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Mass formulas for local Galois representations and quotient singularities II: Dualities and resolution of singularities

Melanie Matchett Wood and Takehiko Yasuda

A total mass is the weighted count of continuous homomorphisms from the absolute Galois group of a local field to a finite group. In the preceding paper, the authors observed that in a particular example two total masses coming from two different weightings are dual to each other. We discuss the problem of how generally such a duality holds and relate it to the existence of simultaneous resolution of singularities, using the wild McKay correspondence and the Poincaré duality for stringy invariants. We also exhibit several examples.

1. Introduction

In [Wood and Yasuda 2015] we found a close relationship between mass formulas for local Galois representations, studied in the number theory, and stringy invariants of singular varieties. In the present paper we discuss dualities of mass formulas in relation with the existence of some kinds of desingularization.

For a local field K and a finite group Γ, let $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ be the absolute Galois group of K and $S_{K,\Gamma}$ be the set of continuous homomorphisms $G_K \to \Gamma$. For a function $c : S_{K,\Gamma} \to \mathbb{R}$, the *total mass* of (K, Γ, c) is defined as

$$M(K, \Gamma, c) := \frac{1}{\sharp\Gamma} \sum_{\rho \in S_{K,\Gamma}} q^{-c(\rho)},$$

with q the cardinality of the residue field of K. For the symmetric group S_n, fixing the standard representation

$$\iota : S_n \hookrightarrow \mathrm{GL}_n(\mathcal{O}_K) \subset \mathrm{GL}_n(K),$$

with $\mathcal{O}_K$ the integer ring of K, we can associate the Artin conductor $\boldsymbol{a}_\iota(\rho)$ to each continuous homomorphism $\rho : G_K \to S_n$. According to Kedlaya [2007], Bhargava's

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mass formula [2007] for étale extensions of a local field is expressed as

$$M(K, S_n, \boldsymbol{a}_\iota) = \sum_{m=0}^{n-1} P(n, n-m) q^{-m},$$

where $P(n, n-m)$ is the number of partitions of the integer n into exactly $n-m$ parts. It was found in [Wood and Yasuda 2015] that for another function $\rho \mapsto \boldsymbol{w}_{2\iota}(\rho)$ originating in the wild McKay correspondence [Yasuda 2013] we have

$$M(K, S_n, -\boldsymbol{w}_{2\iota}) = \sum_{m=0}^{n-1} P(n, n-m) q^{m}.$$

Here 2ι stands for the direct sum of two copies of the representation ι. The right-hand sides of the two formulas are interchanged by replacing q with q^{-1}: this is what we call a *duality*. It is then natural to ask how generally the duality holds: what about other groups and other representations? There exists yet another function $\boldsymbol{v}_\tau$ on $S_{K,\Gamma}$ for each representation τ of Γ over $\mathcal{O}_K$. It was shown in [Wood and Yasuda 2015] that if τ is a permutation representation, then we have $\boldsymbol{a}_\tau = \boldsymbol{v}_{2\tau}$. It turns out that the function $\boldsymbol{v}_\tau$ is more appropriate when discussing dualities for nonpermutation representations. The most basic question we would like to ask is: *when are the total masses $M(K, \Gamma, \boldsymbol{v}_\tau)$ and $M(K, \Gamma, -\boldsymbol{w}_\tau)$ dual to each other in the same way as $M(K, S_n, \boldsymbol{a}_\iota)$ and $M(K, S_n, -\boldsymbol{w}_{2\iota})$*? We will observe that the duality does not always hold, but is closely related to the existence of a *simultaneous resolution of singularities* or to *equisingularities*.

To discuss the duality more rigorously, we need to consider total masses for all unramified extensions of the given local field K. For each integer $r > 0$, let K_r be the unramified extension of K of degree r. For a representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we can naturally generalize functions $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$ to ones on $S_{K_r,\Gamma}$, which we continue to denote by the same symbols. We then regard total masses $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$ as functions in the variable $r \in \mathbb{N}$. When they belong to a certain class of nice functions (which we call *admissible*), we can define their dual functions $\mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau))$ and $\mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau))$ in such a way that the dual of the function q^r is q^{-r}. A duality which we are interested in is now expressed by the equality

$$\mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)) = M(K_r, \Gamma, -\boldsymbol{w}_\tau). \tag{1-1}$$

We call it the *strong duality*.

It is the *wild McKay correspondence* which relates total masses to singularities. Given a representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we have the associated Γ-action on the affine space $\mathbb{A}^d_{\mathcal{O}_K}$ and the quotient scheme $X := \mathbb{A}^d_{\mathcal{O}_K}/\Gamma$, which is a normal $\mathbb{Q}$-Gorenstein variety over $\mathcal{O}_K$. In general, for a normal $\mathbb{Q}$-Gorenstein variety Y

over $\mathcal{O}_K$, we can define the *stringy point count* $\sharp_{\mathrm{st}}(Y)$ as a volume of the $\mathcal{O}_K$-point set $Y(\mathcal{O}_K)$ (see [Yasuda 2014b]). This is an analogue of the stringy E-function defined in [Batyrev and Dais 1996; Batyrev 1998] and is a generalization of the number of k-points on a smooth $\mathcal{O}_K$-variety. For a k-point $y \in Y(k)$, we can similarly define the *stringy point count along* $\{y\}$ (*or the stringy weight of* y), denoted by $\sharp_{\mathrm{st}}(Y)_y$, so that we have

$$\sharp_{\mathrm{st}}(Y) = \sum_{y \in Y(k)} \sharp_{\mathrm{st}}(Y)_y.$$

Yasuda [2014b] proved that if the representation τ is faithful and the morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one, then

$$\sharp_{\mathrm{st}}(X) = M(K, \Gamma, \boldsymbol{v}_\tau) q^d \quad \text{and} \quad \sharp_{\mathrm{st}}(X)_o = M(K, \Gamma, -\boldsymbol{w}_\tau),$$

where o is the origin of $X(k)$. This is a version of the wild McKay correspondence discussed in [Yasuda 2013; 2014a; 2016; Wood and Yasuda 2015]. Special cases were previously proved in [Yasuda 2014a; Wood and Yasuda 2015]. If X has a nice resolution, then $\sharp_{\mathrm{st}}(X)$ and $\sharp_{\mathrm{st}}(X)_o$ are explicitly computed in terms of resolution data. Using it, we can deduce a few properties of the functions $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$: obtained properties depend on what sort of resolution exists.

Remark 1.1. The assumption that the morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one is just for notational simplicity. We may drop this assumption by considering the pair of X and a $\mathbb{Q}$-divisor on it rather than the variety X itself.

Let us think of X as a family of singular varieties over $\operatorname{Spec} \mathcal{O}_K$. If X admits a kind of simultaneous resolution, then

$$\frac{\sharp_{\mathrm{st}}(X) - \sharp_{\mathrm{st}}(X)_o}{q^r - 1}$$

satisfies a certain self-duality, which can be understood as the Poincaré duality of stringy invariants proved in [Batyrev and Dais 1996; Batyrev 1998] over complex numbers. Using the wild McKay correspondence, we can transform this duality into the form:

$$\begin{aligned} M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{rd} &- M(K_r, \Gamma, -\boldsymbol{w}_\tau) \\ &= \mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) \cdot q^{rd} - \mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)). \end{aligned} \tag{1-2}$$

We call it the *weak duality* between $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$, which is indeed weaker than the strong duality (1-1).

There are three possibilities: both dualities hold, both dualities fail, and the weak one holds but the strong one does not. Examples we compute show that each possibility indeed occurs. In the tame case, the strong duality holds. For permutation representations, the weak duality holds in all examples we compute, but the strong duality does not generally hold. When $\mathcal{O}_K = \mathbb{F}_q[\![t]\!]$ and the given representation τ is defined over $\mathbb{F}_q$, the strong duality holds in all computed examples. However, when the representation is not defined over $\mathbb{F}_q$, then the weak duality tends to fail. These examples suggest that we may think of the two dualities as a test of how equisingular the family X is.

