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The Lannes–Zarati homomorphism and decomposable elements

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Let X be a pointed CW–complex. The generalized conjecture on spherical classes states that *the Hurewicz homomorphism* $H\colon \pi_*(Q_0X) \to H_*(Q_0X)$ *vanishes on classes of* $\pi_*(Q_0X)$ *of Adams filtration greater than* 2. Let $\varphi_s^M\colon \mathrm{Ext}^s_{\mathcal{A}}(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)^*$ denote the s^{th} Lannes–Zarati homomorphism for the unstable $\mathcal{A}$–module M. When $M = \widetilde{H}^*(X)$, this homomorphism corresponds to an associated graded of the Hurewicz map. An algebraic version of the conjecture states that the s^{th} Lannes–Zarati homomorphism, φ_s^M, vanishes in any positive stem for $s > 2$ and for any unstable $\mathcal{A}$–module M.

We prove that, for M an unstable $\mathcal{A}$–module of finite type, the s^{th} Lannes–Zarati homomorphism, φ_s^M, vanishes on decomposable elements of the form $\alpha\beta$ in positive stems, where $\alpha \in \mathrm{Ext}^p_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta \in \mathrm{Ext}^q_{\mathcal{A}}(M,\mathbb{F}_2)$ with either $p \geq 2$, $q > 0$ and $p+q = s$, or $p = s \geq 2$, $q = 0$ and $\mathrm{stem}(\beta) > s-2$. Consequently, we obtain a theorem proved by Hưng and Peterson in 1998. We also prove that the fifth Lannes–Zarati homomorphism for $\widetilde{H}^*(\mathbb{RP}^\infty)$ vanishes on decomposable elements in positive stems.

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1 Introduction and statement of results

Let X be a pointed CW–complex. Let $Q_0X = \Omega_0^\infty S^\infty X$ be the basepoint component of $QX = \Omega^\infty S^\infty X$. It is a classical unsolved problem to compute the image of the Hurewicz homomorphisms

$$H\colon \pi_*^S(X) = \pi_*(Q_0X) \to H_*(Q_0X).$$

Here and throughout the paper, homology and cohomology are taken with coefficients in $\mathbb{F}_2$, the field of two elements. The classical conjecture on spherical classes for $X = S^0$ states that the Hopf invariant-one and the Kervaire invariant-one classes are the only elements in $\pi_*^S(S^0) \cong \pi_*(Q_0S^0)$ detected by the Hurewicz homomorphism. Nguyễn H V Hưng states the generalized conjecture on spherical classes as follows (see Hưng and Tuấn [14]).

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Conjecture 1.1 Let X be a pointed CW–complex. Then the Hurewicz homomorphism $H\colon \pi_*(Q_0X) \to H_*(Q_0X)$ vanishes on classes of $\pi_*(Q_0X)$ of Adams filtration greater than 2.

(See Curtis [4], Snaith and Tornehave [21] and Wellington [22] for a discussion with $X = S^0$.)

An algebraic version of this problem goes as follows.

Let $P_s = \mathbb{F}_2[x_1, \dots, x_s]$ be the polynomial algebra on s indeterminates $x_1, \dots, x_s$, each of degree 1. Let the general linear group $\mathrm{GL}_s = \mathrm{GL}(s, \mathbb{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ both act on P_s in the usual way. The Dickson algebra of s variables, D_s, is the algebra of invariants

$$D_s := \mathbb{F}_2[x_1, \dots, x_s]^{\mathrm{GL}_s}.$$

As the action of $\mathcal{A}$ and that of GL_s on P_s commute with each other, D_s is an algebra over $\mathcal{A}$.

Let M be an unstable $\mathcal{A}$–module. The Singer construction R_sM of M is the D_s–submodule of $P_s \otimes M$ generated by $\mathrm{St}_s\,M$, where St_s denotes the Steenrod homomorphism defined as follows. Given a homogeneous element $z \in M$ of degree $|z|$, we set for convention $\mathrm{St}_0(z) = z$, and define by induction

$$\mathrm{St}_1(x; z) = \sum_{i=0}^{|z|} x^{|z|-i} \otimes \mathrm{Sq}^i(z),$$
$$\mathrm{St}_s(x_1, \dots, x_s; z) = \mathrm{St}_1(x_1; \mathrm{St}_{s-1}(x_2, \dots, x_s; z)).$$

Note that R_sM is an $\mathcal{A}$–submodule of $P_s \otimes M$. (See Lannes and Zarati [16, Definition-Proposition 2.4.1].)

Let us denote by

$$\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(M, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i{}^*$$

the s^{th} Lannes–Zarati homomorphism for an unstable $\mathcal{A}$–module M, defined in [16]. Here $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i{}^*$ is the $\mathbb{F}_2$–dual of $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i$. When $M = \widetilde{H}^*(X)$, this homomorphism corresponds to an associated graded of the Hurewicz map. The proof of this assertion is unpublished, but it is sketched by Lannes [15] and by Goerss [7].

