Algebra & Number Theory

Volume 12

2018

No. 2

Quadric surface bundles over surfaces and stable rationality

Stefan Schreieder

msp

ALGEBRA AND NUMBER THEORY 12:2 (2018)
dx.doi.org/10.2140/ant.2018.12.479

Quadric surface bundles over surfaces and stable rationality

Stefan Schreieder

We prove a general specialization theorem which implies stable irrationality for a wide class of quadric surface bundles over rational surfaces. As an application, we solve, with the exception of two cases, the stable rationality problem for any very general complex projective quadric surface bundle over $\mathbb{P}^2$, given by a symmetric matrix of homogeneous polynomials. Both exceptions degenerate over a plane sextic curve, and the corresponding double cover is a K3 surface.

1. Introduction

Recently, Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b; 2016c; 2017] found the first three examples of families of quadric surface bundles over $\mathbb{P}^2$, where the very general member is not stably rational. In each case, the degeneration locus is a plane octic curve. Smooth quadric surface bundles over rational surfaces typically deform to smooth bundles with a section, hence to smooth rational fourfolds. This allowed them to produce the first examples of smooth nonrational varieties that deform to rational ones.

In [Schreieder 2018], we introduced a variant of the method of Voisin [2015] and Colliot-Thélène and Pirutka [2016a], which allowed us to disprove stable rationality via a degeneration argument where a universally CH_0-trivial resolution of the special fiber is not needed. The purpose of this paper is to show that one can use this technique to simplify the arguments in [Hassett et al. 2016b; 2016c; 2017] and to apply them to large classes of quadric surface bundles.

The main result is the following general specialization theorem without resolutions; see Section 1.1 below for what it means that a variety specializes to another variety.

Theorem 1. *Let X and Y be complex projective varieties of dimension four. Suppose that X specializes to Y and that there is a morphism $f : Y \to S$ to a rational surface S, such that*

(1) *the generic fiber of f is a smooth quadric surface Q over $K = \mathbb{C}(S)$,*

(2) *the discriminant $d \in K^*/(K^*)^2$ of Q is nontrivial, and*

(3) $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) \neq 0$.

Then X is not stably rational.

MSC2010: primary 14E08, 14M20; secondary 14J35, 14D06.

Keywords: rationality problem, stable rationality, decomposition of the diagonal, unramified cohomology, Brauer group, Lüroth problem.

Since $H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mathbb{Z}/2) = H^2_{nr}(K(Y)/\mathbb{C}, \mathbb{Z}/2)$, the assumptions in the above theorem concern only the generic fiber of f. In particular, f need not be flat and there is no assumption on the singularities of Y at points which do not dominate S. A universally CH_0-trivial resolution of Y is not needed. For a more general version which works also if the discriminant of Q is possibly trivial, X and Y have arbitrary dimension, and the generic fiber of f is only stably birational to Q, see Theorem 9 below.

The second unramified cohomology group in item (3) coincides with the 2-torsion subgroup of the Brauer group of any resolution of singularities of Y. Pirutka [2016, Theorem 3.17] computed this group explicitly for any quadric surface over $\mathbb{C}(\mathbb{P}^2)$ which satisfies (2). This gives rise to many examples to which the above theorem applies. In this paper we will apply it only to a single example of Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b, Proposition 11].

The proof of Theorem 1 uses results of Pirutka [2016] on the unramified cohomology of quadric surfaces over $\mathbb{C}(\mathbb{P}^2)$, together with our aforementioned method from [Schreieder 2018], which builds on [Voisin 2015; Colliot-Thélène and Pirutka 2016a].

To give an application of Theorem 1, let us consider a generically nondegenerate line bundle valued quadratic form $q : \mathscr{E} \to \mathcal{O}_{\mathbb{P}^2}(n)$, where $\mathscr{E} = \bigoplus_{i=0}^3 \mathcal{O}_{\mathbb{P}^2}(-r_i)$ is split and such that the quadratic form q_s on the fiber $\mathscr{E}_s$ is nonzero for all $s \in \mathbb{P}^2$. Then, $X = \{q = 0\} \subset \mathbb{P}(\mathscr{E})$ defines a quadric surface bundle over $\mathbb{P}^2$. We may also regard q as a symmetric matrix $A = (a_{ij})$, where a_{ij} is a global section of $\mathcal{O}_{\mathbb{P}^2}(r_i + r_j + n)$. Locally over $\mathbb{P}^2$, X is given by

$$\sum_{i,j=0}^{3} a_{ij} z_i z_j = 0, \tag{1}$$

where z_i denotes a local coordinate which trivializes $\mathcal{O}_{\mathbb{P}^2}(-r_i) \subset \mathscr{E}$.

If X is smooth, its deformation type depends only on the integers $d_i := 2r_i + n$; we call any such quadric surface bundle of type (d_0, d_1, d_2, d_3). The degeneration locus of $X \to \mathbb{P}^2$ is a plane curve of degree $\sum_i d_i$, which is always even. If some d_i is negative, then $a_{ii} = 0$ and so $X \to \mathbb{P}^2$ admits a section; hence, X is rational. We may thus from now on restrict ourselves to the case $d_i \geq 0$ for all i.

Corollary 2. *Let d_0, d_1, d_2, and d_3 be nonnegative integers of the same parity, and let $X \to \mathbb{P}^2$ be a very general complex projective quadric surface bundle of type (d_0, d_1, d_2, d_3). If $\sum_i d_i \neq 6$, then*

(1) *X is rational if $\sum_i d_i \leq 4$ or if $d_i = d_j = 0$ for some $i \neq j$ and*

(2) *X is not stably rational otherwise.*

As we will see in the proof, the bundles in item (1) of the above corollary have a rational section, and so already the generic fiber of X over $\mathbb{P}^2$ is rational.

