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**Arithmetic degrees and dynamical degrees of endomorphisms on
surfaces**

Yohsuke Matsuzawa, Kaoru Sano and Takahiro Shibata



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For a dominant rational self-map on a smooth projective variety defined over a number field, Kawaguchi and Silverman conjectured that the (first) dynamical degree is equal to the arithmetic degree at a rational point whose forward orbit is well-defined and Zariski dense. We prove this conjecture for surjective endomorphisms on smooth projective surfaces. For surjective endomorphisms on any smooth projective varieties, we show the existence of rational points whose arithmetic degrees are equal to the dynamical degree. Moreover, if the map is an automorphism, there exists a Zariski dense set of such points with pairwise disjoint orbits.

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1. Introduction

Let k be a number field, X a smooth projective variety over \bar{k} , and $f: X \dashrightarrow X$ a dominant rational self-map on X over \bar{k} . Let $I_f \subset X$ be the indeterminacy locus of f . Let $X_f(\bar{k})$ be the set of \bar{k} -rational points P on X such that $f^n(P) \notin I_f$ for every $n \geq 0$. For $P \in X_f(\bar{k})$, its *forward f -orbit* is defined as $\mathcal{O}_f(P) := \{f^n(P) : n \geq 0\}$.

Let H be an ample divisor on X defined over \bar{k} . The (first) *dynamical degree* of f is defined by

$$\delta_f := \lim_{n \rightarrow \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n}.$$

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The first dynamical degree of a dominant rational self-map on a smooth complex projective variety was first defined by Dinh and Sibony [2004; 2005]. Dang [2017] and Truong [2015] gave algebraic definitions of dynamical degrees.

The *arithmetic degree*, introduced by Silverman [2014], of f at a \bar{k} -rational point $P \in X_f(\bar{k})$ is defined by

$$\alpha_f(P) := \lim_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}$$

if the limit on the right-hand side exists. Here, $h_H : X(\bar{k}) \rightarrow [0, \infty)$ is the (absolute logarithmic) Weil height function associated with H , and we put $h_H^+ := \max\{h_H, 1\}$.

Then we have two types of quantity concerned with the iteration of the action of f . It is natural to consider the relation between dynamical degrees and arithmetic degrees. In this direction, Kawaguchi and Silverman formulated the following conjecture.

Conjecture 1.1 (The Kawaguchi–Silverman conjecture [2016b, Conjecture 6]). *For every \bar{k} -rational point $P \in X_f(\bar{k})$, the arithmetic degree $\alpha_f(P)$ exists. Moreover, if the forward f -orbit $\mathcal{O}_f(P)$ is Zariski dense in X , the arithmetic degree $\alpha_f(P)$ is equal to the dynamical degree δ_f , i.e., we have*

$$\alpha_f(P) = \delta_f.$$

Remark 1.2. Let X be a complex smooth projective variety with $\kappa(X) > 0$, $\Phi : X \dashrightarrow W$ the Iitaka fibration of X , and $f : X \dashrightarrow X$ a dominant rational self-map on X . Nakayama and Zhang [2009, Theorem A] proved that there exists an automorphism $g : W \rightarrow W$ of finite order such that $\Phi \circ f = g \circ \Phi$. This implies that any dominant rational self-map on a smooth projective variety of positive Kodaira dimension does not have a Zariski dense orbit. So the latter half of [Conjecture 1.1](#) is meaningful only for smooth projective varieties of nonpositive Kodaira dimension. However, we do not use their result in this paper.

When f is a dominant *endomorphism* (i.e., f is defined everywhere), the existence of the limit defining the arithmetic degree was proved in [Kawaguchi and Silverman 2016a]. But in general, the convergence is not known. It seems difficult at the moment to prove [Conjecture 1.1](#) in full generality.

In this paper, we prove [Conjecture 1.1](#) for any endomorphism on any smooth projective surface.

Theorem 1.3. *Let k be a number field, X a smooth projective surface over \bar{k} , and $f : X \rightarrow X$ a surjective endomorphism on X . Then [Conjecture 1.1](#) holds for f .*

As by-products of our arguments, we also obtain the following two cases for which [Conjecture 1.1](#) holds:

Theorem 1.4 ([Theorem 3.6](#)). *Let k be a number field, X a smooth projective irrational surface over \bar{k} , and $f : X \dashrightarrow X$ a birational automorphism on X . Then [Conjecture 1.1](#) holds for f .*

Theorem 1.5 ([Theorem 3.7](#)). *Let k be a number field, X a smooth projective toric variety over \bar{k} , and $f : X \rightarrow X$ a toric surjective endomorphism on X . Then [Conjecture 1.1](#) holds for f .*

Lin [2018] gives a precise description of the arithmetic degrees of toric self-maps on toric varieties.

As we will see in the proof of [Theorem 1.3](#), there does not always exist a Zariski dense orbit for a given self-map. For instance, a self-map cannot have a Zariski dense orbit if it is a self-map over a variety of positive Kodaira dimension. So it is also important to consider whether a self-map has a \bar{k} -rational point whose orbit has full arithmetic complexity, that is, whose arithmetic degree coincides with the dynamical degree. We prove that such a point always exists for any surjective endomorphism on any smooth projective variety.

Theorem 1.6. *Let k be a number field, X a smooth projective variety over \bar{k} , and $f : X \rightarrow X$ a surjective endomorphism on X . Then there exists a \bar{k} -rational point $P \in X(\bar{k})$ such that $\alpha_f(P) = \delta_f$.*

If f is an automorphism, we can construct a “large” collection of points whose orbits have full arithmetic complexity.

Theorem 1.7. *Let k be a number field, X a smooth projective variety over \bar{k} , and $f : X \rightarrow X$ an automorphism. Then there exists a subset $S \subset X(\bar{k})$ which satisfies all of the following conditions:*

- (1) For every $P \in S$, $\alpha_f(P) = \delta_f$.
- (2) For $P, Q \in S$ with $P \neq Q$, $\mathcal{O}_f(P) \cap \mathcal{O}_f(Q) = \emptyset$.
- (3) S is Zariski dense in X .

Remark 1.8. Kawaguchi, Silverman, and the second author proved [Conjecture 1.1](#) in the following cases:

- (1) f is an endomorphism and the Néron–Severi group of X has rank one [[Kawaguchi and Silverman 2014](#), Theorem 2(a)].
- (2) f is the extension to \mathbb{P}^N of a regular affine automorphism on \mathbb{A}^N [[Kawaguchi and Silverman 2014](#), Theorem 2(b)].
- (3) X is a smooth projective surface and f is an automorphism on X [[Kawaguchi 2008](#), Theorem A; [Kawaguchi and Silverman 2014](#), Theorem 2(c)].
- (4) f is the extension to \mathbb{P}^N of a monomial endomorphism on \mathbb{G}_m^N and $P \in \mathbb{G}_m^N(\bar{k})$ [[Silverman 2014](#), Proposition 19].
- (5) X is an abelian variety. Note that any rational map between abelian varieties is automatically a morphism [[Kawaguchi and Silverman 2016a](#), Corollary 31; [Silverman 2017](#), Theorem 2].
- (6) f is an endomorphism and X is the product $\prod_{i=1}^n X_i$ of smooth projective varieties, with the assumption that each variety X_i satisfies one of the following conditions [[Sano 2016](#), Theorem 1.3]:
 - The first Betti number of $(X_i)_{\mathbb{C}}$ is zero and the Néron–Severi group of X_i has rank one.
 - X_i is an abelian variety.
 - X_i is an Enriques surface.
 - X_i is a $K3$ surface.

- (7) f is an endomorphism and X is the product $X_1 \times X_2$ of positive dimensional varieties such that one of X_1 or X_2 is of general type. (In fact, there do not exist Zariski dense forward f -orbits on such $X_1 \times X_2$.) [Sano 2016, Theorem 1.4]

Notation. Throughout this paper:

- We fix a number field k .
- A *variety* always means an integral separated scheme of finite type over \bar{k} .
- A *divisor* on a variety X means a divisor on X defined over \bar{k} .
- An *endomorphism* on a variety X means a morphism from X to itself defined over \bar{k} . A *noninvertible endomorphism* is a surjective endomorphism which is not an automorphism.
- A *curve* or *surface* simply means a smooth projective variety of dimension 1 or 2, respectively, unless otherwise stated.
- For any curve C , the genus of C is denoted by $g(C)$.
- When we say that P is a point of X or write as $P \in X$, it means that P is a \bar{k} -rational point of X .
- The Néron–Severi group of a smooth projective variety X is denoted by $\text{NS}(X)$. It is well-known that $\text{NS}(X)$ is a finitely generated abelian group. We put $\text{NS}(X)_{\mathbb{R}} := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.
- The symbols \equiv , \sim , $\sim_{\mathbb{Q}}$ and $\sim_{\mathbb{R}}$ mean algebraic equivalence, linear equivalence, \mathbb{Q} -linear equivalence, and \mathbb{R} -linear equivalence, respectively.
- Let X be a smooth projective variety and $f: X \dashrightarrow X$ a dominant rational self-map. A point $P \in X_f(\bar{k})$ is called *preperiodic* if the forward f -orbit $\mathcal{O}_f(P)$ of P is a finite set. This is equivalent to the condition that $f^n(P) = f^m(P)$ for some $n, m \geq 0$ with $n \neq m$.
- Let f, g and h be real-valued functions on a domain S . The equality $f = g + O(h)$ means that there is a positive constant C such that $|f(x) - g(x)| \leq C|h(x)|$ for every $x \in S$. The equality $f = g + O(1)$ means that there is a positive constant C' such that $|f(x) - g(x)| \leq C'$ for every $x \in S$.

