Algebraic & Geometric Topology 18 (2018) 2875–2895

The action of matrix groups on aspherical manifolds

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Let $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ be the special linear group and M^r be a closed aspherical manifold. It is proved that when $r < n$, a group action of $\mathrm{SL}_n(\mathbb{Z})$ on M^r by homeomorphisms is trivial if and only if the induced group homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi_1(M))$ is trivial. For (almost) flat manifolds, we prove a similar result in terms of holonomy groups. In particular, when $\pi_1(M)$ is nilpotent, the group $\mathrm{SL}_n(\mathbb{Z})$ cannot act nontrivially on M when $r < n$. This confirms a conjecture related to Zimmer's program for these manifolds.

57S25, 57S20; 57S17

1 Introduction

Let $\mathrm{SL}_n(\mathbb{Z})$ be the special linear group over the integers. The linear transformations of $\mathrm{SL}_n(\mathbb{Z})$ on the Euclidean space $\mathbb{R}^n$ induces a natural group action on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Note that T^n is an aspherical manifold, ie the universal cover is contractible. It is believed that this action is minimal in the following sense:

Conjecture 1.1 *Any group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on a closed aspherical r–manifold M^r by homeomorphisms factors through a finite group if $r < n$.*

This conjecture is related to Zimmer's program concerning group action of lattices in Lie groups on manifolds (see the survey articles Fisher [15], Weinberger [29] and Zimmer and Morris [33] for more details). A relevant conjecture is proposed by Farb and Shalen [14]: any smooth action of a finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ $n \geq 3$ on a compact r–manifold factors through a finite group action if $r < n-1$. Compared with Farb and Shalen's conjecture, Conjecture 1.1 considers topological actions and the condition is generalized to $r < n$, but only for aspherical manifolds. When $M = S^1$, Conjecture 1.1 is already proved by Witte [30]. Weinberger [28] confirms the conjecture when M is a torus. For $C^{1+\beta}$–group actions of a finite-index subgroup in $\mathrm{SL}_n(\mathbb{Z})$, one of the results proved by Brown, Rodriguez Hertz and Wang [8] confirms Conjecture 1.1 for surfaces when $r < n-1$. For C^2–group actions of cocompact lattices, Brown, Fisher and Hurtado [7] confirm Conjecture 1.1 when $r < n-1$. Note that the C^0–actions could be very different from smooth actions. It seems very few other cases have been confirmed (for group actions preserving additional structures, many results have been obtained; see [15; 33]).

For a group G, denote by $\mathrm{Out}(G)$ the outer automorphism group. Our first result is the following:

Theorem 1.2 *Let M^r be an aspherical manifold. A group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on M^r with $r \leq n-1$ by homeomorphisms is trivial if and only if the induced group homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi_1(M))$ is trivial. In particular, Conjecture 1.1 holds if the set of group homomorphisms is*

$$\mathrm{Hom}\big(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Out}(\pi_1(M))\big) = 1.$$

An obvious application is the following:

Corollary 1.3 *Any group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on an aspherical manifold M^k with $k \leq n-1$ by homotopic-to-the-identity homeomorphisms is trivial.*

For aspherical manifolds with finitely generated nilpotent fundamental groups (eg Nil-manifolds), we confirm Conjecture 1.1 as follows:

Theorem 1.4 *Let M^r be an aspherical manifold. If the fundamental group $\pi_1(M)$ is finitely generated nilpotent, any group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on M^r with $r \leq n-1$ by homeomorphisms is trivial.*

We now study group actions on (almost) flat manifolds. Recall that a closed manifold M is almost flat if for any $\varepsilon > 0$ there is a Riemannian metric g_ε on M such that $\mathrm{diam}(M, g_\varepsilon) < 1$ and g_ε is ε–flat.

Theorem 1.5 *Let M^r be a closed almost flat manifold with holonomy group Φ. A group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on M^r with $r \leq n-1$ by homeomorphisms is trivial if and only if the induced group homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\Phi)$ is trivial. In particular, Conjecture 1.1 holds if the set of group homomorphisms is*

$$\mathrm{Hom}(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Out}(\Phi)) = 1.$$

Surprisingly, the proof of Theorem 1.5 will use knowledge in algebraic K–theory (Steinberg groups $\mathrm{St}_n(\mathbb{Z})$ and $K_2(\mathbb{Z})$, especially). The usual Zimmer's program is stated for any lattices in high-rank semisimple Lie groups. However, Theorems 1.2, 1.4

and 1.5 cannot hold for general lattices. For example, the congruence subgroup $\Gamma(n, p)$, which is defined as the kernel of $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/p)$ for a prime p, has a nontrivial finite cyclic quotient group (see [22, Theorem 1.1]). The group $\Gamma(n, p)$ could act on S^1 through the cyclic group by rotations.

In order to confirm Conjecture 1.1, it's enough to show that every group homomorphism from $\mathrm{SL}_n(\mathbb{Z})$ to the outer automorphism group of the fundamental (or holonomy) group is trivial, by Theorems 1.2 and 1.5. Actually, Conjecture 1.1 could be confirmed in this way for many other manifolds in addition to manifolds with nilpotent fundamental groups proved in Theorem 1.4. These include the following:

- Flat manifolds with abelian holonomy group (see Corollary 6.4 for a more general result).
- Almost flat manifolds with dihedral, symmetric or alternating holonomy group (see Lemma 6.5).
- Flat manifolds of dimension $r \leq 5$ (see Corollary 6.6).

The article is organized as follows. In Section 2, we study the group action of a Steinberg group on spheres and acyclic manifolds. In Section 3, we give a proof of Theorem 1.2. Theorem 1.4 is proved in Section 4. In Section 5, we study group actions on flat manifolds and Theorem 1.5 is proved. In the last section, we give some applications to flat manifolds with special holonomy groups.

2 The action of Steinberg groups on spheres and acyclic manifolds

2.1 Steinberg group

For a unitary associative ring R, the Steinberg group $\mathrm{St}_n(R)$ with $n \geq 3$ is generated by $x_{ij}(r)$ for $1 \leq i, j \leq n$ and $r \in R$ subject to the relations

(i) $x_{ij}(r_1) \cdot x_{ij}(r_2) = x_{ij}(r_1 + r_2)$;

(ii) $[x_{ij}(r_1), x_{jk}(r_2)] = x_{ik}(r_1 r_2)$;

(iii) $[x_{ij}(r_1), x_{pq}(r_2)] = 1$ if $i \neq p$ and $j \neq q$.

Let $R = \mathbb{Z}$, the integers. There is a natural group homomorphism $f\colon \mathrm{St}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z})$ mapping $x_{ij}(r)$ to the matrix $e_{ij}(r)$, which is a matrix with ones along the diagonal, r in the $(i, j)^{\text{th}}$ position and zeros elsewhere. Let $\omega_{ij}(-1) = x_{ij}(-1)x_{ji}(1)x_{ij}(-1)$,

$h_{ij} = \omega_{ij}(-1)\omega_{ij}(-1)$ and $a = h_{12}^2$. We call a a Steinberg symbol, denoted by $\{-1,-1\}$ usually.

Lemma 2.1 (Milnor [24, Theorem 10.1]) *For $n \geq 3$, the group $\mathrm{St}_n(\mathbb{Z})$ is a central extension*

$$1 \to K \to \mathrm{St}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}) \to 1,$$

where K is the cyclic group of order 2 generated by $a = (x_{12}(-1)x_{21}(1)x_{12}(-1))^4$.

Lemma 2.2 *For distinct integers i, j, s and t, we have the following:*

(i) $[h_{ij}, h_{st}] = 1$.