In particular, when $\mathcal{O}_K = \mathbb{F}_q[\![t]\!]$ and τ is defined over $\mathbb{F}_q$, the associated quotient scheme $X = \mathbb{A}^d_{\mathbb{F}_q[\![t]\!]}$ would be equisingular in any reasonable sense. If there exists a nice resolution of the $\mathbb{F}_q$-variety $X \otimes_{\mathcal{O}_K} \mathbb{F}_q$, we obtain an equally nice simultaneous resolution of X just by base change. Therefore, if one believes that there exists as nice a resolution in positive characteristic as in characteristic zero, then at least the weak duality (1-2) would hold for such τ. If, on the contrary, one finds an example such that weak duality fails, this would give an example of an $\mathbb{F}_q$-variety not admitting any nice resolution. However, every example we have computed satisfies strong duality.

The study in this paper thus gives rise to interesting open problems related to dualities and equisingularities and their interaction (see Questions 4.7, 4.8 and 5.2, and Section 6). Further studies are required.

The paper is organized as follows. In Section 2, we define admissible functions and their duals. In Section 3, we discuss properties of stringy point counts. In Section 4, we define total masses and discuss how the wild McKay correspondence relates dualities of total masses with equisingularities. In Section 5, we exhibit several examples. Section 6 contains concluding remarks.

Convention and notation. We denote the set of positive integers by $\mathbb{N}$. A *local field* means a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ for a prime number p. For a local field K, we denote its integer ring by $\mathcal{O}_K$ and its residue field by k. We denote the characteristic of k by p and the cardinality of k by q. For $r \in \mathbb{N}$, we denote the unramified extension of K of degree r by K_r and its residue field (that is, the extension of k of degree r) by k_r. For a variety X defined over either $\mathcal{O}_K$ or k, we put

$$\sharp X := \sharp X(k),$$

the cardinality of k-points of X. More generally, for $r \in \mathbb{N}$, we put

$$\sharp^r X := \sharp X(k_r).$$

2. Admissible functions and their duals

In this section, we set the foundation to discuss dualities of total masses and stringy point counts by introducing *admissible functions.*

Definition 2.1. We call a function $f : \mathbb{N} \to \mathbb{C}$ *admissible* if there exist numbers $0 \neq c \in \mathbb{Q}$, $n_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{C}$ $(1 \le i \le m)$ such that for every $r \in \mathbb{N}$,

$$f(r) = \frac{1}{q^{cr}-1} \sum_{i=1}^{l} n_i \alpha_i^r .$$

We denote the set of admissible functions by AF. For an admissible function $f(r)$ as above, we define its *dual function* $\mathbb{D} f : \mathbb{N} \to \mathbb{C}$ by

$$(\mathbb{D} f)(r) := \frac{1}{q^{-cr}-1} \sum_{i=1}^{l} n_i \alpha_i^{-r} .$$

It is easy to see that the set of admissible functions is closed under addition and multiplication, thus AF has a natural ring structure. For $f \in \mathrm{AF}$, the dual $\mathbb{D} f$ is also an admissible function. Therefore $\mathbb{D}$ gives an involution of the set AF. Moreover it is easily checked that $\mathbb{D} : \mathrm{AF} \to \mathrm{AF}$ is a ring isomorphism.

A typical admissible function is a rational function in $q^{1/n}$ for some $n \in \mathbb{N}$ with a denominator of the form $q^{cr} - 1$. In that case, the dual function is obtained just by substituting q^{-1} for q. Another example is given as follows: let $k = \mathbb{F}_q$ and $k_r := \mathbb{F}_{q^r}$. For a k-variety X the function

$$r \mapsto \sharp^r X := \sharp X(k_r)$$

is admissible from the Grothendieck–Lefschetz trace formula.

Lemma 2.2. *The dual function $\mathbb{D} f$ does not depend on the choice of the expression of $f(r)$ as $1/(q^{cr} - 1) \sum_{i=1}^{l} n_i \alpha_i^r$.*

Proof. Let

$$\frac{1}{q^{dr}-1} \sum_{j=1}^{l'} m_j \beta_j^r$$

be another such expression of $f(r)$. For every $r \in \mathbb{N}$, we have

$$(q^{dr} - 1) \cdot \sum_{i=1}^{l} n_i \alpha_i^r = (q^{cr} - 1) \cdot \sum_{j=1}^{l'} m_j \beta_j^r .$$

It suffices to show that for every $r \in \mathbb{N}$,

$$(q^{-dr} - 1) \cdot \sum_{i=1}^{l} n_i \alpha_i^{-r} = (q^{-cr} - 1) \cdot \sum_{j=1}^{l'} m_j \beta_j^{-r} .$$

This follows from the following claim:

Claim 2.3. *Suppose that for every r, we have*

$$\sum_{i=1}^{l} n_i \alpha_i^r = \sum_{j=1}^{l'} m_j \beta_j^r .$$

Suppose also that α_i, n_i, β_j and m_j are nonzero, that $\alpha_1, \dots, \alpha_l$ are distinct, and so are $\beta_1, \dots, \beta_{l'}$. Then we have $l = l'$, $\alpha_i = \beta_i$ and $n_i = m_i$, up to permutation.

In turn, this claim follows from:

Claim 2.4. *Suppose that $\alpha_1, \dots, \alpha_l$ are distinct nonzero complex numbers and that for every r,*

$$\sum_{i=1}^{l} n_i \alpha_i^r = 0.$$

Then $n_i = 0$ for every i.

The last claim follows from the regularity of the Vandermonde matrix

$$(\alpha_i^r)_{1 \le i \le l,\, 0 \le r \le l-1}. \qquad \square$$

For a smooth proper k-variety X of pure dimension d, the Poincaré duality for the l-adic cohomology shows that the function $r \mapsto \sharp^r X$ satisfies

$$\sharp^r X = q^{dr} \cdot \mathbb{D}(\sharp^r X).$$

3. Stringy point counts

Stringy point counts. Let K be a local field, that is, a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. We denote its residue field by k and its integer ring by $\mathcal{O}_K$. An *$\mathcal{O}_K$-variety* means an integral separated flat $\mathcal{O}_K$-scheme of finite type such that there exists an open dense subscheme $U \subset X$ which is smooth over $\mathcal{O}_K$. For an $\mathcal{O}_K$-variety X, we denote by X_k the closed fiber $X \otimes_{\mathcal{O}_K} k$ and by X_K the generic fiber $X \otimes_{\mathcal{O}_K} K$. For a constructible subset $C \subset X_k$, we let $X(\mathcal{O}_K)_C$ be the set of $\mathcal{O}_K$-points $\operatorname{Spec} \mathcal{O}_K \to X$ sending the closed point into C.

Suppose now that X is normal. From [Kollár 2013, Definition 1.5], X has a canonical sheaf $\omega_{X/\mathcal{O}_K}$. If d is the relative dimension of X over $\mathcal{O}_K$, then $\omega_{X/\mathcal{O}_K}$ coincides with $\Omega^d_{X/\mathcal{O}_K} = \bigwedge^d \Omega_{X/\mathcal{O}_K}$ on the $\mathcal{O}_K$-smooth locus of X. The canonical divisor $K_X = K_{X/\mathcal{O}_K}$ is defined as a divisor such that $\omega_{X/\mathcal{O}_K} = \mathcal{O}_X(K_X)$, which is determined up to linear equivalence. Let us suppose also that X is $\mathbb{Q}$-Gorenstein, that is, K_X is $\mathbb{Q}$-Cartier. Then, for a constructible subset $C \subset X_k$, we can define the *stringy point count along C*,

$$\sharp_{\mathrm{st}}(X)_C \in \mathbb{R}_{\ge 0} \cup \{\infty\},$$

as a certain p-adic volume of $X(\mathcal{O}_K)_C$. For details, see [Yasuda 2014b]. However, what is important for our purpose is not the definition but the explicit formula in the next subsection.