Outline of this paper. In Section 2, we recall the definitions and some properties of dynamical and arithmetic degrees. In Section 3, at first we recall some lemmata about reduction for Conjecture 1.1, which were proved in [Sano 2016; Silverman 2017]. Then, we prove the birational invariance of arithmetic degree. As its corollary, we prove Theorem 1.4 by reducing to the automorphism case, using minimal models. We also prove Theorem 1.5. In Section 4, by using the Enriques classification of smooth projective surfaces, we reduce Theorem 1.3 to three cases, i.e., the case of \mathbb{P}^1 -bundles, hyperelliptic surfaces, and surfaces of Kodaira dimension one. In Section 5 we recall fundamental properties of \mathbb{P}^1 -bundles over curves. In Sections 6, 7, and 8, we prove Theorem 1.3 in each case explained in Section 4. Finally, in Section 9, we prove Theorems 1.6 and 1.7. In the proof of Theorem 1.6, we use a nef \mathbb{R} -divisor D that satisfies $f^*D \equiv \delta_f D$.

2. Dynamical degree and arithmetic degree

Let H be an ample divisor on a smooth projective variety X . The (first) dynamical degree of a dominant rational self-map $f: X \dashrightarrow X$ is defined by

$$\delta_f := \lim_{n \rightarrow \infty} ((f^n)^* H \cdot H^{\dim X - 1})^{1/n}.$$

The limit defining δ_f exists, and δ_f does not depend on the choice of H [Dinh and Sibony 2005, Corollary 7; Guedj 2005, Proposition 1.2]. Note that if f is an endomorphism, we have $(f^n)^* = (f^*)^n$ as a linear self-map on $\text{NS}(X)$. But if f is merely a rational self-map, then $(f^n)^* \neq (f^*)^n$ in general.

Remark 2.1 [Dinh and Sibony 2005, Proposition 1.2(iii); Kawaguchi and Silverman 2016b, Remark 7]. Let $\rho((f^n)^*)$ be the spectral radius of the linear self-map $(f^n)^*: \text{NS}(X)_{\mathbb{R}} \rightarrow \text{NS}(X)_{\mathbb{R}}$. The dynamical degree δ_f is equal to the limit $\lim_{n \rightarrow \infty} (\rho((f^n)^*))^{1/n}$. Thus we have $\delta_{f^n} = \delta_f^n$ for every $n \geq 1$.

Let $X_f(\bar{k})$ be the set of points P on X such that f is defined at $f^n(P)$ for every $n \geq 0$. The arithmetic degree of f at a point $P \in X_f(\bar{k})$ is defined as follows. Let

$$h_H: X(\bar{k}) \rightarrow [0, \infty)$$

be the (absolute logarithmic) Weil height function associated with H [Hindry and Silverman 2000, Theorem B3.2]. We put

$$h_H^+(P) := \max\{h_H(P), 1\}.$$

We call

$$\bar{\alpha}_f(P) := \limsup_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n} \quad \text{and} \quad \underline{\alpha}_f(P) := \liminf_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}$$

the upper arithmetic degree and the lower arithmetic degree of f at P , respectively. It is known that $\bar{\alpha}_f(P)$ and $\underline{\alpha}_f(P)$ do not depend on the choice of H [Kawaguchi and Silverman 2016b, Proposition 12]. If $\bar{\alpha}_f(P) = \underline{\alpha}_f(P)$, the limit

$$\alpha_f(P) := \lim_{n \rightarrow \infty} h_H^+(f^n(P))^{1/n}$$

is called the arithmetic degree of f at P .

Remark 2.2. Let D be a divisor on X and f a dominant rational self-map on X . Take $P \in X_f(\bar{k})$. Then we can easily check that

$$\bar{\alpha}_f(P) \geq \limsup_{n \rightarrow \infty} h_D^+(f^n(P))^{1/n} \quad \text{and} \quad \underline{\alpha}_f(P) \geq \liminf_{n \rightarrow \infty} h_D^+(f^n(P))^{1/n}.$$

So when these limits exist, we have

$$\alpha_f(P) \geq \lim_{n \rightarrow \infty} h_D^+(f^n(P))^{1/n}.$$

Remark 2.3. When f is an endomorphism, the existence of the limit defining the arithmetic degree $\alpha_f(P)$ was proved by Kawaguchi and Silverman [2016a, Theorem 3]. But it is not known in general.

Remark 2.4. The inequality $\bar{\alpha}_f(P) \leq \delta_f$ was proved by Kawaguchi and Silverman, and the third author [Kawaguchi and Silverman 2016b, Theorem 4; Matsuzawa 2016, Theorem 1.4]. Hence, in order to prove Conjecture 1.1, it is enough to prove the opposite inequality $\underline{\alpha}_f(P) \geq \delta_f$.

3. Some reductions for Conjecture 1.1

Reductions. We recall some lemmata which are useful to reduce the proof of some cases of Conjecture 1.1 to easier cases.

Lemma 3.1. *Let X be a smooth projective variety and $f : X \rightarrow X$ a surjective endomorphism. Then Conjecture 1.1 holds for f if and only if Conjecture 1.1 holds for f^t for some $t \geq 1$.*

Proof. See [Sano 2016, Lemma 3.3]. □

Lemma 3.2 [Silverman 2017, Lemma 6]. *Let $\psi : X \rightarrow Y$ be a finite morphism between smooth projective varieties. Let $f_X : X \rightarrow X$ and $f_Y : Y \rightarrow Y$ be surjective endomorphisms on X and Y , respectively. Assume that $\psi \circ f_X = f_Y \circ \psi$.*

- (i) *For any $P \in X(\bar{k})$, we have $\alpha_{f_X}(P) = \alpha_{f_Y}(\psi(P))$.*
- (ii) *Assume that ψ is surjective. Then Conjecture 1.1 holds for f_X if and only if Conjecture 1.1 holds for f_Y .*

Proof. (i) Take any point $P \in X(\bar{k})$. Let H be an ample divisor on Y . Then ψ^*H is an ample divisor on X . Hence we have

$$\begin{aligned} \alpha_{f_X}(P) &= \lim_{n \rightarrow \infty} h_{\psi^*H}^+(f_X^n(P))^{1/n} \\ &= \lim_{n \rightarrow \infty} h_H^+(\psi \circ f_X^n(P))^{1/n} \\ &= \lim_{n \rightarrow \infty} h_H^+(f_Y^n \circ \psi(P))^{1/n} \\ &= \alpha_{f_Y}(\psi(P)). \end{aligned}$$

(ii) For a point $P \in X(\bar{k})$, the forward f_X -orbit $\mathcal{O}_{f_X}(P)$ is Zariski dense in X if and only if the forward f_Y -orbit $\mathcal{O}_{f_Y}(\psi(P))$ is Zariski dense in Y since ψ is a finite surjective morphism. Moreover we have $\dim X = \dim Y$. So we obtain

$$\begin{aligned} \delta_{f_X} &= \lim_{n \rightarrow \infty} ((f_X^n)^* \psi^* H \cdot (\psi^* H)^{\dim X - 1})^{1/n} \\ &= \lim_{n \rightarrow \infty} (\psi^*(f_Y^n)^* H \cdot (\psi^* H)^{\dim Y - 1})^{1/n} \\ &= \lim_{n \rightarrow \infty} (\deg(\psi)((f_Y^n)^* H \cdot H^{\dim Y - 1}))^{1/n} \\ &= \delta_{f_Y}. \end{aligned}$$

Therefore the assertion follows. □

Birational invariance of the arithmetic degree. We show that the arithmetic degree is invariant under birational conjugacy.

Lemma 3.3. *Let $\mu: X \dashrightarrow Y$ be a birational map of smooth projective varieties. Take Weil height functions h_X and h_Y associated with ample divisors H_X and H_Y on X and Y , respectively. Then there are constants $M \in \mathbb{R}_{>0}$ and $M' \in \mathbb{R}$ such that*

$$h_X(P) \geq Mh_Y(\mu(P)) + M'$$

for any $P \in X(\bar{k}) \setminus I_\mu(\bar{k})$.

Proof. Replacing H_Y by a positive multiple, we may assume that H_Y is very ample. Take a smooth projective variety Z and a birational morphism $p: Z \rightarrow X$ such that p is isomorphic over $X \setminus I_\mu$ and $q = \mu \circ p: Z \rightarrow Y$ is a morphism. Let $\{F_i\}_{i=1}^r$ be the collection of prime p -exceptional divisors. We take H_Y as not containing $q(F_i)$ for any i , so q^*H_Y does not contain F_i for any i . Then $E = p^*p_*q^*H_Y - q^*H_Y$ is an effective divisor contained in the exceptional locus of p . Take a sufficiently large integer N such that $NH_X - p_*q^*H_Y$ is very ample. Then, for $P \in X(\bar{k}) \setminus I_\mu$, we have

$$\begin{aligned} h_X(P) &= \frac{1}{N}(h_{NH_X - p_*q^*H_Y}(P) + h_{p_*q^*H_Y}(P)) + O(1) \\ &\geq \frac{1}{N}h_{p_*q^*H_Y}(P) + O(1) \\ &= \frac{1}{N}h_{p^*p_*q^*H_Y}(p^{-1}(P)) + O(1) \\ &= \frac{1}{N}h_{q^*H_Y}(p^{-1}(P)) + h_E(p^{-1}(P)) + O(1) \\ &= \frac{1}{N}h_Y(\mu(P)) + h_E(p^{-1}(P)) + O(1). \end{aligned}$$

We know that $h_E \geq O(1)$ on $Z(\bar{k}) \setminus \text{Supp } E$ [Hindry and Silverman 2000, Theorem B.3.2(e)]. Since $\text{Supp } E \subset p^{-1}(I_\mu)$, $h_E(p^{-1}(P)) \geq O(1)$ for $P \in X(\bar{k}) \setminus I_\mu$. Finally, we obtain that

$$h_X(P) \geq (1/N)h_Y(\mu(P)) + O(1) \quad \text{for } P \in X(\bar{k}) \setminus I_\mu. \quad \square$$