(ii) $[h_{ij}, h_{is}] = a$.

(iii) *The subgroup $\langle h_{ij}, h_{is}\rangle$ is isomorphic to the quaternion group Q_8.*

Proof (i) follows from the third Steinberg relation easily. (ii) is Milnor [24, Lemma 9.7, page 74]. A direct computation shows that h_{ij}, h_{is} and $h_{ij}h_{is}$ are all elements of order 4. Considering (ii), $\langle h_{ij}, h_{is}\rangle$ is isomorphic to the quaternion group. □

Denote by q: $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/n)$ or $\mathrm{St}_n(\mathbb{Z}) \to \mathrm{St}_n(\mathbb{Z}/n)$ the group homomorphism induced by the ring homomorphism $\mathbb{Z} \to \mathbb{Z}/n$ for some integer n. Let $\mathrm{SL}_n(\mathbb{Z}, n\mathbb{Z}) = \ker(\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/n))$ and $\mathrm{St}_n(\mathbb{Z}, n\mathbb{Z}) = \ker(\mathrm{St}_n(\mathbb{Z}) \to \mathrm{St}_n(\mathbb{Z}/n))$ be the congruence subgroups.

Lemma 2.3 *Let N be a normal subgroup of $\mathrm{St}_n(\mathbb{Z})$. If $f(N)$ contains $\mathrm{SL}_n(\mathbb{Z}, 2\mathbb{Z})$, then N contains the element a. In particular, the normal subgroup generated by h_{ij} with $n \geq 3$ or $h_{ij}h_{st}$ with $n \geq 5$ contains the element a for distinct integers i, j, s and t.*

Proof Recall that $e_{ij}(r)$ is a matrix with ones along the diagonal, r in the $(i,j)^{\text{th}}$ position and zeros elsewhere. Since $f(N)$ contains $\mathrm{SL}_n(\mathbb{Z}, 2\mathbb{Z})$, we have $x_{pq}(2)$ or $a \cdot x_{pq}(2) \in N$ for any integers $p \neq q \in [1, n]$ (the interval). Note that $\mathrm{St}_n(\mathbb{Z}, 2\mathbb{Z})$ is normally generated by $x_{pq}(2)$ (see Magurn [23, 13.18, page 448]). Therefore, $\mathrm{St}_n(\mathbb{Z}, 2\mathbb{Z})$ or $a \cdot \mathrm{St}_n(\mathbb{Z}, 2\mathbb{Z}) \subset N$. However, it is known that the Steinberg symbol $q(a) \in \mathrm{St}_n(\mathbb{Z}/2)$ is trivial (see Milnor [24, Corollary 9.9, page 75]). Thus, $a \in a \cdot \mathrm{St}_n(\mathbb{Z}, 2\mathbb{Z}) = \mathrm{St}_n(\mathbb{Z}, 2\mathbb{Z}) \subset N$. The image $f(h_{ij}) = \mathrm{diag}(1, \dots, -1, \dots, -1, \dots, 1)$ (or $f(h_{ij}h_{st})$) normally generates the congruence subgroup $\mathrm{SL}_n(\mathbb{Z}, 2\mathbb{Z})$ (see Ye [32]). The proof is finished. □

2.2 Homology manifolds

The generalized manifolds studied in this section are following Bredon's book [5]. Let $L=\mathbb{Z}$ or $\mathbb{Z}/p$. All homology groups in this section are Borel–Moore homology with compact supports and coefficients in a sheaf $\mathcal{A}$ of modules over a principal ideal domain L. The homology groups of X are denoted by $H_*^c(X;\mathcal{A})$ and the Alexander–Spanier cohomology groups (with coefficients in L and compact supports) are denoted by $H_c^*(X;L)$. We define $\dim_L X=\min\{n \mid H_c^{n+1}(U;L)=0 \text{ for all open } U\subset X\}$. If $L=\mathbb{Z}/p$, we write $\dim_p X$ for $\dim_L X$. For an integer $k\geq 0$, let $\mathcal{O}_k$ denote the sheaf associated to the presheaf $U\mapsto H_k^c(X,X\backslash U;L)$.

Definition 2.4 An n–dimensional homology manifold over L (denoted by n–hm_L) is a locally compact Hausdorff space X with $\dim_L X<+\infty$ and $\mathcal{O}_k(X;L)=0$ for $p\neq n$ and $\mathcal{O}_n(X;L)$ locally constant with stalks isomorphic to L. The sheaf $\mathcal{O}_n$ is called the orientation sheaf.

There is a similar notion of cohomology manifold over L, denoted by n–cm_L (see [5, page 373]). For a prime p, denote by $\dim_p X$ the cohomological dimension of X.

Definition 2.5 If X is a compact m–hm_L and $H_*^c(X;L)\cong H_*^c(S^m;L)$, then X is called a generalized m–sphere over L. Similarly, if $H_0^c(X;L)=L$ and $H_k^c(X;L)=0$ for $k>0$, then X is said to be L–acyclic.

We will need the following lemmas. The first is a combination of [5, Corollaries 19.8 and 19.9, page 144] (see also [6, Theorem 4.5]).

Lemma 2.6 *Let p be a prime and X be a locally compact Hausdorff space of finite dimension over $\mathbb{Z}_p$. Suppose that $\mathbb{Z}_p$ acts on X with fixed-point set F.*

(i) *If $H_*^c(X;\mathbb{Z}_p)\cong H_*^c(S^m;\mathbb{Z}_p)$, then $H_*^c(F;\mathbb{Z}_p)\cong H_*^c(S^r;\mathbb{Z}_p)$ for some r with $-1\leq r\leq m$. If p is odd, then $r-m$ is even.*

(ii) *If X is $\mathbb{Z}_p$–acyclic, then F is $\mathbb{Z}_p$–acyclic* (in particular, nonempty and connected).

The following is a relation between dimensions of the fixed-point set and the whole space (see Borel [3, Theorem 4.3, page 182]):

Lemma 2.7 *Let G be an elementary p–group operating on a first-countable cohomology n–manifold X mod p. Let $x\in X$ be a fixed point of G on X and let $n(H)$*

be the cohomology dimension mod p of the component of x in the fixed-point set of a subgroup H of G. If $r = n(G)$, we have

$$n - r = \sum_H (n(H) - r),$$

where H runs through the subgroups of G of index p.

The following lemma is proved by Bredon [4, Theorem 7.1]:

Lemma 2.8 *Let G be a group of order 2 operating effectively on an n–cm over $\mathbb{Z}$, with nonempty fixed points. Let F_0 be a connected component of the fixed-point set of G and $r = \dim_2 F_0$. Then $n - r$ is even (respectively odd) if and only if G preserves (respectively reverses) the local orientation around some point of F_0.*

2.3 Steinberg group acting on $\mathbb{R}^n$ and S^n

Lemma 2.9 *Let X be a generalized m–sphere over $\mathbb{Z}/2$ (resp. a $\mathbb{Z}/2$–acyclic m–$\mathrm{hm}_{\mathbb{Z}/2}$). Suppose that τ is an involution of X and F is a closed τ–invariant submanifold. If F containing $\mathrm{Fix}(\tau)$ is an $(m-1)$–sphere (resp. a $\mathbb{Z}/2$–acyclic $(m-1)$–$\mathrm{hm}_{\mathbb{Z}/2}$), then $X \backslash F$ has two $\mathbb{Z}/2$–acyclic components and τ interchanges them.*

Proof The proof is exactly the same as that of Bridson and Vogtmann [6, Lemma 4.11], where $F = \mathrm{Fix}(\tau)$. □

We now study the group action of Steinberg groups $\mathrm{St}_n(\mathbb{Z})$ on spheres and acyclic manifolds. Compared with the proof of actions of $\mathrm{SL}_n(\mathbb{Z})$, there are not enough involutions in $\mathrm{St}_n(\mathbb{Z})$. Note that the element $h_{ij} \in \mathrm{St}_n(\mathbb{Z})$ corresponding to the involution $\mathrm{diag}(1, \dots, -1, \dots, -1, \dots, 1) \in \mathrm{SL}_n(\mathbb{Z})$ is of order 4. Moreover, h_{ij} and h_{is} do not commute with each other. All these facts make the study of the actions of $\mathrm{St}_n(\mathbb{Z})$ difficult and the proof for the action of $\mathrm{SL}_n(\mathbb{Z})$ presented in [6] could not be carried to study that of $\mathrm{St}_n(\mathbb{Z})$ easily.