Up to reordering, the only cases left open by the above corollary are types $(1, 1, 1, 3)$ and $(0, 2, 2, 2)$. The former corresponds to blow-ups of cubic fourfolds containing a plane, see e.g. [Auel et al. 2017b], and the latter are Verra fourfolds [Camere et al. 2017; Iliev et al. 2017], i.e., double covers of $\mathbb{P}^2 \times \mathbb{P}^2$, branched along a hypersurface of bidegree $(2, 2)$. In both exceptions, the degeneration locus of the quadric bundle is a sextic curve in $\mathbb{P}^2$, and so the associated double cover is a K3 surface, see e.g. [Auel et al. 2015].

Specializing to $a_{33} = 0$ in (1) shows that all examples in the above corollary deform to smooth quadric surface bundles with a section, hence to smooth rational fourfolds.

Many quadric surface bundles over $\mathbb{P}^2$ are birational to fourfolds which arise naturally in projective geometry, see e.g. [Schreieder 2018, §3.5]. For instance, Corollary 2 implies that

(I) a very general complex hypersurface of bidegree $(d, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational if $d \geq 2$,

(II) a very general complex hypersurface $X \subset \mathbb{P}^5$ of degree $d+2$ and with multiplicity d along a 2-plane is not stably rational if $d \geq 2$, and

(III) a double cover $X \xrightarrow{2:1} \mathbb{P}^4$, branched along a very general complex hypersurface $Y \subset \mathbb{P}^4$ of even degree $d+2$ and with multiplicity d along a line, is not stably rational if $d \geq 2$.

The case $d = 2$ in items (I) and (III) corresponds to the aforementioned results in [Hassett et al. 2016b; 2016c]. For stable rationality properties of smooth hypersurfaces and double covers, see [Beauville 2016; Colliot-Thélène and Pirutka 2016a; 2016b; Hassett et al. 2016c; Okada 2016; Totaro 2016; Voisin 2015]; for results on conic bundles, see [Ahmadinezhad and Okada 2018; Artin and Mumford 1972; Auel et al. 2016; Beauville 2016; Böhning and von Bothmer 2018; Hassett et al. 2016a; Voisin 2015].

In [Schreieder 2018], we studied rationality properties of quadric bundles with arbitrary fiber dimensions. Our uniform treatment sufficed to prove (I) and (II) for $d \geq 5$, and (III) for $d \geq 8$. On the other hand, the results in [Schreieder 2018] left open infinitely many cases in Corollary 2. For instance, the types $(1, 1, d_2, d_3)$ and $(0, 2, d_2, d_3)$ with $d_2 \leq 7$ and arbitrary d_3 are not covered by [Schreieder 2018] and there are more cases which were not accessible; see [Schreieder 2018, Remark 36].

Our method applies also to quadric surface bundles over other rational surfaces S. We treat in this paper the case $S = \mathbb{P}^1 \times \mathbb{P}^1$ and obtain similar results as those in Corollary 2 above; see Corollaries 11 and 12 below.

1.1. ***Conventions and notations.*** All schemes are separated. A variety is an integral scheme of finite type over a field. A property is said to hold at a very general point of a scheme, if it holds at all closed points outside a countable union of proper closed subsets.

Let k be an algebraically closed field. We say that a variety X over a field L specializes (or degenerates) to a variety Y over k, if there is a discrete valuation ring R with residue field k and fraction field F with an injection of fields $F \hookrightarrow L$, together with a flat proper morphism $\mathcal{X} \to \operatorname{Spec} R$ of finite type, such that Y is isomorphic to the special fiber $Y \simeq \mathcal{X} \times_R k$ and $X \simeq \mathcal{X} \times_R L$ is isomorphic to a base change of the generic fiber. If $\mathcal{Y} \to B$ is a flat proper morphism of complex varieties with integral fibers, then for any closed points $0, t \in B$ with t very general, the fiber Y_t specializes to Y_0 in the above sense [Schreieder 2018, Lemma 8].

A morphism $f : X \to Y$ of varieties over a field k is universally CH_0-trivial, if $f_* : \mathrm{CH}_0(X \times L) \xrightarrow{\simeq} \mathrm{CH}_0(Y \times L)$ is an isomorphism for all field extensions L of k.

A quadric surface bundle is a flat morphism $f : X \to S$ between projective varieties such that the generic fiber is a smooth quadric surface; the degeneration locus is given by all $s \in S$ such that $f^{-1}(s)$ is singular. If f is not assumed flat, then we call X a weak quadric surface bundle over S. Quadric surface bundles over surfaces have been studied in detail in [Auel et al. 2015].

We denote by $\mu_2 \subset \mathbb{G}_m$ the group of second roots of unity. If X is a proper variety over a field k of characteristic different from 2, the unramified cohomology group $H^i_{nr}(k(X)/k, \mu_2^{\otimes i})$ is the subgroup of all elements of the Galois cohomology group $H^i(k(X), \mu_2^{\otimes i})$ which have trivial residue at all discrete valuations of rank one on $k(X)$ over k [Colliot-Thélène and Ojanguren 1989]. This is a stable birational invariant of X [Colliot-Thélène and Ojanguren 1989, Proposition 1.2]. If X is smooth and proper over k, then $H^i_{nr}(k(X)/k, \mu_2^{\otimes i})$ coincides with the subgroup of elements of $H^i(k(X), \mu_2^{\otimes i})$ that have trivial residue at any codimension-one point of X [Colliot-Thélène 1995, Theorem 4.1.1].