Theorem 3.4. *Let $f: X \dashrightarrow X$ and $g: Y \dashrightarrow Y$ be dominant rational self-maps on smooth projective varieties and $\mu: X \dashrightarrow Y$ a birational map such that $g \circ \mu = \mu \circ f$.*

- (i) *Let $U \subset X$ be a Zariski open subset such that $\mu|_U: U \rightarrow \mu(U)$ is an isomorphism. Then $\bar{\alpha}_f(P) = \bar{\alpha}_g(\mu(P))$ and $\underline{\alpha}_f(P) = \underline{\alpha}_g(\mu(P))$ for $P \in X_f(\bar{k}) \cap \mu^{-1}(Y_g(\bar{k}))$ such that $\mathcal{O}_f(P) \subset U(\bar{k})$.*
- (ii) *Take $P \in X_f(\bar{k}) \cap \mu^{-1}(Y_g(\bar{k}))$. Assume that $\mathcal{O}_f(P)$ is Zariski dense in X and both $\alpha_f(P)$ and $\alpha_g(\mu(P))$ exist. Then $\alpha_f(P) = \alpha_g(\mu(P))$.*

Proof. (i) Using Lemma 3.3 for both μ and μ^{-1} , there are constants $M_1, L_1 \in \mathbb{R}_{>0}$ and $M_2, L_2 \in \mathbb{R}$ such that

$$M_1h_Y(\mu(P)) + M_2 \leq h_X(P) \leq L_1h_Y(\mu(P)) + L_2 \tag{*}$$

for $P \in U(\bar{k})$. The claimed equalities follow from (*).

(ii) Since $\mathcal{O}_f(P)$ is Zariski dense in X , we can take a subsequence $\{f^{n_k}(P)\}_k$ of $\{f^n(P)\}_n$ contained in U . Using $(*)$ again, it follows that

$$\alpha_f(P) = \lim_{k \rightarrow \infty} h_X^+(f^{n_k}(P))^{1/n_k} = \lim_{k \rightarrow \infty} h_Y^+(g^{n_k}(\mu(P)))^{1/n_k} = \alpha_g(\mu(P)). \quad \square$$

Remark 3.5. Silverman [2014] dealt with a height function on \mathbb{G}_m^n induced by an open immersion $\mathbb{G}_m^n \hookrightarrow \mathbb{P}^n$ and proved Conjecture 1.1 for monomial maps on \mathbb{G}_m^n . It seems that it has not been checked in the literature that the arithmetic degrees of endomorphisms on quasiprojective varieties does not depend on the choice of open immersions to projective varieties. Now by Theorem 3.4, the arithmetic degree of a rational self-map on a quasiprojective variety at a point does not depend on the choice of an open immersion of the quasiprojective variety to a projective variety. Furthermore, by the birational invariance of dynamical degrees, we can state Conjecture 1.1 for rational self-maps on quasiprojective varieties, such as semiabelian varieties.

Applications of the birational invariance. In this subsection, we prove Theorems 1.4 and 1.5 as applications of Theorem 3.4.

Theorem 3.6 (Theorem 1.4). *Let X be an irrational surface and $f: X \dashrightarrow X$ a birational automorphism on X . Then Conjecture 1.1 holds for f .*

Proof. Take a point $P \in X_f(\bar{k})$. If $\mathcal{O}_f(P)$ is finite, the limit $\alpha_f(P)$ exists and is equal to 1. Next, assume that the closure $\overline{\mathcal{O}_f(P)}$ of $\mathcal{O}_f(P)$ has dimension 1. Let Z be the normalization of $\overline{\mathcal{O}_f(P)}$ and $\nu: Z \rightarrow X$ the induced morphism. Then an endomorphism $g: Z \rightarrow Z$ satisfying $\nu \circ g = f \circ \nu$ is induced. Take a point $P' \in Z$ such that $\nu(P') = P$. Then $\alpha_g(P') = \alpha_f(P)$ since ν is finite by Lemma 3.2 (i). It follows from [Kawaguchi and Silverman 2016a, Theorem 2] that $\alpha_g(P')$ exists (note that their theorem holds for possibly nonsurjective endomorphisms on possibly reducible normal varieties). Therefore $\alpha_f(P)$ exists.

Finally, assume that $\mathcal{O}_f(P)$ is Zariski dense. If $\delta_f = 1$, then $1 \leq \underline{\alpha}_f(P) \leq \bar{\alpha}_f(P) \leq \delta_f = 1$ by Remark 2.4, so $\alpha_f(P)$ exists and $\alpha_f(P) = \delta_f = 1$. So we may assume that $\delta_f > 1$. Since X is irrational and $\delta_f > 1$, $\kappa(X)$ must be nonnegative [Diller and Favre 2001, Theorem 0.4, Proposition 7.1 and Theorem 7.2]. Take a birational morphism $\mu: X \rightarrow Y$ to the minimal model Y of X and let $g: Y \dashrightarrow Y$ be the birational automorphism on Y defined as $g = \mu \circ f \circ \mu^{-1}$. Then g is in fact an automorphism since, if g has indeterminacy, Y must have a K_Y -negative curve. It is obvious that $\mathcal{O}_g(\mu(P))$ is also Zariski dense in Y . Since $\mu(\text{Exc}(\mu))$ is a finite set, there is a positive integer n_0 such that $\mu(f^n(P)) = g^n(\mu(P)) \notin \mu(\text{Exc}(\mu))$ for $n \geq n_0$. So we have $f^n(P) \notin \text{Exc}(\mu)$ for $n \geq n_0$. Replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset X \setminus \text{Exc}(\mu)$. Applying Theorem 3.4 (i) to P , it follows that $\alpha_f(P) = \alpha_g(\mu(P))$. We know that $\alpha_g(\mu(P))$ exists since g is a morphism. So $\alpha_f(P)$ also exists. The equality $\alpha_g(\mu(P)) = \delta_g$ holds as a consequence of Conjecture 1.1 for automorphisms on surfaces (see Remark 1.8(3)). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. □

Theorem 3.7 (Theorem 1.5). *Let X be a smooth projective toric variety and $f: X \rightarrow X$ a toric surjective endomorphism on X . Then Conjecture 1.1 holds for f .*

Proof. Let $\mathbb{G}_m^d \subset X$ be the torus embedded as an open dense subset in X . Then $f|_{\mathbb{G}_m^d} : \mathbb{G}_m^d \rightarrow \mathbb{G}_m^d$ is a homomorphism of algebraic groups by assumption. Let $\mathbb{G}_m^d \subset \mathbb{P}^d$ be the natural embedding of \mathbb{G}_m^d to the projective space \mathbb{P}^d and $g : \mathbb{P}^d \dashrightarrow \mathbb{P}^d$ be the induced rational self-map. Then g is a monomial map.

Take $P \in X(\bar{k})$ such that $\mathcal{O}_f(P)$ is Zariski dense. Note that $\alpha_f(P)$ exists since f is a morphism. Since $\mathcal{O}_f(P)$ is Zariski dense and $f(\mathbb{G}_m^d) \subset \mathbb{G}_m^d$, there is a positive integer n_0 such that $f^n(P) \in \mathbb{G}_m^d$ for $n \geq n_0$. By replacing P by $f^{n_0}(P)$, we may assume that $\mathcal{O}_f(P) \subset \mathbb{G}_m^d$. Applying [Theorem 3.4](#) (i) to P , it follows that $\alpha_f(P) = \alpha_g(P)$.

The equality $\alpha_g(P) = \delta_g$ holds as a consequence of [Conjecture 1.1](#) for monomial maps (see [Remark 1.8\(4\)](#)). Since the dynamical degree is invariant under birational conjugacy, it follows that $\delta_g = \delta_f$. So we obtain the equality $\alpha_f(P) = \delta_f$. □

4. Endomorphisms on surfaces

We start to prove [Theorem 1.3](#). Since [Conjecture 1.1](#) for automorphisms on surfaces is already proved by Kawaguchi (see [Remark 1.8\(3\)](#)), it is sufficient to prove [Theorem 1.3](#) for *noninvertible* endomorphisms, that is, surjective endomorphisms which are not automorphisms.

Let $f : X \rightarrow X$ be a noninvertible endomorphism on a surface. We divide the proof of [Theorem 1.3](#) according to the Kodaira dimension of X :

(I) $\kappa(X) = -\infty$; we need the following result due to Nakayama.

Lemma 4.1 [[Nakayama 2002](#), Proposition 10]. *Let $f : X \rightarrow X$ be a noninvertible endomorphism on a surface X with $\kappa(X) = -\infty$. Then there is a positive integer m such that $f^m(E) = E$ for any irreducible curve E on X with negative self-intersection.*

Let $\mu : X \rightarrow X'$ be the contraction of a (-1) -curve E on X . By [Lemma 4.1](#), there is a positive integer m such that $f^m(E) = E$. Then f^m induces an endomorphism $f' : X' \rightarrow X'$ such that $\mu \circ f^m = f' \circ \mu$. Using [Lemma 3.1](#) and [Theorem 3.4](#), the assertion of [Theorem 1.3](#) for f follows from that for f' . Continuing this process, we may assume that X is relatively minimal.

When X is irrational and relatively minimal, X is a \mathbb{P}^1 -bundle over a curve C with $g(C) \geq 1$.

When X is rational and relatively minimal, X is isomorphic to \mathbb{P}^2 or the Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ for some $n \geq 0$ with $n \neq 1$. Note that [Conjecture 1.1](#) holds for surjective endomorphisms on projective spaces (see [Remark 1.8\(1\)](#)).

(II) $\kappa(X) = 0$; for surfaces with nonnegative Kodaira dimension, we use the following result due to Fujimoto.