The Hopf invariant-one and the Kervaire invariant-one classes are represented by certain permanent cycles in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, respectively, on which the Lannes–Zarati homomorphisms are nonzero (see Adams [1], Browder [3] and Lannes and Zarati [16]). Hưng stated the so-called algebraic version of the generalized conjecture on spherical classes for $M = \widetilde{H}^*(S^0) = \mathbb{F}_2$ in [9] and for any unstable $\mathcal{A}$–module M in [14].

Conjecture 1.2 (the generalized algebraic spherical class conjecture) The Lannes–Zarati homomorphism

$$\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)_i{}^*$$

vanishes in any positive stem i for $s > 2$, and for any unstable $\mathcal{A}$–module M.

The conjecture was established for the case $M = \widetilde{H}^*(S^0)$ with $s = 3$, 4 and 5, respectively, in Hưng [10; 11] and Hưng, Quỳnh and Tuấn [13]. That the Lannes–Zarati homomorphism for $M = \widetilde{H}^*(S^0)$ vanishes for $s > 2$ on decomposable elements in $\mathrm{Ext}_{\mathcal{A}}^s(\mathbb{F}_2,\mathbb{F}_2)$ was proved in [12]. The conjecture was also established for the case $M = \widetilde{H}^*(\mathbb{RP}^\infty)$ with $s = 3, 4$ in [14].

One of the main results of the paper is the following theorem:

Theorem 1.3 *Let M be an unstable $\mathcal{A}$–module of finite type. Then the s^{th} Lannes–Zarati homomorphism for M,*

$$\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)_i^*,$$

vanishes on the elements of the form $\alpha\beta$ in any positive stem i, where $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p \geq 2$, $q > 0$ and $p+q = s$, or $p = s \geq 2$, $q = 0$ and $\mathrm{stem}(\beta) > s-2$.

Theorem 1.3 gives evidence supporting Conjecture 1.2, in particular providing a result valid for all unstable $\mathcal{A}$–modules of finite type M.

Using Theorem 1.3 for the case $M = \mathbb{F}_2$, we obtain the following theorem, which was first proved in [12]:

Theorem 1.4 (Hưng and Peterson [12]) *The s^{th} Lannes–Zarati homomorphism for $\mathbb{F}_2$,*

$$\varphi_s^{\mathbb{F}_2}\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(\mathbb{F}_2,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)_i^*,$$

vanishes on the decomposable elements in any positive stem i for $s \geq 3$.

In [12], Hưng and Peterson proved Theorem 1.4 by showing that $\varphi_* = \bigoplus_s \varphi_s^{\mathbb{F}_2}$ is a homomorphism of algebras and, more importantly, that the product of the nonunital algebra $\bigoplus_{s>0}(\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)^*$ is trivial, except for the case $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*$. The methods used to prove Theorem 1.3 are different from the methods of Hưng and Peterson. The important new ingredient is the usage of the chain level representation of the dual of the Lannes–Zarati homomorphism (see Theorem 2.1). Moreover, the advantage of using the chain level representation of the dual of the Lannes–Zarati homomorphism is that the proof of Theorem 1.3 is short and elementary. The proof of Theorem 1.3 is based upon the key Lemma 3.3.

Hưng and the author [14] established a relation between the Lannes–Zarati homomorphisms for $\widetilde{H}^*(\mathbb{RP}^\infty)$ and for $\widetilde{H}^*(S^0)$. The relation comes from the so-called algebraic Kahn–Priddy theorem (see [17, Theorem 1.1]). By using the algebraic Kahn–Priddy theorem, Hưng and Tuấn showed that if $\varphi_{s-1}^{\widetilde{H}^*(\mathbb{RP}^\infty)}$ vanishes in positive stems, then so does $\varphi_s^{\widetilde{H}^*(S^0)}$, for $s \geq 1$ (see [14, Proposition 10.2]). So, Conjecture 1.2 with $M = \widetilde{H}^*(\mathbb{RP}^\infty)$ is interesting. In this paper, by using Theorem 1.3 and the fact that $\varphi_5^{\mathbb{F}_2}$ and $\varphi_4^{\widetilde{H}^*(\mathbb{RP}^\infty)}$ vanish in positive stems (see [13, Theorem 1.4; 14, Theorem 1.8]), we obtain the following proposition:

Proposition 1.5 *The fifth Lannes–Zarati homomorphism for $\widetilde{H}^*(\mathbb{RP}^\infty)$,*

$$\varphi_5^{\widetilde{H}^*(\mathbb{RP}^\infty)}\colon \operatorname{Ext}_{\mathcal{A}}^{5,5+i}(\widetilde{H}^*(\mathbb{RP}^\infty), \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_5 \widetilde{H}^*(\mathbb{RP}^\infty))_i^*,$$

vanishes on the decomposable elements in any positive stem i.

Note that $\operatorname{Ext}_{\mathcal{A}}^*(M, \mathbb{F}_2)$ is a module over $\operatorname{Ext}_{\mathcal{A}}^*(\mathbb{F}_2, \mathbb{F}_2)$ (see Section 2); the notation of the submodule of decomposables is the usual one.

