ is naturally a quotient of $R^{\square}_{\bar\rho|_{G_{K_\infty}}}$ by [Emerton et al. 2015, Lemma 7.4.3], it suffices to compute the Zariski closure of $\bigcup_{\tau \in T_{J,I}} \operatorname{Spec} R^\tau[p^{-1}]$ in $\operatorname{Spec} R^{\square}_{\bar\rho|_{G_{K_\infty}}}$. Let R be the ring

$$\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}, X_\alpha, X_{\alpha'}]\!]/(g_i(J, \varnothing))_i$$

and consider the deformation $\mathcal{M}_R = \prod_i R((v))\mathfrak{e}^i \oplus R((v))\mathfrak{f}^i$ of $\mathcal{M}$ defined by

$$f - i \notin S_{\bar\rho} : \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}-1}(v + p - Y_{i-1})\mathfrak{e}^i + v^{c_{f-i}}(X_{i-1} + [a_{i-1}])\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}(X_{i-1} + [a_{i-1}])^{-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f - i \in S_{\bar\rho} \setminus J : \begin{cases} \varphi(\mathfrak{e}^{i-1}) \ = v^{c_{f-i}-1}(v + p - X_{i-1}Y_{i-1})\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) \ = -Y_{i-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f - i \in J : \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + (v + p - X_{i-1}Y_{i-1})\mathfrak{f}^i, \end{cases}$$

with the usual modification at $i = 0$. Define the deformation functor $D^\square$ by $D^\square(A) = \{(\psi : R \to A, b_A)\}/\cong$ for A a complete local Noetherian $\mathcal{O}$-algebra, where b_A is a basis for the free rank two A-module $\mathbb{V}^*(\psi^*(\mathcal{M}_R))$ whose reduction modulo $\mathfrak{m}_A$ gives $\bar\rho$. Then the natural map $D^\square \to \operatorname{Spf} R$ is a $\widehat{\mathrm{GL}}_2$-torsor and is thus formally smooth of dimension 4. Let $D^\square$ be $\operatorname{Spf} R^\square$. One can rescale $\mathfrak{e}^0$ and $\mathfrak{f}^0$ by units, and rescale the other basis vectors appropriately so that the coefficients in the definition of φ which are 1 remain 1. This gives a $\hat{\boldsymbol{G}}_m^2$-action on R, and orbits give isomorphic φ-modules. We claim that the natural map $\operatorname{Spf} R^\square/\hat{\boldsymbol{G}}_m^2 \to \operatorname{Spf} R^\square_{\bar\rho|_{G_{K_\infty}}}$ is a closed embedding. It suffices to show injectivity on reduced tangent spaces.

Suppose that t is a reduced tangent vector of $\operatorname{Spf} R^\square/\hat{\boldsymbol{G}}_m^2$ which maps to zero in $\operatorname{Spf} R^\square_{\bar\rho|_{G_{K_\infty}}}$. By formal smoothness, we can extend this to a map $t : R^\square \to \mathbb{F}[\varepsilon]/(\varepsilon^2)$. Let $\mathcal{M}_t$ be $\mathcal{M}_R \otimes_{R,t} \mathbb{F}[\varepsilon]/(\varepsilon^2)$ so that $\mathcal{M}_t$ and $\mathcal{M} \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]/(\varepsilon^2)$ are isomorphic. Let M_i (resp. $M_{t,i}$) be the matrices such that $\varphi(\mathfrak{e}^i \otimes_R \mathbb{F}, \mathfrak{f}^i \otimes_R \mathbb{F}) = M_i(\mathfrak{e}^{i+1} \otimes_R \mathbb{F}, \mathfrak{f}^{i+1} \otimes_R \mathbb{F})$ (resp. $\varphi(\mathfrak{e}^i \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2), \mathfrak{f}^i \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2)) = M_{t,i}(\mathfrak{e}^{i+1} \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2), \mathfrak{f}^{i+1} \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2))$). Then there are matrices $D_i \in \mathrm{GL}_2(\mathbb{F}((v)))$ such that

$$(\mathrm{id}_3 + \varepsilon D_i) M_i \varphi(\mathrm{id}_3 - \varepsilon D_{i-1}) = M_{t,i}$$

for all $i \in \mathbb{Z}/f$, where id_3 is the 3×3 identity matrix (we can assume without loss of generality that the terms without ε are id_3 by multiplying by their inverses). We first claim that $D_i \in \mathrm{GL}_2(\mathbb{F}[\![v]\!])$ for all $i \in \mathbb{Z}/f$. For each i, let $k_i \in \mathbb{Z}$ be the minimal integer such that $v^{k_i} D_i \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$. Then $v^{c_{f-1-i}+k_i}\varphi(\mathrm{id}_3 - \varepsilon D_{i-1}) = v^{c_{f-1-i}+k_i} M_i^{-1}(\mathrm{id}_3 - \varepsilon D_i) M_{t,i} \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$, and thus $c_{f-1-i} + k_i \geq p k_{i-1}$. Since $c_{f-1-i} < p-1$, $k_i \geq 2 + p(k_{i-1}-1)$. If $k_{i-1} \geq n \geq 1$, then $k_i \geq n+1$, from which we derive the contradiction that $k_i \geq n$ for every $n \in \mathbb{N}$. Hence $k_i \leq 0$ for all i.

