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Homological stability for diffeomorphism groups of high-dimensional handlebodies

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We prove a homological stability theorem for the diffeomorphism groups of high-dimensional manifolds with boundary, with respect to forming the boundary connected sum with the product $D^{p+1} \times S^q$ for $|q-p| < \min\{p-2, q-3\}$. In a recent joint paper with B Botvinnik, we prove that there is an isomorphism

$$\operatorname*{colim}_{g\to\infty} H_*(\mathrm{BDiff}((D^{n+1} \times S^n)^{\natural g}, D^{2n}); \mathbb{Z}) \cong H_*(Q_0 BO(2n+1)\langle n\rangle_+; \mathbb{Z})$$

in the case that $n \geq 4$. By combining this "stable homology" calculation with the homological stability theorem of this paper, we obtain the isomorphism

$$H_k(\mathrm{BDiff}((D^{n+1} \times S^n)^{\natural g}, D^{2n}); \mathbb{Z}) \cong H_k(Q_0 BO(2n+1)\langle n\rangle_+; \mathbb{Z})$$

in the case that $k \leq \frac{1}{2}(g-4)$.

57R15, 57R50, 57R65, 57S05

1 Introduction

1.1 Main result

Let M be a smooth, compact, m–dimensional manifold with nonempty boundary. Fix an $(m-1)$–dimensional disk $D^{m-1} \hookrightarrow \partial M$. We denote by $\mathrm{Diff}(M, D^{m-1})$ the group of self-diffeomorphisms of M that restrict to the identity on a neighborhood of D^{m-1}, topologized in the C^∞–topology. We let $\mathrm{BDiff}(M, D^{m-1})$ denote the *classifying space* of the topological group $\mathrm{Diff}(M, D^{m-1})$. Let V be another smooth, compact, m–dimensional manifold with nonempty boundary. There is a homomorphism $\mathrm{Diff}(M, D^{m-1}) \to \mathrm{Diff}(M \natural V, D^{m-1})$ defined by extending a diffeomorphism identically over the boundary connect-summand V. This homomorphism induces a map between the classifying spaces, and iterating this map yields the direct system

$$\mathrm{BDiff}(M, D^{m-1}) \to \mathrm{BDiff}(M \natural V, D^{m-1}) \to \cdots \to \mathrm{BDiff}(M \natural V^{\natural g}, D^{m-1}) \to \cdots. \tag{1}$$

In this paper we study the homological properties of this direct system in the case when V is a high-dimensional handlebody $D^{p+1} \times S^q$. Our main result can be viewed as an

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analogue of the homological stability theorems of Harer [6], or the theorem of Galatius and Randal-Williams [4], for high-dimensional manifolds with boundary.

Below we state our main theorem. For $p, q, g \in \mathbb{Z}_{\geq 0}$, let $V^g_{p,q}$ denote the g–fold boundary connected sum, $(D^{p+1} \times S^q)^{\natural g}$. Let $d(\pi_q(S^p))$ denote the *generating set length* of the homotopy group $\pi_q(S^p)$, which is the quantity

$$d(\pi_q(S^p)) = \min\{k \in \mathbb{N} \mid \text{there exists an epimorphism } \mathbb{Z}^{\oplus k} \to \pi_q(S^p)\}.$$

We let $\kappa(\partial M)$ and $\kappa(M, \partial M)$ denote the degrees of connectivity of ∂M and $(M, \partial M)$, respectively. The main result of this paper is stated below:

Theorem 1.1 *Let M be a compact manifold of dimension m with nonempty boundary. Let p and q be positive integers with $p + q + 1 = m$ such that the inequalities*

$$(2) \quad |q - p| < \min\{p - 2, q - 3\} \quad \text{and} \quad |q - p| < \min\{\kappa(\partial M) - 1, \kappa(M, \partial M) - 2\}$$

are satisfied. Then the homomorphism induced by the maps in (1),

$$H_k(\mathrm{BDiff}(M \natural V^g_{p,q}, D^{m-1}); \mathbb{Z}) \to H_k(\mathrm{BDiff}(M \natural V^{g+1}_{p,q}, D^{m-1}); \mathbb{Z}),$$

is an isomorphism when $k \leq \frac{1}{2}\big(g - d(\pi_q(S^p)) - 3\big)$ and an epimorphism when $k \leq \frac{1}{2}\big(g - d(\pi_q(S^p)) - 1\big)$.