For $r \in \mathbb{N}$, let K_r be the unramified extension of K of degree r and let k_r be its residue field. We then define $\sharp^r_{\mathrm{st}}(X)_C$ to be the stringy point count $X \otimes_{\mathcal{O}_K} \mathcal{O}_{K_r}$ along the preimage of C in $(X \otimes_{\mathcal{O}_K} \mathcal{O}_{K_r}) \otimes_{\mathcal{O}_{K_r}} k_r = X \otimes_{\mathcal{O}_K} k_r$. We often regard $\sharp^r_{\mathrm{st}}(X)_C$ as a function in r. When X is smooth over $\mathcal{O}_K$, then $\sharp^r_{\mathrm{st}}(X)_C$ is equal to the number of k_r-points of X contained in C.

An explicit formula and the Poincaré duality. Let X be a normal $\mathbb{Q}$-Gorenstein variety over either $\mathcal{O}_K$ or k. For a proper birational morphism $f : Y \to X$ with Y normal, the relative canonical divisor

$$K_{Y/X} := K_Y - f^* K_X$$

is uniquely determined as a $\mathbb{Q}$-divisor on Y supported in the exceptional locus of f. We are especially interested in the case where Y is regular and $K_{Y/X}$ is a *simple normal crossing divisor*, a notion that we define on regular $\mathcal{O}_K$-varieties as follows.

Definition 3.1. Let Y be a regular variety over $\mathcal{O}_K$ and $E \subset Y$ a reduced closed subscheme of pure codimension one. We call E a *simple normal crossing divisor* if

- for every irreducible component E_0 of E and for every closed point $x \in E_0$ where Y is smooth over $\mathcal{O}_K$, the completion of E_0 at x is irreducible, and
- for every closed point $x \in Y$ where Y is smooth over $\mathcal{O}_K$, the support of E is, in a certain Zariski neighborhood, defined by a product $y_{i_1} \cdots y_{i_m}$ for a regular system of parameters $y_0 = \varpi, y_1, \dots, y_d \in \widehat{\mathcal{O}}_{Y,y}$ with ϖ a uniformizer of K and $0 \le i_1 < \cdots < i_m \le d$.

We call a $\mathbb{Q}$-divisor on Y a *simple normal crossing divisor* if its support is one.

The reason why we look only at closed points where Y is $\mathcal{O}_K$-smooth is that there is no $\mathcal{O}_K$-point passing through a closed point where Y is not $\mathcal{O}_K$-smooth, and hence such a point is irrelevant to stringy point counts.

Definition 3.2. Let $f : Y \to X$ be a proper birational morphism of varieties over either $\mathcal{O}_K$ or k such that X is normal and $\mathbb{Q}$-Gorenstein:

- We call f a *resolution* if Y is regular.
- We call f a *WL* (*weak log*) *resolution* if f is a resolution and the support of $K_{Y/X}$ is simple normal crossing.
- When f is defined over $\mathcal{O}_K$, we call f an *SWL* (*simultaneous weak log*) *resolution* if f is a WL resolution and Y is smooth over $\mathcal{O}_K$ or k.

Remark 3.3. The word *weak* indicates that we look only at the support of $K_{Y/X}$ rather than the whole exceptional locus. In particular, every resolution $Y \to X$ with $K_{Y/X} = 0$ is a WL resolution, whatever the exceptional locus is.

Remark 3.4. We cannot usually expect that an SWL resolution exists, unless X is thought of as an *equisingular* family over $\mathcal{O}_K$ in some sense. If $\mathcal{O}_K = k[[t]]$ and if $X = X_0 \otimes_k k[[t]]$ for some k-variety X_0, then X would be equisingular in any reasonable sense. In this situation, an SWL resolution of X exists if and only if a WL resolution of X_0 exists.

For a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X and a WL resolution $f : Y \to X$, we can uniquely decompose $K_{Y/X}$ as

$$K_{Y/X} = \sum_{h=1}^{l} a_h A_h + \sum_{i=1}^{m} b_i B_i + \sum_{j=1}^{n} c_j C_j, \tag{3-1}$$

where all the A_h, B_i, C_j are prime divisors on Y, the coefficients a_h, b_i, c_j are nonzero rational numbers, every A_h is contained in X_k and Y is $\mathcal{O}_K$-smooth at the generic point of A_h, every B_i is contained in X_k and Y is *not* $\mathcal{O}_K$-smooth at the generic point of B_i, and every C_j dominates $\operatorname{Spec} \mathcal{O}_K$. We denote by A_h° the locus in A_h where Y is $\mathcal{O}_K$-smooth. For a subset $J \subset \{1, \dots, n\}$, we define

$$C_J^\circ := \bigcap_{j \in J} C_j \Big\backslash \bigcup_{j \notin J} C_j.$$

For a constructible subset W of a k-variety Z, let $\sharp^r(W)$ denote the number of k_r-points of W.

Proposition 3.5 (An explicit formula, [Yasuda 2014b]). *Let $f : Y \to X$ be a WL resolution of a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X and write $K_{Y/X}$ as in (3-1). Let Y_{sm} be the $\mathcal{O}_K$-smooth locus of Y:*

(1) *We have $\sharp^r_{\mathrm{st}}(X)_C < \infty$ for all $r \in \mathbb{N}$ if and only if $c_j > -1$ for every j such that $C_j \cap f^{-1}(C) \cap Y_{\mathrm{sm}} \neq \varnothing$.*

(2) *If the two equivalent conditions of (1) hold, then*

$$\sharp^r_{\mathrm{st}}(X)_C = \sum_{h=1}^{l} q^{-ra_h} \sum_{J \subset \{1,\dots,n\}} \sharp^r(f^{-1}(C) \cap A_h^\circ \cap C_J^\circ) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1}.$$

Proof. In [Yasuda 2014b] it was proved that $\sharp_{\mathrm{st}}(X)_C$ is equal to $\sharp_{\mathrm{st}}(Y, -K_{Y/X})_{f^{-1}(C)}$, the stringy point count of the pair $(Y, -K_{Y/X})$ along $f^{-1}(C)$. The proposition follows from the explicit formula for the stringy point count of a pair with a simple normal crossing divisor in the same paper. $\square$

In particular, if K has characteristic zero and if the generic fiber X_K has only log terminal singularities (for instance, quotient singularities), then $\sharp^r(X)_C < \infty$ for every r and C. The following is a direct consequence of the proposition.

Corollary 3.6. *Suppose that X is $\mathbb{Q}$-Gorenstein, that there exists a WL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X)_C < \infty$ for every $r > 0$. Then the function $\sharp^r_{\mathrm{st}}(X)_C$ in the variable r is admissible.*

The following result was proved by Batyrev–Dais [1996] and Batyrev [1998] in a slightly different setting.

Corollary 3.7 (The Poincaré duality). *Let X be a d-dimensional proper normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety. Suppose that there exists an SWL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X) < \infty$ for every $r \in \mathbb{N}$. We then have*

$$\sharp^r_{\mathrm{st}}(X) = \mathbb{D}(\sharp^r_{\mathrm{st}}(X)) \cdot q^{dr}.$$

Proof. We follow arguments in the proof of [Batyrev 1998, Theorem 3.7]. Since f is an SWL resolution, if we write $K_{Y/X}$ as in (3-1) then the terms $\sum a_h A_h$ and $\sum b_i B_i$ do not appear. Therefore Proposition 3.5 reads

$$\sharp^r_{\mathrm{st}}(X) = \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J^\circ) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1}.$$

Putting

$$C_J := \bigcap_{j \in J} C_j,$$

we have $C_J := \bigsqcup_{J' \supset J} C^\circ_{J'}$ and $\sharp^r(C_J) = \sum_{J' \supset J} \sharp^r(C^\circ_{J'})$. From this and the inclusion-exclusion principle (or the Möbius inversion formula see, for instance, [Stanley 1997]), we deduce

$$\sharp^r(C_J^\circ) = \sum_{J' \supset J} (-1)^{\sharp J' - \sharp J} \sharp^r(C_{J'}).$$

Hence

$$\begin{aligned}
\sharp^r_{\mathrm{st}}(X) &= \sum_{J \subset \{1,\dots,n\}} \Bigl(\sum_{J' \supset J} (-1)^{\sharp J' - \sharp J} \sharp^r(C_{J'}) \Bigr) \prod_{j \in J} \frac{q^r - 1}{q^{r(1+c_j)} - 1} \\
&= \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J) \prod_{j \in J} \Bigl(\frac{q^r - 1}{q^{r(1+c_j)} - 1} - 1 \Bigr) \\
&= \sum_{J \subset \{1,\dots,n\}} \sharp^r(C_J) \prod_{j \in J} \Bigl(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \Bigr).
\end{aligned}$$

Now we have

$$\mathbb{D}\Bigl(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \Bigr) = \frac{q^{-r} - q^{r(-1-c_j)}}{q^{r(-1-c_j)} - 1} = \frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)} - 1} \cdot q^{-r}.$$

Since $\sharp^r(C_J) = \sharp^r(C_J \otimes_K k)$ and $C_J \otimes_K k$ are proper smooth k-varieties of pure dimension $d - \sharp J$, the Poincaré duality for the l-adic cohomology gives

$$\mathbb{D}(\sharp^r(C_J)) = \sharp^r(C_J) \cdot q^{r(\sharp J - d)}.$$

Since $\mathbb{D} : \mathrm{AF} \to \mathrm{AF}$ is a ring homomorphism, the corollary follows. □

Set-theoretically free $\mathbb{G}_m$-actions.

