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A refinement of Betti numbers and homology in the presence of a continuous function II: The case of an angle-valued map

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For $f: X \to \mathbb{S}^1$ a continuous angle-valued map defined on a compact ANR X, κ a field and any integer $r \geq 0$, one proposes a refinement δ^f_r of the Novikov–Betti numbers of the pair (X, ξ_f) and a refinement $\hat{\delta}^f_r$ of the Novikov homology of (X, ξ_f), where ξ_f denotes the integral degree one cohomology class represented by f. The refinement δ^f_r is a configuration of points, with multiplicity located in $\mathbb{R}^2/\mathbb{Z}$ identified to $\mathbb{C} \setminus 0$, whose total cardinality is the r^{th} Novikov–Betti number of the pair. The refinement $\hat{\delta}^f_r$ is a configuration of submodules of the r^{th} Novikov homology whose direct sum is isomorphic to the Novikov homology and with the same support as of δ^f_r. When $\kappa = \mathbb{C}$, the configuration $\hat{\delta}^f_r$ is convertible into a configuration of mutually orthogonal closed Hilbert submodules of the L_2–homology of the infinite cyclic cover of X defined by f, which is an $L^\infty(\mathbb{S}^1)$–Hilbert module. One discusses the properties of these configurations, namely robustness with respect to continuous perturbation of the angle-values map and the Poincaré duality and one derives some computational applications in topology. The main results parallel the results for the case of real-valued map but with Novikov homology and Novikov–Betti numbers replacing standard homology and standard Betti numbers.

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1 Introduction

This paper is a sequel of [2] (which considers the case of real-valued map) but can be read independently of [2]. Here we treat the case of an angle-valued continuous map $f\colon X \to \mathbb{S}^1$ and complete results from Burghelea and Haller [4]. In this paper, without any additional specifications, an angle-valued map assumes that the space X is a compact ANR (in particular a space homeomorphic to a finite simplicial complex or a compact Hilbert cube manifold) and the map f is continuous. The map f determines a degree one integral cohomology class $\xi_f \in H^1(X;\mathbb{Z})$.

We fix a field κ and an integer r, with $r = 0, 1, 2, \dim X$, and provide first a configuration δ^f_r of finitely many points with specified multiplicity located in the space $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}$,[1] which can be identified to the punctured plane $\mathbb{C}\setminus 0$. It will be shown that the set of points (counted with multiplicity) of this configuration has cardinality equal to the Novikov–Betti number $\beta^N_r(X;\xi_f)$. In view of the identification of $\mathbb{T}$ with $\mathbb{C}\setminus 0$ the configuration δ^f_r can be also interpreted as a monic[2] polynomial $P^f_r(z)$, with complex coefficients and nonzero free term of degree equal to the Novikov–Betti number, whose roots are the points of the configuration δ^f_r.

We refine the configuration δ^f_r to the configuration $\hat{\delta}^f_r$ of $\kappa[t^{-1},t]$–free modules indexed by $z \in \mathbb{C}\setminus 0$, each one a quotient of split free submodules of the r^{th} Novikov homology of $(X;\xi_f)$, and in the case $\kappa = \mathbb{C}$ to the configuration $\hat{\hat{\delta}}^f_r$ of closed Hilbert submodules of L_2–homology of $\tilde{X}$, the infinite cyclic cover of X defined by ξ_f. All configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ are maps with finite support defined on $\mathbb{C}\setminus 0$ with $\delta^f_r(z)$ a nonpositive integer, $\hat{\delta}^f_r(z)$ a free $\kappa[t,t^{-1}]$–module and $\hat{\hat{\delta}}^f_r(z)$ an $L^\infty(\mathbb{S}^1)$–Hilbert module with $\hat{\delta}^f_r(z)$ of rank and $\hat{\hat{\delta}}^f_r(z)$ of von Neumann dimension equal to $\delta^f_r(z)$.

In this paper, for a fixed field κ, the Novikov homology $H^N_r(X;\xi)$ is a free $\kappa[t^{-1},t]$–module. Novikov [12] and most authors — see for example Pajitnov [13] — regard Novikov homology as a $\kappa[t^{-1},t]\!]$–vector space, where $\kappa[t^{-1},t]\!]$ denotes the field of Laurent power series with coefficients in κ, obtained by *extending the scalars* from $\kappa[t^{-1},t]$ to $\kappa[t^{-1},t]\!]$.[3] If the Novikov homology is regarded as a vector space over this field then $\hat{\delta}^f_r$ is a configuration of vector subspaces, and this is entirely in analogy with the case of a real-valued map treated in [2].

[1] $\mathbb{R}^2$ is equipped with the action $\mu(n,(a,b)) = (a + 2\pi n, b + 2\pi n)$.
[2] The monomial of highest degree has coefficient 1.
[3] The vector space is equal to $H^N(X;\xi) \otimes_{\kappa[t^{-1},t]} \kappa[t^{-1},t]\!]$.

The results about the configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ are formulated in Theorems 1.1, 1.2 and 1.3 and are formally similar to Theorems 4.1, 4.2 and 4.3 in [2], but conceptually more complex and technically more difficult to conclude. In comparison with [2] there are however a number of differences and new features which deserve to be pointed out:

- The location of the points in the support of the configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ is the space $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}$, identified to the punctured complex plane $\mathbb{C} \setminus 0$ by the map $\langle a, b\rangle \mapsto z = e^{ia+(b-a)}$, and not $\mathbb{R}^2 = \mathbb{C}$ as in [2].

- The Betti numbers $\beta_r(X)$ in [2] are replaced by the Novikov–Betti numbers $\beta^N_r(X;\xi)$ or by L_2–Betti numbers $\beta^{L_2}_r(\widetilde{X})$, with $\widetilde{X}$ the infinite cyclic cover defined by $\xi = \xi_f$, and the homology $H_r(X)$ is replaced by the Novikov homology of (X,ξ) or by the L_2–homology of $\widetilde{X}$.

- For $z = \langle a, b\rangle \in \operatorname{supp}\delta^f_r$ the configuration $\hat{\delta}^f_r$ takes as value $\hat{\delta}^f_r\langle a, b\rangle = \hat{\delta}^f_r(z)$, a free $\kappa[t^{-1}, t]$–module which is a quotient $\widehat{\mathbb{F}}_r(z)/\widehat{\mathbb{F}}'_r(z)$ of split free submodules $\widehat{\mathbb{F}}'_r(z) \subseteq \widehat{\mathbb{F}}_r(z) \subseteq H^N_r(X;\xi)$. The configuration $\hat{\delta}^f_r$ is derived from a *configuration of pairs* $\tilde{\delta}^f_r(z) := (\widehat{\mathbb{F}}_r(z), \widehat{\mathbb{F}}'_r(z))$ *of submodules of* $H^N_r(X,\xi_f)$, a concept explained in Section 2.

- In the case $\kappa = \mathbb{C}$, the ring of Laurent polynomials $\mathbb{C}[t^{-1}, t]$ has a natural completion to the finite von Neumann algebra $L^\infty(S^1)$ and $H^N_r(X;\xi_f)$ to an $L^\infty(\mathbb{S}^1)$–Hilbert module. The Hilbert module structure, although unique up to isomorphism, depends on a chosen $\mathbb{C}[t^{-1}, t]$–compatible Hermitian inner product on $H^N_r(X;\xi_f)$ — see Section 2 — which always exists. With respect to a given $\mathbb{C}[t^{-1}, t]$–compatible inner product, the free module $H^N_r(X;\xi)$ can be canonically converted into the $L^\infty(\mathbb{S}^1)$–Hilbert module $H^{L_2}_r(\widetilde{X})$, $\widetilde{X}$ the infinite cyclic cover associated with ξ, and the configuration $\hat{\delta}^f_r(z)$ into a configuration of mutually orthogonal closed Hilbert submodules $\hat{\hat{\delta}}^f_r(z) \subseteq H^{L_2}_r(\widetilde{X})$ with $\sum_{z\in\operatorname{supp}\delta^f_r}\hat{\hat{\delta}}^f_r(z) = H^{L_2}_r(\widetilde{X})$. This conversion is referred to below as the *von Neumann completion* and is described in Section 2. A Riemannian metric on X when X is the underlying space of a closed smooth manifold, or a triangulation of X when X is homeomorphic to a finite simplicial complex, provides a canonical inner product which leads to the familiar L_2–homology, $H^{L_2}_r(\widetilde{X})$. This is a particular case of a construction described in Lück [9].

- The refinement of the Poincaré duality stated in Theorem 1.3 is derived from the Poincaré duality between Borel–Moore homology and cohomology of the open manifold $\widetilde{M}$.

The paper ends with a few topological applications, Observation 1.4 and Theorem 1.5.

In Section 2 one recalls the definition of various spaces of configurations (of points with multiplicity, of submodules and of pairs of submodules, of mutually orthogonal closed Hilbert submodules of a Hilbert module) and of the relevant topologies on these spaces. The configurations referred to above, δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ for $f\colon X\to\mathbb{S}^1$ an angle-valued map, are defined in Section 3 and all have the same support located in $\mathbb{T}$ or $\mathbb{C}\setminus 0$. A point in $\mathbb{T}$ will be denoted by $\langle a,b\rangle$ and one in $\mathbb{C}\setminus 0$ by z.

To formulate the results, for the reader's convenience we recall some concepts and notation.

For an angle-valued map $f\colon X\to\mathbb{S}^1$ one denotes by $\xi_f\in H^1(X;\mathbb{Z})$ the integral cohomology class represented by f and by $\tilde{f}\colon\tilde{X}\to\mathbb{R}$ an infinite cyclic cover or *lift* of the map f. In Section 2 one also defines the concepts of weakly tame and tame maps as well as of homologically regular and homologically critical values. The simplicial maps are always tame and then weakly tame. Informally, for a weakly tame map a *homologically regular* value is a complex number $z=e^{ia+b}\in\mathbb{S}^1$ such that the homology of the level of z' in a small neighborhood of z is unchanged and a homologically critical value is a complex number $z=e^{ia+b}\in\mathbb{S}^1$ which is not a homologically regular value.

For $\xi\in H^1(X;\mathbb{Z})$ one denotes by:

- $C_\xi(X,\mathbb{S}^1)$ the space of continuous maps in the homotopy class defined by ξ equipped with the compact open topology.
- $\pi\colon\tilde{X}\to X$ an infinite cyclic cover defined by ξ_f, or by f.

For a specified field κ one denotes by:

- $H^N_r(X;\xi)$ the Novikov homology in dimension r with respect to the field κ.
- $\beta^N_r(X;\xi)$ the r^{th} Novikov–Betti number; see Section 2 for definitions.

In case $\kappa=\mathbb{C}$ the L_2–homology of $\tilde{X}$ in dimension r will be denoted by $H^{L_2}_r(\tilde{X})$. This is an $L^\infty(\mathbb{S}^1)$–Hilbert module. In this case the von Neumann dimension of $H^{L_2}_r(\tilde{X})$ equals the Novikov–Betti number with respect to the field $\mathbb{C}$.

The main results of this paper are collected in the following theorems:

Theorem 1.1 (topological results) (1) *The configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ have the same support. If f is weakly tame and $\delta^f_r(z)\neq 0$ with $z=e^{ia+(b-a)}$, then both a and b are homological critical values of $\tilde{f}$.*

(2) (a) $\sum_{z\in\mathbb{C}\setminus 0}\delta^f_r(z)=\beta^N_r(X;\xi_f)$.

(b) $\bigoplus_{z\in\mathbb{C}\setminus 0}\hat\delta^f_r(z)\simeq H^N_r(X;\xi_f)$.

(c) *When* $\kappa=\mathbb{C}$ *a* $\mathbb{C}[t^{-1},t]$*–compatible inner product on* $H^N_r(X;\xi_f)$ (see Section 2 for the definition) *converts* $\hat\delta^f_r$ *into a configuration* $\hat{\hat\delta}{}^f_r$ *of closed Hilbert submodules of* $H^{L_2}_r(\tilde X)$ *which satisfy* $\sum_{z\in\mathbb{C}\setminus 0}\hat{\hat\delta}{}^f_r(z)=H^{L_2}_r(\tilde X)$, *and* $\hat{\hat\delta}{}^f_r(z)\perp\hat{\hat\delta}{}^f_r(z')$ *for* $z\neq z'$.

(3) *If* X *is a good ANR* (see Section 2 for the definition), *in particular homeomorphic to a finite simplicial complex or to a compact Hilbert cube manifold, then for an open and dense set of maps* f *in* $C_\xi(X,\mathbb{S}^1)$ *one has* $\delta^f_r(z)=0$ *or* 1.

Here and below $=$ denotes equality or canonical isomorphism and $\simeq$ indicates the existence of an isomorphism.

Items (1) and (2a) were first established in Burghelea and Haller [4] but only for tame maps and by different methods.

Anticipating Section 2 we denote by $\mathrm{Conf}_k(X)$ the set of configurations of k points in X and by $\mathcal{CONF}_V(X)$ the set of configurations of *subspaces* of the module V indexed by the points of X. If V is a κ–vector space, a *subspace* means a genuine vector subspace, if V is a free module, a *subspace* means a free split submodule and if V is a Hilbert module, a *subspace* means a closed Hilbert submodule. In view of this notation, (2a) indicates that $\delta^f_r\in\mathrm{Conf}_{\beta^N_r(X;\xi_f)}(\mathbb{T})$, $T=\mathbb{C}\setminus 0$, which can be identified to the $\beta^N_r(X;\xi_f)$–fold symmetric product of $\mathbb{C}\setminus 0$, hence to the space of degree $\beta^N_r(X;\xi_f)$ monic polynomials with nonzero free coefficient, hence to $\mathbb{C}^{\beta^N_r(X;\xi_f)-1}\times(\mathbb{C}\setminus 0)$, and (2b) implies that any family of splittings as defined in Section 3 makes $\hat\delta^f_r$ an element in $\mathcal{CONF}_V(\mathbb{T})$, with V the free $\kappa[t^{-1},t]$–module $H^N_r(X;\xi_f)$. Item (2c) states that $\hat{\hat\delta}{}^f_r$ is an element in $\mathcal{CONF}^O_{H^{L_2}_r(\tilde X)}(\mathbb{T})$, the space configurations of mutually orthogonal closed Hilbert submodules of the $L^\infty(\mathbb{S}^1)$–Hilbert $H^{L_2}_r(\tilde X)$.

Associated to ξ there is the infinite cyclic cover $\pi\colon\tilde X\to X$, a principal $\mathbb{Z}$–covering unique up to isomorphism, such that any continuous $f\colon X\to\mathbb{S}^1$ has lifts $\tilde f\colon\tilde X\to\mathbb{R}$ (ie $\mathbb{Z}$–equivariant maps which induce, by passing to $X=\tilde X/\mathbb{Z}$, the map f; see Section 2) unique up to an additive constant of the form $2\pi k$. For two lifts $\tilde f$ of f and $\tilde g$ of g denote by $D(\tilde f,\tilde g)=\sup_{\tilde x\in\tilde X}|\tilde f(\tilde x)-\tilde g(\tilde x)|$ and denote by $D(f,g)$ the minimal of $D(\tilde f,\tilde g)$ over all possible lifts of f and g; see Section 2. $D(f,g)$ provides a metric on $C_\xi(X,\mathbb{S}^1)$.

Theorem 1.2 (stability) *Suppose X is a compact ANR and $\xi \in H^1(X;\mathbb{Z})$.*

(1) *The assignment*

$$f \rightsquigarrow \delta^f_r = P^f_r(z) \in \mathbb{C}^{\beta^N_r(X;\xi)-1} \times (\mathbb{C}\setminus 0) \quad \text{for } f \in C_\xi(X,\mathbb{S}^1)$$

is a continuous map.

Moreover, with respect to the canonical metric $\underline{D}$ provided by the identification of the space of configurations with the $\beta^N_r(X;\xi)$–fold symmetric product of $\mathbb{T}$, one has the estimate

$$\underline{D}(\delta^f, \delta^g) < 2D(f,g).$$

(2) *If $\kappa = \mathbb{C}$ and the spaces of configurations $\mathcal{CONF}_{H^{L_2}_r(\tilde{X})}(\mathbb{C}\setminus 0)$ is equipped with either the* fine *or the* natural *collision topology* (see Section 2 for the definitions) *then the assignment $f \rightsquigarrow \hat{\hat{\delta}}^f_r$ is continuous.*

Item (1) was first established in [4] for X homeomorphic to a simplicial complex.

Theorem 1.3 (Poincaré duality) *Suppose M is a closed topological manifold of dimension n which is κ–orientable*[4] *and $f\colon M \to \mathbb{S}^1$ an angle-valued map with $\xi_f \neq 0$. Then one has the following:*

(1) $\delta^f_r\langle a,b\rangle = \delta^f_{n-r}\langle b,a\rangle$; *equivalently, $\delta^f_r(z) = \delta^f_{n-r}(\tau z)$ with $\tau(z) = z|z|^{-2}e^{i\ln|z|}$.*

(2) *The Poincaré duality between Borel–Moore homology of $\tilde{M}$ and the cohomology of $\tilde{M}$ induces the isomorphisms of $\kappa[t^{-1},t]$–modules from $H^N_r(M;\xi)$ to $H^N_{n-r}(M;\xi)$ which intertwine $\hat{\delta}^f_r\langle a,b\rangle$ and $\hat{\delta}^f_{n-r}\langle b,a\rangle$. Precisely, a collection of compatible N–splittings $\mathcal{S}_r$* (see Definition 3.10), *additional data which always exist, provide the canonical isomorphisms of $\kappa[t^{-1},t]$–modules*[5]

$$\begin{aligned} \mathrm{PD}^S_r\colon & H^N_r(M;\xi_f) \to H^N_{n-r}(M;\xi_f), \\ \mathrm{PD}^S_r\langle a,b\rangle\colon & \hat{\delta}^{\tilde{f}}_r\langle a,b\rangle \to \hat{\delta}^{\tilde{f}}_{n-r}\langle b,a\rangle, \\ I^S_r\colon & \bigoplus_{\langle a,b\rangle \in \mathbb{T}} \hat{\delta}_r\langle a,b\rangle \to H^N_r(M;\xi_f) \end{aligned}$$

such that the diagram

[4]This theorem will be verified only in the case M is homeomorphic to a finite simplicial complex.

[5]The finiteness of the rank of $H^N_r(M;\xi_f)$ implies that $\hat{\delta}^f_r\langle a,b\rangle = 0$ for all but finitely many pairs $\langle a,b\rangle$.

$$\begin{tikzcd}
\bigoplus_{\langle a,b\rangle\in\mathbb{T}} \hat{\delta}_r\langle a,b\rangle \arrow[r, "\bigoplus \mathrm{PD}^S_r\langle a,b\rangle"] \arrow[d, "I^S_r"'] & \bigoplus_{\langle a,b\rangle\in\mathbb{T}} \hat{\delta}_{n-r}\langle b,a\rangle \arrow[d, "I^S_{n-r}"'] \\
H^N_r(M;\xi_f) \arrow[r, "\mathrm{PD}^S_r"] & H^N_{n-r}(M;\xi_f)
\end{tikzcd}$$

is commutative.

(3) *For $\kappa = \mathbb{C}$ a $C[t^{-1},t]$–compatible Hermitian inner product on $H^N_r(M;\xi_f)$* (see Section 2 for the definition) *provides canonical compatible N–splittings $\mathcal{S}_r$ such that the von Neumann completion* (described in Section 2) *leads to the canonical isomorphisms*

$$\mathrm{PD}^{L_2}_r\colon H^{L_2}_r(\tilde{M}) \to H^{L_2}_{n-r}(\tilde{M}),$$
$$\mathrm{PD}^{L_2}_r\langle a,b\rangle\colon \hat{\hat{\delta}}^{L_2}_r\langle a,b\rangle \to \hat{\hat{\delta}}^{L_2}_{n-r}\langle b,a\rangle,$$
$$I^{L_2}_r\colon \bigoplus_{\langle a,b\rangle\in\mathbb{T}} \hat{\hat{\delta}}_r\langle a,b\rangle \to H^{L_2}_r(\tilde{M}),$$

which make the diagram

$$\begin{tikzcd}
\bigoplus_{\langle a,b\rangle\in\mathbb{T}} \hat{\hat{\delta}}_r\langle a,b\rangle \arrow[r, "\bigoplus \mathrm{PD}^{L_2}_r\langle a,b\rangle"] \arrow[d, "I^{L_2}_r"'] & \bigoplus_{\langle a,b\rangle\in\mathbb{T}} \hat{\hat{\delta}}_{n-r}\langle b,a\rangle \arrow[d, "I^{L_2}_{n-r}"'] \\
H^{L_2}_r(\tilde{M}) \arrow[r, "\mathrm{PD}^{L_2}_r"] & H^{L_2}_{n-r}(\tilde{M})
\end{tikzcd}$$

commutative. A Riemannian metric or a triangulation of a closed smooth or topological manifold provides canonical $\mathbb{C}[t^{-1},t]$–compatible Hermitian inner products on $H^N_r(M;\xi_f)$ and therefore the isomorphisms claimed in item (3).

Item (1) was first established in [4].