The above theorem implies that the homology of the direct system (1) stabilizes in the case $V = V^1_{p,q}$. In the special case where $p = q$ we can also identify the homology of the limiting space. This was done in joint work with B Botvinnik [1]. To state this result we need to introduce some notation.

Notational Convention 1.2 Let X be a topological space and let $k \in \mathbb{Z}_{\geq 0}$.

- We let $X\langle k\rangle$ denote the *k–connected cover of X*, which is defined to be the k^{th} stage in the *Whitehead tower* associated to X. It is uniquely (up to homotopy) characterized by the following properties: (a) $\pi_i(X\langle k\rangle) = 0$ for all $i \leq k$; (b) there exists a map $X\langle k\rangle \to X$ that induces an isomorphism $\pi_j(X\langle k\rangle) \cong \pi_j(X)$ for all $j > k$.
- We define $X\langle k\rangle_+$ to be the pointed space given by the disjoint union $X\langle k\rangle \sqcup \{\mathrm{pt}\}$ with basepoint given by the additional point.
- We let $QX\langle k\rangle_+$ denote the infinite loopspace of the suspension spectrum associated to the space $X\langle k\rangle_+$, ie $QX\langle k\rangle_+ = \Omega^\infty\Sigma^\infty X\langle k\rangle_+$.
- We let $Q_0X\langle k\rangle_+$ denote the path component of $QX\langle k\rangle_+$ that contains the constant loop.

In [1] we construct a map

$$\operatorname*{colim}_{g\to\infty} \mathrm{BDiff}((D^{n+1}\times S^n)^{\natural g}, D^{2n}) \to Q_0 BO(2n+1)\langle n\rangle_+,$$

which we prove induces an isomorphism in $H_*(-,\mathbb{Z})$ in the case that $n\geq 4$. Combining this homological equivalence with Theorem 1.1, we obtain a corollary which lets us compute the homology of the classifying space $\mathrm{BDiff}((D^{n+1}\times S^n)^{\natural g}, D^{2n})$ in low degrees relative to g.

Corollary 1.3 *Let $2n+1\geq 9$. Then there is an isomorphism*

$$H_k(\mathrm{BDiff}((D^{n+1}\times S^n)^{\natural g}, D^{2n});\mathbb{Z}) \cong H_k(Q_0 BO(2n+1)\langle n\rangle_+;\mathbb{Z})$$

when $k\leq \frac{1}{2}(g-4)$.

In addition to the classifying spaces of diffeomorphism groups of manifolds, we prove an analogous homological stability theorem for the moduli spaces of manifolds equipped with tangential structures; see Theorem 8.2. The precise statement of this theorem requires a number of preliminary definitions and so we hold off on stating this result until Section 8, where the theorem is proven.

1.2 Ideas behind the proof

The proof of our main theorem follows the strategy developed by Galatius and Randal-Williams [4], used to prove homological stability for the diffeomorphism groups $\mathrm{Diff}((S^n\times S^n)^{\#g}, D^{2n})$ for $g\in\mathbb{N}$. The technique is also inspired by the homological stability theorem of Harer [6] for the mapping class groups of surfaces, and furthermore follows the general proof scheme for proving homological stability formalized by Randal-Williams and Wahl [17]. Given a compact manifold triad $(M,\partial_0 M,\partial_1 M)$ and integers $p+q+1=\dim(M)$, we construct a semisimplicial space $K^{\partial}_{\bullet}(M)_{p,q}$, which admits a continuous action of the diffeomorphism group $\mathrm{Diff}(M,\partial_0 M)$. Roughly, an l–simplex of $K^{\partial}_{\bullet}(M)_{p,q}$ is given by a list of embeddings, $\phi_0,\dots,\phi_l\colon (V^1_{p,q},\partial V^1_{p,q})\to(M,\partial_1 M)$, for which $\phi_i(V^1_{p,q})\cap\phi_j(V^1_{p,q})=\varnothing$ for all $i\neq j$. Nearly all of the technical work of this paper is devoted to showing that the connectivity of the geometric realization $|K^{\partial}_{\bullet}(M)_{p,q}|$ increases linearly — with a slope of $\frac{1}{2}$ — in the number of boundary connect-summands of $V^1_{p,q}$ contained in M (see Theorem 3.6). Once high-connectivity is established, the homological stability theorem follows by applying the same spectral sequence argument used in [4] or [17]. Most of the simplicial and homotopy-theoretic techniques that we use come from [4], but the geometric–topological aspects of the paper require new constructions and arguments. We describe some of these new features below.