2. Second unramified cohomology of quadric surfaces

Let K be a field of characteristic different from 2. It will be convenient to identify the Galois cohomology group $H^i(K, \mu_2^{\otimes i})$ with the étale cohomology group $H^i_{\text{ét}}(\operatorname{Spec}(K), \mu_2^{\otimes i})$. We also use the identification $H^1(K, \mu_2) \simeq K^*/(K^*)^2$, induced by the Kummer sequence. For $a, b \in K^*$, we denote by $(a, b) \in H^2(K, \mu_2^{\otimes 2})$ the cup product of the classes given by a and b. If S is a normal variety over a field k and with fraction field $k(S) = K$, then for any $\alpha \in H^2(K, \mu_2^{\otimes 2})$, the ramification divisor $\operatorname{ram}(\alpha) \subset S$ is given by (the closure of) all codimension-one points $x \in S^{(1)}$ with $\partial^2_x \alpha \neq 0$. Here, $\partial^2_x : H^2(K, \mu_2^{\otimes 2}) \to H^1(\kappa(x), \mu_2)$ denotes the residue induced by the local ring $\mathcal{O}_{S,x} \subset K$.

To any nondegenerate quadratic form q over K, one associates the discriminant $\operatorname{discr}(q) \in K^*/(K^*)^2$ and the Clifford invariant $\operatorname{cl}(q) \in H^2(K, \mu_2^{\otimes 2})$. If q has even dimension, then the discriminant $\operatorname{discr}(q)$ depends only on the quadric hypersurface $Q = \{q = 0\}$ and the Clifford invariant satisfies $\operatorname{cl}(\lambda \cdot q) = \operatorname{cl}(q) + (\lambda, \operatorname{discr}(q))$ for all $\lambda \in K^*$ [Lam 1973, Chapter 5, (3.16)]. If Q is a surface, then up to similarity, $q \simeq \langle 1, -a, -b, abd \rangle$ for some $a, b, d \in K^*$. In this case, $\operatorname{discr}(q) = d$ and $\operatorname{cl}(q) = (-a, -b) + (ab, d)$. We will need the following [Arason 1975; Kahn et al. 1998, Corollary 8]:

Theorem 3. *Let K be a field with $\operatorname{char}(K) \neq 2$, and let $f : Q \to \operatorname{Spec} K$ be a smooth projective quadric surface over K. Denote by $d \in K^*/(K^*)^2$ the discriminant of Q and by $\beta \in H^2(K, \mu_2^{\otimes 2})$ the Clifford invariant of some quadratic form q with $Q = \{q = 0\}$. Then*

$$f^* : H^2(K, \mu_2^{\otimes 2}) \to H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$$

is an isomorphism if d is nontrivial. If $d \in (K^*)^2$, then $\ker(f^*) = \{1, \beta\}$.

Pirutka [2016, Theorem 3.17] computed the unramified cohomology group $H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2})$ of a smooth quadric surface Q with nonzero discriminant over the function field of a smooth complex surface. The following reflects one half of her result:

Theorem 4 (Pirutka). *Let $f : Q \to \operatorname{Spec} K$ be a smooth projective quadric surface over the function field K of some smooth surface S over $\mathbb{C}$. Let $d \in K^*/(K^*)^2$ denote the discriminant and $\beta \in H^2(K, \mu_2^{\otimes 2})$ the Clifford invariant of some quadratic form q with $Q = \{q = 0\}$. If for some $\alpha \in H^2(K, \mu_2^{\otimes 2})$ the pullback $f^*(\alpha) \in H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$ is unramified over $\mathbb{C}$, then the following holds:*

($*$) *If the residue $\partial^2_x\alpha$ at some codimension-one point $x \in S^{(1)}$ is nonzero, then*

(a) $\partial^2_x\alpha = \partial^2_x\beta$ *and*

(b) *d becomes a square in the fraction field of the completion $\widehat{\mathcal{O}_{S,x}}$.*

Proof. The condition on d is by Hensel's lemma equivalent to asking that, up to multiplication by a square, d is a unit in $\mathcal{O}_{S,x}$ whose image in $\kappa(x)$ is a square. The theorem follows therefore from [Pirutka 2016, §3.6.2]. (In [Pirutka 2016, Theorem 3.17], the assumption that d is not a square is only used to invoke bijectivity of f^* via Theorem 3; the assumption that $\mathrm{ram}(\beta)$ is a simple normal crossing divisor on S is only used in [Pirutka 2016, §3.6.1].) □

Remark 5. Up to replacing S by some blow-up, one can always assume that $\mathrm{ram}(\beta)$ is a simple normal crossing divisor on S. Under this assumption, the analysis of Pirutka [2016, §3.6.1] shows that the following converse of the above theorem is also true: if $\alpha \in H^2(K, \mu_2^{\otimes 2})$ is such that condition ($*$) holds, then $f^*\alpha \in H^2_{nr}(K(Q)/K, \mu_2^{\otimes 2})$ is unramified over $\mathbb{C}$; nontriviality can be checked via Theorem 3.

The result of Pirutka [2016, Theorem 3.17] applies to the following important example, due to Hassett, Pirutka, and Tschinkel [Hassett et al. 2016b, Proposition 11]; for a reinterpretation in terms of conic bundles, see [Auel et al. 2016].

Proposition 6 (Hassett, Pirutka, and Tschinkel). *Let $K = \mathbb{C}(x, y)$ be the function field of $\mathbb{P}^2$, and consider the quadratic form $q = \langle y, x, xy, F(x, y, 1)\rangle$ over K, where*

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz).$$

If $f : Q \to \operatorname{Spec} K$ denotes the corresponding projective quadric surface over K, then

$$0 \neq f^*((x, y)) \in H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2}).$$

3. A vanishing result

The following general vanishing result is the key ingredient of this paper.