We next claim that if $f-1-i \notin S_{\bar\rho}$ for some $i \in \mathbb{Z}/f$, then $t(Y_i) = 0$. Suppose for the sake of contradiction that $f-1-i \notin S_{\bar\rho}$ and $t(Y_i) \neq 0$. Let $N_i \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$ be such that $\varepsilon N_i = M_{t,i} - M_i$. Then by the formulas for M_i and $M_{t,i}$, the first (resp. second) entry in the top row of N_i is exactly divisible by $v^{c_{f-1-i}-1}$ (resp. v^0). On the other hand, since $D_i M_i - M_i \varphi(D_{i-1}) = N_i$, the first (resp. second) entry in the top row of N_i is divisible by $v^{c_{f-1-i}}$ (resp. v), which is a contradiction. Thus t is a reduced tangent vector of

$$(\operatorname{Spf} R^\square/(Y_i : f-1-i \notin S_{\bar\rho}))/\hat{\boldsymbol{G}}_m^2.$$

Let τ be the tame inertial type such that $\sigma(\tau) = R_{w_0}(\mu - w_0\eta)$. Then the natural map from the quotient of

$$\operatorname{Spf} R^\square/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\}, \{X_i Y_i : f-1-i \in S_{\bar\rho}\}) \tag{3-2}$$

by $\hat{\boldsymbol{G}}_m^2$ to $\operatorname{Spf} R^\tau/\varpi$ is formally smooth by Proposition 3.4. In fact, it is an isomorphism since the domain and codomain are both of dimension $f+4$ over $\mathbb{F}$. Indeed, for the codomain this follows from [Kisin 2008, Theorem 3.3.4] and p-flatness, while for the domain we see directly that (3-2) has dimension $f+6$. Since the map

$$\operatorname{Spf} R^\square/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\}, \{X_i Y_i : f-1-i \in S_{\bar\rho}\}) \to \operatorname{Spf} R^\square/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\})$$

is an isomorphism on reduced tangent spaces, t is a reduced tangent vector of $\operatorname{Spf} R^\tau$. Since $\operatorname{Spf} R^\tau \to \operatorname{Spf} R^\square_{\bar\rho|_{G_{K_\infty}}}$ is injective on reduced tangent spaces again by [Emerton et al. 2015, Lemma 7.4.3], t is zero.

Finally, since R is p-flat, it suffices to show that if $\#(\{\pm\omega_i\} \cap I) = 1$ for all $i \in \mathbb{Z}/f$, then $\mathbb{V}^*(\mathcal{M}/(g_i(J,I))_i)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of the unique type τ in $T_{J,I}$. This follows from Proposition 3.4. □

4. Patching functors and multiplicity one

Let $\bar\rho : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation. Again, $\bar\rho$ is either an extension of

$$\mathrm{nr}_{\alpha'}\, \omega_f^{\sum_{i=0}^{f-1} \mu_{2,i} p^i} \quad \text{by} \quad \mathrm{nr}_\alpha\, \omega_f^{\sum_{i=0}^{f-1} \mu_{1,i} p^i}$$

or is

$$\mathrm{nr}_\alpha \operatorname{Ind}_{G_{K_2}}^{G_K} \omega_{2f}^{\sum_{i=0}^{f-1} \mu_{1,i} p^i + p^f \sum_{i=0}^{f-1} \mu_{2,i} p^i}$$

for some dominant p-restricted character $\mu_{\bar\rho} = (\mu_{1,i}, \mu_{2,i})_i \in X^*(T)$ and some α and $\alpha' \in \mathbb{F}^\times$.

Definition 4.1. We say that a dominant p-restricted $\mu \in X^*(T)$ is generic if $2 < \langle \mu, \beta \rangle < p-3$. We say that $\bar\rho$ is generic if $\mu_{\bar\rho}$ is generic or if $\bar\rho$ is semisimple and 1-generic in the sense of [Le et al. 2016b, Definition 4.1].