The paper is divided into three sections and organized as follows. Background and references are provided in Section 2. Theorems 1.3 and 1.4 and Proposition 1.5 are proved in Section 3.

2 Background

We start this section by sketching briefly Singer's invariant-theoretic description of the lambda algebra.

Let T_s be the Sylow 2–subgroup of GL_s consisting of all upper triangular $s \times s$ matrices with 1 on the main diagonal. The T_s–invariant ring, $M_s = P_s^{T_s}$, is called the

Mùi algebra. In [19], Mùi shows that $P_s^{T_s}$ is a polynomial algebra

$$P_s^{T_s} = \mathbb{F}_2[V_1, \dots, V_s],$$

on elements V_k of degree 2^{k-1}, where

$$V_i = V_i(x_1, \dots, x_i) = \prod_{a_j \in \mathbb{F}_2} (a_1 x_1 + \dots + a_{i-1} x_{i-1} + x_i).$$

Recall that the Dickson algebra D_s was computed in [5]:

$$D_s = \mathbb{F}_2[Q_{s,0}, \dots, Q_{s,s-1}].$$

Here the Dickson invariant $Q_{s,i}$ of degree $2^s - 2^i$ can inductively be defined by

$$Q_{s,i} = Q_{s-1,i-1}^2 + Q_{s-1,i} V_s,$$

where, by convention, $Q_{s,s} = 1$ and $Q_{s,i} = 0$ for $i < 0$ (see [5; 19]). (For the action of Steenrod algebra on V_i and $Q_{s,i}$, see [8].)

Let $L(s) \subset P_s$ be the multiplicative subset generated by all the nonzero linear forms in P_s. Let $(P_s)_{L(s)}$ be the localization given by inverting all the nonzero linear forms in P_s. Using the results of Dickson [5] and Mùi [19], Singer notes in [20] that

$$\begin{aligned}
\Delta_s &:= ((P_s)_{L(s)})^{T_s} = \mathbb{F}_2[V_1^{\pm 1}, \dots, V_s^{\pm 1}], \\
\Gamma_s &:= ((P_s)_{L(s)})^{\mathrm{GL}_s} = \mathbb{F}_2[Q_{s,s-1}, \dots, Q_{s,1}, Q_{s,0}^{\pm 1}].
\end{aligned}$$

Further, he sets

$$v_1 = V_1, v_k = V_k / V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtains

$$\Delta_s = \mathbb{F}_2[v_1^{\pm 1}, \dots, v_s^{\pm 1}],$$

with $\deg v_i = 1$ for every i.

Singer defines Γ_s^+ to be the $\mathbb{F}_2$–subspace of $\Gamma_s = D_s[Q_{s,0}^{-1}]$ spanned by all monomials $\gamma = Q_{s,s-1}^{i_{s-1}} \cdots Q_{s,0}^{i_0}$ with $i_{s-1}, \dots, i_1 \geq 0, i_0 \in \mathbb{Z}$, and $i_0 + \deg \gamma \geq 0$. He also shows in [20] that the homomorphism

$$\partial_s \colon \Delta_s \otimes N \to \Delta_{s-1} \otimes N, \quad \partial_s(v_1^{j_1} \cdots v_s^{j_s} \otimes z) = v_1^{j_1} \cdots v_{s-1}^{j_{s-1}} \otimes \mathrm{Sq}^{j_s+1} z,$$

maps $\Gamma_s^+ \otimes N$ to $\Gamma_{s-1}^+ \otimes N$. Here N is an arbitrary left $\mathcal{A}$–module. Moreover, it is a differential on $\Gamma^+ N = \bigoplus_s (\Gamma_s^+ \otimes N)$. He also proves that

$$H_s(\Gamma^+ N) \cong \operatorname{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, N).$$

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [2]. It is bigraded by putting $\operatorname{bideg}(\lambda_i) = (1, i)$, where $\lambda_i \in \Lambda^{1,i}$. Singer proves in [20] that the $\mathbb{F}_2$–linear map

$$\ell_s \colon \Gamma_s^+ \to (\Lambda^s)^*, \quad v_1^{j_1} \cdots v_s^{j_s} \mapsto (\lambda_{j_1} \cdots \lambda_{j_s})^*,$$

is an isomorphism for each $s \geq 0$. Here the duality $*$ is taken with respect to the basis of admissible monomials of Λ. Recall that for each $s \geq 1$, a basis for Λ^s is given by the set of admissible monomials

$$\{\lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_s} \mid 0 \leq j_1,\ j_1 \leq 2j_2,\ \ldots,\ j_{s-1} \leq 2j_s\},$$

while Λ^0 is spanned by the unit (see [20]).

