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A GENERALIZATION OF CHENG'S THEOREM*

PETER LI† AND JIAPING WANG‡

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0. Introduction. In this paper, we prove a generalization of a theorem of S.Y. Cheng on the upper bound of the bottom of the L^2 spectrum for a complete Riemannian manifold. In [C], Cheng proved a comparison theorem for the first Dirichlet eigenvalue of a geodesic ball. By taking the radius of the ball to infinity, he obtained an estimate for the bottom of the L^2 spectrum. In particular, he showed that if M^n is an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by $-(n-1)K$ for some constant $K > 0$, then the bottom of the L^2 spectrum, $\lambda_1(M)$, is bounded by

$$\lambda_1(M) \leq \frac{(n-1)^2 K}{4}.$$

This upper bound of $\lambda_1(M)$ is sharp as it is achieved by the hyperbolic space form $\mathbb{H}^n$. Observe that Cheng's theorem can be stated in the following equivalent form.

Cheng's Theorem. *Let M^n be a complete Riemannian manifold of dimension n. If $\lambda_1(M) > 0$ and there exists a constant $A \geq 0$ such that the Ricci curvature of M satisfies*

$$Ric_M \geq -A\lambda_1(M), \tag{0.1}$$

then A must be bounded by

$$A \geq \frac{4}{n-1}.$$

In a previous paper [LW] of the authors, they consider complete Riemannian manifolds on which there is a nontrivial weight function $\rho(x) \geq 0$ for all $x \in M$, such that, the weighted Poincaré inequality

$$\int_M |\nabla\phi|^2 \, dV \geq \int_M \rho\, \phi^2 \, dV$$

is valid for all functions $\phi \in C_c^\infty(M)$. Note that if $\lambda_1(M) > 0$ then $\lambda_1(M)$ can be used as a weight function by the variational characterization of $\lambda_1(M)$, namely,

$$\inf_{\phi \in C_c^\infty(M)} \frac{\int_M |\nabla\phi|^2 \, dV}{\int_M \phi^2 \, dV} = \lambda_1(M).$$

With this point of view, a weight function ρ can be thought of as a pointwise generalization of $\lambda_1(M)$. It was pointed out in [LW] that manifolds possessing a weighted Poincaré inequality is equivalent to being nonparabolic - those admitting a positive

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† Department of Mathematics, University of California, Irvine, CA 92697-3875, USA (pli@math.uci.edu). The author was partially supported by NSF grant DMS-0503735.
‡ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA (jiaping@math.umn.edu). The author was partially supported by NSF grant DMS-0706706.

Green's function for the Laplacian. The main purpose of the short note is to prove the following generalization of Cheng's theorem for manifolds with a weighted Poincaré inequality.

Theorem 1. *Let M^n be a complete Riemannian manifold of dimension n. Suppose there is a nontrivial weight function $\rho(x) \geq 0$ such that the weighted Poincaré inequality*

$$\int_M |\nabla \phi|^2(x)\, dV \geq \int_M \phi^2(x)\, \rho(x)\, dV$$

holds for all test function $\phi \in C_c^\infty(M)$. Assume that the Ricci curvature of M is bounded below by

$$Ric_M(x) \geq -A\,\rho$$

for some constant $A \geq 0$. If, in addition, there exists $\frac{1}{2} < \alpha \leq 1$ such that the conformal metric $\rho^{2\alpha}\, ds^2$ is complete, then A must be bounded by

$$A \geq \frac{4}{n-1}.$$

Let us remark that when $\rho = \lambda_1(M)$, the metric $\lambda_1(M)^{2\alpha}\, ds^2$ is complete for all $\alpha > 0$, hence Theorem 1 is exactly Cheng's theorem stated as above. Moreover, we observe that on $\mathbb{R}^n$ for $n \geq 3$, the function

$$\rho(x) = \frac{(n-2)^2}{4}\, r^{-2}(x),$$

where $r(x)$ is the Euclidean distance to the origin, is a weight function. The condition on the completeness of the conformal metric $\rho^{2\alpha}\, ds^2$ is equivalent to the condition

$$\int_1^\infty r^{-2\alpha}\, dr = \infty,$$

hence the conformal metric is complete if and only if $\alpha \leq \frac{1}{2}$. In this case, since the inequality between the Ricci curvature and the weight function is automatically satisfied for all $A \geq 0$, this indicates that the condition on the completeness of the $\rho^{2\alpha}\, ds^2$ is necessary and sharp.