The semisimplicial space $K^{\partial}_{\bullet}(M)_{p,q}$ can be viewed as a "relative version" of the semisimplicial space constructed in our previous work [15], used to prove homological stability for the diffeomorphism groups $\mathrm{Diff}((S^q \times S^p)^{\# g})$ for $g \in \mathbb{N}$. Our proof that the space $|K^{\partial}_{\bullet}(M)_{p,q}|$ is highly connected follows a similar strategy to what was employed in [15]. The idea is to map $K^{\partial}_{\bullet}(M)_{p,q}$ to another simplicial object, $L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))$, that is constructed entirely out of algebraic data associated to the manifold M, which we call the *Wall form* associated to M (see Section 5). Wall forms were initially defined in [15] to study closed manifolds and their diffeomorphisms. In this paper we have to use a relative version of the Wall form used to study manifold pairs $(M, \partial M)$ and diffeomorphisms $M \to M$ that are nontrivial on the boundary. High-connectivity of the complex $L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))$ follows from our work in [15] as well, and so it remains to prove that the map $\Theta\colon |K^{\partial}_{\bullet}(M)_{p,q}| \to |L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))|$ is highly connected.

One of the main ingredients used in our proof that the map Θ is highly connected is a new disjunction result for embeddings of high-dimensional manifolds with boundary; this is Theorem A.1, stated in the appendix. If $(P, \partial P), (Q, \partial Q) \subset (M, \partial M)$ are submanifold pairs, Theorem A.1 gives conditions for when one can find an ambient isotopy $\Psi_t\colon (M, \partial M) \to (M, \partial M)$ with $\Psi_0 = \mathrm{Id}_M$ such that $\Psi_1(P) \cap Q = \varnothing$. Theorem A.1 can be viewed as a generalization of the disjunction results of Wells [20] and Hatcher and Quinn [7], and thus could be of independent interest.

Plan of the paper

In Section 2 we recast the maps of the direct system (1) as maps arising from concatenation with a relative cobordism between two compact manifold pairs. This will enable us to restate Theorem 1.1 in a slightly more general form that will be easier for us to prove. In Section 3 we construct the main semisimplicial space, $K^{\partial}_{\bullet}(M)_{p,q}$. In Section 4 we define certain algebraic invariants associated to a manifold with boundary. Section 5 is devoted to a recollection of some results from our previous work in [15] regarding Wall forms. In Section 6 we use the results developed throughout the rest of the paper to prove that $|K^{\partial}_{\bullet}(M)_{p,q}|$ is highly connected. In Section 7 we show how to obtain Theorem 2.7 (and thus Theorem 1.1) using the high-connectivity of $|K^{\partial}_{\bullet}(M)_{p,q}|$. In

Section 8 we show how to obtain an analogue of Theorem 2.7 for the moduli spaces of handlebodies equipped with tangential structures. In the appendix we prove a disjunction theorem for embeddings of high-dimensional manifolds with boundary.

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2 Relative cobordism and stabilization

To prove Theorem 1.1, we will need to recast the direct system (1) as arising from concatenation with a relative cobordism between two compact manifold pairs. Doing this will make our constructions consistent with the cobordism categories considered in [5; 1]. Below we introduce some definitions and terminology that we will use throughout the paper.