Suppose N is a left $\mathcal{A}$–module which is finitely generated in every degree. Let N^* be the $\mathbb{F}_2$–dual of N which is a right $\mathcal{A}$–module by transposing the left $\mathcal{A}$–module on N. The tensor product $\Lambda \otimes N^*$ is bigraded by

$$(\Lambda \otimes N^*)^{s,t} = \sum_k \Lambda^{s,t-k} \otimes N_k^*.$$

For any sequence $I = (i_1, \ldots, i_s)$ of nonnegative integers, we write λ_I to denote $\lambda_{i_1} \cdots \lambda_{i_s} \in \Lambda$. For $m^* \in N^*$, we write $\lambda_I m^*$ to denote $\lambda_I \otimes m^* \in \Lambda \otimes N^*$ and let $m^* = 1m^*$. So $\Lambda \otimes N^*$ is a bigraded differential left Λ–module with the action of Λ on it given by

$$\lambda_J(\lambda_I m^*) = \lambda_J \lambda_I m^*,$$

where J is a sequence of nonnegative integers. Moreover, the differential of $\Lambda \otimes N^*$ is given by

$$\delta(\lambda_I m^*) = \delta(\lambda_I) m^* + \sum_{j \geq 0} \lambda_I \lambda_j m^* \operatorname{Sq}^{j+1}.$$

(For the differential δ on the lambda algebra, see [2; 18].) Then $\operatorname{Ext}_{\mathcal{A}}^{s,s+t}(N, \mathbb{F}_2) = H^{s,t}(\Lambda \otimes N^*, \delta)$ (see [2; 18]). By means of the differential, one recognizes that the left action of Λ on $\Lambda \otimes N^*$ induces a left action of $\operatorname{Ext}_{\mathcal{A}}^{*,*} := \operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ on $\operatorname{Ext}_{\mathcal{A}}^{*,*}(N, \mathbb{F}_2)$. Hence, the latter becomes a left $\operatorname{Ext}_{\mathcal{A}}^{*,*}$–module.

In the remaining part of this section, we recall some results used to prove the main results in this paper.

Theorem 2.1 (Hưng and Tuấn [14]) *Let M be an unstable $\mathcal{A}$–module. Then, for any $s \geq 0$, the map*

$$(\widetilde{\varphi_s^M})^*\colon R_s M \to \Gamma_s^+ M, \quad c\,\mathrm{St}_s(z) \mapsto c\,Q_{s,0}^{|z|} \otimes z,$$

for $c \in D_s$ and a homogeneous element z of degree $|z|$ in M, is a chain-level representation of the dual of the Lannes–Zarati homomorphism

$$(\varphi_s^M)^*\colon (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)_i \to \mathrm{Tor}^{\mathcal{A}}_{s,s+i}(\mathbb{F}_2, M).$$

This map is natural with respect to $\mathcal{A}$–homomorphisms of unstable $\mathcal{A}$–modules.

As $R_s M$ is a free D_s–module (see [16, Definition-Proposition 2.4.1]), the map is well defined.

An element in D_s is called $\mathcal{A}$–decomposable if it is in $\bar{\mathcal{A}} D_s$, where $\bar{\mathcal{A}}$ denotes the augmentation ideal of the Steenrod algebra $\mathcal{A}$.

Giambalvo and Peterson showed in [6] a sufficient condition for a monomial in D_s to be $\mathcal{A}$–decomposable as follows:

Theorem 2.2 (see [6, Corollary 4.8]) *Let $s \geq 2$ and assume that $I = (i_0, \dots, i_{s-1})$ is a s–tuple of nonnegative integers and $Q^I = Q_{s,0}^{i_0} \cdots Q_{s,s-1}^{i_{s-1}} \in D_s$ with $i_0 > s-2$. Then Q^I is $\mathcal{A}$–decomposable.*

3 On the vanishing of the Lannes–Zarati homomorphism on decomposable elements

The goal of this section is to prove Theorems 1.3 and 1.4 and Proposition 1.5.

In [20], Singer defines an algebra isomorphism $\psi_{p,q}\colon \Delta_s \to \Delta_p \otimes \Delta_q$ by

$$\psi_{p,q}(v_i) = \begin{cases} v_i \otimes 1 & \text{if } 1 \leq i \leq p, \\ 1 \otimes v_{i-p} & \text{if } p+1 \leq i \leq s, \end{cases}$$

for any pair of nonnegative integers p and q for which $p+q = s$. Here we understand $\Delta_0 = \mathbb{F}_2$, $\psi_{s,0}(x) = x \otimes 1$ and $\psi_{0,s}(x) = 1 \otimes x$. Then, he shows that

$$\psi_{p,q}(Q_{s,i}) = \sum_{j \geq 0} Q_{p,0}^{2^q - 2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j} \tag{3.0.1}$$

for each i with $0 \le i < s$. Suppose $c = Q_{s,0}^{t_0} \cdots Q_{s,s-1}^{t_{s-1}} \in D_s$; then

$$\begin{aligned}
\psi_{p,q}(c) &= \prod_{i=0}^{s-1} \psi_{p,q}(Q_{s,i})^{t_i} && \text{(since } \psi_{p,q} \text{ is an algebra isomorphism)} \\
&= \prod_{i=0}^{s-1} \Bigl(\sum_{j=0}^{\min\{i,q\}} Q_{p,0}^{2^q-2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j} \Bigr)^{t_i} && \text{(by (3.0.1))} \\
&= \prod_{i=0}^{s-1} \sum_{\substack{|\alpha_i|=t_i \\ d_i=1}} Q^{\alpha_i} && \text{(by the binomial theorem),}
\end{aligned}$$

where

$$d_i = \frac{t_i!}{k_0^{(i)}! \cdots k_{\min\{i,q\}}^{(i)}!}, \quad \alpha_i = (k_0^{(i)}, \dots, k_{\min\{i,q\}}^{(i)}), \quad |\alpha_i| = k_0^{(i)} + \cdots + k_{\min\{i,q\}}^{(i)}$$

and

$$Q^{\alpha_i} = \prod_{j=0}^{\min\{i,q\}} (Q_{p,0}^{2^q-2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j})^{k_j^{(i)}}.$$