Definition 3.8. An action of the group scheme $\mathbb{G}_m = \mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}]$ on a k-variety Z is said to be *set-theoretically free* if for every field extension k'/k and every point $x \in Z(k')$, the stabilizer subgroup scheme $\operatorname{Stab}(x) \subset \mathbb{G}_{m,k'}$ consists of a single point.

A set-theoretically free $\mathbb{G}_m$-action naturally appears as the $\mathbb{G}_m$-action on the quotient variety $(\mathbb{A}^d_k/\Gamma) \setminus \{o\}$ with the origin o removed for a linear action of a finite group Γ on an affine space $\mathbb{A}^d_k$. In general, the $\mathbb{G}_m$-action may have nonreduced stabilizer subgroups and not be free. See [Yasuda 2014a] for such an example. If X is the quotient variety $Z/\mathbb{G}_m$ for a set-theoretically free $\mathbb{G}_m$-action on a smooth variety Z and if X is proper, then the function $\sharp^r X$ satisfies the same Poincaré duality as a smooth proper variety does. Note that X is not necessarily smooth because of nonreduced stabilizers. To prove the duality, we need to use Artin stacks.

Based on [Olsson 2015; Laszlo and Olsson 2008a; 2008b], Sun [2012] defined étale cohomology groups $H^i(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$ and $H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$ for Artin stacks $\mathcal{X}$ of finite type over k and a prime number $l \neq p$. For $r \in \mathbb{N}$, we define

$$\sharp^r(\mathcal{X}) := \sum_i (-1)^i \operatorname{Tr}(F^r \mid H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l))$$

with F^r the r-iterated Frobenius action. This is a generalization of $\sharp^r X = \sharp X(k_r)$ for a k-variety X.

Proposition 3.9. *Let $\mathcal{X}$ be an Artin stack of finite type over k with finite diagonal, and $\overline{\mathcal{X}}$ its coarse moduli space. Then we have*

$$\sharp^r(\mathcal{X}) = \sharp^r(\overline{\mathcal{X}}).$$

Moreover, if $\mathcal{X}$ is smooth and proper of pure dimension d over k, then

$$\sharp^r(\mathcal{X}) = \mathbb{D}(\sharp^r(\mathcal{X})) \cdot q^{rd}.$$

Proof. In the proof of [Sun 2012, Proposition 7.3.2], Sun proved that

$$H^i_c(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l) = H^i_c(\overline{\mathcal{X}} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)$$

and that if $\mathcal{X}$ is smooth and proper, then we have the Poincaré duality,

$$H^i(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l)^\vee = H^{2d-i}(\mathcal{X} \otimes_k \bar{k}, \overline{\mathbb{Q}}_l) \otimes \overline{\mathbb{Q}}_l(d).$$

The proposition is a direct consequence of these results. $\square$

Let Z be a k-variety endowed with a set-theoretically free $\mathbb{G}_m$-action. Then the quotient stack $\mathcal{W} := [Z/\mathbb{G}_m]$ is an Artin stack with finite diagonal and hence admits a coarse moduli space, which we write as $W = Z/\mathbb{G}_m$.

Proposition 3.10. *We have*

$$\sharp^r(\mathcal{W}) = \sharp^r(W) = \frac{\sharp^r(Z)}{q^r - 1}.$$

Proof. The left equality was proved in the last proposition. We show the right equality. For every field extension k'/k and every point $x \in Z(k')$, we have

$$\mathrm{Stab}(x) = \operatorname{Spec} k'[t, t^{-1}]/(t^{p^a} - 1)$$

for some nonnegative integer a. Let $U \subset Z$ be the locus where this number a takes the minimum value, and let $\overline{U} \subset W$ be its image. The algebraic spaces U and $\overline{U}$ are open dense subspaces of Z and W respectively. If we put

$$H := \operatorname{Spec} k[t, t^{-1}]/(t^{p^a} - 1),$$

then the given set-theoretically free $\mathbb{G}_m$-action on U induces a free action of the quotient group scheme $\mathbb{G}_m/H$, which is isomorphic to $\mathbb{G}_m$, on U. This action makes the projection $U \to \overline{U}$ a $\mathbb{G}_m$-torsor. From Hilbert's Theorem 90 [Milne 1980, p. 124], every $\mathbb{G}_m$-torsor is Zariski locally trivial, hence

$$\sharp^r(U) = (q^r - 1) \cdot \sharp^r(\overline{U}).$$

It is now easy to show $\sharp^r(Z) = (q^r - 1) \cdot \sharp^r(W)$ by induction. $\square$

Definition 3.11. Let X be an $\mathcal{O}_K$-variety with a $\mathbb{G}_{m,\mathcal{O}_K}$-action. We suppose that the induced $\mathbb{G}_{m,k}$-action on X_k is set-theoretically free. For the quotient stack $[X/\mathbb{G}_{m,\mathcal{O}_K}]$, we put

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) := \frac{\sharp^r_{\mathrm{st}}(X)}{q^r - 1}.$$

Definition 3.12. For a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X endowed with a $\mathbb{G}_{m,\mathcal{O}_K}$-action, we call a resolution $f : Y \to X$ an *ESWL* (*equivariant simultaneous weak log*) *resolution* if f is an SWL resolution and $\mathbb{G}_{m,\mathcal{O}_K}$-equivariant (that is, Y also has a $\mathbb{G}_{m,\mathcal{O}_K}$-action so that f is $\mathbb{G}_{m,\mathcal{O}_K}$-equivariant).

Proposition 3.13. *Let X be a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety X endowed with a $\mathbb{G}_{m,\mathcal{O}_K}$-action. Suppose that the induced action of $\mathbb{G}_{m,k}$ on X_k is set-theoretically free and the quotient stack $[X_k/\mathbb{G}_{m,k}]$ is proper over k. Suppose also that there exists an ESWL resolution $f : Y \to X$ and that $\sharp^r_{\mathrm{st}}(X) < \infty$ for all r. Then*

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) = \mathbb{D}(\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}])) \cdot q^{r(d-1)}$$

with d the dimension of X over $\mathcal{O}_K$.

Proof. From Proposition 3.5, if we write $K_{Y/X} = \sum_{j=1}^{n} c_j C_j$, then we obtain

$$\sharp^r_{\mathrm{st}}([X/\mathbb{G}_{m,\mathcal{O}_K}]) = \sum_{J\subset\{1,\dots,n\}} \frac{\sharp^r(C_J^\circ)}{q^r-1} \prod_{i\in J} \frac{q^r-1}{q^{r(1+c_j)}-1}$$
$$= \sum_{J\subset\{1,\dots,n\}} \frac{\sharp^r(C_J)}{q^r-1} \prod_{j\in J} \left(\frac{q^r - q^{r(1+c_j)}}{q^{r(1+c_j)}-1} \right)$$

as in the proof of Corollary 3.7. Each $C_J \cap Y_k$ is stable under the $\mathbb{G}_{m,k}$-action and the action on $C_J \cap Y_k$ is set-theoretically free. Hence $[(C_J \cap Y_k)/\mathbb{G}_{m,k}]$ is a smooth proper Artin k-stack of dimension $d-1-\sharp J$ with finite diagonal. From Proposition 3.9,

$$\frac{\sharp^r(C_J)}{q^r-1} = \mathbb{D}\left(\frac{\sharp^r(C_J)}{q^r-1}\right) \cdot q^{r(d-1-\sharp J)}.$$

Following the proof of Corollary 3.7 we obtain the result. □

4. Total masses of local Galois representations

Total masses. Let K be a local field and $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ its absolute Galois group. For a finite group Γ, we define $S_{K,\Gamma}$ to be the set of continuous homomorphisms $G_K \to \Gamma$. To each representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$, we can associate, among others, three functions $S_{K,\Gamma} \to \mathbb{R}$ denoted by $\boldsymbol{a}_\tau$, $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$.