Lemma 4.2 [[Fujimoto 2002](#), Lemma 2.3 and Proposition 3.1]. *Let $f : X \rightarrow X$ be a noninvertible endomorphism on a surface X with $\kappa(X) \geq 0$. Then X is minimal and f is étale.*

So X is either an abelian surface, a hyperelliptic surface, a K3 surface, or an Enriques surface. Since f is étale, we have $\chi(X, \mathcal{O}_X) = \deg(f)\chi(X, \mathcal{O}_X)$. Now $\deg(f) \geq 2$ by assumption, so $\chi(X, \mathcal{O}_X) = 0$ [[Fujimoto 2002](#), Corollary 2.4]. Hence X must be either an abelian surface or a hyperelliptic surface

because K3 surfaces and Enriques surfaces have nonzero Euler characteristics. Note that [Conjecture 1.1](#) is valid for endomorphisms on abelian varieties (see [Remark 1.8\(5\)](#)).

(III) $\kappa(X) = 1$; this case will be treated in [Section 8](#).

(IV) $\kappa(X) = 2$; the following fact is well known.

Lemma 4.3. *Let X be a smooth projective variety of general type. Then any surjective endomorphism on X is an automorphism. Furthermore, the group of automorphisms $\text{Aut}(X)$ on X has finite order.*

Proof. See [[Fujimoto 2002](#), Proposition 2.6], [[Iitaka 1982](#), Theorem 11.12], or [[Matsumura 1963](#), Corollary 2]. □

So there is no noninvertible endomorphism on X . As a summary, the remaining cases for the proof of [Theorem 1.3](#) are the following:

- Noninvertible endomorphisms on \mathbb{P}^1 -bundles over a curve.
- Noninvertible endomorphisms on hyperelliptic surfaces.
- Noninvertible endomorphisms on surfaces of Kodaira dimension 1.

These three cases are studied in [Sections 5–8](#) below.

Remark 4.4. Fujimoto and Nakayama gave a complete classification of surfaces which admit noninvertible endomorphisms (see [[Fujimoto 2002](#), Proposition 3.3], [[Fujimoto and Nakayama 2008](#), Theorem 1.1], [[Fujimoto and Nakayama 2005](#), Appendix to Section 4], and [[Nakayama 2002](#), Theorem 3]).

5. Some properties of \mathbb{P}^1 -bundles over curves

In this section, we recall and prove some properties of \mathbb{P}^1 -bundles (see [[Hartshorne 1977](#), Chapter V.2] or [[Homma 1992; 1999](#)] for details). In this section, let X be a \mathbb{P}^1 -bundle over a curve C . Let $\pi : X \rightarrow C$ be the projection.

Proposition 5.1. *We can represent X as $X \cong \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a locally free sheaf of rank 2 on C such that $H^0(\mathcal{E}) \neq 0$ but $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for all invertible sheaves \mathcal{L} on C with $\deg \mathcal{L} < 0$. The integer $e := -\deg \mathcal{E}$ does not depend on the choice of such \mathcal{E} . Furthermore, there is a section $\sigma : C \rightarrow X$ with image C_0 such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$.*

Proof. See [[Hartshorne 1977](#), Proposition 2.8]. □

Lemma 5.2. *The Picard group and the Néron–Severi group of X have the following structure:*

$$\text{Pic}(X) \cong \mathbb{Z} \oplus \pi^* \text{Pic}(C) \quad \text{and} \quad \text{NS}(X) \cong \mathbb{Z} \oplus \pi^* \text{NS}(C) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Furthermore, the image C_0 of the section $\sigma : C \rightarrow X$ in [Proposition 5.1](#) generates the first direct factor of $\text{Pic}(X)$ and $\text{NS}(X)$.

Proof. See [[Hartshorne 1977](#), V, Proposition 2.3]. □

Lemma 5.3. *Let $F \in \text{NS}(X)$ be a fiber $\pi^{-1}(p) = \pi^* p$ over a point $p \in C(\bar{k})$, and e the integer defined in Proposition 5.1. Then the intersection numbers of generators of $\text{NS}(X)$ are as follows:*

$$F \cdot F = 0, \quad F \cdot C_0 = 1, \quad C_0 \cdot C_0 = -e.$$

Proof. It is easy to see that the equalities $F \cdot F = 0$ and $F \cdot C_0 = 1$ hold. For the last equality, see [Hartshorne 1977, V, Proposition 2.9]. □

We say that f *preserves fibers* if there is an endomorphism f_C on C such that $\pi \circ f = f_C \circ \pi$. In our situation, since there is a section $\sigma : C \rightarrow X$, f preserves fibers if and only if, for any point $p \in C$, there is a point $q \in C$ such that $f(\pi^{-1}(p)) \subset \pi^{-1}(q)$.

The following lemma appears in [Amerik 2003, p.18] in a more general form. But we need it only in the case of \mathbb{P}^1 -bundles on a curve, and the proof in the general case is similar to our case. So we deal only with the case of \mathbb{P}^1 - bundles on a curve.

Lemma 5.4. *For any surjective endomorphism f on X , the iterate f^2 preserves fibers.*

Proof. By the projection formula, the fibers of $\pi : X \rightarrow C$ can be characterized as connected curves having intersection number zero with any fiber $F_p = \pi^* p$, $p \in C$. Hence, to check that the iterate f^2 sends fibers to fibers, it suffices to show that $(f^2)^*(\pi^* \text{NS}(C)_{\mathbb{R}}) = \pi^* \text{NS}(C)_{\mathbb{R}}$. Now $\dim \text{NS}(X)_{\mathbb{R}} = 2$ and the set of the numerical classes in X with self-intersection zero forms two lines, one of which is $\pi^* \text{NS}(C)_{\mathbb{R}}$, and f^* fixes or interchanges them. So $(f^2)^*$ fixes $\pi^* \text{NS}(C)_{\mathbb{R}}$. □

The following might be well-known, but we give a proof for the reader’s convenience.

Lemma 5.5. *A surjective endomorphism f preserves fibers if and only if there exists a nonzero integer a such that $f^* F \equiv aF$. Here, F is the numerical class of a fiber.*

Proof. Assume $f^* F \equiv aF$. For any point $p \in C$, we set $F_p := \pi^{-1}(p) = \pi^* p$. If f does not preserve fibers, there is a point $p \in C$ such that $f(F_p) \cdot F > 0$. Now we can calculate the intersection number as follows:

$$0 = F \cdot aF = F \cdot (f^* F) = F_p \cdot (f^* F) = (f_* F_p) \cdot F = \deg(f|_{F_p}) \cdot (f(F_p) \cdot F) > 0.$$

This is a contradiction. Hence f preserves fibers.

Next, assume that f preserves fibers. Write $f^* F = aF + bC_0$. Then we can also calculate the intersection number as follows:

$$b = F \cdot (aF + bC_0) = F \cdot f^* F = (f_* F) \cdot F = \deg(f|_F) \cdot (F \cdot F) = 0.$$

Further, by the injectivity of f^* , we have $a \neq 0$. The proof is complete. □

Lemma 5.6. *If \mathcal{E} splits, i.e., if there is an invertible sheaf \mathcal{L} on C such that $\mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L}$, the invariant e of $X = \mathbb{P}(\mathcal{E})$ is nonnegative.*

Proof. See [Hartshorne 1977, V, Example 2.11.3]. □

Lemma 5.7. *Assume that $e \geq 0$. Then for a divisor $D = aF + bC_0 \in \text{NS}(X)$, the following properties are equivalent:*

- D is ample.
- $a > be$ and $b > 0$.

In other words, the nef cone of X is generated by F and $eF + C_0$.

Proof. See [Hartshorne 1977, V, Proposition 2.20]. □

We can prove a result stronger than Lemma 5.4 as follows.

Lemma 5.8. *Assume that $e > 0$. Then any surjective endomorphism $f: X \rightarrow X$ preserves fibers.*

Proof. By Lemma 5.5, it is enough to prove $f^*F \equiv aF$ for some integer $a > 0$. We can write $f^*F \equiv aF + bC_0$ for some integers $a, b \geq 0$.

Since we have

$$aF + bC_0 = (a - be)F + b(eF + C_0)$$

and f preserves the nef cone and the ample cone, either of the equalities $a - be = 0$ or $b = 0$ holds.

We have

$$0 = \deg(f)(F \cdot F) = (f_*f^*F) \cdot F = (f^*F) \cdot (f^*F) = (aF + bC_0) \cdot (aF + bC_0) = 2ab - b^2e = b(2a - be).$$

So either of the equalities $b = 0$ or $2a - be = 0$ holds.