So, for $c \in D_s$, we have $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$ with $Q^I \in D_p$ and $Q^J \in D_q$.

Lemma 3.1 *Suppose $c \in D_s$ and $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$, $p+q=s$. Then each Q^I has the form $Q_{p,0}^{i_0} \cdots Q_{p,p-1}^{i_{p-1}}$, where $i_1 = n_1 + 2m_1, \dots, i_{p-1} = n_{p-1} + 2m_{p-1}$ and $i_0 \ge (2^q-1)(n_1 + \cdots + n_{p-1})$ for $n_1, \dots, n_{p-1}, m_1, \dots, m_{p-1}$ nonnegative integers.*

Proof Suppose $c = Q_{s,0}^{t_0} \cdots Q_{s,s-1}^{t_{s-1}} \in D_s$. From the above calculation, we see that each Q^I has the form $Q_{p,0}^{i_0} \cdots Q_{p,p-1}^{i_{p-1}}$ with

$$\begin{aligned}
i_0 &\ge (2^q-1)(k_0^{(0)} + k_0^{(1)} + \cdots + k_0^{(s-1)}), \\
i_1 &= k_0^{(1)} + 2^1 k_1^{(2)} + \cdots + 2^q k_q^{(q+1)}, \\
i_2 &= k_0^{(2)} + 2^1 k_1^{(3)} + \cdots + 2^q k_q^{(q+2)}, \\
&\vdots \\
i_{p-1} &= k_0^{(p-1)} + 2^1 k_1^{(p)} + \cdots + 2^q k_q^{(s-1)}.
\end{aligned}$$

Set $n_i = k_0^{(i)}$ and $m_i = \sum_{j=1}^q 2^{j-1} k_j^{(i+j)}$ for $1 \le i \le p-1$. Then $i_1 = n_1 + 2m_1, \dots, i_{p-1} = n_{p-1} + 2m_{p-1}$ and $i_0 \ge (2^q-1)(n_1 + \cdots + n_{p-1})$.

The lemma follows. □

Suppose N is an $\mathcal{A}$–module of finite type. By ambiguity of notation, the following $\mathbb{F}_2$–linear map is also denoted by the same notation as the isomorphism $\ell_s\colon \Gamma_s^+ \to (\Lambda^s)^*$ (see [20, page 689]):

$$\ell_s\colon \Gamma_s^+ \otimes N \to (\Lambda^s \otimes N^*)^*, \quad v_1^{j_1}\cdots v_s^{j_s} \otimes z \mapsto \langle z,\cdot\rangle\langle \ell_s(v_1^{j_1}\cdots v_s^{j_s}),\cdot\rangle.$$

This map is an $\mathbb{F}_2$–isomorphism for each $s \geq 0$.

The following lemma was first proved for $N = \mathbb{F}_2$ by Singer in [20, page 689].

Lemma 3.2 *The diagram*

$$\begin{tikzcd} \Gamma_s^+ \otimes N \arrow[r, "\ell_s"] \arrow[d, "\partial"] & (\Lambda^s \otimes N^*)^* \arrow[d, "\delta^*"] \\ \Gamma_{s-1}^+ \otimes N \arrow[r, "\ell_{s-1}"] & (\Lambda^{s-1} \otimes N^*)^* \end{tikzcd}$$

commutes for $s \geq 1$. Here, N is an $\mathcal{A}$–module of finite type.

Proof Use an argument similar to the proof of [20, Proposition 8.2]. □

Suppose N is an $\mathcal{A}$–module of finite type. Let $\langle\cdot,\cdot\rangle$ be the usual dual paring $\mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, N)\otimes \mathrm{Ext}_{\mathcal{A}}^s(N,\mathbb{F}_2) \to \mathbb{F}_2$. We note that this dual paring is induced in homology by the dual paring $(\Gamma_s^+ \otimes N)\otimes(\Lambda^s \otimes N^*) \to \mathbb{F}_2$ that allows us to identify $\Gamma_s^+ \otimes N$ with the dual of $\Lambda^s \otimes N^*$, as mentioned in Lemma 3.2. We also denote by $\langle\cdot,\cdot\rangle$ the dual paring $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)\otimes(\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)^* \to \mathbb{F}_2$ for M an unstable $\mathcal{A}$–module.

Let N be an $\mathcal{A}$–module. Suppose α is an element in $\mathrm{Ext}_{\mathcal{A}}^{s,t}(N,\mathbb{F}_2)$. Then, $\mathrm{stem}(\alpha)$ is given by $\mathrm{stem}(\alpha) = t-s$.