Theorem 1.3(2) implies the following:

Observation 1.4 *If M is a compact manifold with boundary ∂M, and $H^N_r(\partial M;\xi_{f_{\partial M}})$* (with $f_{\partial M}$ the restriction of f to ∂M) *vanishes for all r, then $H^N_r(M;\xi_f) \simeq H^N_{n-r}(M;\xi_f)$.*

If X is connected and $u \in \bar{\kappa}\setminus 0$ with $\bar{\kappa}$ the algebraic closure of κ, denote by (ξ,u) the local coefficient system defined by the rank-one κ–representation

$$(\xi,u)\colon \pi_1(X,x_0) \to H_1(X;\mathbb{Z}) \xrightarrow{\xi} \mathbb{Z} \xrightarrow{u^\bullet} \bar{\kappa}\setminus 0,$$

where ξ is interpreted as a group homomorphism $\xi\colon H_1(M.\mathbb{Z}) \to \mathbb{Z}$ and u as the homomorphism $u^\bullet(n) = u^n$. Denote by $H_r(X; (\xi, u))$ the homology with coefficients in (ξ, u) and by $\beta_r(X; (\xi, u))$ the dimension of this κ–vector space.

For $\xi \in H^1(M; \mathbb{Z})$ the set of Jordan cells $\mathcal{J}_r(X; \xi)$ was defined in Burghelea and Dey [3] and from a different perspective of relevance in the theorem below, discussed in Burghelea and Haller [4] and Burghelea [1]. One denotes by $\mathcal{J}_r(X; \xi)(u)$ the set of Jordan cells (λ, k) with $\lambda = u \in \overline{\kappa} \setminus 0$.

Theorem 1.5 *Suppose $(M^n, \partial M)$ is a compact manifold with boundary $\xi \in H^1(M; \mathbb{Z})$ such that $\beta_r^N(\partial M, \xi|_{\partial M}) = 0$ for all r. Suppose that M retracts by deformation to a simplicial complex of dimension $\leq \left[\frac{n}{2}\right]$, where $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$ and $\chi(M)$ is the Euler–Poincaré characteristic with respect to the field κ. Then we have:*

(1) *If $n = 2k$ then one has*

$$\text{(1a)}\qquad \beta_r^N(X; \xi) = \begin{cases} 0 & \text{if } r \neq k, \\ (-1)^k \chi(M) & \text{if } r = k, \end{cases}$$

$$\text{(1b)}\qquad \beta_r(X) = \begin{cases} \mathcal{J}_{r-1}(X; \xi)(1) + \mathcal{J}_r(X; \xi)(1) & \text{if } r \neq k, \\ \mathcal{J}_{k-1}(X; \xi)(1) + \mathcal{J}_k(X; \xi)(1) + (-1)^k \chi(M) & \text{if } r = k, \end{cases}$$

$$\text{(1c)}\qquad \beta_r(X; \hat{u}\xi) = \begin{cases} \mathcal{J}_{r-1}(X; \xi)(u) + \mathcal{J}_r(X; \xi)(1/u) & \text{if } r \neq k, \\ \mathcal{J}_{k-1}(X; \xi)(u) + \mathcal{J}_k(X; \xi)(1/u) + (-1)^k \chi(M) & \text{if } r = k. \end{cases}$$

(2) *If $n = 2k + 1$ then one has*

$$\text{(2a)}\qquad \beta_r^N(X; \xi) = 0,$$

$$\text{(2b)}\qquad \beta_r(X) = \mathcal{J}_{r-1}(X; \xi)(1) + \mathcal{J}_r(X; \xi)(1),$$

$$\text{(2c)}\qquad \beta_r(X; (\xi, u)) = \mathcal{J}_{r-1}(X; \xi)(1/u) + \mathcal{J}_r(X; \xi)(u).$$

(3) *If $V^{n-1} \subset M^n$ is a compact proper submanifold* (ie $V \pitchfork \partial M$,[6] and $V \cap \partial M = \partial V$) *whose Poincaré dual cohomology class is ξ_f and $H_r(V) = 0$, then the set of Jordan cells $J_r(M, \xi)$ is empty for $r > 0$ and $J_0(M, \xi) = \{(\lambda; 1), \lambda = 1\}$.*

As pointed out to us by L Maxim, the complement $X = \mathbb{C}^n \setminus V$ of a complex hypersurface $V \subset \mathbb{C}^n, V := \{(z_1, z_2, \dots, z_n) \mid f(z_1, z_2, \dots, z_n) = 0\}$ regular at infinity, equipped with the canonical class $\xi_f \in H^1(X; \mathbb{Z})$ defined by $f\colon X \to \mathbb{C} \setminus 0$ is an

[6] Here $\pitchfork$ means transversal.

example of an open manifold which has as compactification a manifold with boundary equipped with a degree one integral cohomology class which satisfies the hypotheses and then the conclusion of Theorem 1.5 above.

Item (1) recovers a calculation of Maxim; see Maxim [11] and Friedl and Maxim [7][7] that the complement of an algebraic hypersurface regular at infinity has vanishing Novikov homologies in all dimensions but n.

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2 Preparatory material

Angles and angle-valued maps An *angle* is a complex number $\theta = e^{it} \in \mathbb{C}$ with $t \in \mathbb{R}$ and the set of all angles is denoted by $\mathbb{S}^1 = \{\theta = e^{it} \mid t \in \mathbb{R}\}$. The space of angles, $\mathbb{S}^1$, is equipped with the distance

$$d(\theta_2, \theta_1) = \inf\{|t_2 - t_1| \mid e^{it_1} = \theta_1,\ e^{it_2} = \theta_2\}.$$

One has $d(\theta_1, \theta_2) \leq \pi$. With this description, $\mathbb{S}^1$ is an oriented one-dimensional manifold with the orientation provided by a specified generator u of $H_1(\mathbb{S}^1; \mathbb{Z})$.

A closed interval in $I \subset \mathbb{S}^1$ with ends the angles $\theta_1 = e^{it_1}$ and $\theta_2 = e^{it_2}$ is the set $I := \{e^{it} \mid t_1 \leq t \leq t_2, t_2 - t_1 < 2\pi\}$.

In this paper all real- or angle-valued maps are proper continuous maps defined on an ANR, hence locally compact in the case of real-valued and compact in the case of angle-valued. Recall that an ANR — see Hu [8] — is a space homeomorphic to a closed subset A of a metrizable space which has an open neighborhood U which retracts to A. Simplicial complexes and finite- or infinite-dimensional manifolds are ANRs.

[7]The Friedl–Maxim results state the vanishing of more general and more sophisticated L_2–homologies and Novikov-type homologies. Such results can be also recovered via the appropriate Poincaré duality isomorphisms.

Infinite cyclic cover For an angle-valued map $f\colon X \to \mathbb{S}^1$ let $f^*\colon H^1(\mathbb{S}^1;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ be the homomorphism induced by f in integral cohomology and let $\xi_f = f^*(u) \in H^1(X;\mathbb{Z})$. The assignment $f \rightsquigarrow \xi_f$ establishes a bijective correspondence between the set of homotopy classes of continuous maps from X to $\mathbb{S}^1$ and $H^1(X;\mathbb{Z})$.

Recall the following:

- An infinite cyclic cover of X is a map $\pi\colon \widetilde{X} \to X$ together with a free action $\mu\colon \mathbb{Z} \times \widetilde{X} \to \widetilde{X}$ such that $\pi(\mu(n,x)) = \pi(x)$ and the map induced by π from $\widetilde{X}/\mathbb{Z}$ to X is a homeomorphism. The infinite cyclic cover $\pi\colon \widetilde{X} \to X$ is said to be *associated to* ξ if any continuous proper map $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$ which satisfies $\tilde{f}(\mu(n,x)) = \tilde{f}(x) + 2\pi n$ induces a map $f\colon X \to \mathbb{R}/2\pi\mathbb{Z} = \mathbb{S}^1$ with ξ_f equal to ξ. The homeomorphisms $\mu(k,\dots)\colon \widetilde{X} \to \widetilde{X}$ are called *deck transformations*.

- For two infinite cyclic covers $\pi_i\colon \widetilde{X}_i \to X$ for $i = 1, 2$, associated to ξ, there exists a homeomorphism $\omega\colon \widetilde{X}_1 \to \widetilde{X}_2$ which intertwines the free actions μ_1 and μ_2 and satisfies $\pi_2 \cdot \omega = \pi_1$.

- Given $\pi\colon \widetilde{X} \to X$, an infinite cyclic cover $\mu\colon \mathbb{Z} \times \widetilde{X} \to \widetilde{X}$ and an angle-valued map $f\colon X \to \mathbb{S}^1$, the map $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$ is a called a *lift of* f if $\tilde{f}(\mu(n,x)) = \tilde{f}(x) + 2\pi n$ and by passing to the quotients $X = \widetilde{X}/\mathbb{Z}$ and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ the map $\tilde{f}$ induces exactly f. A lift $\tilde{f}$ provides the pullback diagram

(3)
$$\begin{tikzcd} \mathbb{R} \arrow[r, "p"] & \mathbb{S}^1 \\ \widetilde{X} \arrow[u, "\tilde{f}"] \arrow[r, "\pi"] & X \arrow[u, "f"'] \end{tikzcd}$$

where $p(t)$ is given by $p(t) = e^{it} \in \mathbb{S}^1$. Two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f differ by a deck transformation, ie there exists $k \in \mathbb{Z}$ with $\tilde{f}_2 = \tilde{f}_1 \cdot \mu(k,\dots)$.

Given $f\colon X \to \mathbb{S}^1$, there exists a canonical infinite cyclic cover $\pi\colon \widetilde{X} \to X$, the pullback of $p\colon \mathbb{R} \to \mathbb{S}^1$ by f, and a canonical lift $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$ of f; explicitly, $\widetilde{X} = \{(x,r) \mid f(x) = p(t)\}$ and $\tilde{f}(x,t) = t$.

Denote by $C_\xi(X;\mathbb{S}^1)$ the set of all continuous angle-valued maps on X in the homotopy class defined by ξ and let $\pi\colon \widetilde{X} \to X$ be an infinite cyclic cover associated to ξ. For $f, g \in C_\xi(X,\mathbb{S}^1)$ and $\tilde{h}$ and $\tilde{l}$ lifts of f and g, let

(1) $D(\tilde{h},\tilde{l}) := \sup_{\tilde{x} \in \widetilde{X}} |\tilde{h}(x) - \tilde{l}(x)|$,

(2) $D(f,g) := \inf\{D(\tilde{h},\tilde{l}) \mid \tilde{h} \text{ a lift of } f \text{ and } \tilde{l} \text{ a lift of } g\}$.

Note that if $d(f(x), g(x)) < \pi$ then $D(f, g) = \sup_{x \in X} d(f(x), g(x))$, where d is the standard metric on $\mathbb{S}^1$, namely $d(u, v) = \inf_{\theta,\theta'} |\theta - \theta'|$ for $u = e^{i\theta}$ and $v = e^{i\theta'}$.

Observation 2.1 (1) *For any maps f and g, there exist lifts $\tilde{f}$ and $\tilde{g}$ such that $D(f, g) = D(\tilde{f}, \tilde{g})$.*

(2) *D is a complete metric which induces the compact open topology on $C_\xi(X, \mathbb{S}^1)$.*

(3) *If $f, g \in C_\xi(X, \mathbb{S}^1)$, $\tilde{f}$ and $\tilde{g}$ are lifts of f and g and $0 = t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1} = 1$ any subdivision of the interval* [0, 1], *then the canonical homotopy $\tilde{f}_t = t\tilde{f} + (1-t)\tilde{g}$ satisfies $D(\tilde{f}, \tilde{g}) = \sum_{0 \le i \le N} D(\tilde{f}_{t_i}, \tilde{f}_{t_{i+1}})$. If f_t denotes the homotopy between f and g induced from $\tilde{f}_t$, then, by passing to the quotient and using $D(f, g) = D(\tilde{f}, \tilde{g})$,*

$$D(f, g) \geq \sum_{0 \le i \le N} D(f_{t_i}, f_{t_{i+1}}). \tag{4}$$

The verifications are straightforward and left to the reader.

A homotopy f_t as in item (3) is referred to as a *canonical homotopy*.

Regular and critical values and tameness Let $f \colon X \to \mathbb{S}^1$ or $\mathbb{R}$ be a proper continuous map.

- The value $s \in \mathbb{S}^1$ or $\mathbb{R}$ is *regular/homologically regular* if there exists a neighborhood U of s such that for any $s' \in U$ the inclusion $f^{-1}(s') \subset f^{-1}(U)$ is a homotopy equivalence/homology equivalence[8] and *critical/homologically critical* if not regular/homologically regular. We denote by $\mathrm{CR}(f)$ the set of critical values and by $\mathrm{CRH}^\kappa(f)$ the set of homologically critical values. Clearly $\mathrm{CRH}^\kappa(f) \subseteq \mathrm{CR}(f)$. Since the field κ will be fixed once and for all, κ will be discarded from notation and we write $\mathrm{CRH}(f)$ instead of $\mathrm{CRH}^\kappa(f)$.

- The map f is *weakly tame* if for any $s \in \mathbb{S}^1$ or $\mathbb{R}$ the subspace $f^{-1}(s)$ is an ANR. This implies that, for any closed interval I in $\mathbb{R}$ or $\mathbb{S}^1$, $f^{-1}(I)$ is an ANR.

- The map is *tame* if it is weakly tame and in addition the set of critical values is discrete and the distance between any two critical values bounded from below by a positive number $\epsilon(f)$.

[8] With respect to a specified field κ.

- The map is *homologically tame* with respect to a specified field if the set of homologically critical values is discrete and the distance between any two such homologically critical values is bounded from below by a positive number.

For an angle-valued map $f\colon X \to \mathbb{S}^1$ consider a lift $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ of f. The map f is weakly tame (resp. tame, homologically tame) if and only if so is $\tilde{f}$. If X is a finite simplicial complex then a map $f\colon X \to \mathbb{S}^1$ is called pl (piecewise linear) if some — and then any — lift $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ is a pl map. An angle $\theta \in \mathbb{S}^1$ is a regular value (resp. critical value, homologically critical value) if $\theta = e^{it}$ with t a regular value (resp. critical value, homologically critical value) for $\tilde{f}$. For technical reasons we will need the following concept:

- A compact ANR X is called a *good ANR* if the set of tame maps (real- or angle-valued maps) is dense in the set of all continuous maps with respect to the compact open topology. In particular any finite simplicial complex is a good ANR in view of the fact that the set of pl maps is dense in the set of continuous maps and each pl real- or angle-valued map is tame.

The von Neumann completion When $\kappa = \mathbb{C}$, the ring of Laurent polynomials $\mathbb{C}[t^{-1}, t]$ — equivalently, the group ring $\mathbb{C}[\mathbb{Z}]$ of the infinite cyclic group $\mathbb{Z}$ — is an algebra with involution $*$ and trace tr defined as follows:

If $a = \sum_{n\in\mathbb{Z}} a_n t^n$ then

$$*(a) := a^* = \sum_{n\in\mathbb{Z}} \overline{a}_n t^{-n}, \quad \operatorname{tr}(a) = a_0,$$

with $\overline{a}$ denoting the complex conjugate of the complex number a.

The algebra $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[t^{-1}, t]$ can be considered as a subalgebra of the algebra of bounded linear operators on the separable Hilbert space

$$l_2(\mathbb{Z}) = \Big\{ a_n, n \in \mathbb{Z} \;\Big|\; \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \Big\}$$

with the Hermitian scalar product $\mu(a, b) = \sum_{n\in\mathbb{Z}} a_n \overline{b}_n$.

The linear operator defined by a Laurent polynomial (an alternative name for an element in $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[t^{-1}, t]$) is given by the multiplication of the Laurent polynomial regarded as a sequence with all but finitely many components equal to zero with a sequence in $l_2(\mathbb{Z})$.

One denotes by $\mathcal{N}$ the *weak closure* of $\mathbb{C}[\mathbb{Z}]$ in the space of bounded operators of the Hilbert space $l_2(\mathbb{Z})$ when each element of $\mathbb{C}[\mathbb{Z}]$ is regarded as such an operator, which is a *finite von Neumann algebra*, with involution and trace extending the ones defined above; see [9].

This algebra $\mathcal{N}$ is referred to below as the von Neumann completion of the group ring $\mathbb{C}[\mathbb{Z}]$ and is isomorphic to the familiar $L^\infty(\mathbb{S}^1)$ via Fourier series transform (whose inverse assigns to a complex-valued function defined on $\mathbb{S}^1$ its Fourier series).

Given a free $\mathbb{C}[t^{-1},t]$–module M, a *$C[t^{-1},t]$–compatible Hermitian inner product* is a map $\mu\colon M\times M\to\mathbb{C}$ which satisfies:

(1) **Linearity** $\mathbb{C}$–linear in the first variable.

(2) **Symmetry** $\mu(x,y)=\overline{\mu}(y,x)$.

(3) **Positivity**

  (a) $\mu(x,x)\in\mathbb{R}_{\geq 0}\subset\mathbb{C}$.

  (b) $\mu(x,x)=0$ if and only if $x=0$.

(4) For any x and y, there exists n such that $\mu(t^n x,y)=0$.

(5) $\mu(tx,ty)=\mu(x,y)$.

Items (1)–(4) make μ a nondegenerate Hermitian inner product on M and items (5) and (6) define the $\mathbb{C}[t^{-1},t]$–compatibility of the Hermitian inner product μ.

An equivalent data is provided by a $\mathbb{C}[t^{-1},t]$–valued inner product — see [9] — which is given by a map $\hat{\mu}\colon M\times M\to\mathbb{C}[t^{-1},t]$ which satisfies:

(1) $\mathbb{C}[t^{-1},t]$–linear in the first variable.

(2) Symmetric in the sense that $\hat{\mu}(x,y)=\hat{\mu}(y,x)^*$ for $x,y\in M$.

(3) Positive definite in the sense that it satisfies

  (a) $\hat{\mu}(x,x)\in\mathbb{C}[t^{-1},t]_+$ with $\mathbb{C}[t^{-1},t]_+$ the set of elements of the form aa^*, and

  (b) $\hat{\mu}(x,x)=0$ if and only if $x=0$.

(4) The map $\tilde{\hat{\mu}}\colon M\to\mathrm{Hom}_{\mathbb{C}[t^{-1},t]}(M,\mathbb{C}[t^{-1},t])$ defined by $\tilde{\hat{\mu}}(y)(x)=\hat{\mu}(x,y)$ is one-to-one.

The relation between μ and $\hat{\mu}$ is given by

(5) $$\hat{\mu}(x,y) = \sum_{n\in\mathbb{Z}} t^n \mu(t^{-n}x, y), \quad \mu(x,y) = \operatorname{tr}\hat{\mu}(x,y).$$

Clearly $\mathbb{C}[t^{-1},t]$–valued inner products exist. Indeed, if $e^1, e^2, \dots, e^k$ is a base of M then

$$\mu\Big(\sum a_i e^i, \sum b_j e^j\Big) := \sum a_i (b_i)^*$$

provides such an inner product.

Note that if M is finitely generated but not free, a map $\hat{\mu}$ as above satisfying all properties but (3b) and instead satisfying "ker $\tilde{\hat{\mu}}$ equals the $\mathbb{C}[t^{-1},t]$–torsion of M", induces a $C[t^{-1},t]$–compatible Hermitian inner product μ on M/TM, where TM is the collection of torsion elements in M.

By completing the $\mathbb{C}$–vector space M (the underlying vector space of the finitely generated $\mathbb{C}[t,t^{-1}]$–module M) with respect to the Hermitian inner product μ one obtains a Hilbert space $\overline{M}$ which is an $\mathcal{N}$–Hilbert module — see [9] — isometric to $l_2(\mathbb{Z})^{\oplus k}$, with k the rank of M.

Two different $\mathbb{C}[t^{-1},t]$–valued inner products, μ_1 and μ_2, lead to the isomorphic (and then also isometric) Hilbert modules $\overline{M}_{\mu_1}$ and $\overline{M}_{\mu_2}$. This justifies discarding μ from the notation.

If one identifies $\mathcal{N}$ to $L^\infty(\mathbb{S}^1)$ and $l_2(\mathbb{Z})^{\oplus k}$ to $L^2(\mathbb{S}^1)^{\oplus k}$ — by interpreting the sequence $\sum_{n\in\mathbb{Z}} a_n t^n$ as the complex-valued function $\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ — the $\mathcal{N}$–module structure on $l_2(\mathbb{Z})^{\oplus k}$ becomes the $L^\infty(\mathbb{S}^1)$–module structure on $(L^2(\mathbb{S}^1))^{\oplus k}$ given by the componentwise multiplication of L^∞–functions with L^2–functions.

If $N \subset M$ is a free split submodule of the finitely generated free $\mathbb{C}[t^{-1},t]$–module M and μ is an $\mathbb{C}[t^{-1},t]$–valued inner product on M, then $\overline{N}_\mu$ is a closed Hilbert submodule of $\overline{M}_\mu$. Moreover, if $N_i' \subseteq N_i \subseteq M$ for $i = 1, 2, \dots$ is a collection of pairs of split submodules then the collection N_i/N_i' is a collection of free modules, which are quotients of submodules of M, and the von Neumann completion process converts N_i' and N_i into closed Hilbert submodules of $\overline{M}$ and each N_i/N_i' into a Hilbert module canonically identified to the orthogonal complement of the kernel of the projection $N_i \to N_i/N_i'$ inside N_i. The process of passing from $(\mathbb{C}[t^{-1},t], M)$ to $(\mathcal{N}, \overline{M})$ referred to above as *von Neumann completion* was considered in [10] for any group ring $\mathbb{C}[\Gamma]$ and finitely generated projective $\mathbb{C}[\Gamma]$–module.