Proposition 7. *Let Y be a smooth complex projective variety, and let S be a smooth complex projective surface. Let $f : Y \dashrightarrow S$ be a dominant rational map whose generic fiber Y_η is stably birational to a smooth quadric surface Q over $K = \mathbb{C}(S)$. Suppose that there is some $\alpha \in H^2(K, \mu_2^{\otimes 2})$, such that $\alpha' := f^*\alpha \in H^2_{nr}(K(Y_\eta)/K, \mu_2^{\otimes 2})$ is unramified over $\mathbb{C}$. Then for any prime divisor $E \subset Y$ which does not dominate S, the restriction of α' to E vanishes:*

$$\alpha'|_E = 0 \in H^2(\mathbb{C}(E), \mu_2^{\otimes 2}).$$

Proof. Since unramified cohomology is a functorial stable birational invariant [Colliot-Thélène and Ojanguren 1989], we may up to replacing Y by $Y \times \mathbb{P}^m$ assume that Y_η is birational to $Q \times \mathbb{P}^r_K$ for some $r \geq 0$. This birational map induces a dominant rational map $Y_\eta \dashrightarrow Q$.

Since Y is smooth, f is defined at the generic point y of E. By [Merkurjev 2008, Propositions 1.4 and 1.7; Schreieder 2018, §5], we may up to replacing S by a different smooth projective model assume that the image $x := f(y) \in S^{(1)}$ is a codimension-one point on S. Consider the local ring $A := \mathcal{O}_{S,x}$, and let $\hat{A}$ be its completion with field of fractions $\widehat{K} := \operatorname{Frac}(\hat{A})$. The local ring $B := \mathcal{O}_{Y,y}$ contains A. We let $\widehat{B}$ be the completion of B and $\hat{L} := \operatorname{Frac}(\widehat{B})$ be its field of fractions. Since $Y_\eta \dashrightarrow Q$ is dominant, inclusion of fields induces the sequence

$$H^2(K, \mu_2^{\otimes 2}) \xrightarrow{\varphi_1} H^2(\widehat{K}, \mu_2^{\otimes 2}) \xrightarrow{\varphi_2} H^2(\widehat{K}(Q), \mu_2^{\otimes 2}) \xrightarrow{\varphi_3} H^2(\hat{L}, \mu_2^{\otimes 2}). \tag{2}$$

Lemma 8. *If some $\gamma \in H^2(K, \mu_2^{\otimes 2})$ satisfies $\partial_x^2 \gamma = 0$, then $\varphi_1(\gamma) = 0 \in H^2(\widehat{K}, \mu_2^{\otimes 2})$.*

Proof. Since $\partial_x^2 \gamma = 0$, the image of γ in $H^2(\widehat{K}, \mu_2^{\otimes 2})$ is contained in $H^2_{\text{ét}}(\operatorname{Spec} \hat{A}, \mu_2^{\otimes 2}) \subset H^2(\widehat{K}, \mu_2^{\otimes 2})$ [Colliot-Thélène 1995, §3.3 and §3.8]. It thus suffices to show that $H^2_{\text{ét}}(\operatorname{Spec} \hat{A}, \mu_2^{\otimes 2})$ vanishes. Since $\hat{A}$ is a henselian local ring, restriction to the closed point gives an isomorphism $H^2_{\text{ét}}(\operatorname{Spec} \hat{A}, \mu_2^{\otimes 2}) \simeq H^2(\kappa(x), \mu_2^{\otimes 2})$ [Milne 1980, Corollary VI.2.7]. By Tsen's theorem, $H^2(\kappa(x), \mu_2^{\otimes 2}) = 0$. This concludes the lemma. $\square$

Since $f^*\alpha$ is unramified, we know that

$$\varphi_3 \circ \varphi_2 \circ \varphi_1(\alpha) \in H^2_{\text{ét}}(\operatorname{Spec} \widehat{B}, \mu_2^{\otimes 2}) \subset H^2(\hat{L}, \mu_2^{\otimes 2}) \tag{3}$$

[Colliot-Thélène 1995, §3.3 and §3.8] and the compatibility of the residue map illustrated in [Colliot-Thélène and Ojanguren 1989, p. 143]. We aim to show that this class vanishes, which is enough to conclude the proposition, because $\alpha'|_E$ is obtained as the restriction of the above class to the closed point $\operatorname{Spec} \mathbb{C}(E)$.

In order to show that (3) vanishes, we choose some quadratic form q with $Q = \{q = 0\}$ and denote by $d \in K^*/(K^*)^2$ and $\beta \in H^2(K, \mu_2^{\otimes 2})$ the discriminant and the Clifford invariant of q, respectively. If $\partial_x^2 \alpha = 0$, then (3) vanishes by Lemma 8. If $\partial_x^2 \alpha \neq 0$, then $\partial_x^2(\alpha - \beta) = 0$ by Theorem 4, because Y_η is stably birational to Q and unramified cohomology is a stable birational invariant. By Lemma 8, it then suffices to show that β maps to zero via (2). By Theorem 4, d becomes a square in $\widehat{K}$, and so the latter follows from Theorem 3, applied to φ_2 in (2). This concludes the proof of the proposition. $\square$

4. Proof of Theorem 1

The following is a generalization of Theorem 1, stated in the introduction. For what it exactly means that a variety specializes to another variety, see Section 1.1 above.