Note that if $\bar\rho$ is generic, then $\bar\rho$ is generic in the sense of [Breuil and Paškūnas 2012, Definition 11.7; Emerton et al. 2015, Definition 2.1.1]. We now assume that $\bar\rho$ is not semisimple and is generic. Then a twist of $\bar\rho$ is of the form in Section 3A.

We now fix a Serre weight $\sigma \in W(\bar\rho)$ ($W(\bar\rho)$ is recalled in Section 3A). Let $\mu \in X^*(T)$ be such that $\sigma \cong F(\mu - \eta)$. If σ is $\sigma_{J(\sigma)}$ with respect to $\mu_{\bar\rho}$, define $w_{J(\sigma)} \in W$ by $w_{J(\sigma), i-1} = \mathrm{id}$ if and only if $i \notin J(\sigma)$ for all $i \in \mathbb{Z}/f$ as in Lemma 3.5. Then we set $S_{\bar\rho}^\sigma$ to be $w(S_{\bar\rho})$ with $w = w_{J(\sigma)}^{-1} \pi(w_{J(\sigma)})$.

Lemma 4.2. *The set $W(\bar\rho)$ is $\{\sigma_J \mid J \subset S_{\bar\rho}^\sigma\}$ where σ_J is defined in terms of μ.*

Proof. This follows from Proposition 3.2 and [Le et al. 2016b, Proposition 2.4]. □

Let $\psi : G_K \to \mathcal{O}^\times$ be an unramified twist of $\omega_f^{\sum_{i \in \mathbb{Z}/f} (\mu_{1,i} + \mu_{2,i} - 1)}$ lifting $\bar\varepsilon^{-1} \det \bar\rho$. Suppose that $M_\infty(\cdot)$ is a minimal fixed determinant patching functor over $\mathcal{O}$ for $\bar\rho^\vee$ with fixed determinant $\psi^\vee$ (see [Emerton et al. 2015, Definition 6.1.3]). (Note that $\mathcal{D}(\bar\rho^\vee)$ in the conventions of [loc. cit., Section 2] is $W(\bar\rho)$ in ours.) Using contragredients, we identify $R_{\bar\rho^\vee}^\square$ with $R_{\bar\rho}^\square$. This identifies R^τ with the (framed) lifting ring of $\bar\rho^\vee$ parametrizing lifts $\rho^\vee$ of type $\tau^\vee$ with $\mathrm{HT}_\kappa(\rho^\vee) = \{-1, 0\}$ for all $\kappa : E \hookrightarrow \mathbb{C}_p$. Note that such lifts of $\bar\rho^\vee$ are called potentially Barsotti–Tate in [loc. cit., Section 7]. Similar identifications are made for multitype (fixed determinant) potentially Barsotti–Tate deformation rings. For an $\mathcal{O}_K[\mathrm{GL}_2(\mathcal{O}_K)]$-module N, we will denote $M_\infty(N \otimes_{\mathcal{O}_K} \mathcal{O})$ by $M'_\infty(N)$, where tensor product is over the map $\mathcal{O}_K \hookrightarrow \mathcal{O}$ in Section 1A.

Lemma 4.3. *The R_∞-module $M'_\infty(R_\mu / \mathrm{Fil}^2_\otimes R_\mu)$ is cyclic.*

Proof. Let τ be the tame type such that $\sigma(\tau) = R_w(\mu - w\eta)$. Then $W(\bar\rho)$ is exactly $\mathrm{JH}(\bar\sigma(\tau))$. Let $\sigma^\circ(\tau) \subset \sigma(\tau)$ be the unique lattice up to homothety with cosocle isomorphic to σ (see [loc. cit., Lemma 4.1.1]). Let $\bar\sigma^\circ(\tau)$ be the reduction of $\sigma^\circ(\tau)$. Then the natural map $R_\mu \twoheadrightarrow \bar\sigma^\circ(\tau)$ induces a map

$$R_\mu / \mathrm{Fil}^2_\otimes R_\mu \twoheadrightarrow \bar\sigma^\circ(\tau) / \mathrm{rad}^2 \bar\sigma^\circ(\tau). \tag{4-1}$$