Configurations of points with multiplicity A configuration of points with multiplicity in X is a map with finite support $\delta\colon X \to \mathbb{Z}_{\geq}0$. The support of δ is the set

$$\operatorname{supp} \delta := \{x \in X \mid \delta(x) \neq 0\}$$

and the cardinality of the support is

$$\#\delta := \sum \delta(x).$$

Denote by $\mathrm{Conf}_n(X)$ the set of configurations of cardinality n. Clearly $\mathrm{Conf}_n(X) = S^n(X) = X^n/\Sigma_n$, the quotient of the n–fold product X^n by the action of the permutation group of n–elements, Σ_n, and this description equips $\mathrm{Conf}_n(X)$ with the quotient topology induced from the topology of the product space X^n. There is an alternative but equivalent way — see below — to describe this topology as *collision topology*.

Configuration of subspaces Let A be a commutative ring with unit, for example a field $\kappa = A$, and V a free module of finite rank, rank $V = n$, and let $\mathcal{S}(V)$ be the set of split submodules of V.

A configuration of subspaces of V indexed by points in X is a map with finite support $\widehat{\delta}\colon X \to \mathcal{S}(V)$ such that

$$\bigoplus i_x\colon \bigoplus \widehat{\delta}(x) \to V,$$

with $i_x\colon \widehat{\delta}(x) \to V$ the inclusion, is an isomorphism. As before,

$$\operatorname{supp} \delta := \{x \in X \mid \widehat{\delta}(x) \neq 0\}.$$

Denote by $\mathcal{CONF}_V(X)$ the set of configurations of such submodules (subspaces if V is a vector space). The configuration $\widehat{\delta}$ is called a *refinement of* $\delta \in \mathrm{Conf}_{\operatorname{rank} V}(X)$ if $\delta(x) = \operatorname{rank} \widehat{\delta}(x)$.

If $\mathcal{S}(V)$ is equipped with a topology then $\mathcal{CONF}_V(X)$ carries a topology, the *collision topology*, defined by specifying for each element a system of *fundamental neighborhoods*.[9]

A fundamental neighborhood of a configuration $\widehat{\omega} \in \mathcal{CONF}_V(X)$ with support $\{x_1, x_2, \dots, x_k\}$ and values $\widehat{\omega}(x_i) = V_i$ is specified by

(a) a collection of disjoint open sets $(U_1, U_2, \dots, U_k)$ of X, each U_i a neighborhood of x_i;

[9]Described in [2] in the case A is a field.

(b) a collection of neighborhoods of $\hat{\delta}(x_i)$ in $\mathcal{S}(V)$, $\mathcal{V}_1 \ni \hat{\delta}(x_1)$, $\mathcal{V}_2 \ni \hat{\delta}(x_2), \dots,$ $\mathcal{V}_k \ni \hat{\delta}(x_k)$, and consists of

$$\left\{\hat{\delta} \in \mathcal{CONF}_V(X) \,\middle|\, \sum_{x \in U_i \cap \operatorname{supp} \hat{\delta}} \hat{\delta}(x) \in \mathcal{V}_i\right\}.$$

If the topology on $\mathcal{S}(V)$ is the discrete topology then the topology on $\mathcal{CONF}_V(V)$ is referred to as the *fine collision topology*.

In the case of configuration of points with multiplicity, the topology on $\mathrm{Conf}_{\operatorname{rank} V}(X)$ can be described in the same way, simply by replacing $\mathcal{S}(V)$ by $\mathbb{Z}_{\geq 0}$ equipped with the discrete topology. Note that the assignment

$$\mathcal{CONF}_V(X) \ni \hat{\delta} \rightsquigarrow \delta \in \mathrm{Conf}_{\operatorname{rank} V}(X)$$

is continuous.

If $A = \kappa$ with $\kappa = \mathbb{R}$ or $\kappa = \mathbb{C}$ and the vector space V (not necessary of finite dimension) is equipped with a Hilbert space structure and $\mathcal{S}(V)$ is the set of closed subspaces, then one can consider on $\mathcal{S}(V)$ the topology induced from the *norm topology* on the space of bounded operators on V. The closed subspaces of V are identified to the self-adjoint projectors. In this case the corresponding topology on $\mathcal{CONF}_V(X)$ is called the *natural collision topology*. If V is a Hilbert space, the subset of configurations with the additional property that $\hat{\delta}(x) \perp \hat{\delta}(y)$ is denoted by $\mathcal{CONF}_V^O(X)$.

Configurations of pairs Let A be a commutative ring with unit, V a free module of finite rank, $\operatorname{rank} V = N$, and let $\mathcal{P}(V)$ be the set of pairs (W, W') with $W \supseteq W'$ split submodules of V. The pair (W, W') is called virtually trivial if $W = W'$.

A *configuration of pairs* of submodules of V parametrized by X is a map $\tilde{\delta}\colon X \to \mathcal{P}(V)$ with finite support

$$\operatorname{supp} \tilde{\delta} := \{x \in X \mid \tilde{\delta}(x) \neq \text{virtually trivial}\}$$

which satisfies the following properties:

(1) The set $\mathcal{A} = \tilde{\delta}(X)$ is finite.

(2) If $\alpha, \beta \in \mathcal{A}$ with $\alpha = (W_\alpha, W'_\alpha)$ and $\beta = (W_\beta, W'_\beta)$, then $W_\alpha \subset W_\beta$ implies $W_\alpha \subseteq W'_\beta$.

(3) For any α one has

$$\sum_{\beta \in \mathcal{A},\, W_\beta \subseteq W_\alpha} \operatorname{rank}(W_\beta / W'_\beta) = \operatorname{rank} W_\alpha \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}} \operatorname{rank}(W_\alpha / W'_\alpha) = \operatorname{rank} V.$$

Any collection of splitting $\{s_\alpha\colon W_\alpha/W'_\alpha \to W_\alpha \subseteq V \mid \alpha \in \mathcal{A}\}$, ie right inverses of the canonical projections $p_\alpha\colon W_\alpha \to W_\alpha/W'_\alpha$, assigns to $\widetilde{\delta}$ the configuration of subspaces $\widehat{\delta}$ defined by $\widehat{\delta}(x) = s_{\widetilde{\delta}(x)}(W_{\widetilde{\delta}(x)}/W'_{\widetilde{\delta}(x)})$. If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and V is a κ–Hilbert space, then the orthogonal complements provides canonical splittings and the associated configuration $\widehat{\delta}$ becomes a configuration of subspaces.

If $A = \mathbb{C}[t^{-1}, t]$ and V is a free A–module of finite rank equipped with a $\mathbb{C}[t^{-1}, t]$–valued inner product, then the von Neumann completion converts A into $L^\infty(\mathbb{S}^1)$, V into a finite-type Hilbert module, hence a Hilbert space, and any configuration of pairs $\widetilde{\delta}$, by the process of von Neumann completion, into a configuration of Hilbert submodules. First one converts $\widetilde{\delta}$ into a configuration of pairs of Hilbert submodules and then, using the Hermitian inner product, one realizes the quotient of each pair as a closed Hilbert submodule. Clearly the space of configurations of Hilbert submodules comes equipped with the natural collision topology as well as the fine collision topology.

Novikov homology Let κ be a field and let $\kappa[t^{-1}, t]$ be the κ–algebra of Laurent polynomials with coefficients in κ. This is a commutative algebra which is an integral domain and a principal ideal domain. For a pair (X, ξ) with $\xi \in H^1(X; \mathbb{Z})$ and X a compact ANR, let $\widetilde{X}$ be the associated infinite cyclic cover and let $\tau\colon \widetilde{X} \to \widetilde{X}$ be the positive generator of the group of deck transformations isomorphic to $\mathbb{Z}$, viewed as a homeomorphism of $\widetilde{X}$. Since X is compact, the κ–vector space $H_k(\widetilde{X})$ is actually a finitely generated $\kappa[t^{-1}, t]$–module whose multiplication by t is given by the linear isomorphism induced by the homeomorphism τ.

Since $\kappa[t^{-1}, t]$ is a principal ideal domain, the collection of torsion elements form a $\kappa[t^{-1}, t]$–submodule $V_r(X; \xi) := \mathrm{Torsion}(H_r(\widetilde{X})) = TH_r(\widetilde{M})$ (usually referred to as *monodromy*) which, as a κ–vector space, is of finite dimension. The quotient module $H_r(\widetilde{X})/TH_r(\widetilde{X})$ is a finitely generated free $\kappa[t^{-1}, t]$–module. In this paper, this free $\kappa[t^{-1}, t]$–module and its rank are called the *Novikov homology* and the *Novikov–Betti number* and are denoted by $H^N_r(X; \xi)$ and $\beta^N_r(X; \xi)$, respectively.[10] Since $\kappa[t^{-1}, t]$ is a principal ideal domain one has $H_r(\widetilde{X}) \simeq H^N_r(X; \xi) \oplus TH_r(\widetilde{X})$.

As pointed out above the $\kappa[t^{-1}, t]$–module $V_r(X; \xi) = TH_r(\widetilde{X})$, which is finitely generated, when regarded as a vector space over κ is of finite dimension and the multiplication by t is actually a κ–linear isomorphism T. In view of the Jordan

[10] Classically, the Novikov homology is the $\kappa[t^{-1}, t]]$–vector space $H_r(\widetilde{X}) \otimes_{\kappa[t^{-1}, t]} \kappa[t^{-1}, t]]$ with $\kappa[t^{-1}, t]]$ the field of Laurent power series; clearly, $\beta^N_r = \dim(H^N_r(\widetilde{X}) \otimes_{\kappa[t^{-1}, t]} \kappa[t^{-1}, t]]) = \mathrm{rank}(H^N_r(\widetilde{X}))$.

decomposition theorem it is completely determined up to isomorphism by the collection of pairs with multiplicity

$$\mathcal{J}_r(X:\xi) := \{(\lambda, k) \mid \lambda \in \overline{\kappa} \setminus 0,\ k \in \mathbb{Z}_{\geq 1}\}.$$

Here $\overline{\kappa}$ denotes the algebraic closure of κ. Recall that any such pair should be interpreted as a $k \times k$ matrix $T(\lambda, k)$ with λ on the diagonal, 1 above the diagonal and zero anywhere else and, by the Jordan decomposition theorem, T (when regarded over $\overline{\kappa}$) is similar to the direct sum of all these matrices $T(\lambda, k)$.

For a fixed $u \in \kappa \setminus 0$, one writes $\mathcal{J}(X;\xi)(u) = \{(\lambda, k) \in \mathcal{J}_r(X;\xi) \mid \lambda = u\}$.

$\kappa[t^{-1}, t]$–Modules A $\kappa[t, t^{-1}]$–module V is actually a κ–vector space V equipped with a κ–linear isomorphism $T\colon V \to V$. The multiplication by t and the isomorphism T are related by the formula $tv := T(v)$. With this observation we define V^*, the $\kappa[t^{-1}, t]$–module whose underlying vector space is the dual of V, $\mathrm{Hom}(V, \kappa)$, and a linear isomorphism T^*, the dual of T. Note that if V is finitely generated $\kappa[t^{-1}, t]$–module then V^* is not finitely generated in general, but only when it is a torsion module.

If $\mathbb{Z}$ acts freely on the set S, $\kappa[S]$ denotes the vector space of κ–valued maps with finite support and $\kappa[\![S]\!]$ denotes the vector space of all κ–valued maps, then:

(1) Both $\kappa[S]$ and $\kappa[\![S]\!]$ are $\kappa[t^{-1}, t]$–torsion-free modules, with $\kappa[t^{-1}, t]$–structure induced by the action of $1 \in \mathbb{Z}$ on S.

(2) $\kappa[\![S]\!]$ is isomorphic to $\kappa[S]^*$ (as $\kappa[t^{-1}, t]$–modules).

(3) A torsion-free $\kappa[t^{-1}, t]$–module is finitely generated if and only if it is isomorphic to $\kappa[S]$ for some free $\mathbb{Z}$–action on some set S with the quotient set $S/\mathbb{Z}$ finite.

(4) If V is a finitely generated torsion $\kappa[t^{-1}, t]$–module then V^* is a finitely generated torsion module and is isomorphic to V.

3 The configurations δ^f_r, $\widetilde{\delta}^f_r$ and $\widehat{\widetilde{\delta}}^f_r$

3.1 Boxes and the maps

Let κ be a fixed field. Consider $h\colon Y \to \mathbb{R}$ a (continuous) proper map with Y an ANR, hence locally compact. For $a, b \in \mathbb{R}$ consider

$$\begin{aligned} \mathbb{I}^h_a(r) &= \mathrm{img}(H_r(h^{-1}(-\infty, a]) \to H_r(Y)), \\ \mathbb{I}^b_h(r) &= \mathrm{img}(H_r(h^{-1}([b, \infty)) \to H_r(Y)), \\ \mathbb{F}^h_r(a, b) &= \mathbb{I}^h_a(r) \cap \mathbb{I}^b_h(r) \subseteq H_r(Y), \end{aligned} \tag{6}$$

and let

$$\begin{aligned}
\mathbb{I}^h_{-\infty}(r) &= \bigcap_{a\in\mathbb{R}} I^h_a(r),\\
\mathbb{I}^\infty_h(r) &= \bigcap_{b\in\mathbb{R}} I^b_h(r),\\
\mathbb{F}^h_r(-\infty,b) &= \mathbb{I}^h_{-\infty}(r)\cap\mathbb{I}^b_h(r)\subseteq H_r(Y),\\
\mathbb{F}^h_r(a,\infty) &= \mathbb{I}^h_a(r)\cap\mathbb{I}^\infty_h(r)\subseteq H_r(Y),\\
\mathbb{F}^h_r(-\infty,\infty) &= \mathbb{I}^h_{-\infty}(r)\cap\mathbb{I}^\infty_h(r)\subseteq H_r(Y).
\end{aligned}\tag{7}$$

Proposition 3.1 *For $-\infty\le a'<a, b<b'\le\infty$ one has*

(1) $\mathbb{F}^h_r(a',b')\subseteq\mathbb{F}^h_r(a,b)$,

(2) $\mathbb{F}^h_r(a',b')=\mathbb{F}^h_r(a',b)\cap\mathbb{F}^h_r(a,b')$,

(3) $\mathbb{F}^h_r(a,b)$ *is a finite-dimensional vector space.*

Proof Items (1) and (2) follow from the definitions. To check item (3), observe that by (1) it suffices to verify the statement for $a\ge b$. If f is weakly tame, the statement follows from the Mayer–Vietoris long exact sequence in homology in view of the finite-dimensionality of $H_r(f^{-1}[b,a])$, a consequence of the fact that $f^{-1}[b,a]$ is a compact ANR. If f is only a proper map, one proceeds as in the proof of [2, Proposition 3.2]. Precisely, one dominates X by a locally compact simplicial complex K and up to a proper homotopy the map f by a simplicial proper map $g\colon K\to\mathbb{R}$ which is weakly tame. The result is true for g by a simple Mayer–Vietoris argument and then is true for f. □

A subset B of $\mathbb{R}^2$ of the form

$$B=(a',a]\times[b,b')$$

with $-\infty\le a'<a, b<b'\le\infty$ is called a *box*. When both a' and b' are finite, the box is called a *finite box*. Let

$$B(a,b,\epsilon):=(a-\epsilon,a]\times[b,b+\epsilon)$$

for $0<\epsilon\le\infty$.

Below we write

$$B+c$$

for the box which is *the (c,c)–translation along the diagonal of the box B*; explicitly, $B+c:=(a'+c,a+c]\times[b+c,b'+c)$, and for $B=(a',a]\times[b,b')$ we denote by

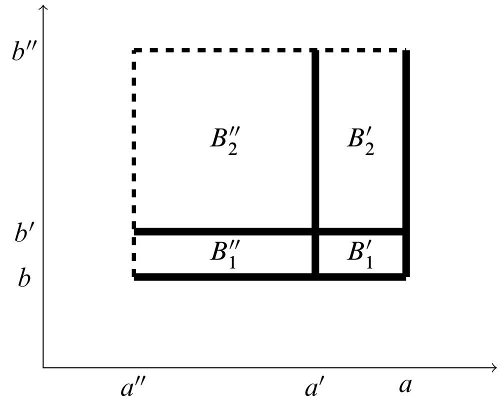

Figure 1

$cB \subset \mathbb{R}^2$ the set

$$cB := B(a,b;\infty) \setminus B.$$

For a box $B = (a',a] \times [b,b')$ let

$$\begin{aligned} \mathbb{F}'^h_r(B) &:= \mathbb{F}^h_r(a',b) + \mathbb{F}^h_r(a,b') \subseteq \mathbb{F}^h_r(a,b) \subseteq H_r(Y), \\ \mathbb{F}^h_r(B) &:= \mathbb{F}^h_r(a,b)/\mathbb{F}'^h_r(B). \end{aligned} \tag{8}$$

Clearly, if $\mathbb{I}^h_{-\infty}(r) = \mathbb{I}^\infty_h(r)$, as will be the case for h a lift of a continuous angle-valued map (see Proposition 3.8), then, for any $a,b \in \mathbb{R}$,

$$\mathbb{F}'^h_r(B(a,b;\infty)) = \mathbb{I}^h_{-\infty}(r) + \mathbb{I}^\infty_h(r) = \mathbb{I}^h_{-\infty}(r) = \mathbb{I}^\infty_h(r).$$

For $-\infty \le a'' < a' < a$ and $b < b' < b'' \le \infty$, consider

$$\begin{aligned} B_1' &:= (a',a] \times [b,b'), & B_2' &:= (a',a] \times [b',b''), \\ B_1 &:= (a'',a] \times [b,b'), & B_2 &:= (a'',a] \times [b',b''), \\ B_1'' &:= (a'',a'] \times [b,b'), & B_2'' &:= (a'',a'] \times [b',b'') \end{aligned}$$

and

$$B := (a'',a]\times; [b,b''); \tag{9}$$

see Figure 1.

One has

$$B_1 = B_1' \sqcup B_1'', \quad B_2 = B_2' \sqcup B_2'', \quad B = B_1 \sqcup B_2.$$

Proposition 3.2 *The inclusions $B_1'' \subseteq B_1 \supseteq B_1'$, $B_2'' \subseteq B_2 \supseteq B_2'$ and $B_1 \subseteq B \supseteq B_2$ induce the short exact sequences*

$$0 \to \mathbb{F}_r^h(B_i'') \xrightarrow{i^{B_i}_{B_i'',r}} \mathbb{F}_r^h(B_i) \xrightarrow{\pi^{B_i'}_{B_i,r}} \mathbb{F}_r^h(B_i') \to 0 \quad \text{for } i = 1, 2,$$
$$0 \to \mathbb{F}_r^h(B_2) \xrightarrow{i^{B}_{B_2,r}} \mathbb{F}_r^h(B) \xrightarrow{\pi^{B_1}_{B,r}} \mathbb{F}_r^h(B_1) \to 0.$$

The proof follows from the definition of $\mathbb{F}_r^h(a,b)$ and Proposition 3.1 above.

Observation 3.3 *If B' and B'' are the boxes $B' = B_1' \sqcup B_2'$ and $B'' = B_1'' \sqcup B_2''$ then one has*

$$i^{B}_{B_2'',r} := i^{B}_{B'',r} \cdot i^{B''}_{B_2'',r} = i^{B}_{B_2,r} \cdot i^{B_2}_{B_2'',r}$$

with $i^{B}_{B_2'',r}$ injective and

$$\pi^{B_1'}_{B,r} := \pi^{B_1'}_{B',r} \cdot \pi^{B'}_{B,r} = \pi^{B_1'}_{B_1,r} \cdot \pi^{B_1}_{B,r}$$

with $\pi^{B_1'}_{B,r}$ surjective.

For $\epsilon' > \epsilon$, the inclusion $B(a,b;\epsilon) \subseteq B(a,b;\epsilon')$ for $\epsilon' > \epsilon$ induces the surjective linear map

$$\pi^{B(a,b;\epsilon)}_{B(a,b;\epsilon'),r} \colon \mathbb{F}_r^h(B(a,b;\epsilon')) \to \mathbb{F}_r^h(B(a,b;\epsilon)).$$

Define

$$\hat{\delta}_r^h(a,b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r^h(B(a,b;\epsilon)).$$

In view of Proposition 3.1, $\hat{\delta}_r^h(a,b)$ is a finite-dimensional vector space.