1. Prelimaries. The proof of Theorem 1 is motivated by the work of X. Cheng [Cg], where she proved that a manifold satisfying the hypothesis of Theorem 1 with $A < \frac{4}{n-1}$ must have only one end. Her approach was different from the authors in [LW] where they also proved various versions of structural theorems for manifolds with property $\mathcal{P}_\rho$. These are manifolds with a weight function ρ such that the conformal metric $ds_\rho^2 = \rho\, ds^2$ is complete. The first part of our argument pretty much follows that of Cheng and so we will refer the reader to [Cg] for some of the detailed but direct computation.

Let M^n be an n-dimensional Riemannian manifold with the metric given by ds^2. Suppose u is a positive function defined on M. We define the new conformal metric by

$$\tilde{ds}^2 = u^2\, ds^2.$$

We will recall some of the computations on a conformal change of metrics. Let $\{\omega_i\}$ be an orthonormal coframe defined on M with respect to ds^2. Then $\{\tilde{\omega}_i = u\, \omega_i\}$ is an orthonormal coframe with respect to $\tilde{ds^2}$. The connection 1-forms with respect to ds^2 and $\tilde{ds^2}$ are related by

$$\tilde{\omega}_{ij} = \omega_{ij} - (\log u)_j\,\omega_i + (\log u)_i\,\omega_j. \tag{1.1}$$

The curvature tensors with respect to ds^2 and $\tilde{ds^2}$ are related by

$$\begin{aligned}\frac{1}{2}\tilde{R}_{ijk\ell}\,\tilde{\omega}_\ell\wedge\tilde{\omega}_k = {}& \frac{1}{2}u^{-2}\,R_{ijk\ell}\,\tilde{\omega}_\ell\wedge\tilde{\omega}_k - u^{-2}\,(\log u)_{jk}\,\tilde{\omega}_k\wedge\tilde{\omega}_i + u^{-2}\,(\log u)_{ik}\,\tilde{\omega}_k\wedge\tilde{\omega}_j\\ &+ u^{-2}\,|\nabla(\log u)|^2\,\tilde{\omega}_i\wedge\tilde{\omega}_j - u^{-2}\,(\log u)_k\,(\log u)_i\,\tilde{\omega}_k\wedge\tilde{\omega}_j\\ &- u^{-2}\,(\log u)_k\,(\log u)_j\,\tilde{\omega}_i\wedge\tilde{\omega}_k,\end{aligned} \tag{1.2}$$

where $(\log u)_{jk}$ denotes the Hessian of $\log u$ in the direction of e_j and e_k with respect to the metric ds^2.

The sectional curvatures and Ricci curvatures are then related by

$$u^2\,\tilde{K}(\tilde{e}_i,\tilde{e}_j) = K(e_i,e_j) - |\nabla(\log u)|^2 + (\log u)_i^2 + (\log u)_j^2 - (\log u)_{ii} - (\log u)_{jj}, \tag{1.3}$$

and

$$u^2\,\tilde{\mathrm{Ric}}_{ii} = \mathrm{Ric}_{ii} - (n-2)|\nabla(\log u)|^2 + (n-2)(\log u)_i^2 - \Delta(\log u) - (n-2)(\log u)_{ii}. \tag{1.4}$$

Let $N\subset M$ be a minimal submanifold of dimension $d<n$ with respect to the $\tilde{ds^2}$ metric. We choose an adapted orthonormal frame so that $\{e_1,\dots,e_d\}$ are tangent to N and $\{e_{d+1},\dots,e_n\}$ are normal to N. In particular, $\{\tilde{e}_\nu = u^{-1}e_\nu\,|\,\nu = d+1,\dots n\}$ are unit normal vectors to N with respect to $\tilde{ds^2}$. The second fundamental forms $h^\nu_{\alpha\beta}$ and $\tilde{h}^\nu_{\alpha\beta}$ corresponding to the metrics ds^2 and $\tilde{ds^2}$, respectively, in the direction of e_ν and $\tilde{e}_\nu$ are given by

$$\tilde{h}^\nu_{\alpha\beta} = u^{-1}\,h^\nu_{\alpha\beta} + u^{-1}\,(\log u)_\nu\,\delta_{\alpha\beta},$$

for $1\le\alpha,\beta\le d$. The minimality condition implies that

$$H^\nu = (\log u)_\nu$$

where H^ν is the mean curvature in the direction of ν with respect to the metric ds^2.