Lemma 3.3 *Let M be an unstable $\mathcal{A}$–module of finite type. Let $c\,\mathrm{St}_s(z)$ be an element of $R_s M$ for $c \in D_s$ and a homogeneous element z of degree $|z|$ in M. Then, for $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$, $p > 0$, $q \geq 0$ and $p+q = s$,*

$$\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = \sum_{\substack{|Q^I Q_{p,0}^{2^q|z|}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle [Q^I Q_{p,0}^{2^q|z|}], \varphi_p^{\mathbb{F}_2}(\alpha)\rangle\langle [Q^J \mathrm{St}_q(z)], \varphi_q^M(\beta)\rangle.$$

Here Q^I and Q^J appear in $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$.

Proof Suppose $\alpha = [x] \in \mathrm{Ext}_{\mathcal{A}}^{p}(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta = [y] \in \mathrm{Ext}_{\mathcal{A}}^{q}(M,\mathbb{F}_2)$, where x is a cycle in Λ^p and y is a cycle in $\Lambda^q \otimes M^*$. Then we have

$$\begin{aligned}
&\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle \\
&\qquad = \langle (\varphi_s^M)^*([c\,\mathrm{St}_s(z)]), \alpha\beta\rangle \\
&\qquad = \langle [c\,Q_{s,0}^{|z|} \otimes z], \alpha\beta\rangle && \text{(by Theorem 2.1)} \\
&\qquad = \langle c\,Q_{s,0}^{|z|} \otimes z, xy\rangle \\
&\qquad = \langle \psi_{p,q}(c\,Q_{s,0}^{|z|}) \otimes z, x \otimes y\rangle && \text{(see [20, page 688])} \\
&\qquad = \Big\langle \sum Q^I Q_{p,0}^{2^q|z|} \otimes Q^J Q_{q,0}^{|z|} \otimes z, x \otimes y\Big\rangle && (\text{since } \psi_{p,q}(Q_{s,0}) = Q_{p,0}^{2^q} \otimes Q_{q,0}) \\
&\qquad = \sum \langle Q^I Q_{p,0}^{2^q|z|}, x\rangle \langle Q^J Q_{q,0}^{|z|} \otimes z, y\rangle.
\end{aligned}$$

We note that $Q^I Q_{p,0}^{2^q|z|}$ and $Q^J Q_{q,0}^{|z|} \otimes z$ are cycles in Γ_p^+ and $\Gamma_q^+ \otimes M$, respectively. So, we get

$$\begin{aligned}
&\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle \\
&\quad = \sum \langle [Q^I Q_{p,0}^{2^q|z|}], \alpha\rangle \langle [Q^J Q_{q,0}^{|z|} \otimes z], \beta\rangle \\
&\quad = \sum \langle (\varphi_p^{\mathbb{F}_2})^*[Q^I Q_{p,0}^{2^q|z|}], \alpha\rangle \langle (\varphi_q^M)^*[Q^J\,\mathrm{St}_q(z)], \beta\rangle && \text{(by Theorem 2.1)} \\
&\quad = \sum_{\substack{|Q^I Q_{p,0}^{2^q|z|}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle (\varphi_p^{\mathbb{F}_2})^*[Q^I Q_{p,0}^{2^q|z|}], \alpha\rangle \langle (\varphi_q^M)^*[Q^J\,\mathrm{St}_q(z)], \beta\rangle \\
&\quad = \sum_{\substack{|Q^I Q_{p,0}^{2^q|z|}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle [Q^I Q_{p,0}^{2^q|z|}], \varphi_p^{\mathbb{F}_2}(\alpha)\rangle \langle [Q^J\,\mathrm{St}_q(z)], \varphi_q^M(\beta)\rangle. && \square
\end{aligned}$$

We recall the following lemma, which was first proved in [12]. We give a proof to make the paper self-contained.

Lemma 3.4 (Hưng and Peterson [12]) *Let $c = Q_{p,0}^{i_0} \cdots Q_{p,p-1}^{i_{p-1}} \in D_p$ with $i_0 > 0$. If $i_m \equiv 0 \pmod 2$ for some $m > 0$, then c is $\mathcal{A}$–decomposable.*

Proof We prove this by induction on the smallest $m > 0$ with $i_m \equiv 0 \pmod 2$. If $m = 1$, then $\mathrm{Sq}^1(Q_{p,0}^{i_0-1} Q_{p,1}^{i_1+1} \cdots Q_{p,p-1}^{i_{p-1}}) = c$. For the induction step,

$$\mathrm{Sq}^{2^{m-1}}(Q_{p,0}^{i_0} \cdots Q_{p,m-1}^{i_{m-1}-1} Q_{p,m}^{i_m+1} \cdots Q_{p,p-1}^{i_{p-1}}) = c + \sum Q^K,$$

where each Q^K has the form $Q_{p,0}^{k_0} \cdots Q_{p,p-1}^{k_{p-1}}$ with $k_{m-1} \equiv 0 \pmod 2$ and $k_0 > 0$. $\square$

The following repeats Theorem 1.3 from the introduction:

Theorem 3.5 *Let M be an unstable $\mathcal{A}$–module of finite type. Then the s^{th} Lannes–Zarati homomorphism for M*

$$\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2)\to(\mathbb{F}_2\otimes_{\mathcal{A}} R_sM)_i^*$$

vanishes on the elements of the form $\alpha\beta$ in any positive stem i, where $\alpha\in\mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta\in\mathrm{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p\ge 2$, $q>0$ and $p+q=s$, or $p=s\ge 2$, $q=0$ and $\mathrm{stem}(\beta)>s-2$.