Let us recall first the definition of a *manifold with corners.* The following definition is lifted from [11] with a few minor changes in terminology.

Definition 2.1 A smooth manifold with corners is a topological manifold X together with a C^∞*–structure with corners*. That is, X is covered by charts, $\phi\colon U \to [0,\infty)^n$, which are homeomorphisms from open sets U onto open subsets of $[0,\infty)^n$. Two charts $(\phi_i, U_i)_{i=0,1}$ are said to be compatible if $\phi_2 \circ \phi_1^{-1}\colon \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is a diffeomorphism. Such a collection of charts covering X is called an atlas (with corners). A C^∞–structure with corners is a maximal atlas.

Let M be an n–dimensional smooth manifold with corners. Let (ϕ, U) be a chart on M. For any $x \in U$ the number of zeros in the n–tuple $\phi(x) \in [0,1)^n$ does not depend on the choice of chart (ϕ, U). We denote the number of zeros in $\phi(x)$ by $c(x)$. The boundary ∂X of X consists of all points $x \in X$ with $c(x) \geq 1$. All points with $c(x) = 0$ belong to the interior. The *depth* of a manifold with corners X is defined to be the number $\mathrm{depth}(X) := \max\{c(x) \mid x \in X\}$.

We refer the reader to [11] for more background on manifolds with corners. In this paper we will only need to work with manifolds with corners with depth 2. We will work with such manifolds with corners that are equipped with some extra structure. The main definition is given below:

Definition 2.2 A *manifold triad* of dimension n is a triple $(W; \partial_0 W, \partial_1 W)$ where

- W is an n–dimensional smooth manifold with corners with $\mathrm{depth}(W) = 2$,
- $\partial_0 W, \partial_1 W \subset \partial W$ are $(n-1)$–dimensional submanifolds,

subject to the conditions

(i) $\partial W = \partial_0 W \cup \partial_1 W$,

(ii) $\partial(\partial_0 W) = \partial_0 W \cap \partial_1 W = \partial(\partial_1 W)$,

(iii) $c(x) = 2$ for all $x \in \partial_0 W \cap \partial_1 W$.

We will write $\partial_{0,1} W := \partial_0 W \cap \partial_1 W$. Since $\mathrm{depth}(W) = 2$, it follows from the conditions above that $\partial_{0,1} W$ is a closed manifold of dimension $n-2$.

Two compact d–dimensional manifold pairs $(M, \partial M)$ and $(N, \partial N)$ are said to be *cobordant* if there exists a $(d+1)$–dimensional compact manifold triad $(W; \partial_0 W, \partial_1 W)$ such that

$$(\partial_0 W, \partial_{0,1} W) = (M \sqcup N, \partial M \sqcup \partial N).$$

The pair $(W, \partial_1 W)$ is then said to be a *relative cobordism* between the pairs $(M, \partial M)$ and $(N, \partial N)$.

Let $(M; \partial_0 M, \partial_1 M)$ be an m–dimensional, compact manifold triad. Let $\mathrm{Diff}(M)$ denote the topological group of self-diffeomorphisms $f\colon M \to M$ with $f(\partial_i M) = \partial_i M$ for $i = 0, 1$. We are mainly interested in the subgroup $\mathrm{Diff}(M, \partial_0 M) \subset \mathrm{Diff}(M)$ consisting of those self-diffeomorphisms that restrict to the identity on a neighborhood of $\partial_0 M$. We will need the following construction:

Construction 2.3 Let $(P, \partial P)$ be an $(m-1)$–dimensional manifold pair and let $(K; \partial_0 K, \partial_1 K)$ be a compact manifold triad such that $\partial_0 K = \partial_0 M \sqcup P$, ie the pair $(K, \partial_1 K)$ is a relative cobordism between $(\partial_0 M, \partial_{0,1} M)$ and $(P, \partial P)$. Let $M \cup_{\partial_0} K$ be the manifold obtained by attaching K to M along the face $\partial_0 M \subset \partial_0 K$. Similarly, let $\partial_1(M \cup_{\partial_0} K) := \partial_1 M \cup_{\partial_{0,1}} \partial_1 K$ be the manifold obtained by attaching $\partial_1 K$ to $\partial_1 M$ along their common boundary $\partial_{0,1} M$. By setting $\partial_0(M \cup_{\partial_0} K) := P$, we obtain a new manifold triad,

$$(M \cup_{\partial_0} K; \partial_0(M \cup_{\partial_0} K), \partial_1(M \cup_{\partial_0} K)).$$