Theorem 9. *Let X be a proper variety which specializes to a complex projective variety Y. Suppose that there is a dominant rational map $f : Y \dashrightarrow \mathbb{P}^2$ with the properties that*

(a) *some Zariski open and dense subset $U \subset Y$ admits a universally CH_0-trivial resolution of singularities $\widetilde{U} \to U$ such that the induced rational map $\widetilde{U} \dashrightarrow \mathbb{P}^2$ is a morphism whose generic fiber is proper over $\mathbb{C}(\mathbb{P}^2)$ and*

(b) *the generic fiber Y_η of f is stably birational to a smooth projective quadric surface $g : Q \to \operatorname{Spec} K$ over $K = \mathbb{C}(\mathbb{P}^2)$, such that there is a class $\alpha \in H^2(K, \mu_2^{\otimes 2})$ whose pullback $g^*\alpha$ is nontrivial and* *unramified over* $\mathbb{C}$:

$$0 \neq g^*\alpha \in H^2_{nr}(K(Q)/\mathbb{C}, \mu_2^{\otimes 2}) = H^2_{nr}(\mathbb{C}(Y)/\mathbb{C}, \mu_2^{\otimes 2}).$$

Then no resolution of singularities of X admits an integral decomposition of the diagonal. In particular, X is not stably rational.

Proof. Since $g^*\alpha \neq 0$ is unramified over $\mathbb{C}$ and unramified cohomology is a stable birational invariant, $\alpha' := f^*\alpha \in H^2(\mathbb{C}(Y), \mu_2^{\otimes 2})$ is a nontrivial class which is unramified over $\mathbb{C}$. By Hironaka's theorem, there exists a resolution of singularities $\tau : \widetilde{Y} \to Y$, such that $\tau^{-1}(U)$ identifies with the resolution of singularities $\widetilde{U}$ of U given in (a), and such that $E := \widetilde{Y} \setminus \widetilde{U}$ is a simple normal crossing divisor in $\widetilde{Y}$. Our assumption on $\widetilde{U}$ then implies that $\tau^{-1}(U) \to U$ is universally CH_0-trivial. Moreover, each component E_i of E is smooth and does not dominate $\mathbb{P}^2$. Therefore, Proposition 7 implies that the nontrivial class α' restricts to zero on E_i for all i and so Theorem 9 follows from the new key technique in [Schreieder 2018, §4]. □

Proof of Theorem 1. Condition (1) in Theorem 1 implies condition (a) in Theorem 9 with $\widetilde{U} = U$. By Theorem 3, conditions (1), (2), and (3) in Theorem 1 imply condition (b) in Theorem 9. Theorem 1 follows therefore from Theorem 9. □

5. Applications

5.1. *Quadric surface bundles over* $\mathbb{P}^2$. If the symmetric matrix $A = (a_{ij})$ in (1) is of diagonal form, i.e., $a_{ij} = 0$ for all $i \neq j$, then we say that the corresponding quadric surface bundle X is given by the quadratic form $q = \langle a_{00}, \dots, a_{33}\rangle$. The condition that X is flat over $\mathbb{P}^2$ means that the a_{ii} have no common zero. If the homogeneous polynomials a_{ii} degenerate and acquire common zeros, then the same formula still defines a weak quadric bundle as long as the a_{ii} are nonzero and have no common factor. We will use such degenerations in the proofs below.

Proof of Corollary 2. In the notation of (1), let $A = (a_{ij})_{0 \le i,j \le 3}$ be the symmetric matrix which corresponds to the very general quadric surface bundle X of type (d_0, d_1, d_2, d_3) over $\mathbb{P}^2$. We may without loss of generality assume $0 \le d_0 \le d_1 \le d_2 \le d_3$. If $d_1 = 0$, then also $d_0 = 0$ and $a_{ij} \in \mathbb{C}$ is constant for $i, j \in \{0, 1\}$. The quadric $\{a_{00}z_0^2 + 2a_{01}z_0z_1 + a_{11}z_1^2 = 0\}$ thus has a point over $\mathbb{C}$ and so $X \to \mathbb{P}^2$ has a section. Hence, X is rational. If $d_i = 1$ for all i, then X is a hypersurface of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ and so projection to the second factor shows that X is rational. Since the d_i have all the same parity, this shows that X is rational if $\sum d_i \le 4$ or $d_1 = 0$.

The case $d_i = 2$ for all i is due to [Hassett et al. 2016b]; a quick proof follows from [Hassett et al. 2016b, Proposition 11] (= Proposition 6 above) and Theorem 1.

It remains to deal with the case where $\sum_i d_i \ge 8$, $d_1 \ge 1$, and $d_3 \ge 3$. Recall that all d_i are either even or odd. Consider the weak quadric surface bundle $Y_i := \{q_i = 0\} \subset \mathbb{P}(\mathscr{E})$ of type (d_0, d_1, d_2, d_3), given by the diagonal forms

$$q_1 := \langle z^{d_0}, x^{d_1}, xyz^{d_2-2}, yz^{d_3-3}F(x,y,z)\rangle,$$
$$q_2 := \langle z^{d_0}, xz^{d_1-1}, x^{d_2-1}y, yz^{d_3-3}F(x,y,z)\rangle,$$
$$q_3 := \langle z^{d_0}, x^{d_1}, yz^{d_2-1}, xyz^{d_3-4}F(x,y,z)\rangle,$$

where F is the quadratic polynomial from Proposition 6.

Note that Y_i is integral, because the entries in the diagonal form are coprime. Consider the natural projection $Y_i \to \mathbb{P}^2$. The generic fiber is a smooth quadric surface Q_i over $K = \mathbb{C}(\mathbb{P}^2)$. Setting $z = 1$ shows that Q_1 is given by the quadratic form $q_1' = \langle 1, x^{d_1}, xy, yF(x,y,1)\rangle$, Q_2 is given by $q_2' = \langle 1, x, x^{d_2-1}y, yF(x,y,1)\rangle$, and Q_3 is given by $q_3' = \langle 1, x^{d_1}, y, xyF(x,y,1)\rangle$.