Proof We will show that $\varphi_s^M(\alpha\beta)=0$, where $\alpha\in\mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$, $\beta\in\mathrm{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p\ge 2$, $q>0$ and $p+q=s$, or $p=s\ge 2$, $q=0$ and $\mathrm{stem}(\beta)>s-2$.

Case 1 $(p\ge 2,\ q>0)$ By Lemma 3.3, for any $c\,\mathrm{St}_s(z)\in R_sM$ with $c\in D_s$, a homogeneous element $z\in M$ of degree $|z|$ and $\psi_{p,q}(c)=\sum Q^I\otimes Q^J$ with $Q^I\in D_p$ and $Q^J\in D_q$, we have

$$\langle[c\,\mathrm{St}_s(z)],\varphi_s^M(\alpha\beta)\rangle=\sum_{\substack{|Q^IQ_{p,0}^{2^q|z|}|=\mathrm{stem}(\alpha)\\ |Q^J\mathrm{St}_q(z)|=\mathrm{stem}(\beta)}}\langle[Q^IQ_{p,0}^{2^q|z|}],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle\langle[Q^J\,\mathrm{St}_q(z)],\varphi_q^M(\beta)\rangle.$$

We see that $\psi_{p,q}(cQ_{s,0}^{|z|})=\sum Q^IQ_{p,0}^{2^q|z|}\otimes Q^JQ_{q,0}^{|z|}$. So, by Lemma 3.1, $Q^IQ_{p,0}^{2^q|z|}$ has the form $Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}$, where $i_0\ge(2^q-2^0)(n_1+\cdots+n_{p-1})$, $i_1=n_1+2m_1,\ldots,$ $i_{p-1}=n_{p-1}+2m_{p-1}$. We will prove that $Q^IQ_{p,0}^{2^q|z|}$ is $\mathcal{A}$–decomposable.

If $i_0=0$, then $0\ge(2^q-2^0)(n_1+\cdots+n_{p-1})$. So, it implies that $n_1=\cdots=n_{p-1}=0$. We get

$$Q^IQ_{p,0}^{2^q|z|}=Q_{p,1}^{2m_1}\cdots Q_{p,p-1}^{2m_{p-1}}=\mathrm{Sq}^{(2^p-2^1)m_1+\cdots+(2^p-2^{p-1})m_{p-1}}(Q_{p,1}^{m_1}\cdots Q_{p,p-1}^{m_{p-1}}).$$

Hence, $Q^IQ_{p,0}^{2^q|z|}\in\bar{\mathcal{A}}D_p$

If $i_0>0$ and one of the nonnegative integers $n_1,\ldots,n_{p-1}$ is even, then by Lemma 3.4 we have $Q^IQ_{p,0}^{2^q|z|}=Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}\in\bar{\mathcal{A}}D_p$.

If $i_0>0$ and all of the nonnegative integers $n_1,\ldots,n_{p-1}$ are odd, then

$$i_0\ge(2^q-2^0)(n_1+\cdots+n_{p-1})\ge p-1>p-2.$$

Hence, by Theorem 2.2, we obtain $Q^IQ_{p,0}^{2^q|z|}=Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}\in\bar{\mathcal{A}}D_p$.

So, we get $\langle[Q^IQ_{p,0}^{2^q|z|}],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle=\langle[0],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle=0$. We conclude that $\varphi_s^M(\alpha\beta)=0$.

Case 2 ($p = s \geq 2$, $q = 0$ and $\mathrm{stem}(\beta) > s-2$) By Lemma 3.3, for any $c\,\mathrm{St}_s(z) \in R_s M$ with $c \in D_s$, a homogeneous element $z \in M$ of degree $|z|$ and $\psi_{s,0}(c) = c \otimes 1$, we have

$$\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = \langle [c\,Q_{s,0}^{|z|}], \varphi_s^M(\alpha)\rangle \langle [z], \beta\rangle.$$

If $|z| \neq \mathrm{stem}(\beta)$, then $\langle [z], \beta\rangle = 0$. So $\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = 0$.

If $|z| = \mathrm{stem}(\beta)$, then $|z| > s-2$ (since $\mathrm{stem}(\beta) > s-2$). By Theorem 2.2, we have $c\,Q_{s,0}^{|z|} \in \bar{\mathcal{A}} D_s$. We conclude that $\langle [c\,Q_{s,0}^{|z|}], \varphi_s^M(\alpha)\rangle = \langle [0], \varphi_s^M(\alpha)\rangle = 0$. Hence, $\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = 0$.

