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In a seminal work Lyubeznik [1997] introduces a category F-finite modules in order to show various finiteness results of local cohomology modules of a regular ring R in positive characteristic. The key notion on which most of his arguments rely is that of a *generator* of an F-finite module. This may be viewed as an R finitely generated representative for the generally nonfinitely generated local cohomology modules. In this paper we show that there is a functorial way to choose such an R-finitely generated representative, called the *minimal root*, thereby answering a question that was left open in Lyubeznik's work. Indeed, we give an equivalence of categories between F-finite modules and a category of certain R-finitely generated modules with a certain Frobenius operation which we call minimal γ-sheaves.

As immediate applications we obtain a globalization result for the parameter test module of tight closure theory and a new interpretation of the generalized test ideals of Hara and Takagi [2004] which allows us to easily recover the rationality and discreteness results for F-thresholds of Blickle et al. [2008].

1. Introduction

Let R be a regular ring of positive characteristic $p > 0$. We denote by $\sigma : R \to R$ the Frobenius map which sends $r \in R$ to its p-th power r^p. We assume that R is F-finite, which means that the Frobenius map σ is a finite morphism. In order to show various finiteness results for the generally nonfinitely generated local cohomology modules $H^i_J(R)$, with J being some ideal of R, Lyubeznik [1997] observes that if one enlarges the ring R by adjoining a new, noncommutative variable representing the Frobenius morphism σ then all local cohomology modules (on which the Frobenius acts in a natural way) are finitely generated over the resulting ring

$$R[\sigma] = \frac{R\{\sigma\}}{\langle \sigma r - r^p\sigma | r \in R\rangle}\,.$$

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Note that the datum of a module over this ring is the same as giving an R-module $\mathcal{M}$ together with a p-linear map $\sigma_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ representing the action of σ. By adjointness of restriction and extension, the p-linear map $\sigma_{\mathcal{M}}$ is equivalent to an R-linear map $\theta : \sigma^*\mathcal{M} \stackrel{\text{def}}{=} \mathcal{M} \otimes_\sigma R \to \mathcal{M}$. This leads to the key definition:

Definition 1.1. A finitely generated $R[\sigma]$-module $\mathcal{M}$ is called a *unit* module if the structural morphism $\theta : \sigma^*\mathcal{M} \to \mathcal{M}$ is an isomorphism.

Lyubeznik [1997] shows many good properties of the category of finitely generated unit $R[\sigma]$-modules.[1] Most notably, they form an abelian category in which every object has finite length. By observing that local cohomology modules are finitely generated unit $R[\sigma]$-modules he is able to conclude many strong finiteness results about them. A systematic study of a globalized version of finitely generated unit modules is undertaken by Emerton and Kisin [2004]. There it is shown that the category of locally finitely generated $R[\sigma]$-modules is (derived) equivalent to the category of étale sheaves of $\mathbb{F}_p$-vector spaces on Spec R.

A prominent role in Lyubeznik's as well as in Emerton and Kisin's study of finitely generated unit $R[\sigma]$-modules has the concept of a *generator* or *root*. A *root* of a finitely generated unit $R[\sigma]$-module $\mathcal{M}$ is a finitely generated R-submodule $M \subseteq \mathcal{M}$ such that the structural map $\theta_{\mathcal{M}}$ induces an inclusion $\gamma : M \subseteq \sigma^*M$ and that $\bigcup \sigma^{e*}M = \mathcal{M}$. Hence one may view the root $\gamma : M \to \sigma^*M$ as a coherent representative of the finitely generated unit $R[\sigma]$-module $\mathcal{M}$. Generalizing slightly we define:

Definition 1.2. A finitely generated R-module M together with an R-linear morphism $\gamma : M \to \sigma^*M$ is called a γ*-module*, or γ-sheaf.

The Frobenius iterates of the map γ form a directed system

$$M \xrightarrow{\gamma} \sigma^*M \xrightarrow{\sigma^*\gamma} \sigma^{2*}M \xrightarrow{\sigma^{2*}\gamma} \dots ,$$

the limit of which we denote by $\mathsf{Gen}M$. One checks easily that γ induces a map $\mathsf{Gen}M \to \sigma^*\mathsf{Gen}M$ which is an isomorphism. By inverting this isomorphism, $\mathsf{Gen}M$ becomes a unit $R[\sigma]$-module. It is shown in [Emerton and Kisin 2004] that a finitely generated unit $R[\sigma]$-module $\mathcal{M}$ is precisely a module which is isomorphic to $\mathsf{Gen}M$ for some γ-module (M, γ).

Of course, different γ-modules may generate isomorphic unit $R[\sigma]$-modules. The obvious question whether there is a unique minimal (in an appropriate sense, see Definition 2.7) γ-module that generates a given unit $R[\sigma]$-module has remained open for a long time. In the case that R is complete, a positive answer was given already in [Lyubeznik 1997, Theorem 3.5]. In [Blickle 2004] this is extended to

[1] In [Lyubeznik 1997] this category is called F-finite modules. We follow here the notation of Blickle and Böckle [$\geq$ 2008] which in turn is taken from the monograph [Emerton and Kisin 2004].

the case that R is local (at least if R is F-finite). The purpose of this article is to show this in general, that is, for any F-finite regular ring R (see Theorem 2.24). A notable point in the proof is that it does not rely on the hard finiteness result [Lyubeznik 1997, Theorem 4.2], but only on the (easier) local case of it which is in some sense proven here *en passant* (see Remark 2.14). Our main result is the following.

Theorem 2.27. *Let R be a regular ring of positive characteristic $p > 0$ such that the Frobenius map is finite. Let $\mathcal{M}$ be a finitely generated unit $R[\sigma]$-module, then there is a unique minimal γ-sheaf M such that* $\mathsf{Gen} M \cong \mathcal{M}$.

Moreover, the functor Gen *induces an equivalence of abelian categories between finitely generated unit $R[\sigma]$-modules and minimal γ-sheaves.*

Our approach to prove this result is not the most direct one imaginable since we essentially develop a theory of minimal γ-sheaves from scratch (Section 2). The benefit is that after this is established, the result on the existence of minimal roots naturally appears as a byproduct. For this reason, it is important to isolate the key point in the argument: For a fixed coherent γ-sheaf M, the order of nilpotency of quotients of M is universally bounded. This is the statement of Proposition 2.11 (local case) and of the main result Theorem 2.22 (general case). The proof of this comes down to checking that decreasing sequences of γ-subsheaves of a fixed coherent γ-sheaf are eventually constant. This is achieved using the Chevalley lemma in the local case, or, via duality, by invoking a key result of Hartshorne and Speiser [1977]. The main difficulty (and the achievement in this paper) however lies in reducing the general case to the local case.

The quite explicit nature of our proof allows us to draw a series of interesting consequences. In particular, the connection to generalized test ideals [Hara and Takagi 2004] which appeared in computing the simplest examples of minimal gamma sheaves is quite surprising at first. In Section 3 we also give some applications of the result on the existence of minimal γ-sheaves. First, we show that the category of minimal γ-sheaves is equivalent to the category γ-crystals of Blickle and Böckle [$\geq$ 2008]. We show that a notion from tight closure theory, namely the parameter test module, is a global object (Proposition 3.3). Statements of this type are notoriously hard in the theory of tight closure, particularly in the light of recent evidence that localization for tight closure might fail in general. Furthermore, we give a concrete description of minimal γ-sheaves in a very simple case (Proposition 3.5), relating it to the generalized test ideals studied in [Blickle et al. 2008]. This viewpoint also recovers (and slightly generalizes, with new proofs) the main results on the rationality and discreteness of jumping numbers of Blickle et al. [2008] and the results on generators of certain D-modules of Alvarez-Montaner et al. [2005]. A similar generalization, however using slightly different (but related, see

Remark 2.12) methods, was recently obtained independently by Katzman et al. [2007].