The first one, $\boldsymbol{a}_\tau$, is called the *Artin conductor*. For $\rho \in S_{K,\Gamma}$, let H be the image of ρ and L/K the associated Galois extension, having H as its Galois group. The Galois group H has the filtration by ramification subgroups with lower numbering

$$H \supset H_0 \supset H_1 \supset \cdots,$$

(see [Serre 1979]). We define

$$\boldsymbol{a}_\tau(\rho) := \sum_{i=0}^{\infty} \frac{1}{(H_0 : H_i)} \operatorname{codim}(K^d)^{H_i}.$$

The second function $\boldsymbol{v}_\tau$ is defined as follows. For $\rho \in S_{K,\Gamma}$ let $\operatorname{Spec} M \to \operatorname{Spec} K$ be the corresponding étale Γ-torsor and $\mathcal{O}_M$ the integer ring of M. Note that the spectrum of the extension L mentioned above is a connected component of $\operatorname{Spec} M$. Then the free $\mathcal{O}_M$-module $\mathcal{O}_M^{\oplus d}$ has two (left) Γ-actions: firstly the action induced from the map

$$\Gamma \xrightarrow{\tau} \mathrm{GL}_d(\mathcal{O}_K) \subset \mathrm{GL}_d(\mathcal{O}_M)$$

and secondly the diagonal action induced from the given Γ-action on $\mathcal{O}_M$. We define the associated *tuning submodule* $\Xi \subset \mathcal{O}_M^{\otimes d}$ to be the subset of those element on which the two actions coincide. It turns out that Ξ is a free $\mathcal{O}_K$-module of rank d. We define

$$\boldsymbol{v}_\tau(\rho) := \frac{1}{\sharp\Gamma} \cdot \operatorname{length}\left(\frac{\mathcal{O}_M^{\oplus d}}{\mathcal{O}_M \cdot \Xi}\right).$$

Note that we may also use $\mathcal{O}_L$ and the subgroup $H = \operatorname{Im}(\rho)$ in the definition of $\boldsymbol{v}_\tau$ instead of $\mathcal{O}_M$ and Γ.

The last function $\boldsymbol{w}_\tau$ is called the *weight function.* Let $\rho \in S_{K,\Gamma}$, M and $\Xi \subset \mathcal{O}_M^{\oplus d}$ be as above. We identify Ξ with the subgroup of Γ-equivariant maps of $\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^{\oplus d}, \mathcal{O}_M) = \mathcal{O}_M^{\oplus d}$. For an $\mathcal{O}_K$-basis $\phi_1, \dots, \phi_d$ of Ξ, we define an $\mathcal{O}_K$-algebra homomorphism $u^* : \mathcal{O}_K[x_1, \dots, x_d] \to \mathcal{O}_M[y_1, \dots, y_d]$ by

$$u^*(x_i) = \sum_{j=1}^{n} \phi_j(x_i) y_j.$$

Let $u : \mathbb{A}^d_{\mathcal{O}_M} \to \mathbb{A}^d_{\mathcal{O}_K}$ be the corresponding morphism of schemes and $o : \operatorname{Spec} k \hookrightarrow \mathbb{A}^d_{\mathcal{O}_K}$ the k-point at the origin. We define

$$\boldsymbol{w}_\tau(\rho) := \dim u^{-1}(o) - \boldsymbol{v}_\tau(\rho).$$

Remark 4.1. Our definition of $\boldsymbol{w}_\tau$ follows the one in [Yasuda 2014b] and is slightly different from the one in the earlier paper [Wood and Yasuda 2015]. Let us consider the left Γ-action on $\mathbb{A}^d_{\mathcal{O}_K}$ induced from the one on $\mathcal{O}_K[x_1, \dots, x_d]$. To be precise, an element $\gamma \in \Gamma$ gives the automorphism of $\mathbb{A}^d_{\mathcal{O}_K}$ corresponding to the automorphism on $\mathcal{O}_K[x_1, \dots, x_d]$ given by the inverse γ^{-1} of γ. In [Wood and Yasuda 2015] we defined

$$\boldsymbol{w}_\tau(\rho) := \operatorname{codim}((\mathbb{A}^d_k)^{H_0}, \mathbb{A}^d_k) - \boldsymbol{v}_\tau(\rho),$$

where H_0 is the zeroth ramification subgroup of H, that is, the inertia subgroup. However, as proved in [Yasuda 2014b], the two definitions coincide in the following three cases with which we are mainly concerned:

(1) The order of the given group Γ is invertible in k.

(2) The given representation τ is a permutation representation.

(3) The given local field K has positive characteristic, and hence $K = k((t))$, and the image of τ is contained in $\mathrm{GL}_d(k)$.

Definition 4.2. For a function $c : S_{K,\Gamma} \to \mathbb{R}$, we define the *total mass* of (K, Γ, c) as

$$M(K, \Gamma, c) := \frac{1}{\sharp\Gamma} \cdot \sum_{\rho :\in S_{K,\Gamma}} q^{-c(\rho)} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

For $r \in \mathbb{N}$, let K_r/K be the unramified extension of degree r. We continue to denote the induced representations $\Gamma \xrightarrow{\tau} \mathrm{GL}_n(\mathcal{O}_K) \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K_r})$ by τ. Then we can consider total masses $M(K_r, \Gamma, \boldsymbol{a}_\tau)$, $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$, and regard them as functions in r.

Kedlaya studied total masses for $c = \boldsymbol{a}_\tau$ and used them to interpret and generalize the mass formula of Bhargava [2007]. Wood [2008] studied total masses for different choices of c. In the context of the wild McKay correspondence, explained below, functions $\boldsymbol{v}_\tau$ and $\boldsymbol{w}_\tau$ occur. For permutation representations, functions $\boldsymbol{a}_\tau$ and $\boldsymbol{v}_\tau$ are related as follows.

Lemma 4.3 [Wood and Yasuda 2015]. *If τ is a permutation representation, then*

$$2\boldsymbol{v}_\tau = \boldsymbol{v}_{\tau\oplus\tau} = \boldsymbol{a}_\tau.$$

The wild McKay correspondence. A representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ defines Γ-actions on the polynomial ring $\mathcal{O}_K[x_1, \dots, x_d]$ as above. Let X be the quotient variety $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma = \operatorname{Spec} \mathcal{O}_K[x_1, \dots, x_d]^\Gamma$, which is a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety. The following theorem, which we call the wild McKay correspondence, connects total masses and stringy point counts. This was proved in [Yasuda 2014a; Wood and Yasuda 2015] for special cases and in [Yasuda 2014b] in full generality.

Theorem 4.4 (The wild McKay correspondence). *Suppose that the representation τ is faithful and the quotient morphism $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one. We have*

$$\sharp^r_{\mathrm{st}}(X) = M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{dr} \quad \textit{and} \quad \sharp^r_{\mathrm{st}}(X)_o = M(K_r, \Gamma, -\boldsymbol{w}_\tau).$$

The next corollary is a direct consequence of this theorem and Proposition 3.5.