The theorem is proved. □

Consequently, when $M = \mathbb{F}_2$, we obtain the following theorem, which was first proved by Hưng and Peterson in [12]. Recall that Hưng and Peterson proved this theorem by showing that $\varphi_* = \bigoplus_s \varphi_s^{\mathbb{F}_2}$ is a homomorphism of algebras and, more importantly, that the product of the nonunital algebra $\bigoplus_{s>0}(\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)^*$ is trivial, except for the case $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*$.

The following repeats Theorem 1.4 from the introduction:

Theorem 3.6 (Hưng and Peterson [12]) *The s^{th} Lannes–Zarati homomorphism*

$$\varphi_s^{\mathbb{F}_2} \colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)_i^*$$

vanishes on the decomposable elements in any positive stem i for $s \geq 3$.

Proof We must show that $\varphi_s^{\mathbb{F}_2}(\alpha\beta) = 0$, where $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2, \mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(\mathbb{F}_2, \mathbb{F}_2)$ with $p > 0$, $q > 0$ and $p + q = s$. Since the algebra $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is commutative, we have left to consider the case $p \geq 2$ and $q > 0$. In this case, by Theorem 3.5, we have $\varphi_s^{\mathbb{F}_2}(\alpha\beta) = 0$.

The theorem is proved. □

For brevity, $\widetilde{H}^*(\mathbb{RP}^\infty)$ will be denoted by $\widetilde{P}$. The following repeats Proposition 1.5 from the introduction:

Proposition 3.7 *The fifth Lannes–Zarati homomorphism for $\widetilde{P}$,*

$$\varphi_5^{\widetilde{P}} \colon \mathrm{Ext}_{\mathcal{A}}^{5,5+i}(\widetilde{P}, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_5 \widetilde{P})_i^*,$$

vanishes on the decomposable elements in any positive stem i.

Proof We must prove that $\varphi_5^{\widetilde{P}}(\alpha\beta)=0$, where $\alpha\in\mathrm{Ext}_{\mathcal{A}}^{p}(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta\in\mathrm{Ext}_{\mathcal{A}}^{q}(\widetilde{P},\mathbb{F}_2)$ with $p>0$, $q\geq 0$ and $p+q=5$. We will consider the following three cases:

Case 1 $(p\geq 2,\ q>0)$ By Theorem 3.5, we have $\varphi_5^{\widetilde{P}}(\alpha\beta)=0$.

Case 2 $(p=5,\ q=0)$ Then, for any $c\,\mathrm{St}_5(z)\in R_5\widetilde{P}$ with $c\in D_5$ and a homogeneous element $z\in\widetilde{P}$ of degree $|z|$, we have $\psi_{5,0}(c)=c\otimes 1$, and

$$\begin{aligned}\langle[c\,\mathrm{St}_5(z)],\varphi_5^{\widetilde{P}}(\alpha\beta)\rangle &= \langle[c\,Q_{5,0}^{|z|}],\varphi_5^{\mathbb{F}_2}(\alpha)\rangle\langle[\mathrm{St}_0(z)],\varphi_0^{\widetilde{P}}(\beta)\rangle \quad \text{(by Lemma 3.3)}\\ &=0,\end{aligned}$$

where the last equality follows from the fact that $\varphi_5^{\mathbb{F}_2}(\alpha)=0$ (see [13, Theorem 1.4]).

Case 3 $(p=1,\ q=4)$ Then, for any $c\,\mathrm{St}_5(z)\in R_5\widetilde{P}$ with $c\in D_5$, a homogeneous element $z\in\widetilde{P}$ of degree $|z|$ and $\psi_{1,4}(c)=\sum Q^I\otimes Q^J$ with $Q^I\in D_1$ and $Q^J\in D_4$, we have

$$\begin{aligned}\langle[c\,\mathrm{St}_5(z)],\varphi_5^{\widetilde{P}}(\alpha\beta)\rangle &= \sum_{\substack{|Q^IQ_{1,0}^{2^4|z|}|=\mathrm{stem}(\alpha)\\ |Q^J\mathrm{St}_4(z)|=\mathrm{stem}(\beta)}} \langle[Q^IQ_{1,0}^{2^4|z|}],\varphi_1^{\mathbb{F}_2}(\alpha)\rangle\langle[Q^J\,\mathrm{St}_4(z)],\varphi_4^{\widetilde{P}}(\beta)\rangle\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 3.3)}\\ &=0,\end{aligned}$$

where the last equality follows from the fact that $\varphi_4^{\widetilde{P}}(\beta)=0$ (see [14, Theorem 1.8]).

The proposition is completely proved. □

Remark 3.8 From the proof of Proposition 3.7, and Theorem 3.5, we can see that for $s\geq 3$, and for any unstable $\mathcal{A}$–module M of finite type, if $\varphi_s^{\mathbb{F}_2}$ and φ_{s-1}^{M} vanish in positive stems, then φ_s^{M} vanishes on the decomposable elements in positive stems.

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