By [Le et al. 2016b, Proposition 3.2], the Jordan–Hölder factors of $R_\mu / \mathrm{Fil}^2_\otimes R_\mu$ appear without multiplicity. Moreover, those Jordan–Hölder factors which are also in $W(\bar\rho)$ are in $\mathrm{JH}(\bar\sigma^\circ(\tau) / \mathrm{rad}^2 \bar\sigma^\circ(\tau))$ by [Emerton et al. 2015, Theorem 5.1.1] (these are exactly the Serre weights σ_J with respect to μ with $J \subset S^{\sigma}_{\bar\rho}$ and $\#J = 1$.). Thus the kernel of the map (4-1) contains no Jordan–Hölder factors in $W(\bar\rho)$. We then see that the induced map

$$M'_\infty(R_\mu/\operatorname{Fil}^2_\otimes R_\mu) \twoheadrightarrow M'_\infty(\bar\sigma^\circ(\tau)/\operatorname{rad}^2 \bar\sigma^\circ(\tau))$$

is an isomorphism. As $M'_\infty(\bar\sigma^\circ(\tau))$ is a cyclic R_∞-module by [Emerton et al. 2015, Theorem 10.1.1], so is $M'_\infty(\bar\sigma^\circ(\tau)/\operatorname{rad}^2 \bar\sigma^\circ(\tau))$. □

Lemma 4.4. *Suppose that $I \subset S$ such that*

$$\#(I \cap \{\pm\omega^{(i)}\}) + \#(S^{\sigma}_{\bar\rho} \cap \{\pm\omega^{(i)}\}) = 1$$

for all i. Let N be a submodule of $\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$, and let $\overline{V}$ be its image in $\operatorname{gr}^k_\otimes R_{\mu,I}$. If $\operatorname{gr}^k_\otimes R_{\mu,I}/\overline{V}$ contains no Serre weights in $W(\bar\rho)$, then

$$(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})/N$$

contains no Jordan–Hölder factors in $W(\bar\rho)$.

Proof. It suffices to show that $\operatorname{gr}^{k+1}_\otimes R_{\mu,I}/\operatorname{gr}^{k+1}_\otimes N$ contains no Jordan–Hölder factors in $W(\bar\rho)$, since by assumption $\operatorname{gr}^k_\otimes R_{\mu,I}/\operatorname{gr}^k_\otimes N$ contains no Jordan–Hölder factors in $W(\bar\rho)$. In fact, it suffices to show that $\operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I}/(N \cap \operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I})$ contains no Jordan–Hölder factors in $W(\bar\rho)$ since $\sum_{|\boldsymbol{k}|=k} \operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I} = \operatorname{gr}^{k+1}_\otimes R_{\mu,I}$.

By Proposition 2.1, a Jordan–Hölder factor of $\operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ has the form $\sigma_{J'}$ with respect to μ where $J' \cap I = \varnothing$ and there is a $j \in \mathbb{Z}/f$ such that if $\boldsymbol{k}(J') = \boldsymbol{k}'$ then $k'_i = k_i$ for all $i \neq j$ and $k'_j = k_j + 1$. Suppose that $\sigma_{J'} \in W(\bar\rho)$. If $k'_j = 2$, then let $J = J' \setminus \{-w_j\omega^{(j)}\}$ (with w defined in the beginning of the section). Otherwise, $J' \cap \{\pm\omega^{(j)}\} = \{w_j\omega^{(j)}\}$ since we assumed that $\sigma_{J'} \in W(\bar\rho)$. In this case, let $J = J' \setminus \{w_j\omega^{(j)}\}$. Then $\sigma_J \in W(\bar\rho)$ and is thus a Jordan–Hölder factor of $N \cap W_{\boldsymbol{k},\boldsymbol{k}+1,I}$. By Proposition 2.3, $\sigma_{J'}$ is a Jordan–Hölder factor of N. □

The following lemma generalizes [Emerton et al. 2015, Lemma 10.1.13], one of the methods used to compute patched modules.

Lemma 4.5. *Let R be a local ring, and $M'' \subset M' \subset M$ be R-modules such that M'/M'' and M' are minimally generated by the same finite number of elements. Then $M'' \subset \mathfrak{m}M$. If, moreover, M is finitely generated over R, then M/M'' and M are minimally generated by the same number of elements.*

Proof. By Nakayama's lemma, that M'/M'' and M' are minimally generated by the same finite number of elements implies that $M'' \subset \mathfrak{m}M'$ and thus $M'' \subset \mathfrak{m}M$. If M is finitely generated, then another application of Nakayama's lemma implies that M/M'' and M are minimally generated by the same number of elements. □

The following proposition generalizes the results and methods of [Hu and Wang 2018; Le et al. 2016b] by combining Lemmas 4.3, 4.4, and 4.5.

Proposition 4.6. *Suppose that $I \subset S$ such that $\#(I \cap \{\pm\omega^{(i)}\}) + \#(S^{\sigma}_{\bar\rho} \cap \{\pm\omega^{(i)}\}) = 1$. Then $M'_\infty(\tilde R_{\mu,I})$ is a cyclic R_∞-module.*

Proof. By Nakayama's lemma, it suffices to show that $M'_\infty(R_{\mu,I})$ is a cyclic R_∞-module. We will show that $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+1}_\otimes R_{\mu,I})$ is a cyclic R_∞-module by induction on k. If $k=1$, then the result follows from Lemma 4.3.