We are pleased to recently have learned that Carl Miller obtained the existence of minimal roots in the case of dimension 1 independently in his PhD thesis [Miller 2007]. He uses the existence of minimal roots in this case as the main tool to obtain a lower bound for the Euler characteristic of a p-torsion étale sheaf on a smooth characteristic p curve, thereby answering a question of Pink (who considered the case where there is no higher wild ramification). The link between the p-torsion sheaf and the coherent minimal γ-sheaf is obtained via the Riemann–Hilbert type correspondence of Emerton and Kisin [2004]. I expect that the results in the present paper could be of great use for similar applications to the study of p-torsion sheaves via coherent sheaves. For example, starting with a suitable p-torsion étale sheaf F, one can now uniquely associate a *coherent* γ-sheaf $\mathcal{M}$ with F, and one may, as in the work of Miller, use invariants of $\mathcal{M}$, such as its degree, as an invariant for F.

The ideas in this paper have two sources. Firstly, the ongoing project of Blickle and Böckle [$\geq$ 2008] lead to a systematic study of γ-sheaves (the notation γ-sheaf is chosen to remind of the notion of a **g**enerator introduced in [Lyubeznik 1997]). Secondly, insight gained from the D-module theoretic viewpoint on generalized test ideals developed in [Blickle et al. 2008] jointly with Mircea Mustaţă and Karen Smith leads to the observation that these techniques can be successfully applied to study γ-sheaves.

Notation. Throughout we fix a *regular* scheme X over a field $k \supseteq \mathbb{F}_q$ of characteristic $p > 0$ (with $q = p^e$ fixed). We further assume that X is F-finite, that is, the Frobenius morphism $\sigma : X \to X$, which is given by sending $f \in \mathcal{O}_X$ to f^q, is a finite morphism.[2] In general, σ is affine. This allows to reduce in many arguments below to the case that X itself is affine and I will do so if convenient. We will use without further mention that because X is regular, the Frobenius morphism $\sigma : X \to X$ is flat such that σ^* is an exact functor (see [Kunz 1969]).

2. Minimal γ–sheaves

We begin with recalling the notion of γ-sheaves and nilpotence.

Definition 2.1. A *γ-sheaf* on X is a pair (M, γ_M) consisting of a quasicoherent $\mathcal{O}_X$-module M and a $\mathcal{O}_X$-linear map $\gamma : M \to \sigma^* M$. A γ-sheaf is called *coherent* if its underlying sheaf of $\mathcal{O}_X$-modules is coherent.

[2]It should be possible to replace the assumption of F-finiteness to saying that if X is a k-scheme with k a field such that the relative Frobenius $\sigma_{X/k}$ is finite. This would extend the results given here to desirable situations such as X of finite type over a field k with $[k : k^q] = \infty$. The interested reader should have no trouble to adjust our treatment to this case.

A γ-sheaf (M, γ) is called *nilpotent* (of order n) if

$$\gamma^n \overset{\text{def}}{=} \sigma^{n*}\gamma \circ \sigma^{(n-1)*}\gamma \circ \ldots \circ \sigma^*\gamma \circ \gamma = 0$$

for some $n > 0$. A γ-sheaf is called *locally nilpotent* if it is the union of nilpotent γ subsheaves.

Maps of γ-sheaves are maps of the underlying $\mathcal{O}_X$-modules such that the obvious diagram commutes. We denote the category of coherent γ-sheaves on X by $\mathbf{Coh}_\gamma(X)$. The following proposition summarizes some properties of γ-sheaves; for proofs and more details see [Blickle and Böckle $\geq$ 2008].

Proposition 2.2. (a) *The set of γ-sheaves forms an abelian category.*

(b) *The coherent, nilpotent and locally nilpotent γ-sheaves are abelian subcategories, also closed under extension.*

Proof. The point in the first statement is that the $\mathcal{O}_X$-module kernel, cokernel and extension of (maps of) γ-sheaves naturally carry the structure of a γ-sheaf. This is really easy to verify so we only give the construction of the γ-structure on the kernel as an illustration. Recall that we assume that X is regular such that σ is flat, hence σ^* is an exact functor. A morphism $\varphi : M \to N$ of γ-sheaves is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\gamma_M"'] & N \arrow[d, "\gamma_N"] \\ \sigma^*M \arrow[r, "\sigma^*\varphi"] & \sigma^*N \end{tikzcd}$$

from which we obtain the induced map $\ker\varphi \to \ker(\sigma^*\varphi)$. Since σ^* is exact, the natural map $\sigma^*(\ker\varphi) \to \ker(\sigma^*\varphi)$ is an isomorphism. Hence the composition

$$\ker\varphi \to \ker(\sigma^*\varphi) \xrightarrow{\cong} \sigma^*(\ker\varphi)$$

equips $\ker\varphi$ with a natural structure of a γ-sheaf.

The second part of Proposition 2.2 is also easy to verify so we leave it to the reader (see the proof of Lemma 2.3 below). □

Lemma 2.3. *A morphism $\varphi : M \to N$ of γ-sheaves is called* nil-injective (*respectively*, nil-surjective, nil-isomorphism) *if its kernel* (*respectively*, *cokernel*, *both*) *is locally nilpotent.*

(a) *If M* (*respectively*, *N*) *is coherent and φ is* nil-injective (*respectively*, nil-surjective) *then* $\ker\varphi$ (*respectively*, $\operatorname{coker}\varphi$) *is nilpotent.*

(b) *Kernel and cokernel of φ are nilpotent* (*of order n and m respectively*) *if and only if there is, for some $k \geq 0$ ($k = n + m$), a map $\psi : N \to \sigma^{k*}M$ such that $\gamma_M^k = \psi \circ \varphi$ and $\gamma_N^k = \sigma^{k*}(\varphi) \circ \psi$.*

(c) *If N is nilpotent of order $\leq n$ (that is, $\gamma_N^n = 0$) and $N' \subseteq N$ contains the kernel of γ_N^i for $1 \leq i \leq n$, then N/N' is nilpotent of order $\leq n - i$.*

Proof. The first statement is clear since X is noetherian. For the second statement consider the diagram obtained from the exact sequence $0 \to K \to M \to N \to C \to 0$:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] \arrow[dl, "\psi"'] \arrow[ddl, dotted] & C \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \sigma^{n*}K \arrow[r] \arrow[d, "0"] & \sigma^{n*}M \arrow[r] \arrow[d] & \sigma^{n*}N \arrow[r] \arrow[d] & \sigma^{n*}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{(n+m)*}K \arrow[r] & \sigma^{(n+m)*}M \arrow[r] & \sigma^{(n+m)*}N \arrow[r] & \sigma^{(n+m)*}C \arrow[r] & 0\,.
\end{tikzcd}
$$

If there is a ψ as indicated, then clearly the leftmost and rightmost vertical arrows of the first row are zero, that is, K and C are nilpotent. Conversely, let $K = \ker \varphi$ be nilpotent of degree n and $C = \operatorname{coker} \varphi$ be nilpotent of degree m. Then the top right vertical arrow and the bottom left vertical arrow are zero. The fact that the top right arrow is zero allows to define ψ as follows: Take $n \in N$, map it down to $\sigma^{n*}N$. Since its image to the right is zero, take any preimage from the left and map that element down in the diagram to $\sigma^{(n+m)*}M$. This procedure defines $\psi(n)$. To show that it is well defined and to see that the two relevant triangles commute is not difficult by using that the bottom left vertical arrow is zero.