If we further assume that N is stable in the $\tilde{ds^2}$ metric, then the stability inequality asserts that, for any normal vector field $T = \sum_\nu \phi^\nu\tilde{e}_\nu$, we have

$$\begin{aligned}0 \le {}& -\int_N \Big\{\sum_\nu\sum_{\alpha,\,\beta}\phi^\nu\,(\tilde{h}^\nu_{\alpha\beta})^2 + \sum_{\nu,\mu}\sum_\alpha \phi^\nu\,\phi^\mu\,\langle\tilde{R}_{\tilde{e}_\alpha\tilde{e}_\nu}\tilde{e}_\mu,\tilde{e}_\alpha\rangle\Big\}\,d\tilde{V}\\ &+ \int_N\Big\{\sum_\alpha\sum_\nu\left(\sum_\mu\phi^\mu\,\langle\tilde{\nabla}_{\tilde{e}_\alpha}\tilde{e}_\mu,\tilde{e}_\nu\rangle\right)^2 + \sum_\nu|\tilde{\nabla}^N\phi^\nu|^2\Big\}\,d\tilde{V},\end{aligned} \tag{1.5}$$

where $\tilde{\nabla}^N$ denotes the gradient on N with respect to the induced metric from $\tilde{ds^2}$.

2. Proof of Theorem 1.

Proof. Now let us consider the case when $N = \gamma$ is a stable geodesic. The second variation formula (1.5) asserts that

$$\int_\gamma \sum_\nu |\tilde\nabla^\gamma \phi^\nu|^2 \, \tilde{ds} \geq \int_\gamma \sum_{\nu,\mu} \phi^\nu \, \phi^\mu \, \langle \tilde R_{\tilde e_1 \tilde e_\nu} \tilde e_\mu, \tilde e_1 \rangle \, \tilde{ds} - \int_\gamma \sum_\nu \left(\sum_\mu \phi^\mu \, \langle \tilde\nabla_{\tilde e_1} \tilde e_\mu, \tilde e_\nu \rangle \right)^2 \tilde{ds}.$$

By choosing orthonormal frame $\{e_2, \dots, e_n\}$ so that they are parallel along the geodesic, and for each e_ν, by choosing $\phi^\mu = 0$ when $\mu \neq \nu$ and $\phi^\nu = \phi$, the above inequality yields

$$\begin{aligned}\int_\gamma |\tilde\nabla^\gamma \phi|^2 \, \tilde{ds} &\geq \int_\gamma \phi^2 \, \tilde K(\tilde e_1, \tilde e_\nu) \, \tilde{ds} - \int_\gamma \phi^2 \, \sum_\nu \langle \tilde\nabla_{\tilde e_1} \tilde e_\nu, \tilde e_\nu \rangle^2 \, \tilde{ds} \\ &= \int_\gamma \phi^2 \, u^{-1} \, \big(K(e_1, e_\nu) - |\nabla(\log u)|^2 + (\log u)_1^2 + (\log u)_\nu^2 \\ &\quad -(\log u)_{11} - (\log u)_{\nu\nu} \big) \, ds,\end{aligned}$$

for all ν. Summing over all $2 \leq \nu \leq n$, we obtain

(2.1)
$$\begin{aligned}(n-1) \int_\gamma u^{-1} \, |\nabla^\gamma \phi|^2 \, ds &\geq \int_\gamma \phi^2 \, u^{-1} \, \mathrm{Ric}_{11} \, ds - (n-1) \int_\gamma \phi^2 \, u^{-1} \, |\nabla(\log u)|^2 \, ds \\ &\quad + (n-1) \int_\gamma \phi^2 \, u^{-1} \, (\log u)_1^2 \, ds + \int_\gamma \phi^2 \, u^{-1} \sum_\nu (\log u)_\nu^2 \, ds \\ &\quad - (n-1) \int_\gamma \phi^2 \, u^{-1} \, (\log u)_{11} \, ds - \int_\gamma \phi^2 \, u^{-1} \, \sum_\nu (\log u)_{\nu\nu} \, ds \\ &= \int_\gamma \phi^2 \, u^{-1} \, \mathrm{Ric}_{11} \, ds - (n-2) \int_\gamma \phi^2 \, u^{-1} \, |\nabla(\log u)|^2 \, ds \\ &\quad + (n-2) \int_\gamma \phi^2 \, u^{-1} \, (\log u)_1^2 \, ds - (n-2) \int_\gamma \phi^2 \, u^{-1} \, (\log u)_{11} \, ds \\ &\quad - \int_\gamma \phi^2 \, u^{-1} \, \Delta(\log u) \, ds.\end{aligned}$$