If d_0 is even, then so is d_2. Multiplying through by y, absorbing squares and reordering the entries thus shows in this case that q_2' is similar to the quadratic form $q = \langle y, x, xy, F(x,y,1)\rangle$ from Proposition 6. If d_0 is odd, then so is d_1 and so q_1' is isomorphic to $\langle 1, x, xy, yF(x,y,1)\rangle$ and q_3' is isomorphic to $\langle 1, x, y, xyF(x,y,1)\rangle$. Again, q_1' and q_3' are both similar to q. Hence, $H^2_{nr}(K(Q_i)/\mathbb{C}, \mu_2^{\otimes 2}) \neq 0$ for $i \equiv d_0 \bmod 2$ by [Hassett et al. 2016b, Proposition 11] (= Proposition 6 above).

Since $d_1, d_2 \geq 1$ and $d_3 \geq 3$, the very general quadric surface bundle $X \subset \mathbb{P}(\mathscr{E})$ as in Corollary 2 degenerates to Y_2. If d_0 is odd, X also degenerates to Y_1 or Y_3, depending on whether $d_2 \geq 3$ or $d_2 = 1$. Depending on the parity of d_0 and the size of d_2, we can choose one of the three degenerations together with Theorem 1 (or 9) to conclude. □

Remark 10. Pirutka informed me that for any total degree $d := \sum_i d_i \geq 8$, one can reprove some cases of Corollary 2 via degenerations to similar quadric surface bundles as in [Hassett et al. 2016b], for which [Pirutka 2016, Theorem 3.17] applies, and for which one can compute universally CH_0-trivial resolutions explicitly [Auel et al. 2017a].

5.2. *Quadric surface bundles over* $\mathbb{P}^1 \times \mathbb{P}^1$. As a second example where Theorem 1 applies, we consider quadric surface bundles X over $\mathbb{P}^1 \times \mathbb{P}^1$ that are given by a line bundle valued quadratic form $q : \mathscr{E} \to \mathscr{O}(m,n)$, where $\mathscr{E} = \bigoplus_{i=0}^{3} \mathscr{O}(-p_i, -q_i)$ is split. Locally, $X := \{q = 0\} \subset \mathbb{P}(\mathscr{E})$ is given by (1) where a_{ij} is a global section of $\mathscr{O}(p_i + p_j + m, q_i + q_j + n)$. If $a_{ij} = 0$ for $i \neq j$, we say that X is given by the quadratic form $q = \langle a_{00}, \dots, a_{33}\rangle$. If the a_{ii} degenerate and acquire common zeros, then the same formulas still define a hypersurface in $\mathbb{P}(\mathscr{E})$ which is a weak quadric surface bundle over $\mathbb{P}^2$ as long as the a_{ii} are nonzero and have no common factor. The deformation type of X depends only on the integers $d_i := m + 2p_i$ and $e_i := n + 2q_i$, and we call $(d_i, e_i)_{0\leq i\leq 3}$ the type of X. Note that the d_i as well as the e_i have the same parity for all i. We say that the type $(d_i, e_i)_{0\leq i\leq 3}$ is lexicographically ordered, if $d_i < d_{i+1}$, or $d_i = d_{i+1}$ and $e_i \leq e_{i+1}$.

Corollary 11. *Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a very general quadric surface bundle of lexicographically ordered type $(d_i, e_i)_{0\leq i\leq 3}$, with $d_i, e_i \geq 0$ and $d_3, e_3 \geq 3$. Then*

(1) *X is rational if $d_2 = 0$, $d_1 = e_1 = e_0 = 0$ or $e_0 = e_1 = e_2 = 0$ and*

(2) *X is not stably rational otherwise.*

All examples in Corollary 11 deform to smooth rational varieties of dimension four; see for instance [Schreieder 2018, §3.5]. The condition $d_3, e_3 \geq 3$ in the above theorem could be replaced by a weaker but more complicated assumption; we collect in Corollary 12 below the remaining cases where our method works.

Proof of Corollary 11. Let $A = (a_{ij})_{0\leq i,j\leq 3}$ be a symmetric matrix, where a_{ij} is a very general global section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(p_i + p_j + m, q_i + q_j + n)$, and consider the corresponding quadric surface bundle X over $\mathbb{P}^1 \times \mathbb{P}^1$. Here the integers $d_i := 2p_i + m$ and $e_i := 2q_i + n$ are assumed to satisfy the assumptions of Corollary 11; i.e., $(d_i, e_i)_{0\leq i\leq 3}$ is lexicographically ordered with $d_i, e_i \geq 0$ and $d_3, e_3 \geq 3$.

If $d_1 = e_1 = e_0 = 0$, then $(a_{ij})_{0\leq i,j\leq 1}$ is a constant matrix and so X has a section. If $d_2 = 0$, then $(a_{ij})_{0\leq i,j\leq 2}$ is a matrix of polynomials which are constant along the first factor. Since any conic bundle over $\mathbb{P}^1$ has a section, X also admits a section. If $e_0 = e_1 = e_2 = 0$, then $(a_{ij})_{0\leq i,j\leq 2}$ is a matrix of polynomials, constant along the second factor, and so X has a section as before. Since X is general and $d_i, e_i \geq 0$, the generic fiber of X over $\mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric surface and so X is rational in each of the above cases.