If we have $b \neq 0$, we have $a - be = 0$ and $2a - be = 0$. So we get $a = 0$. But since $e \neq 0$, we obtain $b = 0$. This is a contradiction. Consequently, we get $b = 0$ and $f^*F \equiv aF$. □

Lemma 5.9. *Fix a fiber $F = F_p$ for a point $p \in C(\bar{k})$. Let f be a surjective endomorphism on X preserving fibers, f_C the endomorphism on C satisfying $\pi \circ f = f_C \circ \pi$, $f_F := f|_F: F \rightarrow f(F)$ the restriction of f to the fiber F . Set $f^*F \equiv aF$ and $f^*C_0 \equiv cF + dC_0$. Then we have $a = \deg(f_C)$, $d = \deg(f_F)$, $\deg(f) = ad$, and $\delta_f = \max\{a, d\}$.*

Proof. Our assertions follow from the following equalities of divisor classes in $\text{NS}(X)$ and of intersection numbers:

$$aF = f^*F = f^*\pi^*p = \pi^*f_C^*p = \pi^*(\deg(f_C)p) = \deg(f_C)\pi^*p = \deg(f_C)F,$$

$$\begin{aligned} \deg(f)F &= f_*f^*F = f_*f^*\pi^*p = f_*\pi^*f_C^*p = f_*\pi^*(\deg(f_C)p) \\ &= \deg(f_C)f_*F = \deg(f_C)\deg(f_F)f(F) = \deg(f_C)\deg(f_F)F \end{aligned}$$

$$\deg(f) = \deg(f)C_0 \cdot F = (f_*f^*C_0) \cdot F = (f^*C_0) \cdot (f^*F) = (cF + dC_0) \cdot aF = ad.$$

The last assertion $\delta_f = \max\{a, d\}$ follows from the equality $\delta_f = \lim_{n \rightarrow \infty} \rho((f^n)^*)^{1/n} = \rho(f^*)$ and from the functoriality of f^* (see Remark 2.1). □

Lemma 5.10. *Using the notation of Lemma 5.9, assume that $e \geq 0$. Then both F and C_0 are eigenvectors of $f^*: \text{NS}(X)_{\mathbb{R}} \rightarrow \text{NS}(X)_{\mathbb{R}}$. Further, if e is positive, then we have $\deg(f_C) = \deg(f_F)$.*

Proof. Set $f^*F = aF$ and $f^*C_0 = cF + dC_0$ in $\text{NS}(X)$. Then we have

$$-ead = -e \deg f = (f_*f^*C_0) \cdot C_0 = (f^*C_0)^2 = (cF + dC_0)^2 = 2cd - ed^2.$$

Hence, we get $c = e(d - a)/2$. We have the following equalities in $\text{NS}(X)$:

$$f^*(eF + C_0) = aeF + (cF + dC_0) = (ae + c)F + dC_0.$$

By the fact that f^*D is ample if and only if D is ample, it follows that $eF + C_0$ is an eigenvector of f^* . Thus, we have

$$de = ae + c = ae + e(d - a)/2 = e(d + a)/2.$$

Therefore, the equality $e(d - a) = 0$ holds. So $c = e(d - a)/2 = 0$ holds.

Further, we assume that $e > 0$. Then it follows that $d - a = 0$. So we have $\deg(f_C) = a = d = \deg(f_F)$. \square

The following lemma is used on page 1650.

Lemma 5.11. *Let \mathcal{L} be a nontrivial invertible sheaf of degree 0 on a curve C with $g(C) \geq 1$, $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$, and $X = \mathbb{P}(\mathcal{E})$. Let C_0 and C_1 be sections corresponding to the projections $\mathcal{E} \rightarrow \mathcal{L}$ and $\mathcal{E} \rightarrow \mathcal{O}_C$. If $\sigma : C \rightarrow X$ is a section such that $(\sigma(C))^2 = 0$, then $\sigma(C)$ is equal to C_0 or C_1 .*

Proof. Note that $e = 0$ in this case and thus $(C_0^2) = 0$. Moreover, $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(1) \otimes \pi^* \mathcal{L}^{-1}$. Set $\sigma(C) \equiv aC_0 + bF$. Then $a = (\sigma(C) \cdot F) = 1$ and $2ab = (\sigma(C))^2 = 0$. Thus $\sigma(C) \equiv C_0$. Therefore, $\mathcal{O}_X(\sigma(C)) \cong \mathcal{O}_X(C_0) \otimes \pi^* \mathcal{N}$ for some invertible sheaf \mathcal{N} of degree 0 on C . Then

$$0 \neq H^0(X, \mathcal{O}_X(\sigma(C))) = H^0(C, \pi_* \mathcal{O}_X(C_0) \otimes \mathcal{N}) = H^0(C, (\mathcal{L} \oplus \mathcal{O}_C) \otimes \mathcal{N})$$

and this implies $\mathcal{N} \cong \mathcal{O}_C$ or $\mathcal{N} \cong \mathcal{L}^{-1}$. Hence $\mathcal{O}_X(\sigma(C))$ is isomorphic to $\mathcal{O}_X(C_0)$ or $\mathcal{O}_X(C_0) \otimes \pi^* \mathcal{L}^{-1} = \mathcal{O}_X(C_1)$. Since \mathcal{L} is nontrivial, we have $H^0(\mathcal{O}_X(C_0)) = H^0(\mathcal{O}_X(C_1)) = \bar{k}$ and we get $\sigma(C) = C_0$ or C_1 . \square

6. \mathbb{P}^1 -bundles over curves

In this section, we prove [Conjecture 1.1](#) for noninvertible endomorphisms on \mathbb{P}^1 -bundles over curves. We divide the proof according to the genus of the base curve.

\mathbb{P}^1 -bundles over \mathbb{P}^1 .

Theorem 6.1. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a \mathbb{P}^1 -bundle over \mathbb{P}^1 and $f : X \rightarrow X$ be a noninvertible endomorphism. Then [Conjecture 1.1](#) holds for f .*

Proof. Take a locally free sheaf \mathcal{E} of rank 2 on \mathbb{P}^1 such that $X \cong \mathbb{P}(\mathcal{E})$ and $\deg \mathcal{E} = -e$ (see [Proposition 5.1](#)). Then \mathcal{E} splits [[Hartshorne 1977](#), V, Corollary 2.14]. When X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the case of $e = 0$, the assertion holds by [[Sano 2016](#), Theorem 1.3]. When X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the case of $e > 0$, the endomorphism f preserves fibers and induces an endomorphism $f_{\mathbb{P}^1}$ on the base curve \mathbb{P}^1 . By [Lemma 5.10](#), we have $\delta_f = \delta_{f_{\mathbb{P}^1}}$. Fix a point $p \in \mathbb{P}^1$ and set $F = \pi^*p$. Let $P \in X(\bar{k})$ be a point

whose forward f -orbit is Zariski dense in X . Then the forward $f_{\mathbb{P}^1}$ -orbit of $\pi(P)$ is also Zariski dense in \mathbb{P}^1 . Now the assertion follows from the following computation.

$$\begin{aligned} \alpha_f(P) &\geq \lim_{n \rightarrow \infty} h_F(f^n(P))^{1/n} = \lim_{n \rightarrow \infty} h_{\pi^*P}(f^n(P))^{1/n} \\ &= \lim_{n \rightarrow \infty} h_P(\pi \circ f^n(P))^{1/n} = \lim_{n \rightarrow \infty} h_P(f_{\mathbb{P}^1}^n \circ \pi(P))^{1/n} = \delta_{f_{\mathbb{P}^1}} = \delta_f. \end{aligned} \quad \square$$

\mathbb{P}^1 -bundles over genus one curves. In this subsection, we prove [Conjecture 1.1](#) for any endomorphisms on a \mathbb{P}^1 -bundle on a curve C of genus one.

The following result is due to Amerik. Note that Amerik in fact proved it for \mathbb{P}^1 -bundles over varieties of arbitrary dimension.

Lemma 6.2. *Let $X = \mathbb{P}(\mathcal{E})$ be a \mathbb{P}^1 -bundle over a curve C . If X has a fiber-preserving surjective endomorphism whose restriction to a general fiber has degree greater than 1, then \mathcal{E} splits into a direct sum of two line bundles after a finite base change. Furthermore, if \mathcal{E} is semistable, then \mathcal{E} splits into a direct sum of two line bundles after an étale base change.*

Proof. See [\[Amerik 2003, Theorem 2 and Proposition 2.4\]](#). □

Lemma 6.3. *Let E be a curve of genus one with an endomorphism $f : E \rightarrow E$. If $g : E' \rightarrow E$ is a finite étale covering of E , there exists a finite étale covering $h : E'' \rightarrow E'$ and an endomorphism $f' : E'' \rightarrow E''$ such that $f \circ g \circ h = g \circ h \circ f'$. Furthermore, we can take h as satisfying $E'' = E$.*

Proof. At first, since E' is an étale covering of E , a genus one curve, E' is also a genus one curve. By fixing a rational point $p \in E'(\bar{k})$ and $g(p) \in E(\bar{k})$, these curves E and E' can be regarded as elliptic curves, and g can be regarded as an isogeny between elliptic curves. Let $h := \hat{g} : E \rightarrow E'$ be the dual isogeny of g . The morphism f is decomposed as $f = \tau_c \circ \psi$ for a homomorphism ψ and a translation map τ_c by $c \in E(\bar{k})$. Fix a rational point $c' \in E(\bar{k})$ such that $[\deg(g)](c') = c$ and consider the translation map $\tau_{c'}$, where $[\deg(g)]$ is the multiplication by $\deg(g)$. We set $f' = \tau_{c'} \circ \psi$. Then we have the following equalities.

$$f \circ g \circ h = \tau_c \circ \psi \circ g \circ \hat{g} = \tau_c \circ \psi \circ [\deg(g)] = \tau_c \circ [\deg(g)] \circ \psi = [\deg(g)] \circ \tau_{c'} \circ \psi = g \circ h \circ f'.$$

This is what we want. □

Proposition 6.4. *Let \mathcal{E} be a locally free sheaf of rank 2 on a genus one curve C and $X = \mathbb{P}(\mathcal{E})$. Suppose [Conjecture 1.1](#) holds for any noninvertible endomorphism on X with $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where \mathcal{L} is a line bundle of degree zero on C . Then [Conjecture 1.1](#) holds for any noninvertible endomorphism on $X = \mathbb{P}(\mathcal{E})$ for any \mathcal{E} .*