The cobordism $(K, \partial_1 K)$ induces a homomorphism

$$\mathrm{Diff}(M, \partial_0 M) \to \mathrm{Diff}(M \cup_{\partial_0} K, P), \quad f \mapsto f \cup \mathrm{Id}_K,$$

defined by extending diffeomorphisms identically over K. This homomorphism in turn induces a map on the level of classifying spaces,

$$\mathrm{BDiff}(M, \partial_0 M) \to \mathrm{BDiff}(M \cup_{\partial_0 M} K, P). \tag{3}$$

This map should be compared to the one from [4, Equation 1.1]. Indeed, they are the same map in the case that $\partial_1 M = \partial_1 K = \varnothing$.

Suppose now that $\partial_0 M$ and $\partial_1 M$ are nonempty. Let p and q be integers such that $p+q+1 = m$. Choose embeddings

$$\partial D^{p+1} \times S^q \hookleftarrow D^{m-1} \hookrightarrow \partial_{0,1} M \times (0,1). \tag{4}$$

Let $K_{p,q}$ then denote the manifold obtained by forming the boundary connected sum of $D^{p+1} \times S^q$ and $\partial_0 M \times [0,1]$ along the embeddings (4). Notice that by construction the boundary of $K_{p,q}$ contains $\partial_0 M \times \{0,1\}$. We then set

$$\begin{aligned} \partial_0 K_{p,q} &:= \partial_0 M \times \{0,1\}, \\ \partial_1 K_{p,q} &:= \partial K_{p,q} \setminus \mathrm{Int}(\partial_0 K_{p,q}) = (\partial_{0,1} M \times [0,1]) \# (S^p \times S^q). \end{aligned}$$

The triple $(K_{p,q}; \partial_0 K_{p,q}, \partial_1 K_{p,q})$ is a manifold triad and $(K_{p,q}, \partial_1 K_{p,q})$ is a relative cobordism between the manifold pairs

$$(\partial_0 M \times \{0\}, \partial_{0,1} M \times \{0\}) \quad \text{and} \quad (\partial_0 M \times \{1\}, \partial_{0,1} M \times \{1\}).$$

We apply Construction 2.3 to this relative cobordism $(K_{p,q}, \partial_1 K_{p,q})$. We form the manifold $M \cup_{\partial_0} K_{p,q}$ and notice that $\partial_0(M \cup_{\partial_0} K_{p,q}) = \partial_0 M$. We define

$$s_{p,q}\colon \mathrm{BDiff}(M, \partial_0 M) \to \mathrm{BDiff}(M \cup_{\partial_0} K_{p,q}, \partial_0 M) \tag{5}$$

to be the map on classifying spaces induced by $(K_{p,q}, \partial_1 K_{p,q})$ from (3). We will refer to this map as the $(p,q)^{th}$ *stabilization map*.

Remark 2.4 The manifold $M \cup_{\partial_0} K_{p,q}$ is diffeomorphic to the boundary connected sum $M \natural V^1_{p,q} = M \natural (D^{p+1} \times S^q)$. By identifying $M \cup_{\partial_0} K_{p,q}$ with $M \natural V^1_{p,q}$, (5) yields the map

$$\mathrm{BDiff}(M, \partial_0 M) \to \mathrm{BDiff}(M \natural V^1_{p,q}, \partial_0 M).$$

In the case that $\partial_0 M = D^{p+q}$, we obtain the maps used in the direct system (1); we take this to be the definition of those maps.