Corollary 4.5. *If X admits a WL resolution and if $M(K_r, \Gamma, \boldsymbol{v}_\tau) < \infty$ (resp. $M(K_r, \Gamma, -\boldsymbol{w}_\tau) < \infty$) for every $r \in \mathbb{N}$, then the function $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ (resp. $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$) is admissible.*

Dualities of total masses. To discuss dualities, let us assume that both functions $M(K_r, \Gamma, \boldsymbol{v}_\tau)$ and $M(K_r, \Gamma, -\boldsymbol{w}_\tau)$ have finite values for all r and are admissible. The quotient variety X associated to a faithful representation $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ has a natural $\mathbb{G}_{m,\mathcal{O}_K}$-action. Let X^* be the complement of the zero section $\operatorname{Spec} \mathcal{O}_K \hookrightarrow X$. The $\mathbb{G}_{m,k}$-action on X^*_k is set-theoretically free. By definition, we have

$$\sharp^r_{\mathrm{st}}([X^*/\mathbb{G}_{m,\mathcal{O}_K}]) = \frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1}.$$

If there exists an ESWL resolution $Y \to X^*$, then from Proposition 3.13, we have

$$\frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1} = \mathbb{D}\left(\frac{\sharp^r_{\mathrm{st}}(X) - \sharp^r_{\mathrm{st}}(X)_o}{q^r - 1}\right) \cdot q^{r(d-1)}. \tag{4-1}$$

Remark 4.6. When $\Gamma = \mathbb{Z}/p\mathbb{Z} \subset \mathrm{GL}_d(k)$ with k a perfect field of characteristic $p > 0$, the motivic version of (4-1) was checked in [Yasuda 2014a, Proposition 6.36], not by using a resolution but by an explicit formula for motivic total masses.

If $\mathbb{A}^d_{\mathcal{O}_K} \to X$ is étale in codimension one, then the wild McKay correspondence, Theorem 4.4, shows that the last equality is equivalent to:

$$M(K_r, \Gamma, \boldsymbol{v}_\tau) \cdot q^{rd} - M(K_r, \Gamma, -\boldsymbol{w}_\tau) \\ = \mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) \cdot q^{rd} - \mathbb{D}(M(K_r, \Gamma, \boldsymbol{v}_\tau)). \quad (4\text{-}2)$$

We call this the *weak duality*. The equality may be regarded as a constraint imposed by the existence of an ESWL resolution of X^*. If this duality fails for some representation τ, then the associated X^* would have no ESWL resolution. In particular, it is interesting to ask the following question.

Question 4.7. Suppose that K has positive characteristic so that $K = k((t))$, that $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ is a faithful representation factoring through $\mathrm{GL}_n(k)$ and that the quotient morphism $\mathbb{A}^d_{\mathcal{O}_K} \to \mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ is étale in codimension one. Then, does the weak duality always hold?

If this has the negative answer and some representation τ does not satisfy (4-2), then the associated k-variety $X_k = \mathbb{A}^d_k/\Gamma$ does not admit any $\mathbb{G}_{m,k}$-equivariant WL resolution.

In some examples, even a stronger equality,

$$\mathbb{D}(M(K_r, \Gamma, -\boldsymbol{w}_\tau)) = M(K_r, \Gamma, \boldsymbol{v}_\tau), \quad (4\text{-}3)$$

holds: we call this the *strong duality*. A special case of this was the duality involving Bhargava's formula and the Hilbert scheme of points observed in [Wood and Yasuda 2015] (see Section 5). Actually, as far as the authors know, even the strong duality holds in all examples satisfying the assumptions of Question 4.7. It is thus natural to ask:

Question 4.8. With the same assumptions as in Question 4.7, does the strong duality always hold?

5. Examples

The tame case. We first consider the tame case. Suppose $p \nmid \sharp\Gamma$. Let $\tau : \Gamma \to \mathrm{GL}_d(\mathcal{O}_K)$ be a representation. Then $S_{K,\Gamma}$ is a finite set and every $\rho \in S_{K,\Gamma}$ factors through the Galois group G_K^{tame} of the maximal tamely ramified extension K^{tame}/K. It is topologically generated by two elements a and b with one relation $bab^{-1} = a^q$ (see [Neukirch et al. 2008, p. 410]). Here a is the topological generator of the inertia subgroup of G_K^{tame}. Therefore $S_{K,\Gamma}$ is in one-to-one correspondence with

$$\{(g, h) \in \Gamma^2 \mid hgh^{-1} = g^q\}.$$

The last set admits the involution $(g, h) \mapsto (g^{-1}, h)$. Let ι be the corresponding involution of $S_{K,\Gamma}$.

Lemma 5.1. *For $\rho \in S_{K,\Gamma}$, we have $\boldsymbol{v}_\tau(\iota(\rho)) = \boldsymbol{w}_\tau(\rho)$.*

Proof. Let K' be the completion of the maximal unramified extension of K. Both ρ and $\iota(\rho)$ define the same cyclic extension L/K' of order $l := \mathrm{ord}(g)$ with $(g, h) \in \Gamma^2$ corresponding to ρ. We can regard the element $a \in G_K^{\mathrm{tame}}$ as a generator of $\mathrm{Gal}(L/K')$. Let $\varpi \in L$ be a uniformizer and $\zeta \in K'$ an l-th root of unity such that $a(\varpi) = \zeta\varpi$.

If $\zeta^{a_1}, \dots, \zeta^{a_d}$ with $0 \le a_i < l$ are the eigenvalues of $g \in \Gamma \subset \mathrm{GL}_d(K')$, then, from [Wood and Yasuda 2015, Lemma 4.3], we have

$$\boldsymbol{v}_\tau(\rho) = \frac{1}{l}\sum_{i=1}^{l} a_i.$$

In the tame case, our definition of $\boldsymbol{w}_\tau$ coincides with the one in [Wood and Yasuda 2015] (see Remark 4.1). Therefore

$$\boldsymbol{w}_\tau(\rho) = \sharp\{i \mid a_i \neq 0\} - \boldsymbol{v}_\tau(\rho) = \frac{1}{l}\sum_{a_i \neq 0} l - a_i = \boldsymbol{v}_\tau(\iota(\rho)). \qquad \square$$

From the lemma,

$$\begin{aligned} M(K, \Gamma, -\boldsymbol{w}_\tau) &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{w}_\tau(\rho)} \\ &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{v}_\tau(\iota(\rho))} \\ &= \frac{1}{\sharp\Gamma}\sum_{\rho \in S_{K,\gamma}} q^{\boldsymbol{v}_\tau(\rho)} \\ &= \mathbb{D}(M(K, \Gamma, \boldsymbol{v}_\tau)). \end{aligned}$$

The strong duality thus holds in the tame case. Moreover it is a term-wise duality: $q^{\boldsymbol{w}_\tau(\rho)}$ and $q^{\boldsymbol{v}_\tau(\iota(\rho))}$ are dual to each other. This is no longer true in wild cases satisfying dualities.

Quadratic extensions: characteristic zero. We consider an unramified extension K of $\mathbb{Q}_2$ and compute some total masses counting quadratic extensions of K. Namely we consider the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$. Let $\mathfrak{m}_K$ be the maximal ideal of $\mathcal{O}_K$, quadratic extensions are divided into 4 classes:

- The trivial extension $K \times K$.
- The unramified field extension of degree two.
- Ramified extensions with discriminant $\mathfrak{m}_K^2$. There are $2(q-1)$ of them.
- Ramified extensions with discriminant $\mathfrak{m}_K^3$. There are $2q$ of them.

For instance, this follows from Krasner's formula [1966] (see also [Serre 1978]).