Proof. By [Lemmas 5.4](#) and [3.1](#), we may assume that f preserves fibers. We can prove [Conjecture 1.1](#) in the case of $\deg(f|_F) = 1$ in the same way as in the case of $g(C) = 0$ since $\deg(f|_F) = 1 \leq \deg(f_C)$. Since we are considering the case of $g(C) = 1$, if \mathcal{E} is indecomposable, then \mathcal{E} is semistable (see [\[Mukai 2003, 10.2\(c\), 10.49\]](#) or [\[Hartshorne 1977, V, Exercise 2.8\(c\)\]](#)). By [Lemma 6.2](#), if $\deg(f|_F) > 1$ and \mathcal{E} is indecomposable, there is a finite étale covering $g : E \rightarrow C$ satisfying that $E \times_C X \cong \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L})$ for

an invertible sheaf \mathcal{L} over E . Furthermore, by [Lemma 6.3](#), we can take E equal to C and there is an endomorphism $f'_C : C \rightarrow C$ satisfying $f_C \circ g = g \circ f'_C$. Then by the universality of cartesian product $X \times_{C,g} C$, we have an induced endomorphism $f' : X \times_{C,g} C \rightarrow X \times_{C,g} C$. By [Lemma 3.2](#), it is enough to prove [Conjecture 1.1](#) for the endomorphism f' . Thus, we may assume that \mathcal{E} is decomposable, i.e., $X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. Then the invariant e is nonnegative by [Lemma 5.6](#). When e is positive, by the same method as the proof of [Theorem 1.3](#) in the case of $g(C) = 0$, the proof is complete. When $e = 0$, we have $\deg \mathcal{L} = 0$ and the assertion holds by the assumption. \square

In the rest of this subsection, we keep the following notation. Let C be a genus one curve and \mathcal{L} an invertible sheaf on C with degree 0. Let $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) = \text{Proj}(\text{Sym}(\mathcal{O}_C \oplus \mathcal{L}))$ and $\pi : X \rightarrow C$ the projection. When \mathcal{L} is trivial, we have $X \cong C \times \mathbb{P}^1$, and by [[Sano 2016](#), Theorem 1.3], [Conjecture 1.1](#) is true for X . Thus we may assume \mathcal{L} is nontrivial. In this case, we have two sections of $\pi : X \rightarrow C$ corresponding to the projections $\mathcal{O}_C \oplus \mathcal{L} \rightarrow \mathcal{L}$ and $\mathcal{O}_C \oplus \mathcal{L} \rightarrow \mathcal{O}_C$. Let C_0 and C_1 denote the images of these sections. Then we have $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$ and $\mathcal{O}_X(C_1) = \mathcal{O}_X(1) \otimes \pi^* \mathcal{L}^{-1}$. Since \mathcal{L} is nontrivial, we have $C_0 \neq C_1$. But since $\deg \mathcal{L} = 0$, C_0 and C_1 are numerically equivalent. Thus $(C_0 \cdot C_1) = (C_0^2) = 0$ and therefore $C_0 \cap C_1 = \emptyset$.

Let f be a noninvertible endomorphism on X such that there is a surjective endomorphism $f_C : C \rightarrow C$ with $\pi \circ f = f_C \circ \pi$.

Lemma 6.5. *When \mathcal{L} is a torsion element of $\text{Pic } C$, [Conjecture 1.1](#) holds for f .*

Proof. We fix an algebraic group structure on C . Since \mathcal{L} is torsion, there exists a positive integer $n > 0$ such that $[n]^* \mathcal{L} \cong \mathcal{O}_C$. Then the base change of $\pi : X \rightarrow C$ by $[n] : C \rightarrow C$ is the trivial \mathbb{P}^1 -bundle $\mathbb{P}^1 \times C \rightarrow C$. Applying [Lemma 6.3](#) to $g = [n]$, we get a finite morphism $h : C \rightarrow C$ such that the base change of $\pi : X \rightarrow C$ by $h : C \rightarrow C$ is $\mathbb{P}^1 \times C \rightarrow C$ and there exists a finite morphism $f'_C : C \rightarrow C$ with $f_C \circ h = h \circ f'_C$. Then f induces a noninvertible endomorphism $f' : \mathbb{P}^1 \times C \rightarrow \mathbb{P}^1 \times C$. By [[Sano 2016](#), Theorem 1.3], [Conjecture 1.1](#) holds for f' . By [Lemma 3.2](#), [Conjecture 1.1](#) holds also for f . \square

Now, let F be the numerical class of a fiber of π . By [Lemma 5.10](#), we have

$$f^* F \equiv aF \quad \text{and} \quad f^* C_0 \equiv bC_0$$

for some integers $a, b \geq 1$. Note that $a = \deg f_C$, $b = \deg f|_F$ and $ab = \deg f$ (see [Lemma 5.9](#)).

Lemma 6.6. (1) *One of the equalities $f(C_0) = C_0$, $f(C_0) = C_1$ or $f(C_0) \cap C_0 = f(C_0) \cap C_1 = \emptyset$ holds. The same is true for $f(C_1)$.*

(2) *If $f(C_0) \cap C_i = \emptyset$ for $i = 0, 1$, then the base change of $\pi : X \rightarrow C$ by $f_C : C \rightarrow C$ is isomorphic to $\mathbb{P}^1 \times C$. In particular, $f_C^* \mathcal{L} \cong \mathcal{O}_C$ and \mathcal{L} is a torsion element of $\text{Pic } C$. The same conclusion holds under the assumption that $f(C_1) \cap C_i = \emptyset$ for $i = 0, 1$.*

Proof. (1) Since $f^* C_i \equiv bC_i$, $C_0 \equiv C_1$ and $(C_0^2) = 0$, we have $(f_* C_i \cdot C_j) = 0$ for every i and j . Thus the assertion follows.

(2) Assume $f(C_0) \cap C_i = \emptyset$ for $i = 0, 1$. Consider the following Cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \pi' \downarrow & & \downarrow \pi \\ C & \xrightarrow{f_C} & C \end{array}$$

Then Y is a \mathbb{P}^1 -bundle over C associated with the vector bundle $\mathcal{O}_C \oplus f_C^* \mathcal{L}$. The pull-backs $C_i = g^{-1}(C_i)$, $i = 0, 1$ are sections of π' . By the projection formula, we have $(C_i^2) = 0$. Let $\sigma : C \rightarrow X$ be the section with $\sigma(C) = C_0$. Since $\pi \circ f \circ \sigma = f_C$, we get a section $s : C \rightarrow Y$ of π' .

Note that $g(s(C)) = f(C_0) \neq C_0, C_1$. Thus $s(C)$, C'_0 and C'_1 are distinct sections of π' . Moreover, by the projection formula, we have $(s(C) \cdot C'_0) = 0$. Thus we have three sections which are numerically equivalent to each other. Then Lemma 5.11 implies $f_C^* \mathcal{L} \cong \mathcal{O}_C$ and $Y \cong \mathbb{P}^1 \times C$. Since $f_C^* : \text{Pic}^0 C \rightarrow \text{Pic}^0 C$ is an isogeny, the kernel of f_C^* is finite and thus \mathcal{L} is a torsion element of $\text{Pic } C$. □

Lemma 6.7. (1) Suppose that

- \mathcal{L} is nontorsion in $\text{Pic } C$,
- $f(C_0) = C_0$ or C_1 , and
- $f(C_1) = C_0$ or C_1 .

Then $f(C_0) = C_0$ and $f(C_1) = C_1$, or $f(C_0) = C_1$ and $f(C_1) = C_0$.

(2) If the equalities $f(C_0) = C_0$ and $f(C_1) = C_1$ hold, then $f_* C_i \sim_{\mathbb{Q}} b C_i$ for $i = 0$ and 1 .

Proof. (1) Assume that $f(C_0) = C_0$ and $f(C_1) = C_0$. Then $f_* C_0 = a C_0$ and $f_* C_1 = a C_0$ as cycles. Since $f_C^* : \text{Pic}^0 C \rightarrow \text{Pic}^0 C$ is surjective, there exists a degree zero divisor M on C such that $f_C^* \mathcal{O}_C(M) \cong \mathcal{L}$. Then $C_1 \sim C_0 - \pi^* f_C^* M$. Hence

$$a C_0 = f_* C_1 \sim (f_* C_0 - f_* \pi^* f_C^* M) = (a C_0 - f_* \pi^* f_C^* M)$$

and

$$0 \sim f_* \pi^* f_C^* M \sim f_* f^* \pi^* M \sim (\deg f) \pi^* M.$$

Thus $\pi^* M$ is torsion and so is M . This implies that \mathcal{L} is torsion, which contradicts the assumption.

The same argument shows that the case when $f(C_0) = C_1$ and $f(C_1) = C_1$ does not occur.

(2) In this case, we have $f_*C_0 \sim aC_0$. We can write $f^*C_0 \sim bC_0 + \pi^*D$ for some degree zero divisor D on C . Thus

$$(\deg f)C_0 \sim f_*f^*C_0 \sim abC_0 + f_*\pi^*D = (\deg f)C_0 + f_*\pi^*D$$

and $f_*\pi^*D \sim 0$. Since $f_C^*: \text{Pic}^0 C \rightarrow \text{Pic}^0 C$ is surjective, there exists a degree zero divisor D' on C such that $f_C^*D' \sim D$. Then

$$0 \sim f_*\pi^*D \sim f_*\pi^*f_C^*D' \sim f_*f^*\pi^*D' \sim (\deg f)\pi^*D'.$$

Hence $\pi^*D' \sim_{\mathbb{Q}} 0$ and $D' \sim_{\mathbb{Q}} 0$. Therefore $D \sim_{\mathbb{Q}} 0$ and $f^*C_0 \sim_{\mathbb{Q}} bC_0$.

Similarly, we have $f^*C_1 \sim_{\mathbb{Q}} bC_1$. □

Lemma 6.8. *Suppose $a < b$. If $f^*C_i \sim_{\mathbb{Q}} bC_i$ for $i = 0, 1$, the line bundle \mathcal{L} is a torsion element of $\text{Pic } C$.*

Proof. Let L be a divisor on C such that $\mathcal{O}_C(L) \cong \mathcal{L}$. Note that $C_1 \sim C_0 - \pi^*L$. Thus

$$f^*\pi^*L \sim f^*(C_0 - C_1) \sim_{\mathbb{Q}} bC_0 - bC_1 \sim b\pi^*L$$

and $f_C^*L \sim_{\mathbb{Q}} bL$ hold.