Define

$$\delta_r^h(a,b) := \dim \hat{\delta}_r^h(a,b).$$

Let $\pi^{(a,b)}_{B,r} \colon \mathbb{F}_r^h(B) \to \hat{\delta}_r^h(a,b)$ be given by

$$\pi^{(a,b)}_{B,r} := \varinjlim_{\epsilon \to 0} \pi^{B(a,b;\epsilon)}_{B,r}.$$

Proposition 3.4 (1) *For $a, b \in \mathbb{R}$ and ϵ small enough, $\hat{\delta}_r^h(a,b) = \mathbb{F}_r^h(B(a,b;\epsilon))$.*

(2) *For any box $B = (a',a] \times [b,b')$ with $-\infty \le a' < a$, $b < b' \le \infty$, the set $\operatorname{supp} \delta_r^h \cap B$ has finite cardinality and one has*

$$\sum_{(a,b) \in B \cap \operatorname{supp} \hat{\delta}_r^h} \delta_r^h(a,b) = \dim \mathbb{F}_r^h(B).$$

(3) *If* h *is weakly tame and* $\widehat{\delta}^h_r(a,b)\neq 0$ *then both* a *and* b *are homological critical values, hence* $\operatorname{supp}\widehat{\delta}^h_r=\operatorname{supp}\delta^h_r\subseteq \mathrm{CRH}(h)\times\mathrm{CRH}(h)\subseteq\mathrm{CR}(h)\times\mathrm{CR}(h)$.

Proof In view of the finite-dimensionality of $\dim\mathbb{F}^h_r(a,b)$ stated in Proposition 3.1(3), for any $a<b$ there are at most finitely many values of α, say $\alpha_1<\alpha_2<\cdots<\alpha_k$ with $a\le\alpha_1\le b$, such that $\dim\mathbb{I}^h_\alpha/\mathbb{I}^h_a$ or $\mathbb{I}^\alpha_h/\mathbb{I}^b_h$ changes. This implies also $\operatorname{supp}\delta^h_r\cap B$ has finite cardinality (hence the first part of (2)). The finite-dimensionality of $\dim\mathbb{F}^h_r(a,b)$ implies that $\dim\mathbb{F}^h(B(a,b;\epsilon))$ stabilizes when $\epsilon\to 0$, which implies (1).

To conclude item (2) entirely, consider $a'=\alpha_0<\alpha_1<\cdots<\alpha_r=a$ and $b=\beta_1<\cdots<\beta_{s+1}=b'$ such that any box $B_{i,j}=(\alpha_{i-1},\alpha_i]\times[\beta_j,\beta_{j+1})$ contains at most one point in $\operatorname{supp}\delta^h_r$. Apply inductively Proposition 3.2 to derive that

$$\sum_{\substack{1\le i\le r\\1\le j\le s}}\dim\mathbb{F}^h(B_{i,j})=\mathbb{F}^h_r(B).$$

If h is weakly tame, then unless both a and b are homologically critical values, $\mathbb{F}^h_r(B(a,b;\epsilon))$ stabilizes to zero, which implies (3). Indeed, in view of the definition one has

$$\begin{aligned}&\dim\mathbb{F}_r(B(a,b;\epsilon))\\&\qquad=\dim\mathbb{F}_r(a,b)+\dim\mathbb{F}_r(a-\epsilon,b-\epsilon)-\dim\mathbb{F}_r(a-\epsilon,b)-\dim\mathbb{F}_r(a,b-\epsilon).\end{aligned}$$

If either a or b are regular values and ϵ is small enough, the right side of the equality vanishes. □

In general, δ^h_r and $\widehat{\delta}^h_r$ are not configurations since their support, although discrete, might not be finite.

Consider the canonical surjective maps

(10) $$\pi_r(a,b)\colon\mathbb{F}^h_r(a,b)\to\widehat{\delta}^h_r(a,b),\quad \pi^B_r(a,b)\colon\mathbb{F}^h_r(a,b)\to\mathbb{F}^h_r(B).$$

Clearly $\pi_r(a,b)=\pi^{(a,b)}_{B,r}\cdot\pi^B_r(a,b)$.

One calls a *splitting* any linear map

(11) $$i_r(a,b)\colon\widehat{\delta}_r(a,b)\to\mathbb{F}^h_r(a,b)\quad\text{or}\quad i^B_r(a,b)\colon\widehat{\delta}_r(a,b)\to\mathbb{F}^h_r(B)$$

which satisfies

$$\pi_r(a,b)\cdot i_r(a,b)=\mathrm{id}\quad\text{or}\quad\pi^{(a,b)}_{B,r}\cdot i^B_r(a,b)=\mathrm{id}.$$

We continue to write $i_r(a,b)$ for its composition with the inclusion $\mathbb{F}^h_r(a,b) \subseteq H_r(Y)$. A splitting $i_r(a,b)$ provides the splitting $i^B_r(a,b)$ defined by

$$i^B_r(a,b) = \pi^B_r(a,b) \cdot i_r(a,b).$$

For $(a,b) \in B'$ with $B' = (a', a_+] \times [b_-, b'')$ and $-\infty \le a' < a \le a_+, b_- \le b < b' \le \infty$, let

$$i^{B'}_r(a,b) \colon \hat{\delta}^f_r(a,b) \to \mathbb{F}^h_r(B')$$

be the composition

$$\hat{\delta}^h_r(a,b) \xrightarrow{i_r(a,b)} \mathbb{F}^h_r(a,b) \xrightarrow{\subseteq} \mathbb{F}^h_r(a_+, b_-) \xrightarrow{\pi^{B'}_r(a',b')} \mathbb{F}^h_r(B').$$

Both linear maps $i_r(a,b)$ and $i^{B'}_r(a,b)$ are injective. The first is injective because $\pi_r(a,b) \cdot i_r(a,b) = \mathrm{id}$. The second is injective because of the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{F}^h_r(a,b) \arrow[r, "\pi^B_r(a,b)"] \arrow[d, "\subseteq"] & \mathbb{F}^h_r(B) \arrow[d, "i^{B'}_{B,r}"] \\
\mathbb{F}^h_r(a_+, b_-) \arrow[r, "\pi^{B'}_r(a_+,b_-)"] & \mathbb{F}^h_r(B')
\end{tikzcd}$$

which implies $i^{B'}_r(a,b) = i^{B'}_{B,r} \cdot i^B_r(a,b)$ with $i^{B'}_{B,r}(a,b)$injective by Observation 3.3 and $i^B_r(a,b)$ injective being a splitting.

One summarizes the above maps by the diagram

$$\begin{tikzcd}
H_r(Y) & \mathbb{F}^h_r(a,b) \arrow[l, tail, "\supseteq"'] \arrow[r, two heads] \arrow[d, two heads, "\pi^B_r(a,b)"'] & \hat{\delta}^h_r(a,b) \arrow[l, dashed, bend right, "i_r(a,b)"'] \arrow[dl, dashed, "i^B_r(a,b)"] \arrow[d, dashed, tail, "i^{B'}_r(a,b)"] \\
& \mathbb{F}^h_r(B) \arrow[ur, two heads, bend left, "\pi^{(a,b)}_{B,r}", "\pi_r(a,b)"'] \arrow[r, dashed, tail, "i^{B'}_{B,r}"] & \mathbb{F}^h_r(B')
\end{tikzcd} \tag{12}$$

where the subdiagrams involving only arrows $\rightarrow$ or only arrows $\dashrightarrow$ are commutative and $i^B_r(a,b) = \pi^B_r(a,b) \cdot i_r(a,b)$.

To simplify the writing, until the end of this section we will write $\bigoplus_{(a,b)}$ and $\bigoplus_{(a,b)\in B}$ instead of $\bigoplus_{(a,b)\in \operatorname{supp} \delta^h_r}$ and $\bigoplus_{(a,b)\in \operatorname{supp} \delta^h_r \cap B}$, respectively.

Choose a collection of splittings $\mathcal{S} = \mathcal{S}_r = \{i_r(a,b) \mid (a,b) \in \operatorname{supp} \delta^h_r\}$.

Define

$$^{\mathcal{S}}\hat{\delta}^h_r(a,b) := i_r(a,b)(\hat{\delta}^h_r(a,b)) \subseteq \mathbb{F}^h_r(a,b) \subseteq H_r(Y)$$

and consider the map

$$^{\mathcal{S}}I_r = \bigoplus_{(a,b)} i_r(a,b)\colon \bigoplus_{(a,b)} \hat{\delta}^h_r(a,b) \to H_r(Y),$$

and for a box B the map

$$^{\mathcal{S}}I^B_r = \bigoplus_{(a,b)\in B} i^B_r(a,b)\colon \bigoplus_{(a,b)\in B} \hat{\delta}^h_r(a,b) \to \mathbb{F}^h_r(B)$$

with $i_r(a,b)$ and $i^B_r(a,b)$ provided by the splittings in the collection $\mathcal{S}$.

Denote by

$$\pi_r\colon H_r(Y) \to H_r(Y)/(\mathbb{I}^h_{-\infty} + \mathbb{I}^\infty_h)$$

the canonical projection.

Proposition 3.5 *Suppose h is a weakly tame map.*

(1) *The linear maps $^{\mathcal{S}}I^B_r$ and $\pi_r \cdot {}^{\mathcal{S}}I_r$ are isomorphisms. Therefore, $^{\mathcal{S}}I_r$ is injective and $^{\mathcal{S}}I_r\big(\bigoplus_{(\alpha,\beta)\in\mathbb{R}^2} \hat{\delta}^h_r(\alpha,\beta)\big) \cap (\mathbb{I}^h_{-\infty}(r) + \mathbb{I}^\infty_h(r)) = 0$.*

(2) *We have*

(2a) $$\sum_{\alpha\le a,\,\beta\ge b} {}^{\mathcal{S}}\hat{\delta}^h_r(\alpha,\beta) + \mathbb{I}^h_{-\infty}(r) + \mathbb{I}^\infty_h(r) = \mathbb{F}^h_r(a,b),$$

(2b) $$\sum_{(\alpha,\beta)\in cB} {}^{\mathcal{S}}\hat{\delta}^h_r(\alpha,\beta) + \mathbb{I}^h_{-\infty}(r) + \mathbb{I}^\infty_h(r) = \mathbb{F}'^h_r(B),$$

(2c) $$\sum_{(\alpha,\beta)\in\mathbb{R}^2} {}^{\mathcal{S}}\hat{\delta}^h_r(\alpha,\beta) + \mathbb{I}^h_{-\infty}(r) + \mathbb{I}^\infty_h(r) = H_r(Y),$$

(2d) $$\sum_{(\alpha,\beta)\in B} {}^{\mathcal{S}}\hat{\delta}^h_r(\alpha,\beta) = \mathbb{F}^h_r(B).$$

Proof (1) To shorten the notation introduce the vector spaces

$$\hat{\mathbb{F}}^h_r(B) := \bigoplus_{(a,b)\in B} \hat{\delta}^h_r(a,b) \quad \text{and} \quad \hat{\mathbb{F}}^h_r := \bigoplus_{(a,b)} \hat{\delta}^h_r(a,b)$$

and for the collection of splittings $\mathcal{S}$ one regards $^{\mathcal{S}}I^B_r$ and $^{\mathcal{S}}I_r$ as maps

$$^{\mathcal{S}}I^B_r\colon \hat{\mathbb{F}}^h_r(B) \to \mathbb{F}^h_r(B), \quad {}^{\mathcal{S}}I_r\colon \hat{\mathbb{F}}^h_r \to H_r(Y).$$

For $B = B_1 \sqcup B_2$ with $B_1 = B_1''$ and $B_2 = B_1'$ or $B_1 = B''$ and $B_2 = B'$ as in Figure 1, one has the commutative diagram

$$\begin{tikzcd}
\mathbb{F}_r^h(B_1) \arrow[r] & \mathbb{F}_r^h(B) \arrow[r] & \mathbb{F}_r^h(B_2) \\
\widehat{\mathbb{F}}_r^h(B_1) \arrow[r] \arrow[u, "{}^{\mathcal{S}}I_r^{B_1}"'] & \widehat{\mathbb{F}}_r^h(B) \arrow[r] \arrow[u, "{}^{\mathcal{S}}I_r^{B}"] & \widehat{\mathbb{F}}_r^h(B_2). \arrow[u, "{}^{\mathcal{S}}I_r^{B_2}"']
\end{tikzcd}$$

First one checks the statement (1) for boxes B with $\operatorname{supp} \delta_r^h \cap B$ consisting of only one element. This is indeed the case by Proposition 3.4(3) for a box $B(a,b;\epsilon)$ with ϵ small enough.

Manipulation with this diagram as in [2], namely a decomposition of B as a disjoint union of smaller boxes and successive applications of Proposition 3.4, permits us to establish inductively the result for any finite box B. The general case and the isomorphism $\pi_r \cdot {}^{\mathcal{S}}I_r$ follows from the case of B a finite box by passing to the projective limit as follows.

Observe that because $\mathbb{F}_r^h(a,b)$ is finite-dimensional, by Proposition 3.1(3) the cardinality of the set $\operatorname{supp} \delta_r^h \cap B(a,b;R)$ remains constant when R is large enough.

Consider

$$\widehat{\mathbb{F}}_r^h(B(a,b;\infty)) := \bigoplus_{(a,b)\in(B(a,b:\infty))} \hat{\delta}_r^h(a,b).$$

Since the set $\operatorname{supp} \delta_r^h \cap B(a,b;R)$ is constant when R is large, one has

$$\widehat{\mathbb{F}}_r^h(B(a,b;\infty)) = \varprojlim_{R\to\infty} \widehat{\mathbb{F}}_r^h(B(a,b;R)).$$

Consider $\mathbb{F}_r^h(B(a,b;\infty)) := \mathbb{F}_r^h(a,b)/(\mathbb{I}_{-\infty}^h(r) \cap \mathbb{I}_h^b(r) + \mathbb{I}_a^h(r) \cap \mathbb{I}_h^\infty(r))$. For the same reason, $\mathbb{F}_r^h(B(a,b;\infty)) = \varprojlim_{R\to\infty} \mathbb{F}_r^h(B(a,b;R))$.

Since ${}^{\mathcal{S}}I_r^{B(a,b;R)}$ is an isomorphism for any R, ${}^{\mathcal{S}}I_r^{B(a,b;R)}$ stabilizes in R and

$${}^{\mathcal{S}}I^{B(a,b;\infty)} := \varprojlim_{R\to\infty} {}^{\mathcal{S}}I_r^{B(a,b;R)},$$

one has that ${}^{\mathcal{S}}I^{B(a,b;\infty)}$ is an isomorphism.

Note that $\mathbb{R}^2 = \bigcup_L B(-L,L;\infty)$ and

$${}^{\mathcal{S}}I_r^{\mathbb{R}^2} = \varinjlim_{L\to-\infty} {}^{\mathcal{S}}I_r^{B(-L,L;\infty)}.$$

Since ${}^{\mathcal{S}}I_r^{B(-L,L;\infty)}$ is an isomorphism for any L, so is ${}^{\mathcal{S}}I_r^{\mathbb{R}^2}$, which is exactly $\pi_r\cdot({}^{\mathcal{S}}I_r)$.

(2) Proposition 3.4 and (1) imply (2). □

An immediate consequence of Proposition 3.5 is the following corollary:

Corollary 3.6 *For a discrete collection of points $(a_i,b_i)\in\mathbb{R}^2$ for $i\in\mathcal{A}$,*

(1) $\bigcap_{i\in\mathcal{A}}\mathbb{F}_r^h(a_i,b_i)=\bigoplus_{\{(\alpha,\beta)\in\bigcap_{i\in\mathcal{A}}B(a_i,b_i;\infty)\}}{}^{\mathcal{S}}\hat{\delta}_r^h(\alpha,\beta)+\mathbb{I}_{-\infty}^h(r)+\mathbb{I}_h^{\infty}(r)$,

(2) $\bigcup_{i\in\mathcal{A}}\mathbb{F}_r^h(a_i,b_i)=\bigoplus_{\{(\alpha,\beta)\in\bigcup_{i\in\mathcal{A}}B(a_i,b_i;\infty)\}}{}^{\mathcal{S}}\hat{\delta}_r^h(\alpha,\beta)+\mathbb{I}_{-\infty}^h(r)+\mathbb{I}_h^{\infty}(r)$.

3.2 Definition and properties of δ_r^f and $\hat{\delta}_r^f$

Let $f\colon X\to\mathbb{S}^1$ be a continuous map, X a compact ANR, $\tilde{f}\colon\tilde{X}\to\mathbb{R}$ a lift of f and κ a fixed field. We apply the previous considerations to $\tilde{f}\colon\tilde{X}\to\mathbb{R}$. In this case we have the deck transformation $\tau\colon\tilde{X}\to\tilde{X}$, which induces the isomorphism $t_r\colon H_r(\tilde{X})\to H_r(\tilde{X})$ and therefore the structure of a $\kappa[t^{-1},t]$–module on this κ–vector space. Recall that for a box $B=(a',a]\times[b,b')$ one denotes by $B+c$ the box $B+c:=(a'+c,a+c]\times[b+c,b'+c)$.

Observation 3.7 (1) *The isomorphism t_r satisfies*

$$t_r(\mathbb{F}_r^{\tilde{f}}(a,b))=\mathbb{F}_r^{\tilde{f}}(a+2\pi,b+2\pi)\quad\text{and}\quad t_r^{-1}(\mathbb{F}_r^h(a,b))=\mathbb{F}_r^h(a-2\pi,b-2\pi).$$

(2) *For any box $B=(a',a]\times[b,b')$, consider the box $B+2\pi$. The isomorphism t_r induces the isomorphisms $t_r(B)\colon\mathbb{F}_r^{\tilde{f}}(B)\to\mathbb{F}_r^{\tilde{f}}(B+2\pi)$ and then $\hat{t}_r(a,b)\colon\hat{\delta}_r^{\tilde{f}}(a,b)\to\hat{\delta}_r^{\tilde{f}}(a+2\pi,b+2\pi)$.*

(3) *$\mathbb{I}_{-\infty}^{\tilde{f}}(r)$ and $\mathbb{I}_{\tilde{f}}^{\infty}(r)$ are invariant with respect to t_r, hence $\kappa[t^{-1},t]$–submodules, therefore*

$$H_r(\tilde{X})/(\mathbb{I}_{\infty}^{\tilde{f}}(r)+\mathbb{I}_{\tilde{f}}^{\infty}(r))$$

is a $\kappa[t^{-1},t]$–module.

Clearly the following diagram with the vertical arrows $t_r(a,b),t_r(B),\hat{t}_r(a,b)$ induced by t_r is commutative:

$$
\begin{tikzcd}
H_r(\widetilde{X}) \arrow[dd, "t_r"] & \mathbb{F}_r^{\tilde f}(a,b) \arrow[l, "\supseteq"'] \arrow[rr, "\pi_r(a{,}b)"] \arrow[dd, "t_r(a{,}b)"] \arrow[dr, "\pi^B_{ab,r}"'] & & \hat{\delta}_r^{\tilde f}(a,b) \arrow[dd, "\hat{t}_r(a{,}b)"] \\
 & & \mathbb{F}_r^{\tilde f}(B) \arrow[ur, "\pi^{(a,b)}_{B,r}"] \arrow[dd, "\mathrm{tr}(B)"] & \\
H_r(\widetilde{X}) & \mathbb{F}_r^{\tilde f}(a+2\pi, b+2\pi) \arrow[l, "\supseteq"'] \arrow[rr, "\pi_r(a+2\pi{,}b+2\pi)"] \arrow[dr, "\pi_r^{B'}(a+2\pi{,}b+2\pi)"'] & & \hat{\delta}_r^{\tilde f}(a+2\pi, b+2\pi) \\
 & & \mathbb{F}_r^{\tilde f}(B+2\pi) \arrow[ur, "\pi^{(a+2\pi,b+2\pi)}_{B+2\pi,r}"'] &
\end{tikzcd}
\tag{13}
$$

Proposition 3.8 $$\mathbb{I}^{\tilde f}_{-\infty}(r) = \mathbb{I}^{\infty}_{\tilde f}(r) = T(H_r(\widetilde{X})).$$

Proof If $x \in T(H_r(\widetilde{X}))$ then there exists an integer $l \in \mathbb{Z}$ and a polynomial $P(t) = \alpha_n t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + \alpha_0$ with $\alpha_i \in \kappa$ and $\alpha_0 \neq 0$ such that $P(t)t^l x = 0$. Let $y = t^l x$. Since $H_r(\widetilde{X}) = \bigcup_b \mathbb{I}^b_{\tilde f}(r)$, one has $y \in \mathbb{I}^b(r)$ for some $b \in \mathbb{R}$. Since $P(t)y = 0$, one concludes that

$$y = -(\alpha_n/\alpha_0)t^{n-1} - \cdots - (\alpha_1/\alpha_0)ty$$

and therefore $y \in \mathbb{I}^{b+2\pi}(r)$.

Repeating the argument, one concludes that $y \in I^{b+2\pi l}$ for any l, hence $y \in \mathbb{I}^{\infty}(r)$. Since $x = t^{-l}y$, one has $x \in \mathbb{I}^{\infty}(r)$. Hence, $T(H_r(\widetilde{X})) \subseteq I^{\infty}(r)$.