For the last part, consider the short exact sequence $0 \to N' \to N \to N/N' \to 0$ and the diagram one obtains by considering $\sigma^{(n-i)*}$ and σ^{n*} of this sequence:

$$
\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\gamma^{n-i}"] & N/N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{(n-i)*}N' \arrow[r] \arrow[d, "0"] & \sigma^{(n-i)*}N \arrow[r] \arrow[d, "\sigma^{(n-i)*}\gamma^i"] & \sigma^{(n-i)*}(N/N') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^{n*}N' \arrow[r] & \sigma^{n*}N \arrow[r] & \sigma^{n*}(N/N') \arrow[r] & 0\,.
\end{tikzcd}
$$

The composition of the middle vertical map is γ_N^n which is zero by assumption. To conclude that the top right vertical arrow is zero one uses the fact that

$$\sigma^{(n-i)^*}N' \supseteq \sigma^{(n-i)*} \ker \gamma^i = \ker\bigl(\sigma^{(n-i)^*}\gamma^i\bigr).$$

With this it is an easy diagram chase to conclude that the top right vertical map is zero. $\square$

Lemma 2.4. *Let $M \xrightarrow{\varphi} N$ be a map of γ-sheaves. Let $N' \subseteq N$ be such that N/N' is nilpotent (hence $N' \subseteq N$ is a nil-isomorphism). Then $M/(\varphi^{-1}N')$ is also nilpotent.*

Proof. $M/(\varphi^{-1}N')$ injects into N/N'. Since the latter is nilpotent, so is the former. □

If (M,γ) is a γ-sheaf, then σ^*M is naturally a γ-sheaf with structural map $\sigma^*\gamma$. Furthermore, the map $\gamma : M \to \sigma^*M$ is then a map of γ-sheaves which is a *nil-isomorphism*, that is, kernel and cokernel are nilpotent. We can iterate this process to obtain a directed system

$$M \xrightarrow{\gamma} \sigma^*M \xrightarrow{\sigma^*\gamma} \sigma^{2*}M \xrightarrow{\sigma^{2*}\gamma} \dots$$

whose limit we denote by $\mathsf{Gen}M$. Clearly $\mathsf{Gen}M$ is a γ-sheaf whose structural map $\gamma_{\mathsf{Gen}M}$ is injective. In fact, it is an isomorphism since clearly $\sigma^*\mathsf{Gen}M \cong \mathsf{Gen}M$. One observes that $\mathsf{Gen}M = 0$ if and only if M is locally nilpotent. Note that even if M is coherent, $\mathsf{Gen}M$ is generally not coherent. Furthermore, let $\overline{M}$ be the image of M under the natural map $M \to \mathsf{Gen}M$. Then, if M is coherent, so is $\overline{M}$ and the map $M \twoheadrightarrow \overline{M}$ is a nil-isomorphism. Since $\overline{M}$ is a γ-submodule of $\mathsf{Gen}M$ whose structural map is injective, the structural map $\overline{\gamma}$ of $\overline{M}$ is injective as well.

Proposition 2.5. *The operation that assigns to each γ-sheaf M its image $\overline{M}$ in $\mathsf{Gen}M$ is an end-exact functor (preserves exactness only at the end of sequences) from $\mathbf{Coh}_\gamma(X)$ to $\mathbf{Coh}_\gamma(X)$. The kernel*

$$M^\circ = \bigcup \ker \gamma_M^i$$

of the natural map $M \to \overline{M}$ is the maximal locally nilpotent subsheaf of M.

Proof. The verification of the statement about M° is left to the reader. One has a functorial map between the exact functors $\mathrm{id} \to \mathsf{Gen}$. An easy diagram chase shows that the image of such a functorial map is an end-exact functor (see for example [Katz 1996, 2.17 Appendix 1]). In the concrete situation we are in, one can also verify this directly: right exactness is clear since $\overline{M}$ is a quotient of M. On the other hand, if $M \subseteq N$ is a γ-subsheaf, then $N^\circ \cap M \subseteq M^\circ$ since the former is clearly locally nilpotent. It follows that $\overline{M} \subseteq \overline{N}$. □

Such γ-submodules with injective structural map enjoy a certain minimality property with respect to nilpotent subsheaves:

Lemma 2.6. *Let (M,γ) be a γ-sheaf. The structural map γ_M is injective if and only if M does not have a nontrivial nilpotent subsheaf.*

Proof. Assume that the structural map of M is injective. This implies that the structural map of any γ-subsheaf of M is injective. But a γ-sheaf with injective structural map is nilpotent if and only it is zero.

Conversely, $\ker \gamma_M$ is a nil-potent subsheaf of M. If γ_M is not injective it is nontrivial. □

2.1. *Definition of minimal γ–sheaves.*

Definition 2.7. A coherent γ-sheaf M is called *minimal* if the following two conditions hold:

(a) M does not have nontrivial nilpotent subsheaves;

(b) M does not have nontrivial nilpotent quotients.

We denote by $\mathbf{Min}_\gamma(X)$ the subcategory of all γ-sheaves consisting of the minimal ones.

A simple consequence of the definition is

Lemma 2.8. *Let M be a γ-sheaf. If M satisfies (a) then any γ-subsheaf of M also satisfies (a). If M satisfies (b), so does any quotient.*

Proof. Immediate from the definition. □

As the preceding Lemma 2.6 shows, (a) is equivalent to the condition that the structural map γ_M is injective. We give a concrete description of the second condition.

Proposition 2.9. *For a coherent γ-sheaf M, the following conditions are equivalent.*

(a) *M does not have nontrivial nilpotent quotients.*

(b) *For any map of γ-sheaves $\varphi : N \to M$, if $\gamma_M(M) \subseteq (\sigma^*\varphi)(\sigma^*N)$ (as subsets of σ^*M) then φ is surjective.*

Proof. I begin with showing the easy direction that (a) implies (b): Note that the condition $\gamma_M(M) \subseteq (\sigma^*\varphi)(\sigma^*N)$ in (b) precisely says that the induced structural map on the cokernel of $N \to M$ is the zero map, thus in particular $M/\varphi(N)$ is a nilpotent quotient of M. By assumption on M, $M/\varphi(N) = 0$ and hence $\varphi(N) = M$.

Let $\pi : M \twoheadrightarrow C$ be such that C is nilpotent. Let $N \subseteq M$ be its kernel. We have to show that $N = M$. The proof is by induction on the order of nilpotency of C (simultaneously for all C). If $C = M/N$ is nilpotent of order 1 this means precisely that $\gamma(M) \subseteq \sigma^*N$; hence by (b) we have $N = M$ as claimed. Now let N be such that the nilpotency order of $C \overset{\text{def}}{=} M/N$ is equal to $n \geq 2$. Consider the γ-submodule $N' = \pi^{-1}(\ker \gamma_C)$ of M. This N' clearly contains N and we have that $M/N' \cong C/(\ker \gamma_C)$. By the previous Lemma 2.3 we conclude that the nilpotency order of M/N' is $\leq n-1$. Thus by induction $N' = M$. Hence

$$M/N = N'/N \cong \ker \gamma_C$$

is of nilpotency order 1. Again by the base case of the induction we conclude that $M = N$. □

By replacing N by its image $\varphi(N)$ in M in item (b) of Proposition 2.9 it follows that it would be enough to consider such φ which are injective.

Corollary 2.10. *A coherent γ-sheaf M is minimal if and only if the following two conditions hold.*

(a) *The structural map of M is injective.*

(b) *If $N \subseteq M$ is a subsheaf such that $\gamma(M) \subseteq \sigma^* N$ then $N = M$.*

The conditions in this corollary are essentially the definition of a *minimal root* of a finitely generated unit $R[\sigma]$-module in [Lyubeznik 1997]. The finitely generated unit $R[\sigma]$-module generated by (M, γ) is of course $\mathrm{Gen} M$. Lyubeznik shows in the case that R is a complete regular ring, that minimal roots exist. In [Blickle 2004, Theorem 2.10] I showed how to reduce the local case to the complete case if R is F-finite. For convenience we give a streamlined argument of the result in the local case in the language of γ-sheaves.

2.2. *Minimal γ–sheaves over local rings.* The difficult part in establishing the existence of a minimal root is to satisfy condition (b) of Definition 2.7. The point is to universally bound the order of nilpotency of any nilpotent quotient of a fixed γ-sheaf M.