Thus, from the following lemma, \mathcal{L} is a torsion element. □

Lemma 6.9. *Let a and b be integers such that $1 \leq a < b$. Let C be a curve of genus one defined over an algebraically closed field k . Let $f_C: C \rightarrow C$ be an endomorphism of $\deg f_C = a$. If L is a divisor on C of degree 0 satisfying*

$$f_C^*L \sim_{\mathbb{Q}} bL,$$

the divisor L is a torsion element of $\text{Pic}^0(C)$

Proof. By the definition of \mathbb{Q} -linear equivalence, we have $f_C^*rL \sim brL$ for some positive integer r . Since the curve C is of genus one, the group $\text{Pic}^0(C)$ is an elliptic curve. Assume the (group) endomorphism

$$f_C^* - [b]: \text{Pic}^0(C) \rightarrow \text{Pic}^0(C)$$

is the 0 map. Then we have the equalities $a = \deg f_C = \deg f_C^* = \deg[b] = b^2$. But this contradicts to the inequality $1 \leq a < b$. Hence the map $f_C^* - [b]$ is an isogeny, and $\text{Ker}(f_C^* - [b]) \subset \text{Pic}^0(C)$ is a finite group scheme. In particular, the order of $rL \in \text{Ker}(f_C^* - [b])(k)$ is finite. Thus, L is a torsion element. □

Remark 6.10. We can actually prove the following. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and $f: X \rightarrow X$ be a surjective morphism over $\overline{\mathbb{Q}}$ with first dynamical degree δ . If an \mathbb{R} -divisor D on X satisfies

$$f^*D \sim_{\mathbb{R}} \lambda D$$

for some $\lambda > \delta$, then one has $D \sim_{\mathbb{R}} 0$.

Sketch of the proof. Consider the canonical height

$$\hat{h}_D(P) = \lim_{n \rightarrow \infty} h_D(f^n(P))/\lambda^n$$

where h_D is a height associated with D [Call and Silverman 1993]. If $\hat{h}_D(P) \neq 0$ for some P , then we can prove $\bar{\alpha}_f(P) \geq \lambda$. This contradicts the fact $\delta \geq \bar{\alpha}_f(P)$ and the assumption $\lambda > \delta$. Thus one has $\hat{h}_D = 0$ and therefore $h_D = \hat{h}_D + O(1) = O(1)$. By a theorem of Serre, we get $D \sim_{\mathbb{R}} 0$ [Serre 1997, 2.9, Theorem]. \square

Proposition 6.11. *Let \mathcal{L} be an invertible sheaf of degree zero on a genus one curve C and $X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$. For any noninvertible endomorphism $f : X \rightarrow X$, Conjecture 1.1 holds.*

Proof. By Lemmas 6.5 and 6.9 we may assume $a \geq b$. In this case, $\delta_f = a$ and Conjecture 1.1 can be proved as in the proof of Theorem 6.1. \square

Proof of Theorem 1.3 for \mathbb{P}^1 -bundles over genus one curves. As we argued at the first of Section 4, we may assume that the endomorphism $f : X \rightarrow X$ is not an automorphism. Then the assertion follows from Propositions 6.4 and 6.11. \square

Remark 6.12. In the above setting, the line bundle \mathcal{L} is actually an eigenvector for f_C^* up to linear equivalence. More precisely, for a \mathbb{P}^1 -bundle $\pi : X = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L}) \rightarrow C$ over a curve C with $\deg \mathcal{L} = 0$ and an endomorphism $f : X \rightarrow X$ that induces an endomorphism $f_C : C \rightarrow C$, there exists an integer t such that $f_C^* \mathcal{L} \cong \mathcal{L}^t$. Indeed, let C_0 and C_1 be the sections defined above. Since $(f^*(C_0) \cdot C_0) = 0$, we can write $\mathcal{O}_X(f^{-1}(C_0)) \cong \mathcal{O}_X(mC_0) \otimes \pi^* \mathcal{N}$ for some integer m and degree zero line bundle \mathcal{N} on C . Since

$$0 \neq H^0(\mathcal{O}_X(f^{-1}(C_0))) = H^0(\mathcal{O}_X(mC_0) \otimes \pi^* \mathcal{N}) = H^0(\text{Sym}^m(\mathcal{O}_C \oplus \mathcal{L}) \otimes \mathcal{N}) = \bigoplus_{i=0}^m H^0(\mathcal{L}^i \otimes \mathcal{N}),$$

we have $\mathcal{N} \cong \mathcal{L}^r$ for some $-m \leq r \leq 0$. Thus $f^* \mathcal{O}_X(C_0) \cong \mathcal{O}_X(mC_0) \otimes \pi^* \mathcal{L}^r$. The key is the calculation of global sections using projection formula. Since $\mathcal{O}_X(C_1) \cong \mathcal{O}_X(C_0) \otimes \pi^* \mathcal{L}^{-1}$, we have $\pi_* \mathcal{O}_X(mC_1) \cong \pi_* \mathcal{O}_X(mC_0) \otimes \mathcal{L}^{-m}$. Moreover, since C_0 and C_1 are numerically equivalent, we can similarly get $f^* \mathcal{O}_X(C_1) \cong \mathcal{O}_X(mC_0) \otimes \pi^* \mathcal{L}^s$ for some integer s . Thus, $f^* \pi^* \mathcal{L} \cong \pi^* \mathcal{L}^{-s}$. Therefore, $\pi^* f_C^* \mathcal{L} \cong \pi^* \mathcal{L}^{-s}$. Since $\pi^* : \text{Pic } C \rightarrow \text{Pic } X$ is injective, we get $f_C^* \mathcal{L} \cong \mathcal{L}^{-s}$.

\mathbb{P}^1 -bundles over curves of genus ≥ 2 . By the following proposition, Conjecture 1.1 trivially holds in this case.

Proposition 6.13. *Let C be a curve with $g(C) \geq 2$ and $\pi : X \rightarrow C$ be a \mathbb{P}^1 -bundle over C . Let $f : X \rightarrow X$ be a surjective endomorphism. Then there exists an integer $t > 0$ such that f^t is a morphism over C , that is, f^t satisfies $\pi \circ f^t = \pi$. In particular, f admits no Zariski dense orbit.*

Proof. By Lemma 5.4, we may assume that f induces a surjective endomorphism $f_C : C \rightarrow C$ with $\pi \circ f = f_C \circ \pi$. Since C is of general type, f_C is an automorphism of finite order and the assertion follows. \square

Remark 6.14. One can also show that any surjective endomorphism over a curve of genus at least two admits no dense orbit by using the Mordell conjecture (Faltings’s theorem).

7. Hyperelliptic surfaces

Theorem 7.1. *Let X be a hyperelliptic surface and $f : X \rightarrow X$ a noninvertible endomorphism on X . Then [Conjecture 1.1](#) holds for f .*

Proof. Let $\pi : X \rightarrow E$ be the Albanese map of X . By the universality of π , there is a morphism $g : E \rightarrow E$ satisfying $\pi \circ f = g \circ \pi$. It is well-known that E is a genus one curve, π is a surjective morphism with connected fibers, and there is an étale cover $\phi : E' \rightarrow E$ such that $X' = X \times_E E' \cong F \times E'$, where F is a genus one curve [[Bădescu 2001](#), Chapter 10]. In particular, X' is an abelian surface. By [Lemma 6.3](#), taking a further étale base change, we may assume that there is an endomorphism $h : E' \rightarrow E'$ such that $\phi \circ h = g \circ \phi$. Let $\pi' : X' \rightarrow E'$ and $\psi : X' \rightarrow X$ be the induced morphisms. Then, by the universality of fiber products, there is a morphism $f' : X' \rightarrow X'$ satisfying $\pi' \circ f' = \pi' \circ h$ and $\psi \circ f' = f \circ \psi$. Applying [Lemma 3.2](#), it is enough to prove [Conjecture 1.1](#) for the endomorphism f' . Since X' is an abelian variety, this holds by [[Kawaguchi and Silverman 2016a](#), Corollary 31] and [[Silverman 2017](#), Theorem 2]. \square

8. Surfaces with $\kappa(X) = 1$

Let $f : X \rightarrow X$ be a noninvertible endomorphism on a surface X with $\kappa(X) = 1$. In this section we shall prove that f does not admit any Zariski dense forward f -orbit. Although this result is a special case of [[Nakayama and Zhang 2009](#), Theorem A] (see [Remark 1.2](#)), we will give a simpler proof of it.

By [Lemma 4.2](#), X is minimal and f is étale. Since $\deg(f) \geq 2$, we have $\chi(X, \mathcal{O}_X) = 0$.

Let $\phi = \phi_{|mK_X|} : X \rightarrow \mathbb{P}^N = \mathbb{P}H^0(X, mK_X)$ be the Iitaka fibration of X and set $C_0 = \phi(X)$. Since f is étale, it induces an automorphism $g : \mathbb{P}^N \rightarrow \mathbb{P}^N$ such that $\phi \circ f = g \circ \phi$ [[Fujimoto and Nakayama 2008](#), Lemma 3.1]. The restriction of g to C_0 gives an automorphism $f_{C_0} : C_0 \rightarrow C_0$ such that $\phi \circ f = f_{C_0} \circ \phi$. Take the normalization $\nu : C \rightarrow C_0$ of C_0 . Then ϕ factors as $X \xrightarrow{\pi} C \xrightarrow{\nu} C_0$ and π is an elliptic fibration. Moreover, f_{C_0} lifts to an automorphism $f_C : C \rightarrow C$ such that $\pi \circ f = f_C \circ \pi$.

So we obtain an elliptic fibration $\pi : X \rightarrow C$ and an automorphism f_C on C such that $\pi \circ f = f_C \circ \pi$. In this situation, the following holds.