Theorem 2.10 *We have the following:*

(i) *Any group action of $\mathrm{St}_n(\mathbb{Z})$ with $n \geq 4$ on a generalized k–sphere M^k over $\mathbb{Z}/2$ with $k \leq n-2$ by homeomorphisms is trivial.*

(ii) *Any group action of $\mathrm{St}_n(\mathbb{Z})$ with $n \geq 4$ on a $\mathbb{Z}/2$–acyclic k–$\mathrm{hm}_{\mathbb{Z}/2}$ M^k with $k \leq n-1$ by homeomorphisms is trivial.*

Proof Let $a = x_{12}(1)x_{21}(-1)x_{12}(1))^4$ as in Lemma 2.1.

Case 1 If a acts trivially on M^k, the group action of $\mathrm{St}_n(\mathbb{Z})$ factors through an action of $\mathrm{SL}_n(\mathbb{Z})$. However, it is proved by Bridson and Vogtmann [6] that the group action of $\mathrm{SL}_n(\mathbb{Z})$ is trivial.

Case 2 Suppose now that a acts nontrivially, ie $\mathrm{Fix}(a) \neq \varnothing$ and $\mathrm{Fix}(a) \neq M^k$. Since $\mathrm{St}_n(\mathbb{Z})$ with $n \geq 3$ is perfect, every element acts by an orientation-preserving homeomorphism. Bredon's result (see Lemma 2.8) shows that $\mathrm{Fix}(a)$ is of even dimension. If $\dim_2 \mathrm{Fix}(a) = k$, then $\mathrm{Fix}(a) = M^k$ by invariance of domain. This is a contradiction to the fact that a acts nontrivially. Therefore,

$$\dim_2 \mathrm{Fix}(a) \leq k - 2.$$

Note that $h_{12}h_{34}$ is of order 2 and $A := \langle a, h_{12}h_{34}\rangle$ is isomorphic to $(\mathbb{Z}/2)^2$. Write $r = \dim_2(\mathrm{Fix}(A))$ and $n(H) = \dim_2(\mathrm{Fix}(H))$ for each nontrivial cyclic subgroup $H < A$. By Borel's formula (see Lemma 2.7),

$$k - r = \sum n(H) - r, \tag{1}$$

where H ranges over the nontrivial subgroups of index 2. Since a is in the center of $\mathrm{St}_n(\mathbb{Z})$, there is a group action of $\mathrm{SL}_n(\mathbb{Z})$ on the acyclic $\mathbb{Z}/2$–manifold or generalized sphere $\mathrm{Fix}(a)$ induced by that of $\mathrm{St}_n(\mathbb{Z})$. This group action is trivial by Bridson and Vogtmann [6]. Thus,

$$n(\langle h_{12}h_{34}\rangle) \geq r = \dim_2 \mathrm{Fix}(a).$$

Case 2.1 If $n(\langle h_{12}h_{34}\rangle) = r$, we have $n(\langle a \cdot h_{12}h_{34}\rangle) = k$ by (1). By invariance of domain, $a \cdot h_{12}h_{34}$ acts trivially on M^k. Take $\omega = h_{12}\omega_{12}(-1)\omega_{34}(-1)$ and $C = \langle a, \omega\rangle$. Note that $\omega^2 = a \cdot h_{12}h_{34}$ and $f(\omega) = f(a \cdot \omega)$ has the form

$$\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix} \in \mathrm{SL}_4(\mathbb{Z}).$$

Therefore, C acts on M^k as a group isomorphic to $(\mathbb{Z}/2)^2$. If $\dim_2 \mathrm{Fix}(\omega) = k$ or $\dim_2 \mathrm{Fix}(a \cdot \omega) = k$, ie ω or $a \cdot \omega$ acts trivially on M^k, the normal subgroup in $\mathrm{St}_n(\mathbb{Z})$ generated by ω or $a \cdot \omega$ contains a by Lemma 2.3. This is a contradiction to the fact that $\mathrm{Fix}(a) \neq M^k$. Therefore, we may assume that $\dim_2 \mathrm{Fix}(\omega) \leq k-2$ and $\dim_2 \mathrm{Fix}(a \cdot \omega) \leq k-2$, considering Lemma 2.8. According to Borel's formula (1), we have $k \geq 4$ and $n \geq 5$. By invariance of domain, $\mathrm{Fix}(h_{12}h_{34}) = M^k$. The normal subgroup in $\mathrm{St}_n(\mathbb{Z})$ generated by $h_{12}h_{34}$ contains a by Lemma 2.3. Thus, $\mathrm{Fix}(a) = M^k$, which is impossible. Similarly, $n(\langle a \cdot h_{12}h_{34}\rangle) \neq r$.

Case 2.2 If $n(\langle h_{12}h_{34}\rangle) - r \geq 2$ and $n(\langle a \cdot h_{12}h_{34}\rangle) - r \geq 2$ (noting that $n(\langle h_{12}h_{34}\rangle) - r$ is even), then

$$k - r = 2(n(\langle h_{12}h_{34}\rangle) - r) \geq 4. \tag{2}$$

(Note that when $n \geq 5$, we have $h_{15}h_{12}h_{34}h_{15}^{-1} = a \cdot h_{12}h_{34}$ by Lemma 2.2 and thus $n(\langle h_{12}h_{34}\rangle) = n(\langle a \cdot h_{12}h_{34}\rangle)$.) When $k = 4$, we have $r = 0$ and $n(\langle h_{12}h_{34}\rangle) = 2$. Therefore, $\mathrm{Fix}(h_{12}h_{34})$ is S^2 or $\mathbb{R}^2$ (see [5, V.16.32, page 388]). If $\mathrm{Fix}(h_{12}h_{34}) = \mathbb{R}^2$, the quaternion group $\langle h_{12}, h_{13}\rangle$ acts on $\mathrm{Fix}(h_{12}h_{34})$ with a global fixed point in $\mathrm{Fix}(a)$. Since any finite group of orientation-preserving homeomorphisms of the plane that fix the origin is cyclic, $\langle h_{12}, h_{13}\rangle$ cannot act effectively. A nontrivial element in $\langle h_{12}, h_{13}\rangle$ will normally generate a group containing a by Lemma 2.3, which is impossible.