The fact γ is a geodesic with respect to the metric $\tilde{ds}^2$ together with (1.1) implies that

$$\begin{aligned}(\log u)_{11} &= (\log u)'' - \nabla_{e_1} e_1 (\log u) \\ &= (\log u)'' - \sum_\nu (\log u)_\nu^2 \\ &= (\log u)'' - |\nabla(\log u)|^2 + ((\log u)')^2,\end{aligned}$$

where prime denotes differentiating with respect to $\frac{\partial}{\partial s} = e_1$. Hence, we have

(2.2)
$$\begin{aligned}&\int_\gamma \phi^2 \, u^{-1} \, (\log u)_{11} \, ds \\ &= 2 \int_\gamma \phi^2 \, u^{-1} \, (\log u)_1^2 \, ds - 2 \int_\gamma \phi \, u^{-1} \, \phi_1 \, (\log u)_1 \, ds - \int_\gamma \phi^2 \, u^{-1} \, |\nabla(\log u)|^2 \, ds.\end{aligned}$$

Using the assumption that M admits the weighted Poincaré inequality

$$\int_M |\nabla\phi|^2\,dV \geq \int_M \phi^2\,\rho\,dV$$

for the weight function ρ, there exists a positive solution v to the equation

$$(\Delta+\rho)v = 0.$$

Letting $u = v^k$, we have

$$\Delta(\log u) = -k\rho - k^{-1}|\nabla(\log u)|^2$$

Substituting this and (2.2) into (2.1), we have

$$\begin{aligned}
&(n-1)\int_\gamma u^{-1}\,(\phi')^2\,ds \\
&\geq \int_\gamma \phi^2\,u^{-1}\,(\mathrm{Ric}_{11}+k\rho)\,ds + k^{-1}\int_\gamma \phi^2\,u^{-1}\,|\nabla(\log u)|^2\,ds \\
&\quad - (n-2)\int_\gamma \phi^2\,u^{-1}\,((\log u)')^2\,ds + 2(n-2)\int_\gamma \phi\,u^{-1}\,\phi_1\,(\log u)'\,ds.
\end{aligned}$$

Setting $\phi = u^{\frac{1}{2}}\,\psi$, we conclude that

(2.3)

$$\begin{aligned}
(n-1)\int_\gamma (\psi')^2\,ds \geq& \int_\gamma \psi^2\,(\mathrm{Ric}_{11}+k\rho)\,ds + k^{-1}\int_\gamma \psi^2\,|\nabla(\log u)|^2\,ds \\
&+ (n-3)\int_\gamma \psi\,\psi_1\,(\log u)'\,ds - \frac{n-1}{4}\int_\gamma \psi^2\,((\log u)')^2\,ds.
\end{aligned}$$

Also, let γ be a geodesic ray, with respect to the metric $d\tilde{s}^2$, emanating from a fixed point $p \in M$ to an end of M. Let us parametrize $\gamma : [0,\infty) \to M$ by arc-length with respect to the metric ds^2. According to (2.3) and the Schwarz inequality, we have

(2.4)

$$\begin{aligned}
2\int_0^\infty (\psi')^2\,ds \geq& \int_0^\infty \psi^2\,(k-A)\rho\,ds + k^{-1}\int_0^\infty \psi^2\,|\nabla(\log u)|^2\,ds \\
&+ (n-3)\int_0^\infty \psi\,\psi'\,(\log u)'\,ds - \frac{n-1}{4}\int_0^\infty \psi^2\,(\log u)_1^2\,ds \\
\geq& \int_0^\infty \psi^2\,(k-A)\rho\,ds + k^{-1}\int_0^\infty \psi^2\,|\nabla(\log u)|^2\,ds \\
&- \frac{(n-3)^2}{4\epsilon}\int_0^\infty (\psi')^2\,ds - \left(\frac{n-1}{4}+\epsilon\right)\int_0^\infty \psi^2\,((\log u)')^2\,ds,
\end{aligned}$$

for any $\epsilon > 0$. If we choose $\epsilon = k^{-1} - \frac{n-1}{4}$, inequality (2.4) can then be written as

$$\left(2 + \frac{(n-3)^2}{4k^{-1}-(n-1)}\right)\int_0^\infty (\psi')^2\,ds \geq (k-A)\int_0^\infty \psi^2\,\rho\,ds. \tag{2.5}$$