Now suppose that $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+1}_\otimes R_{\mu,I})$ is a cyclic R_∞-module. Let $\mathfrak{J}$ be

$$\{J \subset S : k(J) = k,\ J \cap I = \varnothing, \sigma_J \in W(\bar\rho)\}.$$

Recall that for each $J \in \mathfrak{J}$,

$$\overline{V}_J \subset \operatorname{Fil}^k_\otimes R_\mu/\operatorname{Fil}^{k+2}_\otimes R_\mu$$

is defined before [Le et al. 2016b, Proposition 3.9] to be the minimal submodule whose image in $\operatorname{gr}^k_\otimes R_\mu$ contains σ_J. Then we let $\overline{V}_{J,I}$ be the image of $\overline{V}_J$ in $R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$. Note that $M'_\infty(\overline{V}_{J,I})$ is a cyclic R_∞-module by Lemma 4.3. Let $\overline{V}$ be $\sum_{J\in\mathfrak{J}} \overline{V}_{J,I} \subset \operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$. By Lemma 4.4, the quotient $(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})/\overline{V}$ does not contain any Jordan–Hölder factors in $W(\bar\rho)$. Thus the natural inclusion $M'_\infty(\overline{V}) \subset M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$ is an equality. In particular,

$$M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$$

is generated by no more than $\#\mathfrak{J}$ elements. On the other hand, $M'_\infty(\operatorname{gr}^k_\otimes R_{\mu,I}) \cong \bigoplus_{J\in\mathfrak{J}} M'_\infty(\sigma_J)$ is generated by (at least) $\#\mathfrak{J}$ elements. By Lemma 4.5 with $M = M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$, $M' = M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$, and $M'' = M'_\infty(\operatorname{gr}^{k+1}_\otimes R_{\mu,I})$, $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$ is a cyclic R_∞-module. □

Proposition 4.7. *The scheme-theoretic support of $M'_\infty(\tilde R_{\sigma,I})$ is* $\operatorname{Spec}\big(R_\infty \widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}} R^{\psi,T_{\sigma,I}}\big)$.

Proof. Since $M'_\infty(\tilde R_{\sigma,I})[p^{-1}]$ is isomorphic to $\bigoplus_{\sigma(\tau)\in T_{\sigma,I}} M'_\infty(\sigma(\tau))$, the scheme-theoretic support of $M'_\infty(\tilde R_{\sigma,I})[p^{-1}]$ is $\bigcup_{\sigma(\tau)\in T_{\sigma,I}} \operatorname{Spec}\big(R_\infty \widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}} R^{\psi,\tau}\big)[p^{-1}]$ by the proof of [Emerton et al. 2015, Theorem 9.1.1]. Since $M'_\infty(\tilde R_{\sigma,I})$ is $\mathcal{O}$-flat by definition of a patching functor, the scheme-theoretic support of $M'_\infty(\tilde R_{\sigma,I})$ is the Zariski closure of that of $M'_\infty(\tilde R_{\sigma,I})[p^{-1}]$. The result now follows from the definition of $\operatorname{Spec} R^{\psi,T_{\sigma,I}}$. □

In order to weaken the hypotheses on I in Proposition 4.6, we compute an integral scheme intersection, of which the following lemma is the key example.

Lemma 4.8. *There is an exact sequence*

$$0 \to \mathcal{O}[\![Y]\!]/(Y(Y-p)) \to \mathcal{O}[\![Y]\!]/(Y) \oplus \mathcal{O}[\![Y]\!]/(Y-p) \to \mathcal{O}[\![Y]\!]/(Y,p) \to 0,$$

where the second and third maps are the sum and difference, respectively, of the natural projections.

Proof. Given a ring R and ideals I and $J \subset R$, the sequence

$$0 \to R/(I\cap J) \to R/I \oplus R/J \to R/(I+J) \to 0,$$

where the second and third maps are the sum and difference, respectively, of the natural projections, is exact. The lemma follows from this exact sequence and the relations $(Y)\cap(Y-p)=(Y(Y-p))$ and $(Y)+(Y-p)=(Y,p)$ in $\mathcal{O}[\![Y]\!]$. □

The following is our main result in the setting of patching functors. Recall that $\bar\rho$ is generic, but not semisimple.