Proposition 2.11. *Let $(R, \mathfrak{m})$ be regular, local and F-finite. Let M be a coherent γ-sheaf and $\{N_i\}_{i \in I}$ be a collection of γ-subsheaves which is closed under finite intersections and such that M/N_i is nilpotent for all i. Then $M/\bigcap N_i$ is nilpotent.*

Proof. Note that if N and N' are γ-subsheaves of M such that M/N and M/N' are nilpotent of order n and n', then $M/(N \cap N')$ is nilpotent of order $\max\{n, n'\}$. Hence, with $\{N_i\}$ the family of *all* $N \subseteq M$ such that M/N is nilpotent, Proposition 2.11 may be equivalently stated:

The order of nilpotency of any nilpotent quotient of M is universally bounded.

By faithful flatness of completion (together with the fact that completion commutes with Frobenius), *order of nilpotency* of quotients of M is preserved by completion. Therefore we may reduce to the case that R is complete.

Let us hence assume that R is complete, local, regular and F-finite. Since R is via σ a free R-module of finite rank, σ^* is nothing but tensorisation with a free R-module of finite rank. Such an operation commutes with the formation of inverse limits such that $\sigma^* \bigcap N_i = \bigcap(\sigma^* N_i)$ and hence $\bigcap N_i$ is a γ-subsheaf of M. Clearly we may replace M by $M/\bigcap N_i$ such that we have $\bigcap N_i = 0$. We may further replace M by its image $\overline{M}$ in $\mathrm{Gen} M$. Thus we may assume that M has injective structural map $\gamma : M \subseteq \sigma^* M$. We have to show that $M = 0$.

By the Artin–Rees Lemma (applied to $M \subseteq \sigma^*M$) there exists $t \geq 0$ such that for all $s > t$,

$$M \cap \mathfrak{m}^s\sigma^*M \subseteq \mathfrak{m}^{s-t}(M \cap \mathfrak{m}^t\sigma^*M) \subseteq \mathfrak{m}^{s-t}M\,.$$

By Chevalley's Theorem in the version of [Lyubeznik 1997, Lemma 3.3], for some $s \gg 0$ (in fact $s \geq t+1$ will suffice) we find N_i with $N_i \subseteq \mathfrak{m}^s M$. Possibly increasing s we may assume that $N_i \not\subseteq \mathfrak{m}^{s+1}M$ (unless, of course $N_i = 0$ in which case $M/N_i = M$ is nilpotent $\Rightarrow M = 0$ since γ_M is injective, and we are done). Combining these inclusions we get

$$\begin{aligned} N_i &\subseteq \sigma^*N_i \cap M \subseteq \sigma^*(\mathfrak{m}^s M) \cap M \\ &\subseteq (\mathfrak{m}^s)^{[q]}\sigma^*M \cap M \subseteq \mathfrak{m}^{sq}\sigma^*M \cap M \\ &\subseteq \mathfrak{m}^{sq-t}M\,. \end{aligned}$$

But since $sq-t \geq s+1$ for our choice of $s \geq t+1$ this is a contradiction (to the assumption $N_i \neq 0$) and the result follows. ☐

Remark 2.12. An alternative way to prove this result is to use Matlis duality and then invoke a result of Hartshorne and Speiser [1977, Proposition 1.11]. Their result states that if U is a cofinite $R[\sigma]$-module then the subset

$$U_{\mathrm{nil}} = \{\, u \in U \,|\, \sigma^n(u) = 0 \text{ for some } n \,\}$$

is annihilated by a fixed power of σ, that is, there is $k \geq 1$ such that $\sigma^k(U_{\mathrm{nil}}) = 0$. If one applies this to the Matlis dual $U = M^\vee$ of M and the union of its σ-nilpotent submodules $(M/N_i)^\vee$ in the above statement, an alternative proof is obtained. This approach via the Hartshorne–Speiser result is used in [Katzman et al. 2007] to study F-thresholds and hence appears to be directly related to the observations we make in Section 3.3 below.

Corollary 2.13. *Let R be regular, local and F-finite and M a coherent γ-sheaf. Then M has a nil-isomorphic subsheaf without nonzero nilpotent quotients* (*that is, satisfying (b) of the definition of minimality*). *In particular, M is nil-isomorphic to a minimal γ-sheaf.*

Proof. Let N_i be the collection of all nil-isomorphic subsheaves of M. Since M is coherent each M/N_i is indeed nilpotent, say of order $\leq n_i$. Since

$$M/(N_i \cap N_j) \subseteq M/N_i \oplus M/N_j\,,$$

it follows that $M/(N_i \cap N_j)$ is nilpotent of order $\leq \max\{n_i, n_j\}$. Hence the collection of nil-isomorphic subsheaves of M is closed under intersection which allows to apply Proposition 2.11 to conclude that $M/\bigcap N_i$ is nilpotent. Hence $N \stackrel{\text{def}}{=} \bigcap N_i$ is the unique smallest nil-isomorphic subsheaf of M. It is clear that N cannot have nonzero nilpotent quotients (since the kernel would be a strict subsheaf of N, nil-isomorphic to M, by Proposition 2.2 (b)).

By first replacing M by $\overline{M}$ we can also achieve that condition (a) of the definition of minimality holds. As condition (a) passes to subsheaves, the smallest nil-isomorphic subsheaf of $\overline{M}$ is the sought after minimal γ-sheaf which is nil-isomorphic to M. □

Remark 2.14. Essentially the same argument as in the proof of Proposition 2.11 shows the following: If R is local and M is a coherent γ-sheaf over R with injective structural map, then any descending chain of γ-submodules of M stabilizes. This was shown (with essentially the same argument) in [Lyubeznik 1997] and implies immediately that γ-sheaves with injective structural map satisfy DCC.

If one tries to reduce the general case of Corollary 2.13 (that is, R not local) to the local case just proven, one encounters the problem of having to deal with the behavior of the infinite intersection $\bigcap N_i$ under localization. This is a source of troubles I do not know how to deal with directly. The solution to this is to take a detour and realize this intersection in a fashion such that each term functorially depends on M and furthermore that this functorial construction commutes with localization. This is explained in the following section.

2.3. ***$D_X^{(1)}$–modules and Frobenius descent.*** Let D_X denote the sheaf of differential operators on X. This is a sheaf of rings on X which locally, on each affine subvariety Spec R, is described as

$$D_R = \bigcup_{i=0}^{\infty} D_R^{(i)}$$

where $D_R^{(i)}$ is the subset of $\operatorname{End}_{\mathbb{F}_q}(R)$ consisting of the operators which are linear over R^{q^i}, the subring of q^i-th powers of elements of R. In particular

$$D_R^{(0)} \cong R, \quad \text{and} \quad D_R^{(1)} = \operatorname{End}_{R^q}(R).$$

Clearly, R itself becomes naturally a left $D_R^{(i)}$-module. Now denote by $R^{(1)}$ the $D_R^{(1)}$-R-bimodule R which has this left $D_R^{(1)}$-module structure and the right R-module structure via Frobenius, that is, for $r \in R^{(1)}$ and $x \in R$ we have $r \cdot x = x^q r$. With this notation we may view $D_R^{(1)} = \operatorname{End}^{\mathrm{r}}_R(R^{(1)})$ as the right R-linear endomorphisms of $R^{(1)}$. Thus we have

$$\sigma^*(\_\_) = R^{(1)} \otimes_R \_\_ : \ R\text{-mod} \to D_R^{(1)}\text{-mod}\,,$$

which makes σ^* into an equivalence of categories from R-modules to $D_R^{(1)}$-modules (because, since σ is flat and R is F-finite, $R^{(1)}$ is a locally free right R-module of finite rank). Its inverse functor is given by

$$\sigma^{-1}(\_\_) = \mathrm{Hom}^{\mathrm{r}}_R(R^{(1)}, R) \otimes_{D^{(1)}_R} \_\_ : \ D^{(1)}_R\text{-mod} \to R\text{-mod}\,. \qquad (2\text{-}1)$$