Assuming that $A < \frac{4}{n-1}$, we can choose $A < k < \frac{4}{n-1}$ to ensure that the coefficients on both sides are positive. In particular, by taking $\psi = s^{\frac{1}{2}}\,\eta$ with

$$\eta(s) = \begin{cases} s & \text{for} \quad 0 \leq s \leq 1 \\ 1 & \text{for} \quad 1 \leq s \leq R \\ \dfrac{2R - s}{R} & \text{for} \quad R \leq s \leq 2R \\ 0 & \text{for} \quad 2R \leq s, \end{cases}$$

we conclude that

$$\begin{aligned} \int_0^\infty (\psi')^2\,ds &= \int_0^1 (\psi')^2 + \int_R^{2R} s\,(\eta')^2\,ds + \int_R^{2R} \eta\,\eta'\,ds + \frac{1}{4}\int_1^{2R} s^{-1}\,\eta^2\,ds \\ &\leq \frac{33}{16} + \frac{1}{4}\log(2R), \end{aligned}$$

for $R > 1$. Hence (2.5) can be written as

$$C_1 + C_2 \log R \geq \int_1^R s\,\rho\,ds. \tag{2.6}$$

On the other hand, for $\frac{1}{2} < \alpha \leq 1$, the Schwarz inequality and (2.6) assert that

$$\begin{aligned} \int_R^{2R} \rho^\alpha\,ds &\leq \left(\int_R^{2R} s\,\rho\,ds\right)^\alpha \left(\int_R^{2R} s^{-\frac{\alpha}{1-\alpha}}\,ds\right)^{1-\alpha} \\ &= \left(\int_R^{2R} s\,\rho\,ds\right)^\alpha \left(R^{\frac{1-2\alpha}{1-\alpha}} - (2R)^{\frac{1-2\alpha}{1-\alpha}}\right)^{1-\alpha} \left(\frac{1-\alpha-1}{2\alpha-1}\right)^{1-\alpha} \\ &\leq C_3\,(\log R)^\alpha\,R^{1-2\alpha}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_1^\infty \rho^\alpha\,ds &= \sum_{i=0}^\infty \int_{2^i}^{2^{i+1}} \rho^\alpha\,ds \\ &\leq C_3 \sum_{i=0}^\infty (\log 2^i)^\alpha\,2^{(1-2\alpha)i} \\ &\leq C_4 \sum_{i=0}^\infty i^\alpha\,2^{(1-2\alpha)i} \\ &< \infty. \end{aligned}$$

In particular, this gives a contradiction if the metric $\rho^{2\alpha}\,ds^2$ is complete.

The following corollary slightly strengthens the aforementioned result of X. Cheng in [Cg].

Corollary 2. *Let M^n be a complete Riemannian manifold of dimension n. Suppose there is a nontrivial weight function $\rho(x) \geq 0$ such that the weighted Poincaré inequality*

$$\int_M |\nabla\phi|^2(x)\,dV \geq \int_M \phi^2(x)\,\rho(x)\,dV$$

holds for all test function $\phi \in C_c^\infty(M)$. *Assume that there exists a constant*

$$A < \frac{4}{n-1},$$

such that, the Ricci curvature of M *is bounded below by:*

$$\text{(1)} \qquad \textit{either} \quad Ric_M(x) \geq -A\,\rho \quad \textit{and} \quad \rho > 0;$$

$$\text{(2)} \qquad \textit{or} \quad Ric_M(x) > -A\,\rho.$$

Then M *must have only one end and is simply connected at infinity.*

Proof. Note that if M has a stable geodesic segment γ with respect to the $\tilde{ds}^2$ metric that can be parametrized by $\gamma : (-\infty, \infty) \to M$ in arc-length with respect to ds^2, then (2.5) will imply that along γ it must satisfy the weighted Poincaré inequality with weight function $\rho(\gamma(s))$. Hence the real line is nonparabolic, which is an obvious contradiction. In particular, this rules out the possibility of M having two ends.