If $\mathrm{Fix}(h_{12}h_{34}) = S^2$, we have $n \geq 6$. Denote by $B = \langle a \cdot h_{12}h_{34}, a \cdot h_{34}h_{56}\rangle \cong (\mathbb{Z}/2)^2$. Note that $\mathrm{Fix}(a) \subset \mathrm{Fix}(B)$. Write $r' = \dim_2(\mathrm{Fix}(B))$ and $n(H) = \dim_2(\mathrm{Fix}(H))$ for each nontrivial cyclic subgroup $H < B$. By Borel's formula (see Lemma 2.7),

$$k - r' = \sum n(H) - r', \tag{3}$$

where H ranges over the nontrivial subgroup in B of index 2. Since any two nontrivial elements in B are conjugate (see [23, 12.20, page 418]), we get

$$k - r' = 3(n(\langle h_{12}h_{34}\rangle) - r') \tag{4}$$

and $r' = 1$. Therefore, $\mathrm{Fix}(a)$ is a submanifold of $\mathrm{Fix}(B)$ of codimension 1. By Lemma 2.9, a permutes the two components of $\mathrm{Fix}(B) \backslash \mathrm{Fix}(a)$. However, $h_{12}^2 = a$ and $h_{12}\mathrm{Fix}(B) = \mathrm{Fix}(B)$. This is impossible.

Similar arguments using (2) and (4) prove the following. When $k = 5$, we have $r = 1, n(\langle h_{12}h_{34}\rangle) = 3$ and $r' = 2$. When $k = 6$, we have $r = 2$, $n(\langle h_{12}h_{34}\rangle) = 4$ and $r' = 3$. When $k = 7$, we have $r = 3, n(\langle h_{12}h_{34}\rangle) = 5$ and $r' = 4$. In all these cases, $\mathrm{Fix}(a)$ is still a submanifold in $\mathrm{Fix}(B)$ of codimension 1. By Lemma 2.9, this is impossible. When $k = 8$, we have either $r = 4$, $n(\langle h_{12}h_{34}\rangle) = 6$ and $r' = 5$ or $r = 0$, $n(\langle h_{12}h_{34}\rangle) = 4$ and $r' = 2$. The former is impossible for the same reason as $k = 7$, while the latter is impossible for the same reason as $k = 4$.

When $k \geq 9$, we have $n \geq 10$. If $k - r' \geq 6$, then $\mathrm{St}_{n-6}(\mathbb{Z})$ generated by all $x_{ij}(r)$ with $7 \leq i \neq j \leq n$ acts on $\mathrm{Fix}(B)$. By Smith theory (see Lemma 2.6), $\mathrm{Fix}(B)$ is still a generalized sphere over $\mathbb{Z}/2$ or a $\mathbb{Z}/2$–acyclic $\mathrm{hm}_{\mathbb{Z}/2}$. An inductive argument shows that $\mathrm{St}_{n-6}(\mathbb{Z})$ acts trivially on $\mathrm{Fix}(B)$. Therefore, $\mathrm{Fix}(B)=\mathrm{Fix}(a)$. However, this is impossible considering formulas (2) and (4). If $k-r'\leq 5$, we have

$$3(n(\langle h_{12}h_{34}\rangle)-r')\leq 5.$$

When $n(\langle h_{12}h_{34}\rangle)-r'=0$, we have $\mathrm{Fix}(B)=M^k$. Then $h_{12}h_{34}$ acts trivially on M. The normal subgroup generated by $h_{12}h_{13}$ contains a (see Lemma 2.3), which means a acts trivially. This is a contradiction. When $n(\langle h_{12}h_{34}\rangle)-r'=1$, we have $k-r'=3$, $k-n(\langle h_{12}h_{34}\rangle)=2$ and $k-r=4$. Therefore, $\mathrm{Fix}(a)$ is a submanifold in $\mathrm{Fix}(B)$ of codimension 1, which is impossible as above by Lemma 2.9.

Case 3 Suppose $\mathrm{Fix}(a)=\varnothing$. According to the Lefschetz fixed-point theorem, this can only happen when M is a generalized sphere of odd dimension.

When $k=1$, we have $M=S^1$ (see [5, V.16.32, page 388]). The group $\langle h_{12},h_{13}\rangle$, which is isomorphic to the quaternion group Q_8 (see Lemma 2.2), acts freely on S^1. However, this is impossible since any finite subgroup of $\mathrm{Homeo}_+(S^1)$ is isomorphic to a subgroup of $\mathrm{SO}(2;\mathbb{R})$ (see Navas [25, Proposition 1.1.1]).

Assume $k=3$. Recall that $A:=\langle a,h_{12}h_{34}\rangle$ is isomorphic to $(\mathbb{Z}/2)^2$. By Smith theory, $(\mathbb{Z}/2)^2$ cannot act freely and thus $\mathrm{Fix}(h_{12}h_{34})$ is not empty. Bredon's result (see Lemma 2.8) shows that $\mathrm{Fix}(h_{12}h_{34})$ is of even dimension. If $\dim_2\mathrm{Fix}(h_{12}h_{34})=3$, then $h_{12}h_{34}$ acts trivially on M. The normal subgroup in $\mathrm{St}_n(\mathbb{Z})$ generated by $h_{12}h_{34}$ contains a, which is a contradiction to the fact that $\mathrm{Fix}(a)=\varnothing$. Therefore, $\mathrm{Fix}(h_{12}h_{34})=S^1$. Note that the quaternion group $\langle h_{12},h_{13}\rangle$ commutes with $h_{12}h_{34}$. Since a acts freely on $\mathrm{Fix}(h_{12}h_{34})$, so does $\langle h_{12},h_{13}\rangle$, which is impossible as well.

When $k=5$, take $B=\langle a\cdot h_{12}h_{34},a\cdot h_{34}h_{56}\rangle\cong(\mathbb{Z}/2)^2$. Since $\mathrm{Fix}(h_{12}h_{34})$ is a generalized sphere over $\mathbb{Z}/2$, $\langle a,ah_{34}h_{56}\rangle$ can not act freely on it. Therefore, $\mathrm{Fix}(B)\neq\varnothing$. By (4), we have

$$k-r'=3(n(\langle ah_{12}h_{34}\rangle)-r').$$

If $n(\langle ah_{12}h_{34}\rangle)=r'$, we have $k=r'$. Thus, $ah_{12}h_{34}$ acts trivially on M. The normal subgroup in $\mathrm{St}_n(\mathbb{Z})$ generated by $ah_{12}h_{34}$ contains a, a contradiction. Therefore, $r'=2$ and $n(\langle ah_{12}h_{34}\rangle)-r'=1$. By Lemma 2.9, a permutes the two components of $\mathrm{Fix}(ah_{12}h_{34})\backslash\mathrm{Fix}(B)$, which is impossible by noting that $h_{12}^2=a$.

When $k=7$, we may assume $n(\langle ah_{12}h_{34}\rangle)\neq r'$ as in the proof of the case when $k=5$. Considering formula (2), we have either $r'=4$ and $n(\langle ah_{12}h_{34}\rangle)=5$ or

$r' = 1$ and $n(\langle ah_{12}h_{34}\rangle) = 3$. For the former, apply Lemma 2.9 to get a contradiction. For the latter, the quaternion group $\langle h_{12}, h_{13}\rangle$ acts on $\mathrm{Fix}(B) = S^1$ freely, which is impossible as in the case of $k = 3$.