For details see [Alvarez-Montaner et al. 2005, Section 2.2]. I want to point out that these constructions commute with localization at arbitrary multiplicative sets. Let S be a multiplicative set of R.[3] We have

$$\begin{aligned} S^{-1} D^{(1)}_R &= S^{-1}\,\mathrm{End}^{\mathrm{r}}_R(R^{(1)}) \\ &= \mathrm{End}^{\mathrm{r}}_{S^{-1}R}\big((S^{[q]})^{-1} R^{(1)}\big) = \mathrm{End}^{\mathrm{r}}_{S^{-1}R}\big((S^{-1}R)^{(1)}\big) \\ &= D^{(1)}_{S^{-1}R}\,. \end{aligned}$$

Furthermore we have for an $D^{(1)}_R$-module M:

$$\begin{aligned} S^{-1}(\sigma^{-1}M) &= S^{-1}\big(\mathrm{Hom}^{\mathrm{r}}_R(R^{(1)}, R) \otimes_{D^{(1)}_R} M\big) \\ &= S^{-1}\,\mathrm{Hom}^{\mathrm{r}}_R(R^{(1)}, R) \otimes_{S^{-1}D^{(1)}_R} S^{-1}M \\ &= \mathrm{Hom}^{\mathrm{r}}_{S^{-1}R}\big((S^{-1}R)^{(1)}, S^{-1}R\big) \otimes_{D^{(1)}_{S^{-1}R}} S^{-1}M \\ &= \sigma^{-1}(S^{-1}M)\,. \end{aligned}$$

These observations are summarized in the following proposition.

Proposition 2.15. *Let X be F-finite and regular. Let U be an open subset (more generally, U is locally given on* Spec R *as* Spec $S^{-1}R$ *for some (sheaf of) multiplicative sets on X). Then*

$$\big(D^{(1)}_X\big)\,\big|_U = D^{(1)}_U$$

and for any sheaf of $D^{(1)}_X$-modules M one has that

$$\begin{aligned} (\sigma^{-1}M)|_U &= \big(\mathrm{Hom}^{\mathrm{r}}\big(\mathcal{O}^{(1)}_X, \mathcal{O}_X\big) \otimes_{D^{(1)}_X} M\big)\,\big|_U \\ &\cong \mathrm{Hom}^{\mathrm{r}}\big(\mathcal{O}^{(1)}_U, \mathcal{O}_U\big) \otimes_{D^{(1)}_U} M|_U = \sigma^{-1}(M|_U) \end{aligned}$$

as $\mathcal{O}_U$-modules.

2.4. ***A criterion for minimality.*** The Frobenius descent functor σ^{-1} can be used to define an operation on γ-sheaves which assigns to a γ-sheaf M its smallest γ-subsheaf N with the property that M/N has the trivial $(= 0)$ γ-structure. This is the opposite of what the functor σ^* does: $\gamma : M \to \sigma^* M$ is a map of γ sheaves such that $\sigma^*M/\gamma(M)$ has trivial γ-structure.

We define the functor σ^{-1}_γ from γ-sheaves to γ-sheaves as follows. Let $M \xrightarrow{\gamma} \sigma^*M$ be a γ-sheaf. Then $\gamma(M)$ is an $\mathcal{O}_X$-submodule of the $D^{(1)}_X$-module σ^*M.

[3] Since $S^{-1}R = (S^{[q]})^{-1}R$ we may assume that $S \subseteq R^q$. This implies that S is in the center of $D^{(1)}_R$ such that localization in this noncommutative ring along S is harmless. With this I mean that we may view the localization of the left R-module $D^{(1)}_R$ at S^{-1} in fact as the localization of $D^{(1)}_R$ at the central multiplicative set $(S^{[q]})^{-1}$

Denote by $D_X^{(1)}\gamma(M)$ the $D_X^{(1)}$-submodule of σ^*M generated by $\gamma(M)$. To this inclusion of $D_X^{(1)}$-modules

$$D_X^{(1)}\gamma(M) \subseteq \sigma^*M\,,$$

we apply the Frobenius descent functor $\sigma^{-1} : D_X^{(1)}\text{-mod} \to \mathbb{O}_X\text{-mod}$ defined above in (2-1) and use that $\sigma^{-1} \circ \sigma^* = \mathrm{id}$ to define

$$\sigma_\gamma^{-1}M \stackrel{\text{def}}{=} \sigma^{-1}\big(D_X^{(1)}\gamma(M)\big) \subseteq \sigma^{-1}\sigma^*M = M\,.$$

In general one has

$$\sigma_\gamma^{-1}(\sigma^*M) = \sigma^{-1}D_X^{(1)}\sigma^*(\gamma)(\sigma^*M) = \gamma(M)$$

since $\sigma^*(\gamma)(\sigma^*M)$ already is a $D_X^{(1)}$-subsheaf of the $D_X^{(2)}$-module $\sigma^*(\sigma^*M) = \sigma^{2*}M$.

By construction,

$$\sigma_\gamma^{-1}M \subseteq M \xrightarrow{\gamma} \gamma(M) \subseteq D_X^{(1)}\gamma(M) = \sigma^*\sigma^{-1}D_X^{(1)}\gamma(M) = \sigma^*\sigma_\gamma^{-1}M$$

such that $\sigma_\gamma^{-1}M$ is a γ-subsheaf of M.

Furthermore, the quotient $M/\sigma_\gamma^{-1}M$ has zero structural map. One makes the following observation.

Lemma 2.16. *Let M be a γ-sheaf. Then $\sigma_\gamma^{-1}M$ is the smallest subsheaf N of M such that $\sigma^*N \supseteq \gamma(M)$.*

Proof. Clearly $\sigma^{-1}M$ satisfies this condition. Let N be as in the statement of the Lemma. Then σ^*N is a $D_X^{(1)}$-subsheaf of σ^*M containing $\gamma(M)$. Hence $D_X^{(1)}\gamma(M) \subseteq \sigma^*N$. Applying σ^{-1} we see that $\sigma^{-1}M \subseteq N$. □

Therefore, the result of the lemma could serve as an alternative definition of σ_γ^{-1} (one would have to show that the intersection of all such N has again the property that $\gamma(M) \subseteq \sigma^* \bigcap N$ but this follows since σ^* commutes with inverse limits since it is locally just tensorisation with a free module of finite rank). The following lemma is the key point in our reduction to the local case. It is an immediate consequence of Proposition 2.15. Nevertheless we include here a proof using only the characterization of Lemma 2.16. Hence one may avoid the appearance of $D^{(1)}$-modules in this paper altogether but I believe it to be important to explain where the ideas for the arguments originated, so $D^{(1)}$-modules are still there.

Lemma 2.17. *Let M be a γ-sheaf and let $S \subseteq \mathbb{O}_X$ be multiplicative set. Then $S^{-1}(\sigma_\gamma^{-1}M) = \sigma_\gamma^{-1}(S^{-1}M)$.*

Proof. This follows from Proposition 2.15. However, this can also be proven using only the characterization in Lemma 2.16. By this we have

$$\sigma^*(S^{-1}(\sigma_\gamma^{-1}M)) = S^{-1}(\sigma^*(\sigma_\gamma^{-1}M)) \supseteq S^{-1}\gamma(M) = \gamma(S^{-1}M)\,, \tag{2-2}$$

which implies that $\sigma_\gamma^{-1}(S^{-1}M) \subseteq S^{-1}(\sigma_\gamma^{-1}M)$ because $\sigma_\gamma^{-1}(S^{-1}M)$ is smallest (by Lemma 2.16) with respect to the inclusion shown in (2-2). To show the converse inclusion, we consider the localization map $\varphi : M \to S^{-1}M$ and $N \subseteq S^{-1}M$ is a submodule such that $\gamma(S^{-1}M) \subseteq \sigma^* N$. Consider the diagram

$$\begin{tikzcd} M/\varphi^{-1}N \arrow[r, hook] \arrow[d, "\gamma"] & S^{-1}M/N \arrow[d, "\gamma"] \\ \sigma^*M/\sigma^*(\varphi^{-1}N) \arrow[r, hook] & \sigma^*(S^{-1}M)/\sigma^*N \end{tikzcd}$$

of which the horizontal arrows are injections (using the exactness of σ^*). By assumption on N, the right vertical arrow is zero, hence also the left vertical arrow. This implies that $\gamma(M) \subseteq \sigma^*(\varphi^{-1}N)$. By the characterization of Lemma 2.16 one concludes that $\sigma_\gamma^{-1}M \subseteq \varphi^{-1}N$ and hence $S^{-1}\sigma_\gamma^{-1}M \subseteq N$. Applying this with $N = \sigma_\gamma^{-1}(S^{-1}M)$ our claim follows. $\square$

Proposition 2.18. *Let M be a γ-sheaf. Then $\sigma_\gamma^{-1}M = M$ if and only if M has no proper nilpotent quotient (that is, M satisfies condition (b) of the definition of minimality).*

If M is coherent, the condition on $x \in X$ that the inclusion $\sigma_\gamma^{-1}(M_x) \subseteq M_x$ is equality is an open condition on X.