Theorem 8.1. *Let X be a surface with $\kappa(X) = 1$, $\pi : X \rightarrow C$ an elliptic fibration, $f : X \rightarrow X$ a noninvertible endomorphism, and $f_C : C \rightarrow C$ an automorphism such that $\pi \circ f = f_C \circ \pi$. Then $f_C^t = \text{id}_C$ for a positive integer t .*

Proof. Let $\{P_1, \dots, P_r\}$ be the points over which the fibers of π are multiple fibers (possibly $r = 0$, i.e., π does not have any multiple fibers). We denote by m_i denotes the multiplicity of the fiber π^*P_i for every i . Then we have the canonical bundle formula:

$$K_X = \pi^*(K_C + L) + \sum_{i=1}^r \frac{m_i - 1}{m_i} \pi^*P_i,$$

where L is a divisor on C such that $\deg(L) = \chi(X, \mathcal{O}_X)$. Then $\deg(L) = 0$ because f is étale and $\deg(f) \geq 2$ (see [Lemma 4.2](#)). Since $\kappa(X) = 1$, the divisor $K_C + L + \sum_{i=1}^r (m_i - 1)/m_i P_i$ must have

positive degree. So we have

$$2(g(C) - 1) + \sum_{i=1}^r \frac{m_i - 1}{m_i} > 0. \tag{**}$$

For any i , set $Q_i = f_C^{-1}(P_i)$. Then $\pi^* Q_i = \pi^* f_C^* P_i = f^* \pi^* P_i$ is a multiple fiber. So $(f_C)|_{\{P_1, \dots, P_r\}}$ is a permutation of $\{P_1, \dots, P_r\}$ since f_C is an automorphism.

We divide the proof into three cases according to the genus $g(C)$ of C :

(1) $g(C) \geq 2$; then the automorphism group of C is finite. So $f_C^t = \text{id}_C$ for a positive integer t .

(2) $g(C) = 1$; by (**), it follows that $r \geq 1$. For a suitable t , all P_i are fixed points of f_C^t . We put the algebraic group structure on C such that P_1 is the identity element. Then f_C^t is a group automorphism on C . So $f_C^{ts} = \text{id}_C$ for a suitable s since the group of group automorphisms on C is finite.

(3) $g(C) = 0$; again by (**), it follows that $r \geq 3$. For a suitable t , all P_i are fixed points of f_C^t . Then f_C^t fixes at least three points, which implies that f_C^t is in fact the identity map. □

Immediately we obtain the following corollary.

Corollary 8.2. *Let $f : X \rightarrow X$ be a noninvertible endomorphism on a surface X with $\kappa(X) = 1$. Then there does not exist any Zariski dense f -orbit.*

Therefore [Conjecture 1.1](#) trivially holds for noninvertible endomorphisms on surfaces of Kodaira dimension 1.

9. Existence of a rational point P satisfying $\alpha_f(P) = \delta_f$

In this section, we prove [Theorems 1.6](#) and [1.7](#). [Theorem 1.6](#) follows from the following lemma. A subset $\Sigma \subset V(\bar{k})$ is called a *set of bounded height* if for some (or, equivalently, any) ample divisor A on V , the height function h_A associated with A is a bounded function on Σ .

Lemma 9.1. *Let X be a smooth projective variety and $f : X \rightarrow X$ a surjective endomorphism with $\delta_f > 1$. Let $D \not\equiv 0$ be a nef \mathbb{R} -divisor such that $f^* D \equiv \delta_f D$. Let $V \subset X$ be a closed subvariety of positive dimension such that $(D^{\dim V} \cdot V) > 0$. Then there exists a nonempty open subset $U \subset V$ and a set $\Sigma \subset U(\bar{k})$ of bounded height such that for every $P \in U(\bar{k}) \setminus \Sigma$ we have $\alpha_f(P) = \delta_f$.*

Remark 9.2. By a Perron–Frobenius type result of [\[Birkhoff 1967, Theorem\]](#), there is a nef \mathbb{R} -divisor $D \not\equiv 0$ satisfying the condition $f^* D \equiv \delta_f D$ since f^* preserves the nef cone.

Proof. Fix a height function h_D associated with D . For every $P \in X(\bar{k})$, the following limit exists [\[Kawaguchi and Silverman 2016b, Theorem 5\]](#):

$$\hat{h}(P) = \lim_{n \rightarrow \infty} \frac{h_D(f^n(P))}{\delta_f^n}.$$

The function \hat{h} has the following properties [Kawaguchi and Silverman 2016b, Theorem 5]:

- (i) $\hat{h} = h_D + O(\sqrt{h_H})$ where H is any ample divisor on X and $h_H \geq 1$ is a height function associated with H .
- (ii) If $\hat{h}(P) > 0$, then $\alpha_f(P) = \delta_f$.

Since $(D^{\dim V} \cdot V) > 0$, we have $(D|_V)^{\dim V} > 0$ and $D|_V$ is big. Thus we can write $D|_V \sim_{\mathbb{R}} A + E$ with an ample \mathbb{R} -divisor A and an effective \mathbb{R} -divisor E on V . Therefore we have

$$\hat{h}|_{V(\bar{k})} = h_A + h_E + O(\sqrt{h_A})$$

where h_A and h_E are height functions associated with A and E and h_A is taken to be $h_A \geq 1$. In particular, there exists a positive real number $B > 0$ such that $h_A + h_E - \hat{h}|_{V(\bar{k})} \leq B\sqrt{h_A}$. Then we have the following inclusions:

$$\begin{aligned} \{P \in V(\bar{k}) \mid \hat{h}(P) \leq 0\} &\subset \{P \in V(\bar{k}) \mid h_A(P) + h_E(P) \leq B\sqrt{h_A(P)}\} \\ &\subset \text{Supp } E \cup \{P \in V(\bar{k}) \mid h_A(P) \leq B\sqrt{h_A(P)}\} \\ &= \text{Supp } E \cup \{P \in V(\bar{k}) \mid h_A(P) \leq B^2\}. \end{aligned}$$

Hence we can take $U = V \setminus \text{Supp } E$ and $\Sigma = \{P \in U(\bar{k}) \mid \hat{h}(P) \leq 0\}$. □

Corollary 9.3. *Let X be a smooth projective variety of dimension N and $f : X \rightarrow X$ a surjective endomorphism. Let C be a irreducible curve which is a complete intersection of ample effective divisors H_1, \dots, H_{N-1} on X . Then for infinitely many points P on C , we have $\alpha_f(P) = \delta_f$.*

Proof. We may assume $\delta_f > 1$. Let D be as in Lemma 9.1. Then $(D \cdot C) = (D \cdot H_1 \cdots H_{N-1}) > 0$ [Kawaguchi and Silverman 2016b, Lemma 20]. Since $C(\bar{k})$ is not a set of bounded height, the assertion follows from Lemma 9.1. □

To prove Theorem 1.7, we need the following theorem which is a corollary of the dynamical Mordell–Lang conjecture for étale finite morphisms.

Theorem 9.4 (Bell, Ghioca and Tucker [2010, Corollary 1.4]). *Let $f : X \rightarrow X$ be an étale finite morphism of smooth projective variety X . Let $P \in X(\bar{k})$. If the orbit $\mathcal{O}_f(P)$ is Zariski dense in X , then any proper closed subvariety of X intersects $\mathcal{O}_f(P)$ in at most finitely many points.*

Proof of Theorem 1.7. We may assume $\dim X \geq 2$. Since we are working over \bar{k} , we can write the set of all proper subvarieties of X as

$$\{V_i \subsetneq X \mid i = 0, 1, 2, \dots\}.$$

By Corollary 9.3, we can take a point $P_0 \in X \setminus V_0$ such that $\alpha_f(P) = \delta_f$. Assume we can construct P_0, \dots, P_n satisfying the following conditions:

- (1) $\alpha_f(P_i) = \delta_f$ for $i = 0, \dots, n$.
- (2) $\mathcal{O}_f(P_i) \cap \mathcal{O}_f(P_j) = \emptyset$ for $i \neq j$.
- (3) $P_i \notin V_i$ for $i = 0, \dots, n$.

Now, take a complete intersection curve $C \subset X$ satisfying the following conditions:

- For $i = 0, \dots, n$, $C \not\subset \mathcal{O}_f(P_i)$ if $\overline{\mathcal{O}_f(P_i)} \neq X$.
- For $i = 0, \dots, n$, $C \not\subset \mathcal{O}_{f^{-1}}(P_i)$ if $\overline{\mathcal{O}_{f^{-1}}(P_i)} \neq X$.
- $C \not\subset V_{n+1}$.

By [Theorem 9.4](#), if $\mathcal{O}_{f^\pm}(P_i)$ is Zariski dense in X , then $\mathcal{O}_{f^\pm}(P_i) \cap C$ is a finite set. By [Corollary 9.3](#), there exists a point

$$P_{n+1} \in C \setminus \left(\bigcup_{0 \leq i \leq n} \mathcal{O}_f(P_i) \cup \bigcup_{0 \leq i \leq n} \mathcal{O}_{f^{-1}}(P_i) \cup V_{n+1} \right)$$

such that $\alpha_f(P_{n+1}) = \delta_f$. Then P_0, \dots, P_{n+1} satisfy the same conditions. Therefore we get a subset $S = \{P_i \mid i = 0, 1, 2, \dots\}$ of X which satisfies the desired conditions. \square

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myohsuke@ms.u-tokyo.ac.jp

*Graduate school of Mathematical Sciences, University of Tokyo,
Komaba, Tokyo, Japan*

ksano@math.kyoto-u.ac.jp

Department of Mathematics, Faculty of Science, Kyoto University, Japan

tshibata@math.kyoto-u.ac.jp

Department of Mathematics, Faculty of Science, Kyoto University, Japan

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