Suppose that $k \geq 9$. If $k - r' \geq 6$, the subgroup $\mathrm{St}_{n-6}(\mathbb{Z})$ generating by all $x_{ij}(r)$ with $7 \leq i \neq j \leq n$ acts trivially on $\mathrm{Fix}(B)$ by an inductive argument. This is a contradiction to the fact that $\mathrm{Fix}(a) = \varnothing$. If $k - r' \leq 5$, then $n(\langle h_{12}h_{34}\rangle) - r' = 0$ or 1. If $n(\langle h_{12}h_{34}\rangle) = r'$, then $k = r'$ and thus $\mathrm{Fix}(B) = M^k$. Then $h_{12}h_{34}$ acts trivially on M. The normal subgroup generated by $h_{12}h_{13}$ contains a (see Lemma 2.3), which is a contradiction to the fact that $\mathrm{Fix}(a) = \varnothing$. If $n(\langle h_{12}h_{34}\rangle) - r' = 1$, the element a permutes $\mathrm{Fix}(h_{12}h_{34}) \backslash \mathrm{Fix}(B)$ by Lemma 2.9. This is impossible by noting that $h_{12}^2 = a$. The whole proof is finished. □

Corollary 2.11 *Any group homomorphism* $f\colon \mathrm{St}_n(\mathbb{Z}) \to \mathrm{GL}_k(\mathbb{Z})$ *with* $n \geq 3$ *and* $k \leq n-1$ *is trivial.*

Proof When $k = 1$, $\mathrm{GL}_k(\mathbb{Z})$ is abelian. Since $\mathrm{St}_n(\mathbb{Z})$ is perfect, f is trivial. When $k = 2$, f has its image in $\mathrm{SL}_2(\mathbb{Z})$. Note that the projective linear group factors as $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, a free product. Thus, $\mathrm{SL}_2(\mathbb{Z})$ does not have nontrivial perfect subgroup (see [2, 5.8, page 48]). This means that f is trivial. The group $\mathrm{GL}_k(\mathbb{Z})$ acts naturally on the Euclidean space $\mathbb{R}^k$ by linear transformations. When $k \geq 3$, Theorem 2.10 implies that the image $\mathrm{Im}\, f$ acts trivially on $\mathbb{R}^k$. Therefore, $\mathrm{Im}\, f = 1$. □

3 Proof of Theorem 1.2

We need the following lemma:

Lemma 3.1 *Denote by Q a quotient group of $\mathrm{SL}_n(\mathbb{Z})$. Let π be a torsion-free abelian group. For any $n \geq 3$, the second cohomology group is*

$$H^2(Q; \pi) = 0.$$

Proof By van der Kallen [19], the second homology group is $H_2(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2$ when $n \geq 5$ and

$$H_2(\mathrm{SL}_3(\mathbb{Z}); \mathbb{Z}) = H_2(\mathrm{SL}_4(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Since $\mathrm{SL}_n(\mathbb{Z})$ is perfect, $H_1(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Z}) = 0$ for any $n \geq 3$. By the universal coefficient theorem, $H^2(\mathrm{SL}_n(\mathbb{Z}); \pi) = 0$ for any $n \geq 3$. Dennis and Stein proved that

$$H_2(\mathrm{SL}_n(\mathbb{Z}/k); \mathbb{Z}) = \mathbb{Z}/2 \quad \text{for } k \equiv 0 \bmod 4,$$

while $H_2(\mathrm{SL}_n(\mathbb{Z}/k);\mathbb{Z})=0$ otherwise (see [13, Corollary 10.2]). By the universal coefficient theorem again, $H^2(\mathrm{SL}_n(\mathbb{Z}/k);\pi)=0$ for any k. Let $f\colon \mathrm{SL}_n(\mathbb{Z})\to Q$ be a surjective homomorphism. By Margulis's normal subgroup theorem, every quotient Q is either $\mathrm{PSL}_n(\mathbb{Z})$ or a finite group. If $\ker f$ is trivial, $Q=\mathrm{SL}_n(\mathbb{Z})$ and thus $H^2(Q;\pi)=0$. If $\ker f$ is nontrivial, the congruence subgroup property [1] implies that Q is a quotient of $\mathrm{SL}_n(\mathbb{Z}/k)$ by a central subgroup K for some nonzero integer k. From the Serre spectral sequence

$$H^p(Q;H^q(K;\pi))\Rightarrow H^{p+q}(\mathrm{SL}_{n+1}(\mathbb{Z}/k);\pi),$$

we have the exact sequence

$$0\to H^1(Q;\pi)\to H^1(\mathrm{SL}_n(\mathbb{Z}/k);\pi)\to H^0(Q;H^1(K;\pi))\\ \to H^2(Q;\pi)\to H^2(\mathrm{SL}_n(\mathbb{Z}/k);\pi).$$

This implies $H^2(Q;\pi)=0$. □

Proof of Theorem 1.2 If $\mathrm{SL}_n(\mathbb{Z})$ acts trivially on M^r, it is obvious that the induced homomorphism $\mathrm{SL}_n(\mathbb{Z})\to\mathrm{Out}(\pi_1(M))$ is trivial. It is enough to prove the other direction. Denote by $\mathrm{Homeo}(M^r)$ the group of homeomorphisms of M^r. Suppose that $f\colon \mathrm{SL}_n(\mathbb{Z})\to\mathrm{Homeo}(M^k)$ is the group homomorphism. We have a group extension

$$(*)\qquad 1\to\pi_1(M)\to G'\to\mathrm{Im}\, f\to 1,$$

where $\widetilde{M}$ is the universal cover of M and G' is a subgroup of $\mathrm{Homeo}(\widetilde{M})$. Note that there is a one-to-one correspondence between the equivalence classes of extensions and the second cohomology group $H^2(\mathrm{Im}\, f;C(\pi))$, where $C(\pi)$ is the center of $\pi_1(M)$ (see [10, Theorem 6.6, page 105]). By the assumption that the group homomorphism $\mathrm{SL}_n(\mathbb{Z})\to\mathrm{Out}(\pi_1(M))$ is trivial, $\mathrm{Im}\, f$ acts trivially on the center $C(\pi)$. Since M is aspherical, $\pi=\pi_1(M)$ is torsion-free and the center $C(\pi)$ is torsion-free as well. By Lemma 3.1, $H^2(\mathrm{Im}\, f;C(\pi))=0$, which implies that the exact sequence $(*)$ is split. Therefore, $\mathrm{Im}\, f$ is isomorphic to a subgroup of G', which implies that the group $\mathrm{SL}_n(\mathbb{Z})$ and thus $\mathrm{St}_n(\mathbb{Z})$ could act on the acyclic manifold $\widetilde{M}$ through $\mathrm{Im}\, f$. However, Bridson and Vogtmann [6] prove that any action $\mathrm{SL}_n(\mathbb{Z})$ on $\widetilde{M}$ is trivial. (for self-containedness, we may apply Theorem 2.10 to get that any group action of $\mathrm{St}_n(\mathbb{Z})$ with $n\geq 4$ on the acyclic manifold $\widetilde{M}$ is trivial. When $n=3$, for each integer $2\leq i\leq n$, denote by A_i the diagonal matrix $\mathrm{diag}(-1,\dots,-1,\dots,1)$, where the second -1 is the i^{th} entry. The subgroup $G:=\langle A_2,A_3\rangle<\mathrm{SL}_3(\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2)^2$. By Smith theory (see Lemma 2.6), the group action of G on $\widetilde{M}$ has a global fixed point. The Borel formula (see Lemma 2.7) implies that the action of G on $\widetilde{M}$ is trivial. Therefore, the group action of $\mathrm{SL}_3(\mathbb{Z})$ on $\widetilde{M}$ factors through the projective linear group $\mathrm{PSL}_3(\mathbb{Z}/2)$. Using Smith theory and the Borel formula once again for the subgroup $(\mathbb{Z}/2)^2 \cong \langle e_{12}(1), e_{13}(1)\rangle < \mathrm{PSL}_3(\mathbb{Z}/2)$, we see that the group action of $\mathrm{PSL}_3(\mathbb{Z}/2)$ and thus $\mathrm{SL}_3(\mathbb{Z})$ on $\widetilde{M}$ is trivial.) This implies $\mathrm{Im}\, f$ is trivial, ie the group action of $\mathrm{SL}_n(\mathbb{Z})$ on M is trivial. □

4 Aspherical manifolds with nilpotent fundamental groups

Let G be a group. Denote by $Z_1 = Z(G)$ the center. Inductively, define $Z_{i+1}(G) = p_i^{-1} Z(G/Z_{i-1}(G))$, where $p_i\colon G \to G/Z_i(G)$ is the quotient group homomorphism. We have the upper central sequence

$$1 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_i \subset \cdots .$$

If $Z_i = G$ for some i, we call G a nilpotent group. For two groups G and H, denote by $\mathrm{Hom}(G, H)$ the set of group homomorphisms from G to H.