The case where $(e_0, e_1, e_2) \neq (0, 0, 0)$, $(d_1, e_0, e_1) \neq (0, 0, 0)$, and $d_2 \neq 0$ is similar to the proof of Corollary 2. The main point is that we can always degenerate X to weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$ whose generic fiber is isomorphic to the example in Proposition 6. To find such a degeneration, we consider coordinates x_0, x_1 and y_0, y_1 on the first and second factors of $\mathbb{P}^1 \times \mathbb{P}^1$, respectively, and consider the bidegree-(2, 2) polynomial

$$h := x_1^2 y_0^2 + x_0^2 y_1^2 + x_0^2 y_0^2 - 2(x_1 y_1 x_0 y_0 + x_1 x_0 y_0^2 + y_1 y_0 x_0^2). \tag{4}$$

We then start with the quadratic form $q = \langle 1, y_1, x_1, x_1 y_1 h\rangle$. Putting $x_0 = y_0 = 1$ shows that the corresponding quadric surface over $K = \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to the one in Proposition 6. The point is that the isomorphism type of this quadric surface does not change if we perform any of the following operations to the quadratic form q:

- multiply some entries with even powers of x_1 and y_1,
- multiply some entries with arbitrary powers of x_0 and y_0, or
- reorder the entries of the quadratic form.

Our aim is to produce a quadratic form of given type $(e_i, d_i)_{0\leq i\leq 3}$ whose entries are coprime, since the latter guarantees that the associated quadratic form defines a weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$. Once this is achieved, Corollary 11 will follow from Proposition 6 and Theorem 1.

By assumption, $d_2 \geq 1$, and if $e_0 = e_1 = 0$, then $d_1 \geq 1$ and $e_2 \geq 1$. This leads to Cases A, B, and C below. We divide into further subcases and provide each time a quadratic form (produced via the above process) with the properties we want. Recall that the d_i, as well as the e_i, have the same parity.

Case A ($e_1 \geq 1$). (1) If d_0 and e_0 are even, then we take

$$\langle x_1^{d_0} y_1^{e_0}, x_0^{d_1} y_0^{e_1-1} y_1, x_0^{d_2-1} x_1 y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd and e_0 is even, then we take

$$\langle x_0^{d_0} y_1^{e_0}, x_0^{d_1} y_0^{e_1-1} y_1, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(3) If d_0 is even and e_0 is odd, then we take

$$\langle x_1^{d_0} y_0^{e_0}, x_0^{d_1} y_1^{e_1}, x_0^{d_2-1} x_1 y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(4) If d_0 and e_0 are odd, then we take

$$\langle x_0^{d_0} y_0^{e_0}, x_0^{d_1} y_1^{e_1}, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

Case B ($e_0 \geq 1$ and $e_1 = 0$; hence, e_i is even for all i). (1) If d_0 is even, then we take

$$\langle x_1^{d_0} y_0^{e_0-1} y_1, x_0^{d_1}, x_0^{d_2-1} x_1 y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd, then we take

$$\langle x_0^{d_0} y_0^{e_0-1} y_1, x_0^{d_1}, x_1^{d_2} y_0^{e_2}, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

Case C ($d_1, e_2 \geq 1$ and $e_0 = e_1 = 0$; hence, e_i is even for all i). (1) If d_0 is even, then we take

$$\langle x_1^{d_0}, x_0^{d_1-1} x_1, x_0^{d_2} y_0^{e_2-1} y_1, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

(2) If d_0 is odd, then we take

$$\langle x_0^{d_0}, x_1^{d_1}, x_0^{d_2} y_0^{e_2-1} y_1, x_0^{d_3-3} y_0^{e_3-3} x_1 y_1 h\rangle.$$

In each of the above cases, putting $x_0 = y_0 = 1$ and reordering the factors if necessary shows that the corresponding weak quadric surface bundle Y over $\mathbb{P}^1 \times \mathbb{P}^1$ has generic fiber which is isomorphic to $\langle 1, y_1, x_1, x_1 y_1 F(x_1, y_1, 1)\rangle$. Corollary 11 therefore follows from [Hassett et al. 2016b, Proposition 11] (see Proposition 6 above) and Theorem 1. $\square$

Corollary 12. *Let $(d_i, e_i)_{0\leq i\leq 3}$ be a lexicographically ordered tuple of pairs of nonnegative integers with $d_i + d_j$ and $e_i + e_j$ even for all i, j. Suppose that one of the following holds*:

(1) $d_1 \geq 1$, $d_3 \geq 2$, $e_1 + e_2 \geq 1$, *and* $e_3 \geq 3$ *or*

(2) $d_1 \geq 1$, $d_3 \geq 2$, $e_0 \geq 1$, $e_1 + e_2 \geq 1$, *and* $e_2 \geq 2$.

Then a very general complex projective quadric surface bundle X over $\mathbb{P}^1 \times \mathbb{P}^1$ of type $(d_i, e_i)_{0\leq i\leq 3}$ is not stably rational.

Proof. We start with the quadratic forms $q_1 := \langle 1, x_1, x_1 y_1, y_1 h\rangle$ and $q_2 := \langle y_1, x_1, x_1 y_1, h\rangle$, where h is as in (4). If condition (1) holds, then we can use q_1 and if (2) holds, then we can use q_2 to obtain, via the procedure explained in the proof of Corollary 11, a quadratic form of type $(d_i, e_i)_{0\leq i\leq 3}$ whose coefficients are coprime. This yields a special fiber to which Theorem 1 applies. The details are similar as in the proof of Corollary 11, and we leave them to the reader. $\square$

Acknowledgements

I am very grateful to O. Benoist for useful correspondence and comments on a previous version. I am very grateful to the referees and to J.-L. Colliot-Thélène for many suggestions which improved the exposition. Thanks to A. Pirutka and C. Camere for useful comments. The results of this article were conceived when I was a member of the SFB/TR 45.