Proof. One direction is clear since $M/\sigma_\gamma^{-1}M$ is a nilpotent quotient of M. We use the characterization in Proposition 2.9. For this let $N \subseteq M$ be such that $\gamma(M) \subseteq \sigma^*N$. As $\sigma_\gamma^{-1}M$ was the smallest subsheaf with this property we obtain $\sigma_\gamma^{-1}M \subseteq N \subseteq M$. Since $M = \sigma_\gamma^{-1}M$ by assumption it follows that $N = M$. Hence, by Proposition 2.9, M does not have nontrivial nilpotent quotients.

By Lemma 2.17, σ_γ^{-1} commutes with localization which means that $\sigma_\gamma^{-1}(M_x) = (\sigma_\gamma^{-1}M)_x$. Hence the second statement follows simply since both M and $\sigma_\gamma^{-1}M$ are coherent (and equality of two coherent modules via a given map is an open condition). $\square$

Lemma 2.19. *The assignment $M \mapsto \sigma_\gamma^{-1}M$ is an end-exact functor on γ-sheaves.*

Proof. Formation of the image of the functorial map $\mathrm{id} \xrightarrow{\gamma} \sigma^*$ of exact functors is end-exact (see for example [Katz 1996, 2.17 Appendix 1]). If M is a $D_X^{(1)}$-module and $A \subseteq B$ are $\mathcal{O}_X$-submodules of M then

$$D_X^{(1)}A \subseteq D_X^{(1)}B.$$

If $M \twoheadrightarrow N$ is a surjection of $D^{(1)}$-modules which induces a surjection on $\mathcal{O}_X$-submodules $A \twoheadrightarrow B$ then, clearly, $D_X^{(1)}A$ surjects onto $D_X^{(1)}B$. Now one concludes by observing that σ^{-1} is an exact functor. $\square$

Lemma 2.20. *Let $N \subseteq M$ be an inclusion of γ-sheaves such that $\sigma^{n*}N \supseteq \gamma^n(M)$ (that is, the quotient is nilpotent of order $\leq n$). Then*

$$\sigma^{(n-1)*}(N \cap \sigma_\gamma^{-1}M) \supseteq \gamma^{n-1}(\sigma_\gamma^{-1}M).$$

Proof. Consider the γ-subsheaf $M' = (\gamma^{n-1})^{-1}(\sigma^{(n-1)*}N)$ of M. One has

$$\sigma^* M' = (\sigma^*\gamma^{n-1})^{-1}(\sigma^{n*}N) \supseteq \gamma(M)$$

by the assumption that $\gamma^n(M) \subseteq \sigma^{n*}N$. Since $\sigma_\gamma^{-1}M$ is minimal with respect to this property we have $\sigma_\gamma^{-1}M \subseteq (\gamma^{n-1})^{-1}(\sigma^{(n-1)*}N)$. Applying γ^{n-1} we conclude that $\gamma^{n-1}(\sigma_\gamma^{-1}M) \subseteq \sigma^{(n-1)*}N$. Since $\sigma_\gamma^{-1}M$ is a γ-sheaf we have

$$\gamma^{n-1}(\sigma_\gamma^{-1}M) \subseteq \sigma^{(n-1)*}(\sigma_\gamma^{-1}M)$$

such that the claim follows. $\square$

2.5. *Existence of minimal γ–sheaves.* For a given γ-sheaf M we can iterate the functor σ_γ^{-1} to obtain a decreasing sequence of γ-subsheaves,

$$\ldots \subseteq M_3 \subseteq M_2 \subseteq M_1 \subseteq M(\xrightarrow{\gamma} \sigma^* M \to \ldots)$$

where $M_i = \sigma_\gamma^{-1}M_{i-1}$. Note that each inclusion $M_i \subseteq M_{i-1}$ is a nil-isomorphism.

Proposition 2.21. *Let M be a coherent γ-sheaf. Then the following conditions are equivalent.*

(a) *M has a nil-isomorphic γ-subsheaf $\underline{M}$ which does not have nontrivial nilpotent quotients (that is, $\underline{M}$ satisfies condition (b) in the definition of minimal γ-sheaf).*

(b) *M has a* unique *smallest nil-isomorphic subsheaf (equivalently, M has a (unique) maximal nilpotent quotient).*

(c) *For some $n \geq 0$, $M_n = M_{n+1}$.*

(d) *There is $n \geq 0$ such that for all $m \geq n$, $M_m = M_{m+1}$.*

Proof. (a) $\Rightarrow$ (b): Let $\underline{M} \subseteq M$ be the nil-isomorphic subsheaf of part (a) and let $N \subseteq M$ be another nil-isomorphic subsheaf of M. By Lemma 2.4 it follows that $\underline{M} \cap N$ is also nil-isomorphic to M. In particular $\underline{M}/(\underline{M} \cap N)$ is a nilpotent quotient of $\underline{M}$ and hence must be trivial. Thus $N \subseteq \underline{M}$ which shows that $\underline{M}$ is the smallest nil-isomorphic subsheaf of M.

(b) $\Rightarrow$ (c): Let N be this smallest subsheaf as in (b). Since each M_i is nil-isomorphic to M, it follows that $N \subseteq M_i$ for all i. Let n be the order of nilpotency of the quotient M/N, that is, $\gamma^n(M) \subseteq \sigma^{n*}N$. Repeated application ($n$ times) of Lemma 2.20 yields that $M_n \subseteq N$. Hence we get $N \subseteq M_{n+1} \subseteq M_n \subseteq N$ which implies that $M_{n+1} = M_n$.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a) is clear by Proposition 2.18. $\square$

This characterization enables us to show the existence of minimal γ-sheaves by reducing to the local case which we proved above.

Theorem 2.22. *Let M be a coherent γ-sheaf. There is a unique nil-isomorphic subsheaf $\underline{M}$ of M which does not have nontrivial nilpotent quotients.*

Remark 2.23. The following proof shows that in the notation of Proposition 2.21, $\underline{M} = M_n$ for $n \gg 0$.