Theorem 4.9. *Suppose that $I\subset S$ such that $\#(I\cap\{\pm\omega^{(i)}\})+\#(S^\sigma_{\bar\rho}\cap\{\pm\omega^{(i)}\})\le 1$. Then $M'_\infty(\tilde R_{\mu,I})$ is a cyclic R_∞-module.*

Proof. We proceed by induction on $k:=f-\#S^\sigma_{\bar\rho}-\#I$. The case $k=0$ follows from Proposition 4.6. Suppose that $k>0$ and that $(I\cup S^\sigma_{\bar\rho})\cap\{\pm\omega^{(j)}\}=\varnothing$. Then there is an exact sequence

$$0\to\tilde R_{\mu,I}\to\tilde R_{\mu,I\cup\{\omega^{(j)}\}}\oplus\tilde R_{\mu,I\cup\{-\omega^{(j)}\}}\to R_{\mu,I\cup\{\pm\omega^{(j)}\}}\to 0,$$

which induces an exact sequence

$$0\to M'_\infty(\tilde R_{\mu,I})\to M'_\infty(\tilde R_{\mu,I\cup\{\omega^{(j)}\}})\oplus M'_\infty(\tilde R_{\mu,I\cup\{-\omega^{(j)}\}})\to M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})\to 0,$$

where the third map is the sum of two surjections by exactness of $M'_\infty(\cdot)$. By the inductive hypothesis and Proposition 4.7, $M'_\infty(\tilde R_{\mu,I\cup\{\omega^{(j)}\}})$ and $M'_\infty(\tilde R_{\mu,I\cup\{-\omega^{(j)}\}})$ are cyclic R_∞-modules with scheme-theoretic support $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}$ and $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}$, respectively. The scheme-theoretic support of $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ is thus a closed subscheme of the intersections of $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}$ and $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}$, which is $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p$ by Theorem 3.6 and Lemma 3.5 (we can assume without loss of generality that μ has the form in Section 3 by twisting). Since $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ is a cyclic R_∞-module, there is a surjection

$$R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\twoheadrightarrow M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}}).$$

Since $\{\pm\omega^{(j)}\}\cap S^\sigma_{\bar\rho}=\varnothing$, from Proposition 2.1 we see that $M'_\infty(R_{\mu,I\cup\{\omega^{(j)}\}})$ and $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ have the same Hilbert–Samuel multiplicity. Thus, both sides of the map $R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\twoheadrightarrow M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ have the same Hilbert–Samuel multiplicity. Since $R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p$ contains no embedded primes, this map is an isomorphism (see the argument of [Le 2018, Lemma 6.1.1]).

In summary, there is an exact sequence

$$0\to M'_\infty(\tilde R_{\mu,I})\to R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}\oplus R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}\to R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\to 0,$$

where the third map is the sum of two surjections. Any lift of a generator under a surjection between two cyclic modules over a local ring is again a generator by Nakayama's lemma. Hence, we can assume that the third map is the difference of the natural projections. Then by Theorem 3.6 and Lemma 3.5, this exact sequence is obtained from taking a completed tensor product with the exact sequence in Lemma 4.8. Hence, we see that $M'_\infty(\tilde R_{\mu,I})\cong R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I}}$, and in particular that $M'_\infty(\tilde R_{\mu,I})$ is a cyclic R_∞-module. □

5. Global results

Let F be a totally real field in which p is unramified. Let $D_{/F}$ be a quaternion algebra which is unramified at all places dividing p and at most one infinite place, and let $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ be a Galois representation. If $D_{/F}$ is indefinite and $K = \prod_w K_w \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ is an open compact subgroup, then there is a smooth projective curve X_K defined over F and we define $S(K, \mathbb{F})$ to be $H^1((X_K)_{/\bar{F}}, \mathbb{F})$. If $D_{/F}$ is definite, then we let $S(K, \mathbb{F})$ be the space of K-invariant continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to \mathbb{F}.$$

Let S be the union of the set of places in F where $\bar{r}$ is ramified, the set of places in F where D is ramified, and the set of places in F dividing p. Let $\mathbb{T}^{S,\mathrm{univ}}$ be the commutative polynomial algebra over $\mathcal{O}$ generated by the formal variables T_w and S_w for each $w \notin S \cup \{w_1\}$ where w_1 is chosen as in [Emerton et al. 2015, Section 6.2]. Then $\mathbb{T}^{S,\mathrm{univ}}$ acts on $S(K, \mathbb{F})$ with T_w and S_w acting by the usual double coset action of

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_w})\right] \quad \text{and} \quad \left[\mathrm{GL}_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w & \\ & \varpi_w \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_w})\right],$$

respectively. Let $\mathbb{T}^{S,\mathrm{univ}} \to \mathbb{F}$ be the map such that the image of $X^2 - T_w X + (\mathbb{N}w) S_w$ in $\mathbb{F}[X]$ is the characteristic polynomial of $\bar{\rho}^\vee(\mathrm{Frob}_w)$, where Frob_w is a geometric Frobenius element at w, and let the kernel be $\mathfrak{m}_{\bar{r}}$.