Lemma 4.1 *Let*

$$1 \to N \to \pi \xrightarrow{q} Q \to 1$$

be a central extension, ie an exact sequence with $N < Z(\pi)$. Suppose that G is a group with the second cohomology group $H^2(G; N) = 0$, where G acts on N trivially. Then

$$\mathrm{Hom}(G, \pi) \xrightarrow{q_*} \mathrm{Hom}(G, Q)$$

is surjective.

Proof This is an easy exercise in homological algebra. For completeness, we give a proof here. The central extension gives a principal fibration $\mathrm{B}N \to \mathrm{B}\pi \to \mathrm{B}Q$ and thus a fibration

$$\mathrm{B}\pi \to \mathrm{B}Q \xrightarrow{h} K(N, 2),$$

where $\mathrm{B}(-)$ is a classifying space and $K(N, 2)$ is a simply connected CW complex with the second homotopy group N and all other homotopy groups trivial (see [2, 8.2, page 64]). Let $\alpha\colon G \to Q$ be any group homomorphism. The composite

$$\mathrm{B}G \xrightarrow{\mathrm{B}\alpha} \mathrm{B}Q \xrightarrow{h} K(N, 2)$$

is null-homotopic, by the assumption that $H^2(G;N)=0$. Therefore, α could be lifted to a group homomorphism $\alpha'\colon G\to\pi$. □

Lemma 4.2 *Let π be a group with center $Z=Z(\pi)$. Suppose that one of the following holds:*

(i) *G is a perfect group with $H_2(G;\mathbb{Z})$ finite, and π and π/Z are torsion-free; or*

(ii) *G is a perfect group with $H_2(G;\mathbb{Z})=0$.*

If the sets of group homomorphisms are

$$\mathrm{Hom}(G,\mathrm{Aut}(Z))=1 \quad \textit{and} \quad \mathrm{Hom}(G,\mathrm{Out}(\pi/Z))=1,$$

then

$$\mathrm{Hom}(G,\mathrm{Out}(\pi))=1.$$

(Here 1 denotes the trivial group homomorphism.)

Proof Considering the quotient group homomorphism $\pi\to\pi/Z$, we have the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Inn}(\pi) \arrow[r] \arrow[d] & \mathrm{Aut}(\pi) \arrow[r] \arrow[d, "f"] & \mathrm{Out}(\pi) \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\pi/Z) \arrow[r] & \mathrm{Aut}(\pi/Z) \arrow[r] & \mathrm{Out}(\pi/Z) \arrow[r] & 1
\end{tikzcd}$$

Note that $\mathrm{Inn}(\pi)=\pi/Z$. By the snake lemma for groups (see [23, 11.8, page 363]), the following sequence is exact:

$$1\to Z(\pi/Z)\to\ker f\to\ker g\to 1. \tag{5}$$

By diagram chase, the action of $\ker g$ on the center $Z(\pi/Z)$ is by inner automorphisms of π/Z and thus trivial. This proves that the previous exact sequence (5) is a central extension. Since

$$\mathrm{Hom}(G,\mathrm{Out}(\pi/Z))=1$$

by assumption, it suffices to prove $\mathrm{Hom}(G,\ker g)=1$. Let $\alpha\colon G\to\ker g$ be any group homomorphism. In case (i), when π/Z is torsion-free, the center $Z(G/Z)$ is torsion-free. Since G is perfect and $H_2(G;\mathbb{Z})$ is finite, we have $H^2(G;Z(\pi/Z))=0$ by the universal coefficient theorem. In case (ii), we also have $H^2(G;Z(\pi/Z))=0$ using the universal coefficient theorem. Lemma 4.1 implies that α could be lifted to a group homomorphism $\alpha'\colon G\to\ker f$.

Let $F\colon \mathrm{Aut}(\pi) \to \mathrm{Aut}(Z)$ be the restriction of automorphisms of π to those of the center Z. Since the image $F(\ker f)$ is a subgroup of $\mathrm{Aut}(Z)$ and

$$\mathrm{Hom}(G, \mathrm{Aut}(Z)) = 1$$

by assumption, the map α' has its image in $\ker F \cap \ker f$. It is well known that $\ker F \cap \ker f$ is isomorphic to $H^1(\pi/Z; Z)$ (see [17, Proposition 5, page 45]). Since G is perfect, α' has a trivial image. This proves that α is trivial and thus $\mathrm{Hom}(G, \mathrm{Out}(\pi)) = 1$. □

Recall that a group G has cohomological dimension k (denoted by $\mathrm{cd}(G) = k$) if the cohomological groups satisfy $H^i(G; A) = 0$ for any $i > k$ and $\mathbb{Z}G$–module A, but $H^n(G; M) \neq 0$ for some $\mathbb{Z}G$–module M. The Hirsch number or Hirsch length of a polycyclic group G is the number of infinite factors in its subnormal series. The following lemma is well known (see Gruenberg [17, page 152]).

Lemma 4.3 *If G is a finitely generated, torsion-free nilpotent group, then $\mathrm{cd}(G) = h(G)$, where $\mathrm{cd}(G)$ is the cohomological dimension and $h(G)$ is the Hirsch number.*

Lemma 4.4 *Let $1 \to Z \to G \to Q \to$ be a central extension with $Z = Z(G)$ the center and G a torsion-free nilpotent group. Then Q is torsion-free.*

Proof It's known that all the quotients Z_i/Z_{i-1} are torsion-free. Suppose that the nilpotency class of G is n, ie $Z_n = G$. Then G/Z_{n-1} is (torsion-free) abelian and we have an exact sequence

$$1 \to Z_{n-1}/Z_{n-2} \to G/Z_{n-2} \to G/Z_{n-1} \to 1.$$

Since both G/Z_{n-1} and Z_{n-1}/Z_{n-2} are finitely generated (note that every subgroup of a finitely generated nilpotent group is finitely generated) and of finite cohomology dimension, G/Z_{n-2} is of finite cohomological dimension and thus torsion-free. Inductively, we prove the lemma. □

Lemma 4.5 *Let G be a finitely generated torsion-free nilpotent group of cohomological dimension k. When $k < n$, the set of group homomorphisms is*

$$\mathrm{Hom}(\mathrm{St}_n(\mathbb{Z}), \mathrm{Out}(G)) = 1,$$

and thus $\mathrm{Hom}(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Out}(G)) = 1$.