Proof. By Proposition 2.21 it is enough to show that the sequence M_i is eventually constant. Let U_i be the subset of X consisting of all $x \in X$ on which

$$(M_i)_x = (M_{i+1})_x \ (= (\sigma_\gamma^{-1} M_i)_x).$$

By Proposition 2.18 U_i is an open subset of X (in this step I use the key observation Proposition 2.15) and $(M_i)|_{U_i} = (M_{i+1})|_{U_i}$. By the functorial construction of the M_i's the equalilty $M_i = M_{i+1}$ for one i implies equalities for all bigger i. It follows that the sets U_i form an increasing sequence of open subsets of X whose union is X itself by Corollary 2.13 and Proposition 2.21. Since X is noetherian, $X = U_i$ for some i. Hence $M_i = M_{i+1}$ so the claim follows by Proposition 2.21. $\square$

Theorem 2.24. *Let M be a coherent γ-sheaf. Then there is a functorial way to assign to M a* minimal *γ-sheaf $M_{\min}$ in the nil-isomorphism class of M.*

Proof. We may first replace M by the nil-isomorphic quotient $\overline{M}$ which satisfies condition (a) of Definition 2.7. Then replace $\overline{M}$ by its minimal nil-isomorphic submodule $\underline{(\overline{M})}$ which also satisfies condition (b) of Definition 2.7 (and condition (a) because (a) is passed to submodules). Thus the assignment

$$M \mapsto M_{\min} \stackrel{\text{def}}{=} \underline{(\overline{M})}$$

is a functor since it is a composition of the functors $M \mapsto \overline{M}$ and $M \mapsto \underline{M}$. $\square$

Proposition 2.25. *If $\varphi\colon M \to N$ is a nil-isomorphism, then $\varphi_{\min}\colon M_{\min} \to N_{\min}$ is an isomorphism.*

Proof. Clearly, $\varphi_{\min}$ is a nil-isomorphism. Since $\ker \varphi_{\min}$ is a nilpotent subsheaf of $M_{\min}$, we have by Definition 2.7 (a) that $\ker \varphi_{\min} = 0$. Since $\operatorname{coker} \varphi_{\min}$ is a nilpotent quotient of $N_{\min}$ it must be zero by Definition 2.7 (b). $\square$

Corollary 2.26. *Let $\mathcal{M}$ be a finitely generated unit $\mathcal{O}_X[\sigma]$-module. Then M has a unique minimal root in the sense of [Lyubeznik 1997].*

Proof. Let M be any root of $\mathcal{M}$, that is, M is a coherent γ-sheaf such that γ_M is injective and $\mathsf{Gen}\, M \cong \mathcal{M}$. Then $M_{\min} = \underline{M}$ is a minimal nil-isomorphic γ-subsheaf of M by Theorem 2.24. By Corollary 2.10 it follows that $M_{\min}$ is the sought after minimal root of $\mathcal{M}$. Uniqueness is clear since the intersection of any two roots is again a root. □

Note that the only assumption needed in this result is that X is F-finite and regular. In particular it does not rely on the finite-length result [Lyubeznik 1997, Theorem 3.2] which assumes that R is of finite type over a regular local ring (however, in [Lyubeznik 1997] F-finiteness is not assumed).

Theorem 2.27. *Let X be regular and F-finite. Then the functor*

$$\mathsf{Gen}\colon \mathbf{Min}_\gamma(X) \to \textit{finitely generated unit } \mathcal{O}_X[\sigma]\textit{-modules}$$

is an equivalence of categories.

Proof. The preceding corollary shows that Gen is essentially surjective. The induced map on Hom sets is injective since a map of minimal γ-sheaves f is zero if and only if its image is nilpotent (since minimal γ-sheaves do not have nilpotent submodules) which is the condition that $\mathsf{Gen}(f) = 0$. It is surjective since any map between $g : \mathsf{Gen}(M) \to \mathsf{Gen}(N)$ is obtained from a map of γ-sheaves $M \to \sigma^{e*} N$ for some $e \gg 0$. But this induces a map $M = M_{\min} \to (\sigma^{e*} N)_{\min} = N_{\min} = N$. □

3. Applications and Examples

In this section we discuss some further examples and applications of the results on minimal γ-sheaves we obtained so far.

3.1. ***γ–crystals.*** The purpose of this section is to quickly explain the relationship of minimal γ-sheaves to γ-crystals which were introduced in [Blickle and Böckle $\geq$ 2008]. The category of γ-crystals is obtained by inverting nil-isomorphisms in $\mathbf{Coh}_\gamma(X)$. In [Blickle and Böckle $\geq$ 2008] it is shown that the resulting category is abelian. One has a natural functor

$$\mathbf{Coh}_\gamma(X) \twoheadrightarrow \mathbf{Crys}_\gamma(X)$$

whose fibers we may think of consisting of nil-isomorphism classes of M. Note that the objects of $\mathbf{Crys}_\gamma(X)$ are the same as those of $\mathbf{Coh}_\gamma(X)$; however a morphism between γ-crystals $M \to N$ is represented by a left-fraction, that is, a diagram of γ-sheaves $M \Leftarrow M' \to M$ where the arrow $\Leftarrow$ is a nil-isomorphism.

On the other hand we just constructed the subcategory of minimal γ-sheaves $\mathbf{Min}_\gamma(X) \subseteq \mathbf{Coh}_\gamma(X)$ and showed that there is a functorial splitting $M \mapsto M_{\min}$ of this inclusion. An immediate consequence of Proposition 2.25 is that if M and N

are in the same nil-isomorphism class, then $M_{\min} \cong N_{\min}$. The verification of this may be reduced to considering the situation

$$M \Leftarrow M' \Rightarrow N$$

with both maps nil-isomorphisms in which case Proposition 2.25 shows that $M_{\min} \cong M'_{\min} \cong N_{\min}$. One has the following Proposition.

Proposition 3.1. *Let X be regular and F-finite. Then the composition*

$$\mathbf{Min}_\gamma(X) \hookrightarrow \mathbf{Coh}_\gamma(X) \twoheadrightarrow \mathbf{Crys}_\gamma(X)$$

is an equivalence of categories whose inverse is given by sending a γ-crystal represented by the γ-sheaf M to the minimal γ-sheaf $M_{\min}$.

Proof. The existence of $M_{\min}$ shows that $\mathbf{Min}_\gamma(X) \to \mathbf{Crys}_\gamma(X)$ is essentially surjective. It remains to show that $\mathrm{Hom}_{\mathbf{Min}_\gamma}(M, N) \cong \mathrm{Hom}_{\mathbf{Crys}_\gamma}(M, N)$. A map $\varphi\colon M \to N$ of minimal γ-sheaves is zero in $\mathbf{Crys}_\gamma$ if and only if $\operatorname{img}\varphi$ is nilpotent. But $\operatorname{img}\varphi$ is a subsheaf of the minimal γ-sheaf N, which by Definition 2.7 (a) has no nontrivial nilpotent subsheaves. Hence $\operatorname{img}\varphi = 0$ and therefore $\varphi = 0$. This shows that the map on Hom sets is injective. The surjectivity follows again by functoriality of $M \mapsto M_{\min}$. □

Corollary 3.2. *Let X be regular and F-finite. The category of minimal γ-sheaves $\mathbf{Min}_\gamma(X)$ is an abelian category. If $\varphi\colon M \to N$ is a morphism then*

$$\ker_{\min}\varphi = (\ker\varphi)_{\min} = \underline{\ker\varphi} \quad \textit{and} \quad \operatorname{coker}_{\min}\varphi = (\operatorname{coker}\varphi)_{\min} = \overline{\operatorname{coker}\varphi}.$$

Proof. Since $\mathbf{Min}_\gamma(X)$ is equivalent to $\mathbf{Crys}_\gamma(X)$ and since the latter is abelian, so is $\mathbf{Min}_\gamma(X)$. Since ker and coker in $\mathbf{Crys}_\gamma(X)$ are represented by the kernel and cokernel of the underlying coherent sheaf the statement about ker and coker in $\mathbf{Min}_\gamma(X)$ follows, where overline and underline are as defined in Proposition 2.5 and Proposition 2.21. □

3.2. *The parameter test module.* We give an application to the theory of tight closure. In [Blickle 2004, Proposition 4.5], it was shown that the parameter test module τ_{ω_A} is the unique minimal root of the intersection homology unit module $\mathscr{L} \subseteq H^{n-d}_I(R)$ if $A = R/I$ is the quotient of the regular local ring R (where $\dim R = n$ and $\dim A = d$). Locally, the parameter test module τ_{ω_A} is defined as the Matlis dual of

$$H^d_m(A)/0^*_{H^d_m(A)}$$

where $0^*_{H^d_m(A)}$ is the tight closure of zero in $H^d_m(A)$. The fact that we are now able to construct minimal γ-sheaves globally allows us to give a global candidate for the parameter test module.

Proposition 3.3. *Let $A = R/I$ be equi-dimensional of dimension d where R is regular and F-finite. Then there is a submodule*

$$L \subseteq \omega_A = \operatorname{Ext}^{n-d}(R/I, R)$$

such that for each $x \in \operatorname{Spec} A$ we have $L_x \cong \tau_{\omega_x}$.