For the rest of the section, suppose that

(1) $\bar{r}$ is modular, i.e., that there exists K such that $S(K, \mathbb{F})_{\mathfrak{m}_{\bar{r}}}$ is nonzero;

(2) $\bar{r}|_{G_{F(\zeta_p)}}$ is absolutely irreducible;

(3) if $p = 5$ then the image of $\bar{r}(G_{F(\zeta_p)})$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5;

(4) $\bar{r}|_{G_{F_w}}$ is generic (Definition 4.1) for all places $w \mid p$; and

(5) $\bar{r}|_{G_{F_w}}$ is nonscalar at all finite places where D ramifies.

Let $v \mid p$ be a place of F, and let $\bar{\rho}$ be $\bar{r}|_{G_{F_v}}$. Let k_v be the residue field of F_v.

We define $S^{\min}$ to be $S(K^v, \otimes_{w \in S, w \neq v} L_w)_{\mathfrak{m}'_{\bar{r}}}$ as in [Emerton et al. 2015, Section 6.5]. We define $M^{\min}$ to be the $\mathbb{F}$-linear dual of $(S^{\min} \otimes_{\mathcal{O}} \mathbb{F})[\mathfrak{m}'_{\bar{r}}]$, factoring out the Galois action in the indefinite case (see [Emerton et al. 2015, Section 6.2]).

Theorem 5.1. *Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ is a Galois representation satisfying (1)-(5). If $\sigma \in W(\bar{\rho})$ and R_σ is the $\mathbb{F}[\mathrm{GL}_2(k_v)]$-projective envelope of σ, then $\mathrm{Hom}_{\mathbb{F}[\mathrm{GL}_2(k_v)]}(R_\sigma, (M^{\min})^*)$ is one-dimensional.*

Proof. The case where $\bar{\rho}$ is semisimple follows from [Le et al. 2016b, Corollary 5.4]. We now assume that $\bar{\rho}$ is not semisimple. Let $\sigma = F(\mu - \eta) \in W(\bar{\rho})$. Identify k_v with a finite field $\mathbb{F}_q$. Then R_σ is $R_\mu \otimes_{\mathbb{F}_q} \mathbb{F}$. Let M_∞ be the minimal fixed determinant patching functor defined in [Emerton et al. 2015, Section 6.5]. By construction, if $\mathfrak{m}_{R_\infty}$ is the maximal ideal of R_∞, then $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(R_\sigma, (M^{\min})^*)$ is the dual of $M_\infty(R_\sigma)/\mathfrak{m}_{R_\infty} = M'_\infty(R_\mu)/\mathfrak{m}_{R_\infty}$, which is one dimensional since $M'_\infty(R_\mu)$ is a cyclic R_∞-module by Theorem 4.9. $\square$

Let $M^{\min}(K_v(1))$ denote the space of coinvariants $(M^{\min})_{K_v(1)}$. Note that $M^{\min}(K_v(1))$ is isomorphic to the dual of $(S(K^v K_v(1), \otimes_{w\in S, w\neq v} L_w) \otimes_{\mathcal{O}} \mathbb{F})[\mathfrak{m}'_{\bar r}]$, factoring out the Galois action in the indefinite case, by a standard spectral sequence argument using that $\mathfrak{m}'_{\bar r}$ is non-Eisenstein.

Corollary 5.2. *Suppose that* $\bar r : G_F \to \mathrm{GL}_2(\mathbb{F})$ *is a Galois representation satisfying* (1)-(5). *Then the* $\mathrm{GL}_2(\mathbb{F}_q)$*-representation* $(M^{\min}(K_v(1)))^*$ *is isomorphic to* $D_0(\bar\rho)$. *In particular,* $(M^{\min}(K_v(1)))^*$ *depends only on* $\bar\rho$ *and is multiplicity free.*

Proof. There is an injection $D_0(\bar\rho) \hookrightarrow (M^{\min}(K_v(1)))^*$ by [Breuil 2014, Proposition 9.3]. Fix an $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-injective hull $(M^{\min}(K_v(1)))^* \hookrightarrow I$. Since

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(R_\sigma, (M^{\min}(K_v(1)))^*)$$

is one-dimensional for all $\sigma \in W(\bar\rho)$ by Theorem 5.1, this injective hull factors through $D_0(\bar\rho)$ by [Breuil and Paškūnas 2012, Theorem 1.1(i)]. Since $D_0(\bar\rho)$ and $(M^{\min}(K_v(1)))^*$ are finite length $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-modules, they must be isomorphic. Finally, note that $D_0(\bar\rho)$ is multiplicity free by [Breuil and Paškūnas 2012, Theorem 1.1(ii)]. □