Proof Let Z be the center of G. Note that $\mathrm{cd}(G) = h(G)$ and $\mathrm{cd}(G/Z) = h(G/Z)$ and $h(G/Z) \leq h(G) - 1$. When G/Z is abelian, $h(G/Z) \leq k$ and $\mathrm{Out}(G/Z) = \mathrm{GL}_{h(G/Z)}(\mathbb{Z})$. Noting that

$$\mathrm{Hom}(\mathrm{St}_n(\mathbb{Z}), \mathrm{GL}_k(\mathbb{Z})) = 1$$

for any $k \leq n-1$ by Corollary 2.11, we have $\mathrm{Hom}(\mathrm{St}_n(\mathbb{Z}), \mathrm{Out}(G/Z)) = 1$. Similarly, we get $\mathrm{Hom}(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Aut}(Z)) = 1$ by noting that Z is torsion-free abelian and $h(Z) \leq k$. Using Lemma 4.2 repeatedly, we have $\mathrm{Hom}(\mathrm{St}_n(\mathbb{Z}), \mathrm{Out}(G)) = 1$. □

Let M be an aspherical manifold with a finitely generated nilpotent fundamental group $\pi_1(M)$. Any group action of $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ on M^k with $k < n$ is trivial, as follows:

Proof of Theorem 1.4 Since M is aspherical, M is a classifying space for $\pi_1(M)$ and thus the cohomological dimension $\mathrm{cd}(\pi_1(M)) \leq r$. By Lemma 4.5, any group homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi_1(M))$ is trivial. By Theorem 1.2, any group action of $\mathrm{SL}_n(\mathbb{Z})$ on M is trivial. □

5 Flat and almost flat manifolds

A closed manifold M is almost flat if for any $\varepsilon > 0$ there is a Riemannian metric g_ε on M such that $\mathrm{diam}(M, g_\varepsilon) < 1$ and g_ε is ε–flat. Let G be a simply connected nilpotent Lie group. Choose a maximal compact subgroup C of $\mathrm{Aut}(G)$. If π is a torsion-free uniform discrete subgroup of the semidirect product $G \rtimes C$, the orbit space $M = \pi \backslash G$ is called an infra-nilmanifold and π is called a generalized Bieberbach group. The group $F := \pi/(\pi \cap G)$ is called the holonomy group of M. When $G = \mathbb{R}^n$, the abelian Lie group, M is called a flat manifold. By Gromov and Ruh [16; 26], every almost flat is diffeomorphic to an infra-nilmanifold. Note that $N := \pi \cap G$ is the unique maximal nilpotent normal subgroup of π.

The automorphism of π is studied by Igodt and Malfait [18], generalizing the corresponding result for flat manifolds obtained by Charlap and Vasquez [11]. Let's recall the relevant results as follows. Let $\psi\colon F \to \mathrm{Out}(N)$ be an injective group homomorphism and

$$1 \to N \to \pi \to F \to 1$$

be a group extension compatible with ψ. The extension determines a cohomology class $a \in H^2(F; N)$ (the set of 2–cohomology classes compatible with ψ). Let

$p\colon \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the natural quotient map and $\mathcal{M}_\psi = p^{-1}(N_{\mathrm{Out}(N)}(F))$, the preimage of the normalizer of $\psi(F)$ in $\mathrm{Out}(N)$. Denote by $\mathcal{M}_{\psi,a}$ the stabilizer of a under the action of $\mathcal{M}_\psi$ on $H^2(F;N)$ by conjugations. Let $A\colon \mathrm{Aut}(\pi) \to \mathrm{Aut}(N)$ be the group homomorphisms of restrictions. For an element $n \in N$, write $\mu(n) \in \mathrm{Aut}(N)$ for the inner automorphism determined by n, ie $\mu(n)(x) = nxn^{-1}$. Denote by $*\colon \mathcal{M}_\psi \to \mathrm{Aut}(F)$ the group homomorphism given by $*(v) = \psi \circ \mu(p(v)) \circ \psi^{-1}$ (see [18, Lemma 3.3]).

We will need the following result (see [18, Theorem 4.6, Theorem 4.8 and its proof]):

Lemma 5.1 *The following sequences are exact:*

$$1 \to Z^1(F; Z(N)) \to \mathrm{Aut}(\pi) \xrightarrow{A} \mathcal{M}_{\psi,a} \to 1$$

and

$$1 \to H^1(F; Z(N)) \to \mathrm{Out}(\pi) \xrightarrow{A} Q \to 1.$$

The quotient Q equals $(\mathcal{M}_{\psi,a}/\mathrm{Inn}(N))/F$ and fits into an exact sequence

$$\begin{aligned}(6)\qquad 1 \to Q_2 = \big((\ker(*) \cap \mathcal{M}_{\psi,a})/\mathrm{Inn}(N)\big)/Z(F)\\ \to Q \to Q_1 = \mathrm{Im}(*|_{\mathcal{M}_{\psi,a}})/\mathrm{Inn}(F) \to 1,\end{aligned}$$

where $(\ker(*) \cap \mathcal{M}_{\psi,a})/\mathrm{Inn}(N)$ is contained in the centralizer $C_{\mathrm{Out}(N)}(F)$.

Proof of Theorem 1.5 If $\mathrm{SL}_n(\mathbb{Z})$ acts trivially on M^r, it is obvious that the induced homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(F)$ is trivial. In order to prove the converse, it's enough to prove $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi)$ is trivial for a given group action of $\mathrm{SL}_n(\mathbb{Z})$ on M, considering Theorem 1.2. We will use the exact sequence (6) in Lemma 5.1. Note that Q_1 is a subgroup of $\mathrm{Out}(F)$. By the assumption that the group homomorphism $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(F)$ is trivial, the composite

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi) \to Q \to Q_1$$

has to be trivial. Therefore, the map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi) \to Q$ has its image in Q_2. Denote by $K = (\ker(*) \cap \mathcal{M}_{\psi,a})/\mathrm{Inn}(N)$ and $Z = Z(F)$ to fit into an exact sequence

$$1 \to Z \to K \to Q_2 \to 1. \qquad (7)$$

Since K is a subgroup of $C_{\mathrm{Out}(N)}(F)$ by Lemma 5.1, the center $Z(F)$ lies in the center of K. Therefore, the exact sequence (7) is a central extension. Let $\mathrm{St}_n(\mathbb{Z})$ be the Steinberg group and denote the composite by

$$\alpha\colon \mathrm{St}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(\pi) \to Q_2.$$

Since $H_2(\mathrm{St}_n(\mathbb{Z});\mathbb{Z})=0$, we have $H^2(\mathrm{St}_n(\mathbb{Z});Z)=0$ by the universal coefficient theorem. Therefore, α could be lifted to be a group homomorphism $\alpha'\colon \mathrm{St}_n(\mathbb{Z})\to K$ by Lemma 4.1. Note that K is a subgroup of $\mathrm{Out}(N)$ and the cohomological dimension of N is at most r. By Lemma 4.5, α' is trivial and thus α is trivial. This implies the group homomorphism $\mathrm{SL}_n(\mathbb{Z})\to\mathrm{Out}(\pi)$ has its image in $H^1(F;Z(N))$. Since $H^1(F;Z(N))$ is abelian and $\mathrm{SL}_n(\mathbb{Z})$ is perfect, the map $\mathrm{SL}_n(\mathbb{Z})\to\mathrm{Out}(\pi)$ has to be trivial. The proof is finished. □

Remark 5.2 In the proof of Theorem 1.5, we use an essential property of the Steinberg group that $H^2(\mathrm{St}_n(\mathbb{Z});A)=0$ for any abelian group A. This does not hold for $\mathrm{SL}_n(\mathbb{Z})$ for $n\geq 3$ since $H_2(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z})=\mathbb{Z}/2$. If the abelian group A is torsion-free (as in the proofs of Theorems 1.2 and 1.4), we still have $H^2(\mathrm{SL}_n(\mathbb{Z});A)=0$ and it is thus not essential to use $\mathrm{St}_n(\mathbb{Z})$.

6 Examples

In this section, we give further applications of Theorems 1.2 and 1.5.