References

[Ahmadinezhad and Okada 2018] H. Ahmadinezhad and T. Okada, "Stable rationality of higher dimensional conic bundles", preprint, 2018. arXiv

[Arason 1975] J. K. Arason, "Cohomologische invarianten quadratischer Formen", *J. Algebra* **36**:3 (1975), 448–491. MR Zbl

[Artin and Mumford 1972] M. Artin and D. Mumford, "Some elementary examples of unirational varieties which are not rational", *Proc. London Math. Soc.* (3) **25** (1972), 75–95. MR Zbl

[Auel et al. 2015] A. Auel, R. Parimala, and V. Suresh, "Quadric surface bundles over surfaces", pp. 31–70 in *A collection of manuscripts written in honour of Alexander S. Merkurjev's sixtieth birthday*, edited by P. Balmer et al., Doc. Math. **7**, Deutsche Mathematiker-Vereinigung, 2015. MR Zbl

[Auel et al. 2016] A. Auel, C. Böhning, H.-C. G. von Bothmer, and A. Pirutka, "Conic bundles with nontrivial unramified Brauer group over threefolds", preprint, 2016. arXiv

[Auel et al. 2017a] A. Auel, C. Böhning, and A. Pirutka, "Stable rationality of quadric and cubic surface bundle fourfolds", preprint, 2017. arXiv

[Auel et al. 2017b] A. Auel, J.-L. Colliot-Thélène, and R. Parimala, "Universal unramified cohomology of cubic fourfolds containing a plane", pp. 29–55 in *Brauer groups and obstruction problems*, edited by A. Auel et al., Progr. Math. **320**, Springer, 2017. MR Zbl

[Beauville 2016] A. Beauville, "A very general sextic double solid is not stably rational", *Bull. Lond. Math. Soc.* **48**:2 (2016), 321–324. MR Zbl

[Böhning and von Bothmer 2018] C. Böhning and H.-C. G. von Bothmer, "On stable rationality of some conic bundles and moduli spaces of Prym curves", *Comm. Math. Helvetici* **93**:1 (2018), 133–155.

[Camere et al. 2017] C. Camere, G. Kapustka, M. Kapustka, and G. Mongardi, "Verra fourfolds, twisted sheaves and the last involution", preprint, 2017. arXiv

[Colliot-Thélène 1995] J.-L. Colliot-Thélène, "Birational invariants, purity and the Gersten conjecture", pp. 1–64 in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), edited by B. Jacob and A. Rosenberg, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., 1995. MR Zbl

[Colliot-Thélène and Ojanguren 1989] J.-L. Colliot-Thélène and M. Ojanguren, "Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford", *Invent. Math.* **97**:1 (1989), 141–158. MR

[Colliot-Thélène and Pirutka 2016a] J.-L. Colliot-Thélène and A. Pirutka, "Hypersurfaces quartiques de dimension 3: non-rationalité stable", *Ann. Sci. Éc. Norm. Supér.* (4) **49**:2 (2016), 371–397. MR Zbl

[Colliot-Thélène and Pirutka 2016b] J.-L. Colliot-Thélène and A. Pirutka, "Cyclic covers that are not stably rational", *Izv. Ross. Akad. Nauk Ser. Mat.* **80**:4 (2016), 35–48. In Russian; translated in *Izv. Math.* **80**:4 (2016), 665–677. MR Zbl

[Hassett et al. 2016a] B. Hassett, A. Kresch, and Y. Tschinkel, "Stable rationality and conic bundles", *Math. Ann.* **365**:3-4 (2016), 1201–1217. MR Zbl

[Hassett et al. 2016b] B. Hassett, A. Pirutka, and Y. Tschinkel, "Stable rationality of quadric surface bundles over surfaces", preprint, 2016. To appear in *Acta Mathematica*. arXiv

[Hassett et al. 2016c] B. Hassett, A. Pirutka, and Y. Tschinkel, "A very general quartic double fourfold is not stably rational", preprint, 2016. To appear in *Algebraic Geometry*. arXiv

[Hassett et al. 2017] B. Hassett, A. Pirutka, and Y. Tschinkel, "Intersections of three quadrics in $\mathbb{P}^7$", preprint, 2017. arXiv

[Iliev et al. 2017] A. Iliev, G. Kapustka, M. Kapustka, and K. Ranestad, “Hyperkähler fourfolds and Kummer surfaces”, preprint, 2017. Zbl arXiv

[Kahn et al. 1998] B. Kahn, M. Rost, and R. Sujatha, “Unramified cohomology of quadrics, I”, *Amer. J. Math.* **120**:4 (1998), 841–891. MR Zbl

[Lam 1973] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin, 1973. MR Zbl

[Merkurjev 2008] A. Merkurjev, “Unramified elements in cycle modules”, *J. Lond. Math. Soc.* (2) **78**:1 (2008), 51–64. MR Zbl

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University, 1980. MR Zbl

[Okada 2016] T. Okada, “Stable rationality of cyclic covers of projective spaces”, preprint, 2016. arXiv

[Pirutka 2016] A. Pirutka, “Varieties that are not stably rational, zero-cycles and unramified cohomology”, preprint, 2016. To appear in *Proceedings of the AMS Algebraic Geometry Summer Institute* (Salt Lake City, 2015). arXiv

[Schreieder 2018] S. Schreieder, “On the rationality problem for quadric bundles”, preprint, 2018. arXiv

[Totaro 2016] B. Totaro, “Hypersurfaces that are not stably rational”, *J. Amer. Math. Soc.* **29**:3 (2016), 883–891. MR Zbl

[Voisin 2015] C. Voisin, “Unirational threefolds with no universal codimension 2 cycle”, *Invent. Math.* **201**:1 (2015), 207–237. MR Zbl

Communicated by Jean-Louis Colliot-Thélène
Received 2017-06-24 Revised 2017-11-08 Accepted 2017-12-18

schreieder@math.lmu.de *Mathematisches Institut, Ludwig-Maximilians-Universität München, München, Germany*

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2018 is US $340/year for the electronic version, and $535/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 12 No. 2 2018