To see that M is simply connected at infinity, we consider any curve $\tau(t)$ parameterized by $t \in (-\infty, \infty)$ satisfying

$$\lim_{t \to \infty} \tau(t) = \infty$$

and

$$\lim_{t \to -\infty} \tau(t) = \infty.$$

One should take the point of view that τ is a curve in $\bar{M} = M \cup M_\infty$ with based point M_∞, where $\bar{M}$ is the one-point compactification of M. Assuming that $\pi_1(M, M_\infty) \neq \{1\}$, let $[\tau]$ be a nontrivial class in $\pi_1(M, M_\infty)$. For any curve $\tau \in [\tau]$, we let γ_t be a minimal geodesic with respect to $\tilde{ds}^2$ joining the points $\tau(-t)$ to $\tau(t)$, which is in the same homotopy class of $\tau|_{[-t,t]}$. Since $[\tau]$ is nontrivial, there exists a sequence of $t_i \to \infty$ such that $\gamma_{t_i} \cap B_p(R) \neq \emptyset$. Indeed, if not, then the curves given by $\eta_t = \tau|_{(\infty,-t]} \cup \gamma_t \cup \tau|_{[t,\infty)}$ will not intersect $B_p(R)$ for t sufficiently large. This will imply that $\eta_t \to M_\infty$ and $[\tau]$ is trivial. So a subsequence of the curves η_t will converge to some limiting curve $\gamma \in [\tau]$. Moreover, γ will be a stable geodesic because it is the limit of minimal geodesics in $B_p(R)$ for all R. Hence, we produced a stable geodesic γ in M which gives a contradiction. $\square$

Let us point out that the above argument is valid if we only assume the weighted Poincaré inequality only holds outside some compact set of M. This strengthened version of Theorem 1 is a generalization of the statement that if $\mathrm{Ric}_M \geq -(n-1)K$ on $M \setminus D$, then the bottom of the essential spectrum of M is bounded from above by $\frac{(n-1)^2 K}{4}$.

THEOREM 3. *Let* M^n *be a complete Riemannian manifold. Suppose there exists a compact set* D *and a weight function* ρ *defined on* $M \setminus D$ *such that*

$$\int_{M \setminus D} |\nabla \phi|^2 \geq \int_{M \setminus D} \rho\, \phi^2$$

for all functions $\phi \in C_c^\infty(M \setminus D)$. *Assume that the Ricci curvature of* M *is bounded below by*

$$Ric_M(x) \geq -A\,\rho$$

on $M \setminus D$ for some constant $A \geq 0$. If there exists $\frac{1}{2} < \alpha \leq 1$ such that the conformal metric $\rho^{2\alpha}\, ds^2$ is complete, then

$$A \geq \frac{4}{n-1}.$$

The same type of argument also give the following corollary.

COROLLARY 4. *Let M^3 be a complete Riemannian manifold of dimension 3. Suppose there is a nontrivial weight function $\rho(x) \geq 0$ such that the weighted Poincaré inequality*

$$\int_M |\nabla \phi|^2(x)\, dV \geq \int_M \phi^2(x)\, \rho(x)\, dV$$

holds for all test function $\phi \in C_c^\infty(M)$. Assume that the Ricci curvature of M is bounded below by

$$Ric_M(x) \geq -\frac{4}{n-1}\,\rho + \bar{\rho}$$

for some nonnegative function $\bar{\rho}$. Then the conformal metric $\bar{\rho}^{2\alpha}\, ds^2$ cannot be complete for any $\alpha > \frac{1}{2}$.

Proof. When $n = 3$, by setting $k = \frac{4}{n-1}$, (2.3) becomes

$$2\int_\gamma (\psi')^2\, ds \geq \int_\gamma \psi^2\, \bar{\rho}\, ds.$$

The proof of the theorem now applies to this case.

An example of the corollary is the hyperbolic 3-space, $\mathbb{H}^3$, whose Ricci curvature is -2. In this case, we know that $\lambda_1 = 1$, hence it is a weight function. We also know that it is not a maximal weight function since

$$1 + 2(\coth r - 1)$$

and

$$\frac{1}{4}\sinh^{-4} r\left(\int_0^r \sinh^{-2} t\, dt\right)^{-2}$$

are also weight functions. The corollary implies that if there is a weight function $\rho = 1 + \bar{\rho}$, then $\bar{\rho}$ cannot be too large in the sense that the metric $\bar{\rho}^{2\alpha}\, ds^2$ cannot be complete. This is certainly the case for the above two weight functions. The corollary also implies that if we deform the metric on $\mathbb{H}^3$ while maintaining the condition $\lambda_1 = 1$, then the Ricci curvature of the new metric cannot be too much smaller than -1.

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