Acknowledgments

Lemma 4.5 originally appeared in [Le et al. ≥ 2019], and we thank Bao Le Hung and Stefano Morra for allowing us to reproduce it here. The idea to use multitype Barsotti–Tate deformation rings grew out of the joint work [loc. cit.]. We thank Bao Le Hung and Stefano Morra for their collaboration and other useful discussions on Kisin modules and étale φ-modules. We thank the anonymous referee for many useful suggestions, comments, and corrections. The debt owed to the work of Christophe Breuil, Matthew Emerton, Toby Gee, Vytautas Paškūnas, and David Savitt will be clear to the reader.

The author was supported by the Simons Foundation under an AMS-Simons travel grant, by the National Science Foundation under the Mathematical Sciences Postdoctoral Research Fellowship No. 1703182, and by the Centre International de Rencontres Mathématiques under the Research in Pairs program No. 1877. We thank CIRM for providing hospitality and excellent working conditions while part of this work was carried out.

References

[Breuil 2014] C. Breuil, "Sur un problème de compatibilité local-global modulo p pour GL_2", *J. Reine Angew. Math.* **692** (2014), 1–76. MR Zbl

[Breuil and Paškūnas 2012] C. Breuil and V. Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **1016**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Buzzard et al. 2010] K. Buzzard, F. Diamond, and F. Jarvis, "On Serre's conjecture for mod ℓ Galois representations over totally real fields", *Duke Math. J.* **155**:1 (2010), 105–161. MR Zbl

[Diamond 1997] F. Diamond, "The Taylor–Wiles construction and multiplicity one", *Invent. Math.* **128**:2 (1997), 379–391. MR Zbl

[Diamond 2007] F. Diamond, "A correspondence between representations of local Galois groups and Lie-type groups", pp. 187–206 in *L-functions and Galois representations* (Durham, UK, 2004), edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR Zbl

[Emerton et al. 2015] M. Emerton, T. Gee, and D. Savitt, "Lattices in the cohomology of Shimura curves", *Invent. Math.* **200**:1 (2015), 1–96. MR Zbl

[Fujiwara 2006] K. Fujiwara, "Deformation rings and Hecke algebras in the totally real case", submitted, 2006. arXiv

[Henniart 2002] G. Henniart, "Sur l'unicité des types pour GL_2", (2002). Appendix to C. Breuil and A. Mézard, "Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$", *Duke Math. J.* **115**:2 (2002), 205–310. MR Zbl

[Herzig 2009] F. Herzig, "The weight in a Serre-type conjecture for tame n-dimensional Galois representations", *Duke Math. J.* **149**:1 (2009), 37–116. MR Zbl

[Hu and Wang 2018] Y. Hu and H. Wang, "Multiplicity one for the p cohomology of Shimura curves: the tame case", *Math. Res. Lett.* **25**:3 (2018), 843–873. MR Zbl

[Jantzen 1987] J. C. Jantzen, *Representations of algebraic groups*, Pure Appl. Math. **131**, Academic Press, Boston, 1987. MR Zbl

[Kisin 2008] M. Kisin, "Potentially semi-stable deformation rings", *J. Amer. Math. Soc.* **21**:2 (2008), 513–546. MR Zbl

[Le 2018] D. Le, "Lattices in the cohomology of $U(3)$ arithmetic manifolds", *Math. Ann.* **372**:1-2 (2018), 55–89. MR Zbl

[Le et al. 2016a] D. Le, B. V. Le Hung, B. Levin, and S. Morra, "Serre weights and Breuil's lattice conjecture in dimension three", preprint, 2016. arXiv

[Le et al. 2016b] D. Le, S. Morra, and B. Schraen, "Multiplicity one at full congruence level", preprint, 2016. arXiv

[Le et al. 2018] D. Le, B. V. Le Hung, B. Levin, and S. Morra, "Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows", *Invent. Math.* **212**:1 (2018), 1–107. MR Zbl

[Le et al. $\geq$ 2019] D. Le, B. V. Le Hung, and S. Morra, "Cohomology of $U(3)$ arithmetic manifolds at full congruence level", in preparation.

Communicated by Marie-France Vignéras
Received 2017-10-19 Revised 2019-02-13 Accepted 2019-05-27

le@math.toronto.edu *Department of Mathematics, University of Toronto, ON, Canada*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2019 is US $385/year for the electronic version, and $590/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 13 No. 8 2019