The proof of the following lemma is similar to that of the corresponding result for $\mathrm{SL}_n(\mathbb{Z})$, proved by Kielak [20, Theorem 2.27].

Lemma 6.1 *Let p be a prime. Then $\mathrm{Hom}(\mathrm{St}_n(\mathbb{Z}),\mathrm{GL}_k(\mathbb{Z}/p))=1$, ie any group homomorphism $g\colon \mathrm{St}_n(\mathbb{Z})\to\mathrm{GL}_k(\mathbb{Z}/p)$ is trivial, for $k<n-1$.*

Proof Let $N=\ker g$. Since the image of N in $\mathrm{SL}_n(\mathbb{Z})$ is of finite index, the map g factors through $g'\colon \mathrm{St}_n(\mathbb{Z}/N)\to\mathrm{GL}_k(\mathbb{Z}/p)$ for some integer N. Note that $\mathrm{St}_n(R_1\times R_2)=\mathrm{St}_n(R_1)\times\mathrm{St}_n(R_2)$ for rings R_1 and R_2. Without loss of generality, we assume that N is a power of a prime number. Let Z be the center of $\mathrm{St}_n(\mathbb{Z}/N)$. Suppose that $\mathrm{GL}_k(\mathbb{Z}/p)$ acts on $(\mathbb{Z}/p)^k$ naturally. We could assume that the action of $\mathrm{Im}\, g'$ on $(\mathbb{Z}/p)^k$ is irreducible. Note that $(\mathbb{Z}/p)^k$ is the intersection of eigenspaces of $g'(v)$ for $v\in Z$ (if necessary, we may consider the algebraic closure of $\mathbb{Z}/p$). After change of basis in $(\mathbb{Z}/p)^k$, we get that $g'(N)$ lies in the center of $\mathrm{GL}_k(\mathbb{Z}/p)$. Therefore, g' induces a map $g''\colon \mathrm{PSL}_n(\mathbb{Z}/N)\to\mathrm{PGL}_k(\mathbb{Z}/p)$. However, it's known that g'' has to be trivial by Landazuri and Seitz [21]. □

Let A be a finite abelian group. For a prime p, define the p–rank $\mathrm{rank}_p(A)$ as the dimension of $A\bigotimes_{\mathbb{Z}/p}\mathbb{Z}/p$, as a vector space over $\mathbb{Z}/p$.

Lemma 6.2 *Let A be a finite abelian group with $\operatorname{rank}_p(A) < n-1$ for every prime p. Then every group homomorphism $\mathrm{St}_n(\mathbb{Z}) \to \operatorname{Aut}(A)$ is trivial for $n \geq 3$.*

Proof Since a group homomorphism preserves p–Sylow subgroups of A, it's enough to prove the theorem for a p–group A. If A is an elementary p–group, $\operatorname{Aut}(A) = \mathrm{GL}_k(\mathbb{Z}/p)$, with $k = \operatorname{rank}_p(A)$. Thus $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Aut}(A))$ is trivial by Lemma 6.1. If A is not elementary, the subgroup A_1 consisting of elements of order p is a characteristic subgroup of A. Inductively, we assume that $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Aut}(A/A_1))$ and $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Aut}(A_1))$ are both trivial. Let

$$B = \{f \in \operatorname{Aut}(A) : f|_{A_1} = \mathrm{id}_{A_1}\}.$$

The group $\ker(\operatorname{Aut}(A) \to \operatorname{Aut}(A/A_1)) \cap B = H^1(A/A_1; A_1)$ is abelian (see [17, Proposition 5, page 45]). Since $\mathrm{St}_n(\mathbb{Z})$ for $n \geq 3$ is perfect, $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Aut}(A))$ is trivial. □

Lemma 6.3 *Let π be a finite nilpotent group with the upper central series $1 = Z_0 < Z_1 < \cdots < Z_k = \pi$. Suppose that $\operatorname{rank}_p(Z_i/Z_{i-1}) < n-1$ for each prime p and each $i = 1, \dots, k$. Then the set of group homomorphisms is $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Out}(\pi)) = 1$.*

Proof When $k = 1$, this is proved in Lemma 6.2. Considering the central extension

$$1 \to Z_i/Z_{i-1} \to \pi/Z_{i-1} \to \pi/Z_i \to 1$$

the statement could be proved inductively using Lemmas 6.2 and 4.2. □

Corollary 6.4 *Let M^r be a closed almost flat manifold with the holonomy group Φ nilpotent satisfying the condition in Lemma 6.3. Then Conjecture 1.1 holds for M. In particular, when M is a closed flat manifold with abelian holonomy group, Conjecture 1.1 holds.*

Proof By Lemma 6.3, $\operatorname{Hom}(\mathrm{St}_n(\mathbb{Z}), \operatorname{Out}(\Phi)) = 1$ and so $\operatorname{Hom}(\mathrm{SL}_n(\mathbb{Z}), \operatorname{Out}(\Phi)) = 1$. Theorem 1.5 implies that Conjecture 1.1 is true. When M is flat, Φ is a subgroup of $\mathrm{GL}_r(\mathbb{Z})$. If Φ is abelian, elements in Φ could be simultaneously diagonalizable in $\mathrm{GL}_r(\mathbb{C})$. Therefore, $\operatorname{rank}_p(\Phi) \leq r-1 < n-1$ for each prime p. □

Lemma 6.5 *Let Φ be a dihedral group D_{2k}, symmetric group S_k or alternating group A_k. Then $\operatorname{Hom}(\mathrm{SL}_n(\mathbb{Z}), \operatorname{Out}(\Phi)) = 1$ for $n \geq 3$.*

Proof It's well known that $\mathrm{Aut}(D_{2k}) = \mathbb{Z}/k \rtimes \mathrm{Aut}(\mathbb{Z}/k)$, which is a solvable group. Therefore, $\mathrm{Out}(D_{2k})$ is solvable. When Φ is S_k or A_k, we have that $\mathrm{Out}(\Phi)$ is abelian. However, $\mathrm{SL}_n(\mathbb{Z})$ is perfect and thus $\mathrm{Hom}(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Out}(\Phi)) = 1$. □

For flat manifolds of low dimensions, we obtain the following:

Corollary 6.6 *Conjecture 1.1 holds for closed flat manifolds M^r of dimension $r \leq 5$.*

Proof The proof depends on the classification of low-dimensional holonomy groups Φ. When $r \leq 3$, $\Phi = \{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/6$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$ (see [31, Corollary 3.5.6, page 118]). They are all abelian. By Corollary 6.4, Conjecture 1.1 is true. When $r = 4$, the nonabelian Φ is D_6, D_8, D_{12} or A_4 (see [9]). By Lemma 6.5 and Theorem 1.5, Conjecture 1.1 is true. When $r = 5$, the nonabelian Φ is D_6, D_8, D_{12}, $D_8 \times \mathbb{Z}/2$, $D_6 \times \mathbb{Z}/3$, $D_{12} \times \mathbb{Z}/2$, A_4, $A_4 \times \mathbb{Z}/2$, $A_4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$, S_4 or $(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/4$ (see [27, Theorem 1] or [12, Theorem 4.2]). By Lemma 6.5, it's enough to consider the Φ with two factors. By Lemmas 6.3 and 4.2, $\mathrm{Hom}(\mathrm{SL}_n(\mathbb{Z}), \mathrm{Out}(\Phi)) = 1$ and thus Conjecture 1.1 holds by Theorem 1.5. □

Acknowledgements This work is supported by NSFC No. 11501459.

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Received: 20 June 2017 Revised: 19 April 2018