Let $x \in \mathbb{I}^{\infty}(r)$. Since $H_r(\widetilde{X}) = \bigcup_a \mathbb{I}^{\tilde f}_a(r)$, it follows that $x \in \mathbb{I}_a(r)$ for some $a \in \mathbb{R}$, and if in addition $x \in \mathbb{I}^{\infty}(r)$ then, by Observation 3.7(3), all $x, t^{-1}x, t^{-2}x, \dots, t^{-l}x, \dots$ are in $\mathbb{I}_a(r) \cap \mathbb{I}^{\infty}(r)$. Since by Proposition 3.1(3) the dimension of $\mathbb{I}_a(r) \cap \mathbb{I}^{\infty}(r)$ is finite, there exist $\alpha_{i_1}, \dots, \alpha_{i_l}$ such that

$$(\alpha_{i_1}t^{-i_1} + \cdots + \alpha_{i_l}t^{-i_l})x = 0.$$

This gives $x \in T(H_r(\widetilde{X}))$. Hence, $\mathbb{I}^{\infty}(r) \subseteq T(H_r(\widetilde{X}))$. Therefore, $\mathbb{I}^{\infty}(r) = T(H_r(\widetilde{X}))$. By a similar argument one concludes that $H_r(\widetilde{X}) = \mathbb{I}_{-\infty}(r)$. □

Recall that

- $H_r^N(X;\xi_f) := H_r(\widetilde{X})/T(H_r(\widetilde{X}))$,
- $\pi(r)\colon H_r(\widetilde{X}) \to H_r^N(X;\xi_f)$ denotes the canonical projection, and

- the κ–vector spaces $H_r(\widetilde{X})$, $T(H_r(\widetilde{X}))$ and $H_r^N(X;\xi_f)$ are $\kappa[t^{-1},t]$–modules with the multiplication by t given by or induced by the isomorphism t_r and $\pi(r)$ is $\kappa[t^{-1},t]$–linear.

In view of Proposition 3.8, $T(H_r(\widetilde{X}))$ is contained in $\mathbb{F}_r^{\tilde f}(a,b)$ and $\mathbb{F}_r'^{\tilde f}(B)$ and then one defines

- ${}^N\mathbb{F}_r^{\tilde f}(a,b) := \mathbb{F}_r^{\tilde f}(a,b)/T(H_r(\widetilde{X}))$ for any $B := (a',a]\times[b,b')$,
- ${}^N\mathbb{F}_r'^{\tilde f}(B) := \mathbb{F}_r'^{\tilde f}(B)/T(H_r(\widetilde{X}))$ and ${}^N\mathbb{F}_r^{\tilde f}(B) := {}^N\mathbb{F}_r^{\tilde f}(a,b)/{}^N\mathbb{F}_r'^{\tilde f}(B)$, and then
- ${}^N\hat\delta_r^{\tilde f}(a,b) = \lim_{\epsilon\to 0}{}^N\mathbb{F}_r^{\tilde f}(B(a,b;\epsilon) = {}^N\mathbb{F}_r^{\tilde f}(B(a,b;\epsilon))$ for ϵ small enough.

Clearly one has:

- ${}^N\mathbb{F}_r'^{\tilde f}(B) \subseteq {}^N\mathbb{F}_r^{\tilde f}(a,b) \subset H_r^N(X;\xi_f)$.
- (1) $\mathbb{F}_r^{\tilde f}(B) = {}^N\mathbb{F}_r^{\tilde f}(B)$.

  (2) $\hat\delta_r^{\tilde f}(a,b) = {}^N\hat\delta_r^{\tilde f}(a,b)$.
- The diagram

$$
\begin{tikzcd}
\hat\delta_r^{\tilde f}(a,b) \arrow[dd, "\hat t_r(a,b)"] & {}^N\mathbb{F}_r^{\tilde f}(a,b) \arrow[l] \arrow[rr] \arrow[dr] \arrow[dd, "{}^N\hat t_r(a,b)"] & & H_r^N(X,\xi_f) \arrow[dd, "{}^N\hat t_r"] \\
& & {}^N\mathbb{F}_r^{\tilde f}(B) = \mathbb{F}_r^{\tilde f}(B) \arrow[ull] \arrow[dd, "{}^N\hat t_r(B)"] & \\
\hat\delta_r^{\tilde f}(a+2\pi,b+2\pi) & {}^N\mathbb{F}_r^{\tilde f}(a+2\pi,b+2\pi) \arrow[l] \arrow[rr] \arrow[dr] & & H_r^N X,\xi_f) \\
& & {}^N\mathbb{F}_r^{\tilde f}(B+2\pi) = \mathbb{F}_r^{\tilde f}(B+2\pi) \arrow[ull] &
\end{tikzcd}
\tag{14}
$$

  is commutative with the vertical arrows isomorphisms.

Recall from the introduction that $\langle\,\rangle\colon \mathbb{R}^2\to\mathbb{T}=\mathbb{R}^2/\mathbb{Z}$ denotes the map which assigns to $(a,b)\in\mathbb{R}^2$ its equivalence class $\langle a,b\rangle\in\mathbb{T}$. One denotes by $\langle K\rangle\subseteq\mathbb{T}$ the image of $K\subseteq\mathbb{R}^2$ by the map $\langle\,\rangle$; in particular, one writes $\langle a,b\rangle$, $\langle B\rangle$ and $\langle cB\rangle$ for the images of (a,b), B and cB.

The box $B=(a-\alpha,a]\times[b,b+\beta)$ is called *small* if $0<\alpha,\,\beta<2\pi$, in which case the restriction of $\langle\,\rangle$ to B is one-to-one; clearly, if B is a small box, so is any $B+c$ and $(B+2\pi k)\cap(B+2\pi k')=\varnothing$ for $k\neq k'$.

For $\langle a,b\rangle \in \mathbb{T}$ and $\langle B\rangle \subseteq \mathbb{T}$ with $B=(a-\alpha,a]\times[b,b+\beta)$ a small box introduce

$$
\begin{aligned}
{}^N\mathbb{F}^f_r\langle a,b\rangle &:= \sum_{k\in\mathbb{Z}} {}^N\mathbb{F}^{\tilde f}_r(a+2\pi k, b+2\pi k) \subseteq H^N_r(X;\xi_f),\\
{}^N\mathbb{F}'^f_r\langle B\rangle &:= \sum_{k\in\mathbb{Z}} {}^N\mathbb{F}'^{\tilde f}_r(B+2\pi k) = ({}^N\mathbb{F}^{\tilde f}_r\langle a',b\rangle + {}^N\mathbb{F}^{\tilde f}_r\langle a,b'\rangle) \subseteq {}^N\mathbb{F}^{\tilde f}_r\langle a,b\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \subseteq H^N_r(X;\xi_f),
\end{aligned}
$$

both $\kappa[t^{-1},t]$–submodules of the free $\kappa[t^{-1},t]$–module $H^N_r(X;\xi_f)$, hence finitely generated free modules, and

$$
\begin{aligned}
\mathbb{F}^f_r\langle B\rangle &:= \bigoplus_{k\in\mathbb{Z}} \mathbb{F}^{\tilde f}_r(B+2\pi k) = \bigoplus_{k\in\mathbb{Z}} {}^N\mathbb{F}^{\tilde f}_r(B+2\pi k),\\
\hat\delta^f_r\langle a,b\rangle &:= \bigoplus_{k\in\mathbb{Z}} \hat\delta^{\tilde f}_r(a+2\pi k, b+2\pi k) = \bigoplus_{k\in\mathbb{Z}} {}^N\hat\delta^{\tilde f}_r(a+2\pi k, b+2\pi k),
\end{aligned}
$$

both ($\mathbb{F}^f_r\langle B\rangle$ and $\hat\delta^f_r\langle a,b\rangle$) free $\kappa[t^{-1},t]$–modules whose multiplication by t is given by the isomorphism $\bigoplus_{k\in\mathbb{Z}} \hat t_r(a+2\pi k, b+2\pi k)$.

Recall that for a set S equipped with an action $\mu\colon \mathbb{Z}\times S\to S$ the κ–vector space $\kappa[S]$, of κ–valued finitely supported maps, has the structure of a $\kappa[t^{-1},t]$–module which is free when the action is free and has a base indexed by the quotient set $S/\mathbb{Z}$. If $S\subset\mathbb{R}^2$ is a discrete subset, invariant to the action $\mu\colon \mathbb{Z}\times\mathbb{R}^2\to\mathbb{R}^2$ given by $\mu(n,(a,b)) = (a+2\pi n, b+2\pi n)$, then $\kappa[S]$ is a free $\kappa[t^{-1},t]$–module with a base indexed by $\langle S\rangle$. For the box $B=(a-\alpha,a]\times[b,b+\beta)\subseteq\mathbb{R}^2$ one denotes by $\hat B$ and $\widehat{cB}$ the subsets, in $\mathbb{R}^2$,

$$
\begin{aligned}
\hat B &:= \bigcup_{z\in\mathbb{Z}} (B+2\pi k)\subset\mathbb{R}^2,\\
\widehat{cB} &:= \bigcup_{k\in\mathbb{Z}} c(B+2\pi k)\subset\mathbb{R}^2
\end{aligned}
\tag{15}
$$

and by $\langle\hat B\rangle\subseteq\mathbb{T}$ and $\langle\hat c B\rangle\subseteq\mathbb{T}$ their images by the map $\langle\,\rangle$.

Clearly the sets $\hat B$, $\widehat{cB}$ and $\operatorname{supp}\delta^{\tilde f}_r$ are invariant to the free action μ with quotient sets $\langle\hat B\rangle$, $\langle\widehat{cB}\rangle$ and $\langle\operatorname{supp}\delta^{\tilde f}_r\rangle = \operatorname{supp}\delta^f_r$, respectively. In view of the above and of Proposition 3.5 one can conclude that the κ–vector spaces

$$
{}^N\mathbb{F}^f_r\langle a,b\rangle,\quad {}^N\mathbb{F}'^f_r\langle B\rangle,\quad {}^N\mathbb{F}^f_r\langle a,b\rangle/{}^N\mathbb{F}'^f_r\langle B\rangle
$$

are free $\kappa[t^{-1},t]$–modules with bases indexed by

$$\langle \widehat{B}(a,b;\infty)\rangle \cap \operatorname{supp}\delta^f, \quad \langle \widehat{cB}\rangle \cap \operatorname{supp}\delta^f \quad \text{and} \quad \langle \widehat{B}\rangle \cap \operatorname{supp}\delta^f,$$

the quotient sets of $\widehat{B}(a,b;\infty)\cap \operatorname{supp}\delta^{\tilde f}_r$, $\widehat{cB}\cap \operatorname{supp}\delta^{\tilde f}_r$ and $\widehat{B}\cap \operatorname{supp}\delta^{\tilde f}_r$, respectively. For any box $B=(a-\alpha,a]\times[b,b+\beta)$ and $k\in\mathbb{Z}$ consider the κ–linear map

$$\iota_r(a,b;k)\colon \frac{{}^N\mathbb{F}^{\tilde f}_r(a+2\pi k,b+2\pi k)}{{}^N\mathbb{F}'^{\tilde f}_r(B+2\pi k)} \to {}^N\mathbb{F}^f_r\langle a,b\rangle/{}^N\mathbb{F}'^f_r\langle B\rangle$$

induced by the inclusion ${}^N\mathbb{F}^{\tilde f}_r(a+2\pi k,b+2\pi k)\subset\bigoplus_{k'\in\mathbb{Z}}{}^N\mathbb{F}^{\tilde f}_r(a+2\pi k',b+2\pi k')$ and let

$$\iota_r\langle a,b\rangle := \bigoplus_{k'\in\mathbb{Z}}\iota_r(a,b;k')\colon {}^N\mathbb{F}^f_r\langle B\rangle \to {}^N\mathbb{F}^f_r\langle a,b\rangle/{}^N\mathbb{F}'^f\langle B\rangle.$$

This map is surjective and, in view of the commutative diagram (14), is $\kappa[t^{-1},t]$–linear. Since $\iota_r(a,b)$ is surjective and both the source and the target are free modules of equal finite rank, it follows that $\iota_r(a,b)$ is an isomorphism.

One summarizes the above observation as Proposition 3.9:

Proposition 3.9 (1) *The $\kappa[t^{-1},t]$–module ${}^N\mathbb{F}^f_r\langle a,b\rangle/{}^N\mathbb{F}'^f_r\langle B\rangle$ is free and of rank $\#(\langle\widehat{B}\rangle\cap\operatorname{supp}\delta^f_r)$. If B is a small box then this rank equals $\#(B\cap\operatorname{supp}\delta^{\tilde f}_r)$.*

(2) *If B is small then the κ–linear map $\iota_r\langle a,b\rangle$ is an isomorphism.*

(3) *For ϵ small enough,*

$$\hat\delta^f\langle a,b\rangle = {}^N\mathbb{F}_r\langle B(a,b;\epsilon)\rangle$$

with $B(a,b;\epsilon)=(a-\epsilon]\times[b,b+\epsilon)$.

Let ${}^N\pi_r(a,b)\colon {}^N\mathbb{F}^{\tilde f}_r(a,b)\to\hat\delta^{\tilde f}_r(a,b)$ be the canonical projection. As in the definition of *splittings* an N–splitting is a linear map ${}^Ni_r(a,b)\colon\hat\delta^{\tilde f}\to{}^N\mathbb{F}^{\tilde f}_r(a,b)$ such that ${}^N\pi_r(a,b)\cdot{}^Ni_r(a,b)=\mathrm{id}$.

Definition 3.10 A collection $\mathcal{S}$ of splittings $i_r(a,b)\colon\hat\delta^{\tilde f}_r(a,b)\to\mathbb{F}^{\tilde f}_r(a,b)$ for $a,b\in\mathbb{R}$ or N–splittings ${}^Ni_r(a,b)\colon\hat\delta^{\tilde f}_r(a,b)\to{}^N\mathbb{F}^{\tilde f}_r(a,b)$ for $a,b\in\mathbb{R}$ is called a *collection of compatible splittings or compatible N–splittings* if

$$\hat t_r(a,b)\cdot i_r(a,b)=i_r(a+2\pi,b+2\pi)\cdot\hat t_r(a,b)$$

or

$$\hat t_r(a,b)\cdot{}^Ni_r(a,b)={}^Ni_r(a+2\pi,b+2\pi)\cdot\hat t_r(a,b).$$

Note that:

(1) The splitting $i_r(a,b)$ of $\pi_r(a,b)$ induces the splitting ${}^N i_r(a,b)$ of ${}^N\pi_r(a,b)$ by composition with the canonical projection
$$\mathbb{F}_r^{\tilde f}(a,b) \to {}^N\mathbb{F}_r^{\tilde f}(a,b) = \mathbb{F}_r^{\tilde f}(a,b)/TH_r(\widetilde M).$$

(2) Collections of compatible splittings and therefore of compatible N–splittings exist. It suffices to start with splittings for pairs (a,b) for $0 \le a < 2\pi$ and extend them for a outside this interval by composing with the appropriate $\hat t^r$ and get compatible splittings and then derive from them compatible N–splittings.

(3) The linear maps $\pi_r(a,b)$ or ${}^N\pi_r(a,b)$ and the collection $\mathcal{S}$ of compatible splittings or compatible N–splittings induce
$$\pi_r\langle a,b\rangle\colon \mathbb{F}_r^f\langle a,b\rangle \to \hat\delta_r^f\langle a,b\rangle \quad \text{and} \quad i_r\langle a,b\rangle\colon \hat\delta_r^f\langle a,b\rangle \to \mathbb{F}_r^f\langle a,b\rangle$$
or
$${}^N\pi_r\langle a,b\rangle\colon {}^N\mathbb{F}_r^f\langle a,b\rangle \to \hat\delta_r^{\tilde f}\langle a,b\rangle \quad \text{and} \quad {}^N i_r\langle a,b\rangle\colon \hat\delta_r^f\langle a,b\rangle \to {}^N\mathbb{F}_r^f\langle a,b\rangle.$$

Item (3) requires some explanation.

To define $\pi_r\langle a,b\rangle$ and ${}^N\pi_r\langle a,b\rangle$ we show first that the linear maps $\pi_r(a,b)$ and ${}^N\pi_r(a,b)$ extend to
$$\sum_{i\in\mathbb{Z}} \mathbb{F}_r^{\tilde f}(a+2\pi i, b+2\pi i) \subseteq H_r(\widetilde M),$$
$$\sum_{i\in\mathbb{Z}} {}^N\mathbb{F}_r^{\tilde f}(a+2\pi i, b+2\pi i) \subset {}^N H_r(M;\xi),$$

respectively. If this is the case, denote these extensions by $\bar\pi_r\langle a,b\rangle$ and ${}^N\bar\pi_r\langle a,b\rangle$. To show this is the case it suffices to verify that if $x_i \in \mathbb{F}_r^f(a+2\pi i, b+2\pi i)$ and $\sum_{i\in\mathbb{Z}} x_i \in T(H_r(\tilde x)$ then $x_i \in \mathbb{F}'^f_r(B(a+2\pi i, b+2\pi i; 2\pi)) + TH_r(\widetilde M)$; this guarantees that $\bigoplus_i \pi_r(a+2\pi i, b+2\pi i)(x_i) = 0$.

Indeed, since
$$B(a+2\pi i, b+2\pi i;\infty) \cap \Big(\bigcup_{j\in\mathbb{Z},\, j\ne i} B(a+2\pi j, b+2\pi j;\infty)\Big) \subseteq cB(a+2\pi i, b+2\pi i; 2\pi),$$

if $x = \sum_{i\in\mathbb{Z}} x_i \in TH_r(\widetilde M)$ with $x_i \in \mathbb{F}_r(a+2\pi i, b+2\pi i)$, in view of Proposition 3.5(2b), one has
$$x_i = x - \sum_{i\ne j} x_j \in TH_r(\widetilde X) + \mathbb{F}'^{\tilde f}_r(B(a+2\pi i, b+2\pi i; 2\pi)).$$

Define $\pi_r\langle a,b\rangle$ and ${}^N\pi_r\langle a,b\rangle$ to be the direct sum of $\overline{\pi}_r\langle a+2\pi i, b+2\pi i\rangle$ and of ${}^N\overline{\pi}_r\langle a+2\pi i, b+2\pi i\rangle$, respectively, over all $i\in\mathbb{Z}$.

Clearly the map $\pi_r\langle a,b\rangle$ is the factorization of

$$\bigoplus_{k\in\mathbb{Z}}\pi_r(a+2\pi k, b+2\pi k)\colon \bigoplus_{k\in\mathbb{Z}}\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k) \to \hat{\delta}_r^f\langle a,b\rangle = \bigoplus \delta_r^{\tilde f}(a+2\pi k, b+2\pi k)$$

by the projection $\pi\colon \bigoplus_{k\in\mathbb{Z}}\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)\to\sum_{k\in\mathbb{Z}}\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)$. A similar observation holds for ${}^N\pi_r\langle a,b\rangle$.

Define $i_r\langle a,b\rangle$ to be the composition of $\bigoplus_{k\in\mathbb{Z}} i_r(a+2\pi k, b+2\pi k)$ with the projection $\pi\colon \bigoplus_{k\in\mathbb{Z}}\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)\to\sum_{k\in\mathbb{Z}}\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)$ and ${}^N i_r\langle a,b\rangle$ to be the composition of $\bigoplus_{k\in\mathbb{Z}} {}^N i_r(a+2\pi k, b+2\pi k)$ with the projection $\pi\colon \bigoplus_{k\in\mathbb{Z}} {}^N\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)\to\sum_{k\in\mathbb{Z}} {}^N\mathbb{F}_r^{\tilde f}(a+2\pi k, b+2\pi k)$.

Then a collection $\mathcal{S}$ of compatible splittings and implicitly of compatible N–splittings defines the κ–linear map

$${}^{\mathcal{S}}I_r^N = \bigoplus_{\langle a,b\rangle\in\mathbb{T}} i_r\langle a,b\rangle\colon \bigoplus_{(a,b)}\hat{\delta}_r^f\langle a,b\rangle\to H_r^N(M;\xi_f),$$

and for a small box B the κ–linear maps

$${}^{\mathcal{S}}I_r^N\langle B\rangle = \bigoplus_{\langle a,b\rangle\in\langle B\rangle} i_r^B\langle a,b\rangle\colon \bigoplus_{\langle a,b\rangle\in\langle B\rangle}\hat{\delta}_r^{\tilde f}\langle a,b\rangle\to {}^N\mathbb{F}_r^{\tilde f}\langle B\rangle.$$

Proposition 3.11 *Both ${}^{\mathcal{S}}I_r^N$ and ${}^{\mathcal{S}}I_r^N\langle B\rangle$ are $\kappa[t^{-1},t]$–isomorphisms.*

Proof Indeed, for a chosen collection $\mathcal{S}$ of compatible N–splittings consider the diagrams

$$\begin{array}{ccc}
\bigoplus_{\langle\alpha,\beta\rangle}\hat{\delta}^f\langle\alpha,\beta\rangle & \xrightarrow{\ {}^{\mathcal{S}}I_r^N\ } & H_r^N(X;\xi) \\
\subseteq\uparrow & & \subseteq\uparrow \\
\bigoplus_{\{\langle\alpha,\beta\rangle,\alpha\le a,\beta\ge b\}}\hat{\delta}^f\langle\alpha,\beta\rangle & \xrightarrow{\ {}^{\mathcal{S}}I_r^N\ } & {}^N\mathbb{F}_r^f\langle a,b\rangle \\
\subseteq\uparrow & & \subseteq\uparrow \\
\bigoplus_{\langle\alpha,\beta\rangle\in\langle cB\rangle}\hat{\delta}^f\langle\alpha,\beta\rangle & \xrightarrow{\ {}^{\mathcal{S}}I_r^N\ } & {}^N\mathbb{F}'^f_r\langle B\rangle
\end{array}\tag{16}$$

and

$$
\begin{tikzcd}
\bigoplus_{\langle \alpha,\beta\rangle, \alpha\le a, \beta\ge b} \hat\delta^f\langle \alpha,\beta\rangle \arrow[r, "{}^S I_r^N"] \arrow[d, "\pi'_r"] & {}^N\mathbb{F}_r^f\langle a,b\rangle \arrow[d, "\pi''_r"] \\
\bigoplus_{\langle \alpha,\beta\rangle\in \langle B\rangle} \hat\delta^f\langle \alpha,\beta\rangle \arrow[r, "{}^S I_r^N\langle B\rangle"] & {}^N\mathbb{F}_r^f\langle B\rangle = \mathbb{F}_r^f\langle B\rangle
\end{tikzcd}
\tag{17}
$$

with π'_r and π''_r the obvious projections.