Proof. Let $\mathscr{L} \subseteq H_I^{n-d}(R)$ be the unique smallest submodule of $H_I^{n-d}(R)$ which agrees with $H_I^{n-d}(R)$ on all smooth points of $\operatorname{Spec} A$. By [Blickle 2004, Theorem 4.1] $\mathscr{L}$ exists and is a unit $\mathcal{O}_X$-submodule of $H_I^{n-d}(R)$. Let L be a minimal generator of $\mathscr{L}$, that is, a coherent minimal γ-sheaf such that $\mathsf{Gen} L = \mathscr{L}$ which exists due to Theorem 2.22. Because of Proposition 2.15 it follows that L_x is also a minimal γ-sheaf and $\mathsf{Gen} L_x \cong \mathscr{L}_x$. But from [Blickle 2004, Proposition 4.5] we know that the unique minimal root of $\mathscr{L}_x$ is $\tau_{\omega_{A_x}}$, the parameter test module of A_x. It follows that $L_x \cong \tau_{\omega_{A_x}}$ by uniqueness. To see that $L \subseteq \operatorname{Ext}^{n-d}(R/I, R)$ we just observe that $\operatorname{Ext}^{n-d}(R/I, R)$ with the map induced by $\sigma^*(R/I) = R/I^{[q]} \to R/I$ is a γ-sheaf which generates $H_I^{n-d}(R)$. Furthermore, the map

$$\operatorname{Ext}^{n-d}(R/I, R) \to \sigma^* \operatorname{Ext}^{n-d}(R/I, R)$$

is injective since it is so locally and in this case the map is dual to the surjection $\sigma^* H^d_{\mathfrak{m}}(R/I) \to H^d_{\mathfrak{m}}(R/I)$ ($d = \dim R/I$) via local duality for the local ring $(R_{\mathfrak{m}}, \mathfrak{m})$. Hence by minimality of L we have the desired inclusion. □

3.3. ***Test ideals and minimal γ–sheaves.*** We consider now the simplest example of a γ-sheaf, namely that of a free rank one R-module M ($\cong R$). That means that via the identification $R \cong \sigma^* R$ the structural map

$$\gamma : M \cong R \xrightarrow{f\cdot} R \cong \sigma^* R \cong \sigma^* M$$

is given by multiplication with an element $f \in R$. It follows that γ^e is given by multiplication by $f^{1+q+\cdots+q^{e-1}}$ under the identification of $\sigma^{e*} R \cong R$. It is an easy exercise to observe that $\mathsf{Gen} M \cong R_f$ with its usual unit $R[F]$-structure.

We will show that the minimal γ-subsheaf of the just described γ-sheaf M can be expressed in terms of generalized test ideals. We recall from [Blickle et al. 2008, Lemma 2.1] that the test ideal of a principal ideal (f) of exponent $\alpha = \frac{m}{q^e}$ is given by

$$\tau(f^\alpha) = \text{the smallest ideal } J \text{ such that } f^m \in J^{[q^e]}.$$

By Lemma 2.2 of op. cit. $\tau(f^\alpha)$ can also be characterized as σ^{-e} of the $D^{(e)}$-module generated by f^m. We set as a shorthand

$$J_e = \tau\big(f^{(1+q+q^2+\cdots+q^{e-1})/q^e}\big)$$

and repeat the definition

$$J_e = \text{the smallest ideal } J \text{ of } R \text{ such that } f^{1+q+q^2+\cdots+q^{e-1}} \in J^{[q^e]}$$

and we set $J_0 = R$. Further recall from Section 2.5 that

$$M_e = \text{smallest ideal } I \text{ of } R \text{ such that } f \cdot M_{e-1} \subseteq I^{[q]}$$

with $M_0 = M$.

Lemma 3.4. *For all $e \geq 0$ one has $J_e = M_e$.*

Proof. The equality is true for $e = 0, 1$ by definition. We first show the inclusion $J_e \subseteq M_e$ by induction on e.

$$\begin{aligned} M_e^{[p^e]} &\supseteq (f \cdot M_{e-1})^{[q^{e-1}]} = \left(f^{q^{e-1}} M_{e-1}^{[q^{e-1}]}\right) \\ &= \left(f^{q^{e-1}} J_{e-1}^{[q^{e-1}]}\right) \supseteq f^{q^{e-1}} \cdot f^{1+q+q^2+\cdots+q^{e-2}} \\ &= f^{1+q+q^2+\cdots+q^{e-1}}, \end{aligned}$$

and since J_e is minimal with respect to this inclusion we have $J_e \subseteq M_e$.

Now we show for all $e \geq 1$ that $f \cdot J_{e-1} \subseteq J_e^{[q]}$. The definition of J_e implies that

$$f^{1+q+\cdots+q^{e-2}} \in (J^{[q^e]} \colon f^{q^{e-1}}) = (J^{[q]} \colon f)^{[q^{e-1}]}$$

which implies that $J_{e-1} \subseteq (J^{[q]} \colon f)$ by minimality of J_{e-1}. Hence $f \cdot J_{e-1} \subseteq J^{[q]}$. Now, we can show the inclusion $M_e \subseteq J_e$ by observing that by induction one has

$$J_e^{[q]} \supseteq f \cdot J_{e-1} \supseteq f \cdot M_{e-1}.$$

which implies by minimality of M_e that $M_e \subseteq J_e$. □

This shows that the minimal γ-sheaf $M_{\min}$, which is equal to M_e for $e \gg 0$ by Proposition 2.21, is just the test ideal $\tau\left(f^{(1+q+q^2+\cdots+q^{e-1})/q^e}\right)$ for $e \gg 0$. As a consequence we have:

Proposition 3.5. *Let M be the γ-sheaf given by*

$$R \xrightarrow{f\cdot} R \cong \sigma^* R.$$

Then $M_{\min} = \tau\left(f^{(1+q+q^2+\cdots+q^{e-1})/q^e}\right)$ for $q \gg 0$. In particular, $M_{\min} \supseteq \tau(f^{\frac{1}{q-1}})$ and the F-pure-threshold of $f \geq \frac{1}{q-1}$ if and only if M is minimal.

Proof. For $e \gg 0$ the increasing sequence of rational numbers $(1 + q + q^2 + \cdots + q^{e-1})/q^e$ approaches $\frac{1}{q-1}$. Hence

$$M_e = \tau\left(f^{(1+q+q^2+\cdots+q^{e-1})/q^e}\right) \supseteq \tau(f^{\frac{1}{q-1}})$$

for all e. If M is minimal, then all M_e are equal and hence the multiplier ideals $\tau(f^{\alpha})$ must be equal to R for all $\alpha \in [0, \frac{1}{q-1})$. In particular, the F-pure-threshold of $f \geq \frac{1}{q-1}$. Conversely, if the F-pure threshold is less than $\frac{1}{q-1}$, then for some e we must have that

$$\tau\big(f^{(1+q+q^2+\cdots+q^{e-1})/q^e}\big) \neq \tau\big(f^{(1+q+q^2+\cdots+q^e)/q^{e+1}}\big)$$

so $M_e \neq M_{e+1}$ which implies that $M \neq M_1$. So M is not minimal. □

Remark 3.6. After replacing f by f^r, this also shows that $\frac{r}{q-1}$ is not an accumulation point of F-thresholds of f for any f in an F-finite regular ring. In [Blickle et al. 2008] this was shown for R essentially of finite type over a local ring since our argument there depended on [Lyubeznik 1997, Theorem 4.2]. Even though D-modules appear in the present article, they only do so by habit of the author; as remarked before, they can easily be avoided.

Remark 3.7. Of course, for $r = q - 1$ this recovers (and slightly generalizes) the main result in [Alvarez-Montaner et al. 2005].

Remark 3.8. I expect that this descriptions of minimal roots can be extended to a more general setting using the modifications of generalized test ideals to modules as introduced in the preprint [Takagi and Takahashi 2007].

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