They are in one-to-one correspondence with elements of $S_{K,\Gamma}$. Let σ be the two-dimensional representation of $\Gamma = \mathbb{Z}/2\mathbb{Z}$ by the transposition and for $n \in \mathbb{N}$, σ_n the direct sum of n copies of σ. From [Wood and Yasuda 2015, Lemma 2.6], the values $\boldsymbol{a}_\sigma(\rho)$, for $\rho \in S_{K,\Gamma}$, are respectively $0, 0, 2, 3$ depending on the corresponding class of quadratic extension. We have

$$\begin{aligned} M(K, \Gamma, \boldsymbol{v}_{\sigma_n}) &= M\bigl(K, \Gamma, \tfrac{1}{2}\boldsymbol{a}_{\sigma_n}\bigr) \\ &= \tfrac{1}{2}(1 + 1 + 2(q-1)q^{-2n/2} + 2q \cdot q^{-3n/2}) \\ &= 1 + q^{-n+1} - q^{-n} + q^{-3n/2+1}. \end{aligned}$$

As for the total mass with respect to $-\boldsymbol{w}_{\sigma_n}$, we have $\boldsymbol{w}_{\sigma_n}(\rho) = n - \boldsymbol{v}_{\sigma_n}(\rho)$ if ρ corresponds to a totally ramified extension, and $\boldsymbol{w}_{\sigma_n}(\rho) = \boldsymbol{v}_{\sigma_n}(\rho)$ otherwise. Therefore,

$$M(K, \Gamma, \boldsymbol{w}_{\sigma_n}) = \tfrac{1}{2}(1 + 1 + 2(q-1)q^{-n+n} + 2q \cdot q^{-3n/2+n}) = q + q^{-n/2+1}.$$

We easily see that the strong duality holds for $n = 1$. For $n > 1$, the strong duality does not hold, but the weak one holds.

This phenomenon seems to be rather general. For a few other examples of permutation representations σ of cyclic groups, the authors checked that the weak duality holds, but the strong duality does not always hold. From these computations as well as the example in Section 5, it is natural to ask:

Question 5.2. Suppose that the given representation τ is a permutation representation. Does the weak duality (4-2) always hold? Does the quotient scheme $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ always admit an ESWL resolution?

Quadratic extensions: characteristic two. Next we consider the case $K = \mathbb{F}_q((t))$ with q a power of two and $\Gamma = \mathbb{Z}/2\mathbb{Z}$. As in the last example, there are exactly two unramified quadratic extensions of K: the trivial one $K \times K$ and the unramified field extension. As for the ramified extensions, for $i \in \mathbb{N}$, there are exactly $2(q-1)q^{i-1}$ extensions with discriminant $\mathfrak{m}_K^{2i}$. There is no extension with discriminant of odd exponent. This again follows from Krasner's formula.

Let σ be the two dimensional Γ-representation by transposition and σ_n the direct sum of n copies of it as above. We have

$$M(K, \Gamma, \boldsymbol{v}_{\sigma_n}) = 1 + \sum_{i=1}^{\infty}(q-1)q^{i-1}q^{-in} = 1 + \frac{(q-1)q^{-n}}{1 - q^{-n+1}},$$

and

$$M(K, \Gamma, -\boldsymbol{w}_{\sigma_n}) = 1 + \sum_{i=1}^{\infty}(q-1)q^{i-1}q^{-in+n} = 1 + \frac{q-1}{1 - q^{-n+1}},$$

We see that the strong duality and hence the weak duality hold for every n. Thus the answer to Question 4.8 in this case is positive.

For an integer $m \geq 0$, let us now consider the representation $\tau_m : \Gamma \to \mathrm{GL}_2(\mathcal{O}_K)$ given by the matrix

$$\begin{pmatrix} 1 & t^m \\ 0 & 1 \end{pmatrix}.$$

We have $\tau_0 \cong \sigma$. If $m > 0$, then τ_m is not defined over $k = \mathbb{F}_q$.

To compute the functions $\boldsymbol{v}_{\tau_m}$ and $\boldsymbol{w}_{\tau_m}$, let L be a ramified quadratic extension of K with discriminant $\mathfrak{m}_K^{2i}$, ι its unique K-involution and ϖ its uniformizer. Then $\delta(\varpi) := \iota(\varpi) - \varpi \in \mathcal{O}_K$ and $v_K(\delta(\varpi)) = i$ with v_K the normalized valuation of K. Therefore the tuning submodule Ξ is computed as follows:

$$\begin{aligned} \Xi &= \{(x, y) \in \mathcal{O}_L^{\oplus 2} \mid x + t^m y = \iota(x),\ y = \iota(y)\} \\ &= \{(x, t^{-m}\delta(x)) \mid x \in \mathcal{O}_L,\ v_K(\delta(x)) \geq m\} \\ &= \{(x, t^{-m}\delta(x)) \mid x \in \mathcal{O}_K \cdot 1 \oplus \mathcal{O}_K \cdot t^a \varpi\} \\ &= \langle (1, 0),\ (t^a\varpi, t^{a-m}\delta(\varpi))\rangle_{\mathcal{O}_K}, \end{aligned}$$

where $a = \sup\{0, m - i\}$. Therefore,

$$\frac{\mathcal{O}_L^{\oplus 2}}{\mathcal{O}_L \cdot \Xi} \cong \frac{\mathcal{O}_L}{\mathcal{O}_L \cdot t^{a-m}\delta(\varpi)} \cong \frac{\mathcal{O}_L}{\mathcal{O}_L \cdot t^{a-m+i}}.$$

For $\rho \in S_{K,\Gamma}$ corresponding to L,

$$\boldsymbol{v}_{\tau_m}(\rho) = a - m + i = \begin{cases} 0 & i \leq m, \\ i - m & i > m. \end{cases}$$

The map $u^* : \mathcal{O}_K[x, y] \to \mathcal{O}_L[X, Y]$ in the definition of $\boldsymbol{w}$ is explicitly given as

$$u^*(x) = X + t^a\varpi Y, \quad u^*(y) = t^{a-m}\delta(\varpi)Y.$$

The induced morphism $\operatorname{Spec} k[X, Y] \to \operatorname{Spec} k[x, y]$ is an isomorphism if $i \leq m$ and a linear map of rank one if $i > m$. Therefore

$$\boldsymbol{w}_{\tau_m}(\rho) = \begin{cases} 0 & i \leq m, \\ 1 - i + m & i > m. \end{cases}$$

To obtain quotient morphisms which are étale in codimension one, we consider the direct sums $\upsilon_{m,n} := \tau_m \oplus \sigma_n$. We have

$$\begin{aligned} M(K, \Gamma, \boldsymbol{v}_{\upsilon_{m,n}}) &= 1 + \sum_{i=1}^{m} (q-1)q^{i-1} \cdot q^{-in} + \sum_{i=m+1}^{\infty} (q-1)q^{i-1} \cdot q^{-i+m-in} \\ &= 1 + (q-1)q^{-n}\frac{1 - q^{(1-n)m}}{1 - q^{1-n}} + (q-1)\frac{q^{m-(m+1)n-1}}{1 - q^{-n}}, \end{aligned}$$

and

$$M(K,\Gamma,-\boldsymbol{w}_{\upsilon_{m,n}}) = 1+\sum_{i=1}^{m}(q-1)q^{i-1}\cdot q^{-in+n}+\sum_{i=m+1}^{\infty}(q-1)q^{i-1}\cdot q^{1-i+m-in+n}$$
$$= 1+(q-1)\frac{1-q^{(1-n)m}}{1-q^{1-n}}+(q-1)\frac{q^{m-mn}}{1-q^{-n}}.$$

One can easily see that the weak duality does not generally hold, for instance, by putting $(m,n)=(1,1)$. From our viewpoint, this is explained by noting that the entry t^m in the matrix defining τ_m causes the degeneration of singularities of $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$, namely it makes the family over $\mathcal{O}_K$ nonequisingular.

Bhargava's formula. We discuss an example from [Wood and Yasuda 2015] and the duality observed there. Let S_n be the n-th symmetric group and $\sigma : S_n \to \mathrm{GL}_d(\mathcal{O}_K)$ the standard permutation representation. According to Kedlaya [2007], the mass formula of Bhargava [2007] for étale extensions of a local field is formulated as

$$M(K,S_n,\boldsymbol{a}_\sigma)=\sum_{m=0}^{n-1}P(n,n-m)\cdot q^{-m}.$$

Here $P(n,n-m)$ denotes the number of partitions of n into exactly $n-m$ parts.