In view of Proposition 3.5 one has:

- $\hat\delta^f\langle \alpha,\beta\rangle$ is a free $\kappa[t^{-1},t]$–module with the multiplication by t given by the isomorphism $\bigoplus_{k\in\mathbb{Z}} \hat t_r(\alpha+2\pi k, \beta+2\pi k)$.
- The vector spaces involved in the above diagrams are all free $\kappa[t^{-1},t]$–modules and in view of the commutativity of diagrams (13) and (14) all arrows in both diagrams are $\kappa[t^{-1},t]$–linear.
- The horizontal arrows in the above diagrams are isomorphisms; in particular, so are ${}^S I_r^N$ and ${}^S I_r^N\langle B\rangle$. □

Proposition 3.12 *${}^N\mathbb{F}_r^f\langle a,b\rangle$ and ${}^N\mathbb{F}'^f_r\langle B\rangle$ are split free submodules of $H_r^N(X;\xi)$, and $\mathbb{F}_r^f\langle B\rangle$ is a quotient of split free submodules, hence also free. In particular, $\hat\delta_r^f\langle a,b\rangle$, which is canonically isomorphic to $\mathbb{F}_r^f\langle B(a,b;\epsilon)\rangle$ for $\epsilon<\epsilon(f)$, is a quotient of split free submodules $\mathbb{F}_r^f\langle a,b\rangle/\mathbb{F}'^f_r\langle B(a,b;\epsilon)\rangle$.*

Definition of δ_r^f, $\tilde\delta_r^f$, $\hat\delta_r^f$, $\hat{\hat\delta}{}_r^f$ and $P_r^f(z)$ In view of Propositions 3.1, 3.2 and 3.9 and of Proposition 3.12, the assignments

(1) $\langle a,b\rangle \rightsquigarrow \delta_r^f\langle a,b\rangle$,

(2) $\langle a,b\rangle \rightsquigarrow \tilde\delta_r^f\langle a,b\rangle = (\mathbb{F}_r^f\langle a,b\rangle, \mathbb{F}'^f_r\langle B(a,b;\epsilon)\rangle)$, with ϵ small,

(3) $\langle a,b\rangle \rightsquigarrow \hat\delta_r^f\langle a,b\rangle$,

(4) $\langle a,b\rangle \rightsquigarrow \hat{\hat\delta}{}_r^f\langle a,b\rangle$, with $\hat{\hat\delta}{}_r^f\langle a,b\rangle$ the von Neumann completion of $\hat\delta_r^f\langle a,b\rangle$

for $\langle a,b\rangle\in\mathbb{T}$, defined by

(1) $\delta_r^f\langle a,b\rangle := \delta^{\tilde f}(a,b)$,

(2) $\tilde\delta_r^f\langle a,b\rangle := (\mathbb{F}_r^f\langle a,b\rangle, \mathbb{F}'^f_r\langle B(a,b;\epsilon)\rangle)$, with ϵ small enough,

(3) $\hat\delta_r^f\langle a,b\rangle = (\mathbb{F}_r^f\langle a,b\rangle/\mathbb{F}'^f_r\langle B(a,b;\epsilon)\rangle)$, with ϵ small enough

are configurations of points with multiplicity, of pairs of submodules of $H_r^N(X;\xi_f)$ and of free $\kappa[t^{-1},t]$–modules.

We use the identification of $\mathbb{T}$ with $\mathbb{C}\setminus 0$ provided by the map $\langle a,b\rangle \mapsto z = e^{ia+(b-a)}$ and if $z_1, z_2, \dots, z_k \in \mathbb{C}\setminus 0$ are the points in the support of δ_r^f, when regarded in $\mathbb{C}\setminus 0$, define the polynomial

$$P_r^f(z) := \prod (z-z_i)^{\delta_r^f(z_i)}.$$

When $\kappa = \mathbb{C}$, the von Neumann completion described in Section 2 converts $\mathbb{C}[t^{-1},t]$ into the von Neumann algebra $L^\infty(\mathbb{S}^1)$ and a $\mathbb{C}[t^{-1},t]$–valued inner product converts $H_r^N(M;\xi)$ into an $L^\infty(\mathbb{S}^1)$–Hilbert module and $\mathbb{F}'^f_r\langle B\rangle$, $\mathbb{F}_r^f\langle B\rangle$ and $\hat{\delta}\langle a,b\rangle$ into Hilbert submodules. The von Neumann completion leads to the configuration $\hat{\hat{\delta}}{}_r^f$ of mutually orthogonal $L^\infty(\mathbb{S}^1)$–Hilbert modules. If X is an underlying space of a closed Riemannian manifold or of a simplicial complex (hence a space equipped with a triangulation), then the additional structure — the Riemannian metric or the triangulation — provides such an inner product.

4 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Item (1) follows from Proposition 3.4(3) and the definitions of δ_r^f, $\hat{\delta}_r^f$ and $\hat{\hat{\delta}}{}_r^f$.

Item (2) follows from the fact that $\pi(r)\cdot {}^S I_r$ is an isomorphism, as established in Proposition 3.5(1) applied to $\tilde{f}$ and from the definitions of δ_r^f, $\hat{\delta}_r^f$ and $\hat{\hat{\delta}}{}_r^f$. The configuration $\hat{\delta}_r^f$ is derived from a configuration of pairs as described in Section 2 with $\hat{\delta}_r^f\langle a,b\rangle = \mathbb{F}_r^f\langle a,b\rangle/\mathbb{F}'^f_r\langle B(a,b;\epsilon)\rangle$ for any $\epsilon < \epsilon(f)$.

For (3), one proceeds as in the proof of Theorem 4.1(4) in [2] in the case X is a compact smooth manifold or a finite simplicial complex. For example, if X is a smooth manifold, possibly with boundary, any angle-valued map is arbitrary close to a Morse angle-valued map f which takes different values on different critical points. Then the same remains true for $\tilde{f}\colon \widetilde{X} \to \mathbb{R}$, an infinite cyclic cover of this Morse map; this guarantees that for the sequence of critical values $\cdots < c_{i-1} < c_i < c_{i+1} < \cdots$, the inclusion-induced linear maps $H_*(\widetilde{X}_{c_{i-1}}) \to H_*(\widetilde{X}_{c_i})$ have cokernel of dimension at most one. As argued in the proof of Theorem 1(4) in [2], this implies that $\delta_r^{\tilde{f}}$ and then δ_r^f takes only 0 or 1 as values. In the same way as in [2], with the help of results on compact Hilbert cube manifolds, eg Theorem 4.2 below, one derives Theorem 1.1(3) in the generality stated. □

Proof of Theorem 1.2 First observe that in view of Observation 4.1 and Theorem 4.2 (stating results about Hilbert cube manifolds) it will suffice to prove Theorem 1.2 for X a finite simplicial complex.

Indeed, if Theorem 1.2 holds for any finite simplicial complex, in view of Observation 4.1 and Theorem 4.2(3), it holds for $K \times I^\infty$, hence, by Theorem 4.2(2), for X a compact Hilbert cube manifold and then again, by Theorem 4.2(1) and Observation 4.1, holds for any compact ANR.

For a continuous map $f\colon X \to \mathbb{S}^1$ and K a compact space denote by $f_K\colon X \times K \to \mathbb{S}^1$ the composition $f_K := f \cdot \pi_X$ with $\pi_X\colon X \times K \to X$ the first factor projection.

Observation 4.1 *If $f\colon X \to \mathbb{S}^1$ is a continuous map with X a compact ANR and K is a contractible compact ANR, then $\delta_r^f = \delta_r^{f_K}$ and $\widehat{\delta}_r^f = \widehat{\delta}_r^{f_K}$.*

The statement follows in a straightforward manner from the definitions of δ_r^f and $\widehat{\delta}_r^f$.

Denote by I^∞ the product of countably many copies of $I = [0,1]$ and write $I^\infty = I^k \times I^{\infty - k}$.

Theorem 4.2 (Chapman and Edwards [5]) (1) *If X is a compact ANR then $X \times I^\infty$ is a compact Hilbert cube manifold, ie locally homeomorphic to I^∞.*

(2) *Any compact Hilbert cube manifold M is homeomorphic to $K \times I^\infty$ for some finite simplicial complex K.*

(3) *If K is a finite simplicial complex, $f\colon K \times I^\infty \to \mathbb{S}^1$ a continuous map and $\epsilon > 0$, then there exists an n and $g\colon K \times I^n \to \mathbb{S}^1$ a pl map such that $\| f - g_{I^{\infty-n}} \| < \epsilon$.*

A proof of items (1) and (2) can be found in [5]. Item (3) is a rather straightforward consequence of the compactness of $K \times I^\infty$ and the approximation of continuous maps by pl maps when the source is a finite simplicial complex (for more details see Proposition 6.5 in [1]).

We proceed now with the verification of Theorem 1.2 for X a finite simplicial complex.

In view of Observation 2.1, the proof of (1) is the same as of Theorem 4.2 in [2] provided we replace $f\colon X \to \mathbb{R}$ by $\tilde{f}\colon \widetilde{X} \to R$, a lift of $f\colon X \to \mathbb{S}^1$ representing ξ. The basic ingredient, Proposition 3.16 (based on Lemmas 3.17 and 3.18) in [2], holds for $h\colon Y \to \mathbb{R}$, with Y a locally compact ANR and h a proper map, instead of $f\colon X \to \mathbb{R}$ continuous with X compact.

For the reader's convenience we restate this proposition in the way it will to be used but beforehand we introduce the notation

$$D(a,b;\epsilon) := B(a+\epsilon, b+\epsilon; 2\epsilon) = (a-\epsilon, a+\epsilon] \times ((b-\epsilon, b+\epsilon].$$

Proposition 4.3 *Let $f\colon X \to \mathbb{S}^1$ be a tame map and $\epsilon < \frac{1}{3}\epsilon(f)$. For any map $g\colon X \to \mathbb{S}^1$ which satisfies $\|f-g\|_\infty < \epsilon$ and a and b critical values of a lift $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ of f, one has*

$$\sum_{x \in D(a,b;2\epsilon)} \delta_r^{\tilde{g}}(x) = \delta_r^{\tilde{f}}(a,b), \tag{18}$$

$$\operatorname{supp} \delta_r^{\tilde{g}} \subset \bigcup_{(a,b) \in \operatorname{supp} \delta_r^{\tilde{f}}} D(a,b;2\epsilon) \tag{19}$$

when $\tilde{g}\colon \tilde{X} \to \mathbb{R}$ is any lift of g.

Moreover, if $\kappa = \mathbb{C}$ and $H_r(\tilde{X})$ is equipped with a Hermitian scalar product, the above statement can be strengthened to

$$x \in D(a,b;2\epsilon) \implies \hat{\delta}_r^{\tilde{g}}(x) \subseteq \hat{\delta}_r^{\tilde{f}}(a,b), \qquad \bigoplus_{x \in D(a,b;2\epsilon)} \hat{\delta}_r^{\tilde{g}}(x) = \hat{\delta}_r^{\tilde{f}}(a,b), \tag{20}$$

with $\hat{\delta}_r^{\tilde{f}}(x) \perp \hat{\delta}_r^{\tilde{f}}(y)$. Here $\hat{\delta}_r^{\tilde{f}}(x)$ or $\hat{\delta}_r^{\tilde{g}}(x)$ denotes the orthogonal complement of $\mathbb{F'}_r^{\tilde{f}}(D(a,b;\epsilon))$ in $\mathbb{F}_r^{\tilde{f}}(a,b)$ or of $\mathbb{F'}_r^{\tilde{g}}(D(a,b;\epsilon))$ in $\mathbb{F}_r^{\tilde{g}}(a,b)$ for ϵ small enough.

The steps in the proof of Theorem 1.2(1) are similar to the steps described in Section 4.2 in [2]. We summarize them below:

(1) For a pair (X,ξ), with X a compact ANR, let $C_\xi(X,\mathbb{S}^1)$ denote the set of maps in the homotopy class defined by ξ equipped with the compact open topology. Note, in view of Observation 2.1, that:
   (a) The compact open topology is induced from the complete metric $D(f,g)$ and $D(f,g) = D(\tilde{f},\tilde{g})$ for appropriate lifts.
   (b) For $f,g \in C_\xi(X,\mathbb{S}^1)$ and any sequence $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$, by Observation 2.1(3), the canonical homotopy f_t from f to g, ie $f_0 = f$ and $f_1 = g$, satisfies

$$D(f,g) \geq \sum_{0 \leq i < N} D(f_{t_{i+1}}, f_{t_i}). \tag{21}$$

(2) For X is a simplicial complex let $\mathcal{U} \subset C_\xi(X,\mathbb{S}^1)$ be the subset of pl maps. One can verify that:

(a) $\mathcal{U}$ is a dense subset in $C_\xi(X, \mathbb{S}^1)$.

(b) If $f, g \in \mathcal{U}$ with $D(f, g) < \pi$ then, for the canonical homotopy f_t each $f_t \in \mathcal{U}$, hence $\epsilon(f_t) > 0$. Then, for any $t \in [0, 1]$, there exists $\delta(t) > 0$ such that $t', t'' \in (t - \delta(t), t + \delta(t))$ implies $D(f_{t'}, f_{t''}) < \frac{1}{3}\epsilon(f_t)$.

Both statements (a) and (b) are argued as in [2].

(3) Consider the space of configurations $\mathrm{Conf}_{b_r}(\mathbb{T})$, $b_r = \beta_r^N(X; \xi)$ viewed as $S^{b_r}(\mathbb{T})$, the b_r fold symmetric product of $\mathbb{T}$ equipped with the induced metric, $\underline{D}$, which is complete. Since any map in $\mathcal{U}$ is tame, in view of Proposition (3.16) in [2], $f, g \in \mathcal{U}$ with $D(f, g) < \frac{1}{3}\epsilon(f)$ imply

$$\underline{D}(\delta_r^f, \delta_r^g) \le 2D(f, g). \tag{22}$$

This suffices to conclude the continuity of the assignment $f \rightsquigarrow \delta_r^f$.

To finalize the proof of Theorem 1.2(1) we check first (Step 1) that the inequality (22) extends to all $f, g \in \mathcal{U}$, second (Step 2) that the inequality (22) extends to all $f, g \in C_\xi(X, \mathbb{S}^1)$ for X a finite simplicial complex, and third (Step 3) that the inequality (22) extends to all $f, g \in C_\xi(X, \mathbb{S}^1)$ for X an arbitrary compact ANR.

Step 1 Start with $f, g \in \mathcal{U}$ and consider the canonical homotopy $\tilde{f}_t = t\tilde{f} + (1-t)\tilde{g}$ for $t \in [0, 1]$ between two lifts $\tilde{f}$ and $\tilde{g}$ of f and g which satisfy $D(f, g) = D(\tilde{f}, \tilde{g})$. Note that each $\tilde{f}_t$ satisfies $\tilde{f}_t(\mu(n, x)) = \tilde{f}_t(x) + 2\pi n$, hence is a lift of a pl map f_t.

Choose a sequence $0 < t_1 < t_3 < t_5 < \cdots < t_{2N-1} < 1$ such that for $i = 1, \dots, 2N-1$ the intervals $(t_{2i-1} - \delta(t_{2i-1}), t_{2i-1} + \delta(t_{2i-1}))$, with $\delta(t_i)$ as in (2b), cover $[0, 1]$ and $(t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1}), t_{2i+1}) \neq \varnothing$. This is possible in view of the compactness of $[0, 1]$.

Take $t_0 = 0$, $t_{2N} = 1$ and $t_{2i} \in (t_{2i-1}, t_{2i-1} + \delta(t_{2i-1})) \cap (t_{2i+1} - \delta(t_{2i+1}), t_{2i+1})$. To simplify the notation abbreviate f_{t_i} to f_i. In view of (2) and (3) above (the inequality (22)) one has that $|t_{2i-1} - t_{2i}| < \delta(t_{2i-1})$ implies $\underline{D}(\delta^{f_{2i-1}}, \delta^{f_{2i}}) < 2D(f_{2i-1}, f_{2i})$ and $|t_{2i} - t_{2i+1}| < \delta(t_{2i+1})$ implies $\underline{D}(\delta^{f_{2i}}, \delta^{f_{2i+1}}) < 2D(f_{2i}, f_{2i+1})$. Then we have

$$\underline{D}(\delta^f, \delta^g) \le \sum_{0 \le i < 2N-1} \underline{D}(\delta^{f_i}, \delta^{f_{i+1}}) \le 2 \sum_{0 \le i < 2N-1} D(f_i, f_{i+1}) \le 2D(f, g)$$

in view of Observation 2.1(3).

Step 2 Suppose X is a simplicial complex. In view of the density of $\mathcal{U}$ and of the completeness of the metrics on $C_\xi(X; \mathbb{S}^1)$ and $\mathrm{Conf}_{b_r}(\mathbb{T})$, the inequality (22)) extends to the entire $C_\xi(X;\mathbb{S}^1)$. Indeed, the assignment $\mathcal{U}\ni f \rightsquigarrow \delta^f_r \in C_{b_r}(\mathbb{R}^2)$ preserves the Cauchy sequences.

Step 3 We verify the inequality (22) for $X=K\times I^\infty$, with K simplicial complex and I^∞ the Hilbert cube. One proceeds exactly as in [2]. Since by Theorem 1.2(2) any compact Hilbert cube manifold is homeomorphic to $K\times I^\infty$ for some finite simplicial complex K, the inequality (22) continues to hold. Since for X a compact ANR

(i) $X\times I^\infty$ is a Hilbert cube manifold by Theorem 1.2(1),

(ii) $I\colon C(X;\mathbb{R})\to C(X\times I^\infty;\mathbb{R})$ defined by $I(f)=\bar{f}_{I^\infty}$ is an isometric embedding, and

(iii) $\delta^f=\delta^{\bar{f}_{I^\infty}}$,

the inequality (22) holds for X a compact ANR.

To check Theorem 1.2(2) we begin with a few observations. If $\kappa=\mathbb{C}$, a Riemannian metric on a closed smooth manifold $M^n=X$, or a triangulation of a compact ANR X, provides a Hermitian scalar product on $H_r(\widetilde{X})$ invariant to the action of the group of deck transformations of the covering $\widetilde{X}\to X$. Ultimately this provides a $\mathbb{C}[t^{-1},t]$–compatible Hermitian inner product on $H^N_r(X;\xi)$ and then a collection of compatible N–splittings ${}^N i_r(a,b)$ for $(a,b)\in\mathbb{R}^2$ and then the collection of compatible N–splittings ${}^N i_r\langle a,b\rangle$ for $\langle a,b\rangle\in\mathbb{T}$ for both f and g. The images of these splittings are the free submodules $\hat{\delta}^f$ and $\hat{\delta}^g$. In view of Proposition 4.3, for a given f, $(a,b)\in \mathrm{CR}(f)\times\mathrm{CR}(f)$, $\epsilon<\epsilon(f)$ and any g with $\|g-f\|_\infty<\frac{1}{3}\epsilon$, the two subspaces of spaces of $H^N_r(X;\xi)$

$$\sum_{(a',b')\in D(a,b;\epsilon)\cap\operatorname{supp}\delta^{\tilde{g}}} {}^N i_r\langle a',b'\rangle(\hat{\delta}^g\langle a',b'\rangle) \quad\text{and}\quad {}^N i_r\langle a,b\rangle(\hat{\delta}^f\langle a,b\rangle)$$

are equal.

The $\mathbb{C}[t^{-1},t]$–compatibility permits us to pass to von Neumann completions and derive the collection of Hilbert submodules $\hat{\hat{\delta}}^f$ and $\hat{\hat{\delta}}^g$ which under the above hypotheses satisfy

$$\sum_{(a',b')\in D(a,b;\epsilon)\cap\operatorname{supp}\delta^{\tilde{g}}} \hat{\hat{\delta}}^g\langle a',b'\rangle) = \hat{\hat{\delta}}^f\langle a,b\rangle.$$

This implies the continuity of the assignment $f\rightsquigarrow\hat{\hat{\delta}}^f_r$ when the space of configurations is equipped with the fine topology and then with the natural topology, hence Theorem 1.2(2). $\square$

5 Proof of Theorem 1.3

We prove Theorem 1.3 for weakly tame maps $f\colon M \to \mathbb{S}^1$, with M a closed topological manifold of dimension n, whose set of nontopological regular values is finite. If the set of such maps is dense in the set of all maps equipped with the compact open topology then, in view of Theorem 1.2, the result holds for all continuous maps. One expects this to always be the case. When the manifold is homeomorphic to a finite simplicial complex, this is indeed the case since a pl map is weakly tame and has finitely many critical values and the set of pl maps is dense in the set of all continuous maps with the compact open topology. For manifolds which have no triangulation, a possible argument for such density is considerably longer and will not be provided in this paper. We were unable to locate a reference in the literature. For f a weakly tame map it will suffice to consider only regular values a and b. This is because for arbitrary pairs c and c' one can find $\epsilon > 0$ small enough such that for $a' = c - \epsilon$, $a = c + \epsilon$, $b = c' - \epsilon$, $b' = c + \epsilon$ and $B = (a', a] \times [b, b')$, in view of Proposition 3.4 one has

$$\hat{\delta}^{\tilde f}_r(c,c') = \hat{\delta}^{\tilde f}_r(a,b) = \mathbb{F}^{\tilde f}(B)\hat{\delta}^{\tilde f}_r(c',c) = \hat{\delta}^{\tilde f}_r(b',a') = \mathbb{F}^{\tilde f}(B').$$

The proof of Theorem 1.3 requires some additional notation and considerations.