Let

$$\tau := \sigma\oplus\sigma : S_n \to \mathrm{GL}_{2n}(\mathcal{O}_K).$$

From Lemma 4.3, we have

$$M(K,S_n,\boldsymbol{v}_\tau)=M(K,S_n,\boldsymbol{a}_\sigma)=\sum_{m=0}^{n-1}P(n,n-m)\cdot q^{-m}.$$

In [Wood and Yasuda 2015], we verified that

$$M(K,S_n,-\boldsymbol{w}_\tau)=\sum_{m=0}^{n}P(n,n-m)\cdot q^{m}.$$

These show that the functions $M(K_r,S_n,-\boldsymbol{w}_\tau)$ and $M(K_r,S_n,\boldsymbol{v}_\tau)$ satisfy the strong duality (4-3). Thus the answer to Question 5.2 is positive in this situation.

The quotient scheme $X=\mathbb{A}^{2n}_{\mathcal{O}_K}/S_n$ associated to τ is nothing but the n-th symmetric product of the affine plane over $\mathcal{O}_K$. It admits a special ESWL resolution, namely the Hilbert–Chow morphism

$$f:\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})\to X,$$

from the Hilbert scheme of n points on $\mathbb{A}^2_{\mathcal{O}_K}$ defined over $\mathcal{O}_K$.

Using a stratification of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ into affine spaces [Ellingsrud and Strømme 1987; Conca and Valla 2008], we can directly show

$$\sharp\,\mathrm{Hilb}^n(\mathbb{A}_k^2) = \sum_{m=0}^{n} P(n, n-m)\cdot q^{2n-m}.$$

These together with Theorem 4.4 gives a new proof of Bhargava's formula without using the mass formula of Serre [1978] unlike [Bhargava 2007; Kedlaya 2007].

Kedlaya's formula. We suppose $p \neq 2$ and let G be the group of signed permutation matrices in $\mathrm{GL}_n(\mathcal{O}_K)$. A signed matrix is a matrix such that every row or column contains one and only one nonzero entry which is either 1 or -1. The group is isomorphic to the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$. Let

$$\iota : G \hookrightarrow \mathrm{GL}_n(\mathcal{O}_K)$$

be the inclusion and $\tau := \iota \oplus \iota$.

Lemma 5.3. *We have $\boldsymbol{a}_\iota = \boldsymbol{v}_\tau$.*

Proof. We construct a representation $\iota' : G \to \mathrm{GL}_{2n}(\mathcal{O}_K)$ by replacing entries of matrices in G as follows: replace 0 with $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, 1 with $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and -1 with $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The obtained ι' is a permutation representation of G and, from Lemma 4.3,

$$\boldsymbol{a}_{\iota'} = \boldsymbol{v}_{\iota'\oplus\iota'}.$$

If $\tau : G \to \mathrm{GL}_n(\mathcal{O}_K)$ is the trivial representation, then $\iota' \cong \iota \oplus \tau$. The facts that $\boldsymbol{a}_\tau \equiv \boldsymbol{v}_\tau \equiv 0$, $\boldsymbol{a}_{\iota\oplus\tau} = \boldsymbol{a}_\iota + \boldsymbol{a}_\tau$ and $\boldsymbol{v}_{\iota\oplus\tau} = \boldsymbol{v}_\iota + \boldsymbol{v}_\tau$, which were proved in [Wood and Yasuda 2015], prove the lemma. □

We then have

$$M(K, G, \boldsymbol{v}_\tau) = M(K, G, \boldsymbol{a}_\iota) = \sum_{j=0}^{n}\sum_{i=0}^{j} P(j,i)P(n-j)q^{i-n}, \tag{5-1}$$

where the right equality is due to Kedlaya [2007, Remark 8.6] and $P(m)$ denotes the number of partitions of m.

We can construct a resolution of $X := \mathbb{A}^{2n}_{\mathcal{O}_K}/G$ as follows. First consider the involution of $\mathbb{A}^2_{\mathcal{O}_K} = \operatorname{Spec} \mathcal{O}_K[x, y]$ sending x to $-x$ and y to $-y$. Let

$$Z := \mathbb{A}^2_{\mathcal{O}_K}/(\mathbb{Z}/2\mathbb{Z}) = \operatorname{Spec} \mathcal{O}_K[x^2, xy, y^2]$$

be the associated quotient scheme. We have a natural isomorphism

$$X \cong S^n Z := Z^n/S_n.$$

Since Z is the trivial family of A_1-singularities over $\operatorname{Spec} \mathcal{O}_K$ (actually it is a toric variety), there exists the minimal resolution $W \to Z$ of Z such that W is $\mathcal{O}_K$-smooth, the exceptional locus is isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$ and $K_{W/Z}=0$. Let $\mathrm{Hilb}^n(W)$ be the Hilbert scheme of n points of W over $\mathcal{O}_K$, which is again smooth over $\mathcal{O}_K$. We have the proper birational morphism

$$f:\mathrm{Hilb}^n(W)\xrightarrow{\text{Hilbert–Chow}} S^nW\to S^nZ\xrightarrow{\sim} X$$

and $K_{\mathrm{Hilb}^n(W)/X}=0$.

We now compute $\sharp f^{-1}(o)(k)$. Let $E\subset W_k$ be the exceptional divisor of $W_k\to Z_k$, which is isomorphic to $\mathbb{P}^1_k$. Each k-point of $f^{-1}(o)$ corresponds to a zero-dimensional subscheme of W_k of length n supported in E. There exists a stratification

$$E=E_1\sqcup E_0$$

such that $E_1\cong\mathbb{A}^1_k$ is a coordinate line of an open subscheme $\mathbb{A}^2_k\cong U\subset W_k$ and $E_0\cong\operatorname{Spec}k$ is the origin of another open subscheme $\mathbb{A}^2_k\cong U'\subset W_k$. Let C_m be the set of zero-dimensional subschemes U of length m supported in E_1 and D_m the set of zero-dimensional subschemes of U' of length m supported in E_0. From [Conca and Valla 2008, Corollary 3.1],

$$\sharp C_m=P(m)q^m \quad\text{and}\quad \sharp D_m=\sum_{i=0}^{m}P(m,i)q^{m-i}.$$

Since $f^{-1}(o)(k)$ decomposes as

$$f^{-1}(o)(k)=\bigsqcup_{j=0}^{n}C_{n-j}\times D_j,$$

we have

$$\begin{aligned}\sharp f^{-1}(o)(k)&=\sum_{j=0}^{n}\left(P(n-j)q^{n-j}\sum_{i=0}^{j}P(j,i)q^{j-i}\right)\\&=\sum_{j=0}^{n}\sum_{i=0}^{j}P(j,i)P(n-j)q^{n-i}.\end{aligned}$$

From the wild McKay correspondence,

$$M(K,G,-\boldsymbol{w}_\tau)=\sum_{j=0}^{n}\sum_{i=0}^{j}P(j,i)P(n-j)q^{n-i}.$$

Comparing this with (5-1), we verify the strong duality for $M(K_r,G,-\boldsymbol{w}_\tau)$ and $M(K_r,G,\boldsymbol{v}_\tau)$.

In a similar way to computing $\sharp f^{-1}(o)(k)$, we can directly compute $\sharp\mathrm{Hilb}^n(W)(k)$ and obtain a new proof of Kedlaya's formula (5-1).

6. Concluding remarks and extra problems

As far as we computed, we observed the following phenomena. The weak duality and the strong duality may fail. Tame representations satisfy both the strong and weak dualities. Permutation representations satisfy the weak duality but not necessarily the strong duality. When $K = \mathbb{F}_q((t))$, representations defined over $\mathbb{F}_q$ satisfy both the strong and weak dualities.

We may interpret these as follows. We might be able to measure by the two dualities how equisingular the family $\mathbb{A}^d_{\mathcal{O}_K}/\Gamma$ is. If the strong duality holds, then the family would be very equisingular. If only the weak duality holds, then the family would be moderately so. If both dualities fail, then it is rather far from being equisingular. It may be interesting to look for a numerical invariant refining this measurement.

As we saw, the weak duality is derived from the Poincaré duality of stringy invariants if there exists an ESWL resolution. What about the strong duality? Is there any geometric interpretation of it?

Although it is still conjectural, there would be the motivic counterparts of total masses, stringy point counts and the McKay correspondence between them (see [Wood and Yasuda 2015; Yasuda 2013; 2014a; 2016]). We may discuss dualities in this motivic context as well. Indeed, some dualities for motivic invariants were verified in [Yasuda 2014a; 2016].

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