Some additional notation and definitions Recall that a topologically regular value is a value s which has a neighborhood U such that $f\colon f^{-1}(U) \to U$ is a topological bundle. If so, any lift (infinite cyclic cover) $\tilde f\colon \tilde M \to \mathbb{R}$ of f has the set of critical values discrete and 2π–periodic.

We use the notation:

(1) $\tilde M_a := \tilde f^{-1}((-\infty, a])$ and $\tilde M^a := \tilde f^{-1}([a, \infty))$ for $a \in \mathbb{R}$.

(2) $\mathbb{I}_a(r) := \operatorname{img}(H_r(\tilde M_a) \to H_r(\tilde M))$ and $\mathbb{I}^a(r) := \operatorname{img}(H_r(\tilde M^a) \to H_r(\tilde M))$.

(3) $\mathbb{F}^{\tilde f}_r(a,b) = \mathbb{I}^{\tilde f}_a \cap \mathbb{I}^b_{\tilde f}$ and $i_r(a,b)\colon \mathbb{F}^f_r(a,b) \subset H_r(\tilde M)$, the inclusion.

In addition consider:

(4) $\mathbb{G}^{\tilde f}_r(a,b) := H_r(\tilde M)/(\mathbb{I}^{\tilde f}_a + \mathbb{I}^b_{\tilde f})$ and $p_r(a,b)\colon H_r(\tilde M) \to \mathbb{G}^{\tilde f}_r(a,b)$, the canonical projection.

(5) For a box $B = (a', a] \times [b, b')$ let:

  (a) $\mathbb{F}^{\tilde f}_r(B) := \operatorname{coker}(\mathbb{F}^{\tilde f}_r(a',b) \oplus \mathbb{F}^{\tilde f}_r(a,b') \to \mathbb{F}^{\tilde f}_r(a,b))$ and $\pi^B_r\colon \mathbb{F}^{\tilde f}_r(a,b) \twoheadrightarrow \mathbb{F}^f_r(B)$ the canonical surjection,

(b) $\mathbb{G}_r^{\tilde f}(B) := \ker(\mathbb{G}_r^{\tilde f}(a',b') \to \mathbb{G}_r^{\tilde f}(a',b) \times_{\mathbb{G}_r^{\tilde f}(a,b)} \mathbb{G}_r^{\tilde f}(a,b')$ and

$$u_r^B\colon \mathbb{G}_r^{\tilde f}(B) \rightarrowtail \mathbb{G}_r^{\tilde f}(a',b')$$

the canonical inclusion.

(c) Since $\mathbb{F}_r^{\tilde f}(B)$ identifies canonically to

$$\mathbb{I}_a(r) \cap \mathbb{I}^b(r)/(\mathbb{I}_{a'}(r) \cap \mathbb{I}^b(r) + \mathbb{I}_a(r) \cap \mathbb{I}^{b'}(r))$$

and $\mathbb{G}^{\tilde f}(B)$ identifies canonically to

$$(\mathbb{I}_{a'}(r) + \mathbb{I}^b(r)) \cap (\mathbb{I}_a(r) + \mathbb{I}^{b'}(r))/(\mathbb{I}_{a'}(r) + \mathbb{I}^{b'}(r)),$$

the inclusion $\mathbb{I}_a(r) \cap \mathbb{I}^b(r) \subseteq (\mathbb{I}_{a'}(r) + \mathbb{I}^b(r)) \cap (\mathbb{I}_a(r) + \mathbb{I}^{b'}(r))$ induces the linear map $\theta_r(B)\colon \mathbb{F}_r^{\tilde f}(B) \to \mathbb{G}_r^{\tilde f}(B)$, which is an isomorphism.

For a verification one can consult [2, Proposition 4.7].

If a is a topologically regular value then $\tilde M_a$ and $\tilde M^a$ are manifolds with compact boundary $f^{-1}(a)$ and denote by

$$(23)\qquad \begin{aligned} H_r^{BM}(\tilde M) &= \varprojlim_{0<l,t\to\infty} H_r(\tilde M, \tilde M_{-l} \sqcup \tilde M^t),\\ H_r^{BM}(\tilde M_a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M_a, \tilde M_{a-l}),\\ H_r^{BM}(\tilde M^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M^{a+l}),\\ H_r^{BM}(\tilde M, \tilde M_a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M_a \sqcup \tilde M^{a+l}),\\ H_r^{BM}(\tilde M, \tilde M^a) &= \varprojlim_{0<l\to\infty} H_r(\tilde M, \tilde M^a \sqcup \tilde M_{a-l}). \end{aligned}$$

The reader will recognize on the left side of the equalities (23) the notation for the Borel–Moore homology vector spaces with coefficients in κ, the right homology to extend the Poincaré duality from compact manifolds to arbitrary finite-dimensional manifolds.

Poincaré duality diagrams for $\tilde M$ One has the following commutative diagrams, whose vertical arrows are isomorphisms, referred to as the Poincaré duality diagrams for nonclosed manifolds:

$$
\begin{array}{ccccc}
H_r^{BM}(\tilde{M}_a) & \xrightarrow{\quad i_a(r)\quad} & H_r^{BM}(\tilde{M}) & \xrightarrow{\quad j_a(r)\quad} & H_r^{BM}(\tilde{M},\tilde{M}_a) \\
\big\downarrow{\scriptstyle \mathrm{PD}^1_a} & & \big\downarrow{\scriptstyle \mathrm{PD}} & & \big\downarrow{\scriptstyle \mathrm{PD}^2_a} \\
H^{n-r}(\tilde{M},\tilde{M}^a) & \xrightarrow{\quad s^a(n-r)\quad} & H^{n-r}(\tilde{M}) & \xrightarrow{\quad r^a(n-r))\quad} & H^{n-r}(\tilde{M}^a) \\
\big\downarrow{=} & & \big\downarrow{=} & & \big\downarrow{=} \\
(H_{n-r}(\tilde{M},\tilde{M}^a))^* & \xrightarrow{(j^a(n-r))^*} & H_{n-r}(\tilde{M})^* & \xrightarrow{(i^a(n-r))^*} & H_{n-r}(\tilde{M}^a)^*
\end{array}
\tag{24}
$$

$$
\begin{array}{ccccc}
H_r^{BM}(\tilde{M}^b) & \xrightarrow{\quad i^b(r)\quad} & H_r^{BM}(\tilde{M}) & \xrightarrow{\quad j^b(r)\quad} & H_r^{BM}(\tilde{M},\tilde{M}^b) \\
\big\downarrow{\scriptstyle \mathrm{PD}^b_1} & & \big\downarrow{\scriptstyle \mathrm{PD}} & & \big\downarrow{\scriptstyle \mathrm{PD}^b_2} \\
H^{n-r}(\tilde{M},\tilde{M}_b) & \xrightarrow{\quad s_b(n-r)\quad} & H^{n-r}(\tilde{M})) & \xrightarrow{\quad r_b(n-r)\quad} & H^{n-r}(\tilde{M}_b) \\
\big\downarrow{=} & & \big\downarrow{=} & & \big\downarrow{=} \\
(H_{n-r}(\tilde{M},\tilde{M}_b))^* & \xrightarrow{(j_b(n-r))^*} & (H_{n-r}(\tilde{M}))^* & \xrightarrow{(i_b(n-r))^*} & (H_{n-r}(\tilde{M}_b))^*
\end{array}
\tag{25}
$$

One can derive these diagrams from the Poincaré duality for compact bordisms $(\tilde{f}^{-1}[a,b], \tilde{f}^{-1}(a), f^{-1}(b))$ when a and b are topologically regular values, by passing to the limit $a \to -\infty$ or $b \to \infty$ with no knowledge about Borel–Moore homology.

The Poincaré duality isomorphism

$$\mathrm{PD}_r^{BM}\colon H_r^{BM}(\tilde{M}) \xrightarrow{\mathrm{PD}_r} H^{n-r}(\tilde{M}) \xrightarrow{=} (H_{n-r}(\tilde{M}))^*$$

we consider below is the composition of the vertical arrows in the middle of diagram (24) or (25).

Note that all three terms of this sequence are $\kappa[t^{-1},t]$–modules and the two arrows are $\kappa[t^{-1},t]$–linear with the multiplication by t given by the linear isomorphism induced by the deck transformation τ_r.

If one uses $H_r^{BM}(\cdot)$ instead of $H_r(\cdot)$, one can also consider ${}^{BM}\mathbb{F}_r^{\tilde{f}}(a,b)$, ${}^{BM}\mathbb{F}_r^{\tilde{f}}(B)$ and ${}^{BM}\hat{\delta}_r^{\tilde{f}}(a,b)$ instead of $\mathbb{F}_r^{\tilde{f}}(a,b)$, $\mathbb{F}_r^{\tilde{f}}(B)$ and $\hat{\delta}_r^{\tilde{f}}(a,b)$. Proposition 5.2(3) will show that ${}^{BM}\mathbb{F}_r^{\tilde{f}}(B)$ and ${}^{BM}\hat{\delta}_r^{\tilde{f}}(a,b)$ are respectively canonically isomorphic to $\mathbb{F}_r^{\tilde{f}}(B)$ and $\hat{\delta}_r^{\tilde{f}}(a,b)$.

Intermediate results With the definitions already given, one has the following proposition:

Proposition 5.1 (1) *For any a and b regular values of $\tilde f$, the Poincaré duality isomorphism restricts to an isomorphism*

$$\mathrm{PD}^{BM}_r(a,b)\colon {}^{BM}\mathbb{F}^{\tilde f}_r(a,b)\to(\mathbb{G}^{\tilde f}_{n-r}(b,a))^*.$$

(2) *For any box $B=(a',a]\times[b,b')$ and $B'=(b,b']\times[a',a))$ with all a, a', b and b' regular values, PD^{BM}_r induces the isomorphisms $\mathrm{PD}^{BM}_r(a,b)$ and $\mathrm{PD}^{BM}_r(B)$, making the diagram below commutative:*

$$\begin{tikzcd}
{}^{BM}\mathbb{F}^{\tilde f}_r(a',b') \arrow[rr, "\mathrm{PD}^{BM}_r(a',b')"] \arrow[d] & & (\mathbb{G}^{\tilde f}_{n-r}(b',a'))^* \arrow[d] & \\
{}^{BM}\mathbb{F}^{\tilde f}_r(a,b) \arrow[rr, "\mathrm{PD}^{BM}_r"] \arrow[dr] \arrow[dd] & & (\mathbb{G}^{\tilde f}_{n-r}(b,a))^* \arrow[dr, "u^*_{n-r}"] \arrow[dd, "p^*_r"] & \\
& {}^{BM}\mathbb{F}^{\tilde f}_r(B) \arrow[rr, "\mathrm{PD}^{BM}_r(B)"] & & (\mathbb{G}^{\tilde f}_{n-r}(B'))^* \arrow[d, "\theta^*_{n-r}"] \\
H^{BM}_r(\widetilde M) \arrow[rr, "\mathrm{PD}^{BM}_r"] & & (H_{n-r}(\widetilde M))^* & (\mathbb{F}^{\tilde f}_{n-r}(B'))^*
\end{tikzcd} \tag{26}$$

Proof (1) In view of diagrams (24) and (25) one has

$$\begin{aligned}\operatorname{img} i_a(r)\cap\operatorname{img} i^b(r)=\ker j_a(r)\cap\ker j^b(r)&\simeq\ker(i^a(n-r))^*\cap\ker(i_b(n-r))^*\\ &\simeq\big(\operatorname{coker}(i_b(n-r)\oplus i^a(n-r))\big)^*=(\mathbb{G}^{\tilde f}_{n-r}(b,a))^*.\end{aligned}$$

The first equality holds by exactness of the first rows in the diagrams, the second by the equality of the top- and bottom-right horizontal arrows, the third by linear algebra duality and the fourth by the definition of $\mathbb{G}_{n-r}$.

(2) Consider the box $B=(a',a]\times[b,b')$ and denote by B' the box $B'=(b,b']\times[a',a)$. Note that the image of the diagram

$${}^{BM}\mathcal{F}(B):=\left\{\begin{tikzcd}
{}^{BM}\mathbb{F}^{\tilde f}_r(a',b') \arrow[r] \arrow[d] & {}^{BM}\mathbb{F}^{\tilde f}_r(a,b') \arrow[d] \\
{}^{BM}\mathbb{F}^{\tilde f}_r(a',b) \arrow[r] & {}^{BM}\mathbb{F}^{\tilde f}_r(a,b)
\end{tikzcd}\right.$$

by ${}^{BM}\mathrm{PD}_r$ is the diagram

$$\mathcal{G}(B')^* := \left\{ \begin{tikzcd} (\mathbb{G}^{\tilde f}_{n-r}(b',a'))^* \arrow[r] \arrow[d] & (\mathbb{G}^{\tilde f}_{n-r}(b',a))^* \arrow[d] \\ (\mathbb{G}^{\tilde f}_{r}(b,a'))^* \arrow[r] & (\mathbb{G}^{\tilde f}_{n-r}(b,a))^* \end{tikzcd} \right.$$

which is the dual of the diagram

$$\mathcal{G}(B') := \left\{ \begin{tikzcd} \mathbb{G}^{\tilde f}_{n-r}(b,a) \arrow[r] \arrow[d] & \mathbb{G}^{\tilde f}_{n-r}(b',a) \arrow[d] \\ \mathbb{G}^{\tilde f}_{r}(b,a') \arrow[r] & \mathbb{G}^{\tilde f}_{n-r}(b',a') \end{tikzcd} \right.$$

Therefore, ${}^{BM}\mathrm{PD}_r$ induces an isomorphism from ${}^{BM}\mathbb{F}^{\tilde f}_r(B) = \mathrm{coker}\,{}^{BM}\mathcal{F}(B)$ to $(\ker(\mathcal{G}(B')))^* = (\mathbb{G}^{\tilde f}_{n-r}(B'))^*$. ◻

From diagram (26) one derives

$$\text{(27)} \qquad \begin{tikzcd} {}^{BM}\mathbb{F}^{\tilde f}_r(a,b) \arrow[rr, "\mathrm{PD}^{BM}_r(a{,}b)"] \arrow[dd] \arrow[dr] & & (\mathbb{G}^{\tilde f}_{n-r}(b,a))^* \arrow[dd, "p^*_r(a{,}b)"] \arrow[dr, "u^*_{n-r}"] & \\ & {}^{BM}\hat\delta^{\tilde f}_r(a,b) \arrow[rr] & & (\hat\delta^{\tilde f}_{n-r}(b,a))^* \\ H^{BM}_r(\tilde M) \arrow[rr, "\mathrm{PD}^{BM}_r"] & & (H_{n-r}(\tilde M))^* & \end{tikzcd}$$

with the horizontal arrows isomorphisms, the vertical arrows injective and the oblique arrows surjective.

Indeed, for $B = (a-\epsilon, a+\epsilon] \times [b-\epsilon, b+\epsilon)$, so $B' = (b-\epsilon, b+\epsilon] \times [a-\epsilon, a+\epsilon)$, and ϵ small enough to have (in view of Proposition 3.4(1)) $\hat\delta^{\tilde f}_r(a,b) = \mathbb{F}^{\tilde f}_r(B)$ and $\hat\delta^{\tilde f}_{n-r}(b,a) = \mathbb{F}^{\tilde f}_r(B')$, the diagram (26) gives rise to the diagram (27).

The key observation for finalizing (1) and (2) is the following proposition:

Proposition 5.2 *The κ–linear maps $\mathbb{F}^{\tilde f}_r(a,b) \to {}^{BM}\mathbb{F}^{\tilde f}_r(a,b)$*

(1) *are compatible with the deck transformations,*

(2) *are surjective,*

(3) *have kernel $C_r(M)$ independent of (a,b), equal to the kernel of the $\kappa[t^{-1},t]$–linear map $H_r(\widetilde{M}) \to H_r^{BM}(\widetilde{M})$ and equal to $T(H_r(\widetilde{M}))$.*

Proof One shows first that one has a natural short exact sequence

$$0 \to C_r(M) \to \mathbb{F}_r^{\tilde{f}}(a,b) \to {}^{BM}\mathbb{F}_r^{\tilde{f}}(a,b) \to 0$$

which is compatible with the action provided by the deck transformations and leaves $C_r(M)$ fixed, and second that $C_r(M)$ is exactly the $\kappa[t^{-1},t]$–torsion of the $H_r(\widetilde{M})$. Precisely, one show that one has the following commutative diagram with exact sequences as rows and $C_r(M) = \mathbb{I}_{-\infty}^{\tilde{f}}(r) + \mathbb{I}_{\tilde{f}}^{\infty}(r)$:

$$\begin{tikzcd}
0 \arrow[r] & C_r(M) \arrow[r] \arrow[d, "t_r = \mathrm{id}"] & \mathbb{F}_r^{\tilde{f}}(a,b) \arrow[r] \arrow[d, "t_r"] & {}^{BM}\mathbb{F}_r^{\tilde{f}}(a,b) \arrow[r] \arrow[d, "t_r^{BM}"] & 0 \\
0 \arrow[r] & C_r(M) \arrow[r] & \mathbb{F}_r^{\tilde{f}}(a+2\pi, b+2\pi) \arrow[r] & {}^{BM}\mathbb{F}_r^{\tilde{f}}(a+2\pi, b+2\pi) \arrow[r] & 0
\end{tikzcd} \tag{28}$$

The proof uses the following diagram, where $-l < a' < a$ and $b < b' < t$:

$$\begin{tikzcd}
H_{r-1}(\widetilde{M}_{-l}) \arrow[d, "="'] & H_r(\widetilde{M}_{a'}, \widetilde{M}_{-l}) \arrow[l] \arrow[d] & H_r(\widetilde{M}_{a'}) \arrow[l] \arrow[d] \arrow[dd, bend right, "i_{a'}(r)"'] & H_r(\widetilde{M}_{-l}) \arrow[l] \arrow[d, "="'] \\
H_{r-1}(\widetilde{M}_{-l}) \arrow[d, "\hat{\imath}_{-l}(r-1)"'] & H_r(\widetilde{M}_a, \widetilde{M}_{-l}) \arrow[l] \arrow[d] & H_r(\widetilde{M}_a) \arrow[l] \arrow[d, "i_a(r)"] & H_r(\widetilde{M}_{-l}) \arrow[l] \arrow[d, "\hat{\imath}_{-l}(r)"'] \\
H_{r-1}(\widetilde{M}_{-l} \sqcup \widetilde{M}^t) & H_r(\widetilde{M}, \widetilde{M}_{-l} \sqcup \widetilde{M}^t) \arrow[l] & H_r(\widetilde{M}) \arrow[l] & H_r(\widetilde{M}_{-l} \sqcup \widetilde{M}^t) \arrow[l] \\
H_{r-1}(\widetilde{M}^t) \arrow[u, "\hat{\imath}^t(r-1)"] & H_r(\widetilde{M}^b, \widetilde{M}^t) \arrow[l] \arrow[u] & H_r(\widetilde{M}^b) \arrow[l] \arrow[u, "i^b(r)"'] & H_r(\widetilde{M}^t) \arrow[l] \arrow[u, "\hat{\imath}^t(r)"] \\
H_{r-1}(\widetilde{M}^t) \arrow[u, "="] & H_r(\widetilde{M}^{b'}, \widetilde{M}^t) \arrow[l] \arrow[u] & H_r(\widetilde{M}^{b'}) \arrow[l] \arrow[u] \arrow[uu, bend left, "i^{b'}(r)"] & H_r(\widetilde{M}^t) \arrow[l] \arrow[u, "="]
\end{tikzcd} \tag{29}$$

In this diagram the horizontal lines are exact sequences. By passing to limits when $l, t \to \infty$, diagram (29) induces the diagram

$$
\begin{tikzcd}
H^{BM}(\widetilde{M}_{a'}) \arrow[d] \arrow[dd, bend right=50, "i^{BM}_{a'}(r)"'] & H(\widetilde{M}_{a'}) \arrow[l] \arrow[d] \\
H^{BM}_r(\widetilde{M}_a) \arrow[d, "i^{BM}_a(r)"] & H_r(\widetilde{M}_a) \arrow[l] \arrow[d, "i_a(r)"'] \\
H^{BM}_r(\widetilde{M}) & H_r(\widetilde{M}) \arrow[l] \\
H^{BM}_r(\widetilde{M}^b) \arrow[u, "(i^{BM})^b(r)"'] & H_r(\widetilde{M}^b) \arrow[l] \arrow[u, "i^b(r)"] \\
H^{BM}(\widetilde{M}^{b'}) \arrow[u] \arrow[uu, bend left=50, "(i^{BM})^{b'}(r)"] & H_r(\widetilde{M}^{b'}) \arrow[l] \arrow[u]
\end{tikzcd}
\tag{30}
$$

which provides the relation between $\mathbb{F}_r(a,b), \mathbb{F}_r(a',b'), H_r(\widetilde{M})$ and their Borel–Moore versions.

Diagram (30) leads to the linear map $\mathbb{F}^{\tilde{f}}_r(a,b) \to {}^{BM}\mathbb{F}^{\tilde{f}}_r(a,b)$ and establishes the compatibility with the deck transformations, hence Proposition 5.2(1).

Since ${}^{BM}\mathbb{F}^{\tilde{f}}_r(a,b) = \mathrm{img}(i^{BM}_a(r)) \cap \mathrm{img}((i^{BM})^b(r))$ and $\mathrm{img}(\hat{\imath}_{-l}(r)) \cap \mathrm{img}(\hat{\imath}^t(r)) = 0$ for any r, l and t, a careful analysis of the projective limit and of the diagram (29) implies that

$$\mathbb{F}^{\tilde{f}}_r(a,b) \to {}^{BM}\mathbb{F}^{\tilde{f}}_r(a,b)$$

is surjective (hence (2) holds), with kernel isomorphic to

$$\varprojlim_{0<l,t\to\infty} \mathrm{img}(H_r(\widetilde{M}_{-l} \sqcup \widetilde{M}^t) \to H_r(\widetilde{M})) = \mathbb{I}_{-\infty}(r) + \mathbb{I}^{\infty}(r) = C_r(M).$$

In view of Proposition 3.8, $C_r(M)$ is equal to $TH_r(\widetilde{M})$, hence (3) holds too. □

The diagram (26) and the above observations induce the following diagram with the first three horizontal arrows isomorphisms and the last arrow (PD^N) injective and $\kappa[t^{-1},t]$–linear (indeed, the first three horizontal arrows are isomorphisms in view of the isomorphism $\mathbb{F}^{\tilde{f}}_r(a,b)/T(H_r(\widetilde{M})) \simeq {}^{BM}\mathbb{F}^{\tilde{f}}_r(a,b)$):

$$
\begin{tikzcd}
\mathbb{F}_r^{\tilde f}(a,b)/T(H_r(\widetilde M)) \arrow[d, equal] & & & \\
{}^{BM}\mathbb{F}_r^{\tilde f}(a,b) \arrow[rr, "\mathrm{PD}_r^{BM}(a,b)"] \arrow[dr] \arrow[ddd] & & (\mathbb{G}_{n-r}^{\tilde f}(b,a))^* \arrow[dr, "u_{n-r}^*"] \arrow[ddd, "p_r^*"] & \\
& \mathbb{F}_r^{\tilde f}(B) \arrow[rr, "\mathrm{PD}_r(B)"] \arrow[d, "\pi_{B,r}^{ab}"] & & (\mathbb{G}_{n-r}^{\tilde f}(B'))^* \\
& \hat\delta_r^{\tilde f}(a,b) \arrow[rr, "\widehat{\mathrm{PD}}_r(a,b)"] & & (\hat\delta_{n-r}^{\tilde f}(b,a))^* \\
H_r^N(M;\xi) \arrow[rr, "\mathrm{PD}_r^N"] & & (H_{n-r}^N(M;\xi))^* &
\end{tikzcd}
\tag{31}
$$

The bottom arrow is the composition

$$
\begin{aligned}
H_r(\widetilde M)/TH_r(\widetilde M) = H_r^N(M;\xi) &\to H_r^{BM}(\widetilde M)/TH_r^{BM}(\widetilde M) \\
&\to (H_{n-r}(\widetilde M))^*/T(H_{n-r}(\widetilde M)^*) \xleftarrow{=} (H_{n-r}^N(M;\xi))^*
\end{aligned}
\tag{32}
$$

with the first arrow $\kappa[t^{-1},t]$–linear and injective in view of Proposition 5.2(3) and the second arrow, $H_r^{BM}(\widetilde M)/T(H_r^{BM}(\widetilde M)) \to H_{n-r}(\widetilde M)^*/T(H_{n-r}(\widetilde M)^*)$ a $\kappa[t^{-1},t]$–linear isomorphism in view of the isomorphism PD: ${}^{BM}H_r(\widetilde M) \to H_{n-r}(\widetilde M)^*$ and $\xleftarrow{=}$ a canonical isomorphism. Indeed, the finite-dimensionality of $T(H_{n-r}(\widetilde M))$ and the isomorphism $H_{n-r}(\widetilde M) \simeq H_{n-r}^N(M;\xi) \oplus T(H_{n-r}(\widetilde M))$ imply that the composition $H_r^N(M;\xi)^* \to H_{n-r}(\widetilde M)^* \to H_{n-r}(\widetilde M)^*/T(H_{n-r}(\widetilde M)^*)$ is a (canonical) isomorphism.

Observation 5.3 (1) *The diagram*

$$
\begin{tikzcd}
\hat\delta_r^{\tilde f}(a,b) \arrow[r, "\widehat{\mathrm{PD}}_r(a,b)"] \arrow[d, "\hat t_r"] & (\hat\delta_{n-r}^{\tilde f} r(b,a))^* \\
\hat\delta_r^{\tilde f}(a+2\pi,b+2\pi) \arrow[r, "\widehat{\mathrm{PD}}_r(a+2\pi,b+2\pi)"] & (\hat\delta_{n-r}^{\tilde f} r(b+2\pi,a+2\pi))^* \arrow[u, "\hat t_r^*"]
\end{tikzcd}
\tag{33}
$$

is commutative,

(2) $\hat\delta^{\tilde f}(a,b)$ *is a finite-dimensional vector space and therefore* $(\hat\delta^{\tilde f}(a,b))^*$ *is a finite-dimensional vector space isomorphic to* $\hat\delta^{\tilde f}(a,b)$.

Define
$$\hat{\delta}^f_r\langle a,b\rangle := \bigoplus_{k\in\mathbb{Z}} \delta^{\tilde f}_r(a+2\pi k, b+2\pi k),$$
which equipped with the isomorphism $\bigoplus_{k\in\mathbb{Z}} \hat{t}_r(a+2\pi k, b+2\pi k)$ is a free $\kappa[t^{-1},t]$–module, and
$$(\hat{\delta}^f_r)^*\langle a,b\rangle := \bigoplus_{k\in\mathbb{Z}} (\delta^{\tilde f}_r(a+2\pi k.b+2\pi k))^*,$$
which equipped with the isomorphism $\bigoplus_{k\in\mathbb{Z}} \hat{t}_r(a+2\pi k, b+2\pi k)$ is a free $\kappa[t^{-1},t]$–module.

Note that $(\hat{\delta}^f_r)^*\langle a,b\rangle$ is not the same as $(\hat{\delta}^f_r\langle a,b\rangle)^*$. Actually,
$$(\hat{\delta}^f_r)^*\langle a,b\rangle \subseteq (\hat{\delta}^f_r\langle a,b\rangle)^*;$$
the first is a finitely generated free $\kappa[t^{-1},t]$ module while the second is in general infinitely generated except in the case of equality, which happens only in the case that $\hat{\delta}^f_r\langle a,b\rangle = 0$.

Finalizing the proof of Theorem 1.3 In view of Observation 5.3(2) and diagram (31) above we have the isomorphism of $\kappa[t^{-1},t]$–modules
$$\widehat{\mathrm{PD}}_r\langle a,b\rangle\colon \hat{\delta}^f_r\langle a,b\rangle \to (\hat{\delta}^f_{n-r})^*\langle b,a\rangle.$$
The choice of compatible splittings "$\mathcal{S}$" provides an isomorphism of $\kappa[t^{-1},t]$–modules
$$\pi(r)\cdot I^{\mathcal{S}}_r\colon \bigoplus_{\langle a,b\rangle} (\hat{\delta}^f_r)^*\langle a,b\rangle \to H^N_r(M;\xi_f)$$
and then establishes the isomorphism of $H^N_r(X;\xi_f)$ to $H^N_{n-r}(M;\xi_f)$ which intertwines $(\hat{\delta}^f_r)^*\langle a,b\rangle$ with $(\hat{\delta}^f_{n-r})^*\langle a,b\rangle$. This establishes Theorem 1.3(1)–(2).

Suppose that $\kappa=\mathbb{R}$ or $\kappa=\mathbb{C}$. Choose a nondegenerate positive definite inner product on $H_r(\widetilde{M})$ which makes t_r an isometry for any r. Such an inner product can be provided by a Riemannian metric on M when M is a closed smooth manifold or by a triangulation of M when M is triangulable, simply by lifting the metric or the triangulation on $\widetilde{M}$.

These inner products provide canonical compatible splittings which realize canonically $\hat{\delta}^{\tilde f}_r(a,b)$ as a subspace of $H_r(\widetilde{M})$ and then of $H^N_r(M;\xi_f)$ and lead to the embedding of $\hat{\delta}^f_r\langle a,b\rangle$ as a sub-$\kappa[t^{-1},t]$–module of $H^N(X;\xi_f)$ (in view of the observation that the images of $\hat{\delta}^{\tilde f}(a,b)$ and $\hat{\delta}^{\tilde f}(a+2\pi, b+2\pi)$ are orthogonal and $\hat{t}_r$ is an isometry).

These canonical splittings make $\pi(r)\cdot I^S_r$ a canonical isomorphism. The inner products on $\hat\delta^{\tilde f}_{r}(a,b)$ induced from the inner product on $H_r(\tilde M)$ canonically identifies $\hat\delta^{\tilde f}_r(a,b)$ to $\hat\delta^{\tilde f}_r(a,b)^*$ then $\hat\delta^f_r\langle a,b\rangle$ to $(\hat\delta^f_r)^*\langle a,b\rangle$ and provides a canonical isomorphism $\widehat{\mathrm{PD}}_r\langle a,b\rangle\colon \hat\delta^f_r\langle a,b\rangle\to\hat\delta^f_{n-r}\langle a,b\rangle$ and then the isomorphism $\widehat{\mathrm{PD}}_r\colon H^N(M,\xi_f)\to H^N(M;\xi_f)_{n-r}$.

In the case $\kappa=\mathbb{C}$, if $\hat{\hat\delta}^f_r\langle a,b\rangle$ denotes the von Neumann completion of $\hat\delta^f_r\langle a,b\rangle$ (note that $H^{L_2}(\tilde M)$ is the von Neumann completion of $H^N(M;\xi_f)$), then $\hat{\hat\delta}^f\langle a,b\rangle$ is a closed Hilbert submodule of $H^{L_2}(\tilde M)$. Moreover, the von Neumann completion leads to the canonical isomorphism of $L^\infty(\mathbb{S}^1)$–Hilbert modules

$$\widehat{\widehat{\mathrm{PD}}}_r\colon H^{L_2}(\tilde M)\to H^{L_2}_{n-r}(\tilde M)$$

which intertwines $\hat{\hat\delta}^f_r\langle a,b\rangle$ with $\hat{\hat\delta}^f_{n-r}\langle b,a\rangle$. This establishes Theorem 1.3(3).

6 Proof of Observation 1.4 and Theorem 1.5

Proof of Observation 1.4 Suppose $X=X_1\cup X_2, Y=X_1\cap X_2$ with X_1, X_2 and Y closed subsets of X with X, X_1, X_2 and Y all compact ANRs. Suppose $\xi\in H^1(X;\mathbb{Z})$ and let ξ_1, ξ_2 and ξ_o be the pullbacks of ξ on X_1, X_2 and Y and let $\tilde X$, $\tilde X_1$, $\tilde X_2$ and $\tilde Y$ be the infinite cyclic cover of ξ, ξ_1, ξ_2 and ξ_0. Note that $\tilde X=\tilde X_1\cup\tilde X_2$ $\tilde X_1\cap\tilde X_2=\tilde Y$ and then the long exact sequence in homology

$$\cdots\to H_r(\tilde Y)\to H_r(\tilde X_1)\oplus H_r(\tilde X_1)\to H_r(\tilde X)\to H_{r-1}(\tilde Y)\to\cdots$$

is a sequence of $\kappa[t^{-1},t]$–modules with all arrows $\kappa[t^{-1},t]$–linear. If $H^N(Y;\xi_0)=0$ then $\beta^N_r(X_1;\xi_1)+\beta^N_r(X_2;\xi_2)=\beta^N_r(X;\xi)$.

We apply this to the double $X=DM=M_1\cup_{\partial M}M_2$ with M_1 equal to M and M_2 equal to the manifold M with the opposite orientation and $\xi_D\in H^1(DM;\mathbb{Z})$ a cohomology class which restricts to ξ on M_1 and M_2, respectively. Clearly $\beta^N_r(DM;\xi_D)=2\beta^N_r(M;\xi)$. Since DM is closed and orientable, and consequently satisfies $\beta^N_r(DM;\xi_D)=\beta^N_{n-r}(DM;\xi_D)$, the statement follows. □

Proof of Theorem 1.5 Items (1) and (2a) follow from Observation 1.4 and the fact that both Betti numbers and Novikov–Betti numbers calculate the same Euler–Poincaré characteristic; see [6] or [13].

Items (1) and (2b–c) follow from Proposition 4.1 in [4], which calculates $H_r(X;(\xi,u))$.

Item (3) follows from Theorem 1.4 in [4]. Indeed the hypotheses imply the existence of a tame map $f\colon M \to \mathbb{S}^1$ with a given angle θ a regular value and $V = f^{-1}(\theta)$. Since the homology of V is trivial in all dimensions but zero, the relation R^θ_r $r \neq 0$ is the trivial relation and $R^\theta_0 = \mathrm{id}_\kappa$. The statement follows from the description of Jordan cells in terms of linear relations R^θ_r given by Theorem 1.4 in [4]. □

As pointed out to us by Maxim, the complement $X = \mathbb{C}^n \setminus V$ of a complex hypersurface $V \subset \mathbb{C}^n, V := \{(z_1, z_2, \dots, z_n) \mid f(z_1, z_2, \dots, z_n) = 0\}$ regular at infinity, equipped with the canonical class $\xi_f \in H^1(X : \mathbb{Z})$ defined by $f\colon X \to \mathbb{C} \setminus 0$ is an example of an open manifold with an integral cohomology class which has as compactification a manifold with boundary with a cohomology class which satisfies the hypotheses above.

Item (1) recovers a calculation of Maxim; see [11; 7][11] that the complement of an algebraic hypersurface regular at infinity has vanishing Novikov homologies in all dimension but n.

Appendix: Poincaré duality for nonclosed manifolds derived from $\tilde{M}$

Consistent with the previous notation let $\tilde{M}(a,b)$ and $\tilde{M}(c)$ denote the compact set $\tilde{f}^{-1}([a,b])$ and $\tilde{f}^{-1}(a)$ which for a, b and c regular values are submanifolds of $\tilde{M}$ (the first, $M(a,b)$, is a manifold with boundary $\partial M(a,b) = M(a) \sqcup M(b)$). We also recall that $\tilde{M}_a = \tilde{f}^{-1}((-\infty, a])$ and $\tilde{M}^b = \tilde{f}^{-1}([b,\infty))$.

Note that the Poincaré duality for bordisms provides the isomorphisms

$$\begin{aligned} &\mathrm{PD}(-l,a)\colon H_r(\tilde{M}(-l,a), \tilde{M}(-l)) \to H^{n-r}(\tilde{M}(-l,a), \tilde{M}(a)) && \text{for } -l<a,\\ &\mathrm{PD}(b,t)\colon H_r(\tilde{M}(b,t), \tilde{M}(t)) \to H^{n-r}(\tilde{M}(b,t), \tilde{M}(b)) && \text{for } t>b,\\ &\mathrm{PD}(-l,t)\colon H_r(\tilde{M}(-l,t), \tilde{M}(-l) \sqcup \tilde{M}(t)) \to H^{n-r}(\tilde{M}(-l,+t)) && \text{for } t,l>0.\end{aligned} \tag{34}$$

Combining with the excision property in homology or cohomology and passing to the limit when $0 < l \to \infty$, and $0 < l,t \to \infty$, one derives the Poincaré duality

[11]The Friedl–Maxim results state the vanishing of more general and more sophisticated L_2–homologies and Novikov-type homologies. They can be also recovered via the appropriate Poincaré duality isomorphisms.

isomorphisms

$$
\begin{aligned}
&\mathrm{PD}^1_a\colon H^{BM}_r(\tilde M_a)\to H^{n-r}(\tilde M,\tilde M^a),\\
&\mathrm{PD}^b_1\colon H^{BM}_r(\tilde M^b)\to H^{n-r}(\tilde M,\tilde M_b),\\
&\mathrm{PD}\colon H^{BM}_r(\tilde M)\to H^{n-r}(\tilde M),\\
&\mathrm{PD}^b_2\colon H^{BM}_r(\tilde M,\tilde M^b)\to H^{n-r}(\tilde M_b),\\
&\mathrm{PD}^2_a\colon H^{BM}_r(\tilde M,\tilde M_a)\to H^{n-r}(\tilde M^a),
\end{aligned}
\tag{35}
$$

where

$$
\begin{gathered}
\mathrm{PD}^1_a=\varprojlim_{l\to\infty}\mathrm{PD}(-l,a),\quad \mathrm{PD}^b_1=\varprojlim_{t\to\infty}\mathrm{PD}(b,t),\\
\mathrm{PD}=\varprojlim_{l\to\infty,l\to\infty}\mathrm{PD}(-l,t)\\
\mathrm{PD}^b_2=\varprojlim_{l\to\infty,t=b}\mathrm{PD}(-l,t),\quad \mathrm{PD}^a_2=\varprojlim_{t\to\infty,-l=a}\mathrm{PD}(-l,t).
\end{gathered}
\tag{36}
$$

These are the Poincaré duality isomorphisms which appear in the diagrams (24) and (25).

For example, in the case of the first isomorphism in (35),

$$
\begin{aligned}
H^{BM}(\tilde M_a)&=\varprojlim_{l\to\infty}H_r(\tilde M(-l,a),\tilde M(-l)),\\
H^{n-r}(\tilde M,\tilde M^a)=H^{n-r}(\tilde M_a,\tilde M(a))&=\varprojlim_{l\to\infty}H^{n-r}(\tilde M(-l,a),\tilde M(a)),
\end{aligned}
$$

where the passage from l to l' with $l'>l$ in the first equality above is derived from the commutative diagram

$$
\begin{tikzcd}
H_r(\tilde M(-l,a),\tilde M(-l)) \arrow[r] \arrow[d, "="'] & H^{n-r}(\tilde M(-l,a),\tilde M(a)) \\
H_r(\tilde M(-l',a),\tilde M(-l',-l)) & \\
H_r(\tilde M(-l',a),\tilde M(-l')) \arrow[r] \arrow[u] & H^{n-r}(\tilde M(-l',a),\tilde M(a)) \arrow[uu]
\end{tikzcd}
$$

References

[1] **D Burghelea**, *Linear relations, monodromy and Jordan cells of a circle valued map*, preprint (2015) arXiv

[2] **D Burghelea**, *A refinement of Betti numbers and homology in the presence of a continuous function, I*, Algebr. Geom. Topol. 17 (2017) 2051–2080 MR

[3] **D Burghelea**, **T K Dey**, *Topological persistence for circle-valued maps*, Discrete Comput. Geom. 50 (2013) 69–98 MR

[4] **D Burghelea**, **S Haller**, *Topology of angle valued maps, bar codes and Jordan blocks*, J. Appl. and Comput. Topology 1 (2017)

[5] **T A Chapman**, *Lectures on Hilbert cube manifolds*, CMBS Regional Conference Series in Mathematics 28, Amer. Math. Soc., Providence, RI (1976) MR

[6] **M Farber**, *Topology of closed one-forms*, Mathematical Surveys and Monographs 108, Amer. Math. Soc., Providence, RI (2004) MR

[7] **S Friedl**, **L Maxim**, *Twisted Novikov homology of complex hypersurface complements*, Math. Nachr. 290 (2017) 604–612 MR

[8] **S-T Hu**, *Theory of retracts*, Wayne State Univ. Press, Detroit (1965) MR

[9] **W Lück**, *Hilbert modules and modules over finite von Neumann algebras and applications to L^2–invariants*, Math. Ann. 309 (1997) 247–285 MR

[10] **W Lück**, *Dimension theory of arbitrary modules over finite von Neumann algebras and L^2–Betti numbers, I: Foundations*, J. Reine Angew. Math. 495 (1998) 135–162 MR

[11] **L Maxim**, *L^2–Betti numbers of hypersurface complements*, Int. Math. Res. Not. 2014 (2014) 4665–4678 MR

[12] **S P Novikov**, *Quasiperiodic structures in topology*, from "Topological methods in modern mathematics" (L R Goldberg, A V Phillips, editors), Publish or Perish, Houston, TX (1993) 223–233 MR

[13] **A V Pajitnov**, *Circle-valued Morse theory*, De Gruyter Studies in Mathematics 32, de Gruyter, Berlin (2006) MR

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