Remark 2.5 In Section 7.1 we give an alternative definition of the $(p,q)^{\text{th}}$ stabilization map (5), using a particular geometric model for the classifying spaces $\mathrm{BDiff}(M, \partial_0 M)$. The proof of our main theorem (Theorem 2.7) will use the latter definition of $s_{p,q}$. It is a simple exercise to show that the two definitions of the $(p-q)^{\text{th}}$ stabilization map agree up to homotopy.

We will now restate Theorem 1.1 in terms of the maps defined above. We need one more preliminary definition. Recall from the previous section the manifold $V^g_{p,q} = (D^{p+1} \times S^q)^{\natural g}$. Choose an embedded $(p+q)$–dimensional closed disk $D \subset \partial V^g_{p,q}$ and set $\partial_0 V^g_{p,q} := D$ and $\partial_1 V^g_{p,q} := \partial V^g_{p,q} \setminus \mathrm{Int}(D)$. By creating a corner at $\partial D \subset \partial V^g_{p,q}$, $V^g_{p,q}$ obtains the structure of a manifold triad. With $\dim(M) = m$ and $p+q+1 = m$ as above, we let $r_{p,q}(M)$ be the integer defined by

$$r_{p,q}(M) = \max\{g \in \mathbb{N} \mid \text{there exists an embedding } (V^g_{p,q}, \partial_1 V^g_{p,q}) \to (M, \partial_1 M)\}. \tag{6}$$

We refer to this quantity as the (p,q)–rank of M. This quantity $r_{p,q}(M)$ is equivalent to the maximal number of boundary connect-summands of $D^{p+1} \times S^q$ that split off of M. This maximum exists as a consequence of the compactness of M. We emphasize that the embeddings $(V^g_{p,q}, \partial_1 V^g_{p,q}) \to (M, \partial_1 M)$ used in (6) need not send the face $\partial_0 V^g_{p,q}$ into $\partial_0 M$.

Remark 2.6 The value $r_{p,q}(M)$ depends on the structure of the triad $(M; \partial_0 M, \partial_1 M)$ and not just the manifold M itself. In particular, switching the roles of $\partial_0 M$ and $\partial_1 M$ in (6) will change the value of the rank $r_{p,q}(M)$.

As in the statement of Theorem 1.1 we will need to assume that the following inequalities are satisfied:

$$|q-p| < \min\{p-2, q-3\}, \quad |q-p| < \min\{\kappa(\partial_1 M) - 1, \kappa(M, \partial_1 M) - 2\}, \tag{7}$$

where recall that $\kappa(\partial_1 M)$ and $\kappa(M, \partial_1 M)$ denote the degrees of connectivity of $\partial_1 M$ and $(M, \partial_1 M)$, respectively. The main theorem that we will prove in this paper is stated below; it is a generalization of Theorem 1.1. Recall the generating set length $d(\pi_q(S^p))$.

Theorem 2.7 *Let $(M; \partial_0 M, \partial_1 M)$ be an m–dimensional, compact, manifold triad with $\partial_0 M$ and $\partial_1 M$ nonempty. Let p and q be positive integers with $p+q+1 = m$*

and suppose that the inequalities of (7) are satisfied. Let $r_{p,q}(M) \geq g$. Then the homomorphism

$$(s_{p,q})_*\colon H_k(\mathrm{BDiff}(M, \partial_0 M); \mathbb{Z}) \to H_k(\mathrm{BDiff}(M \cup_{\partial_0} K_{p,q}, \partial_0 M); \mathbb{Z})$$

is an isomorphism when $k \leq \frac{1}{2}\big(g - 3 - d(\pi_q(S^p))\big)$ and an epimorphism when $k \leq \frac{1}{2}\big(g - 1 - d(\pi_q(S^p))\big)$.

3 The complex of embedded handles

In this section we construct a semisimplicial space that is analogous to [4, Definition 5.1]. The definition requires a preliminary geometric construction. For integers p, q and g, let $(V^g_{p,q}; \partial_0 V^g_{p,q}, \partial_1 V^g_{p,q})$ be the $(p+q+1)$–dimensional manifold triad defined in the previous section. For each $g \in \mathbb{N}$ we let $W^g_{p,q}$ denote the face $\partial_1 V^g_{p,q} \cong (S^p \times S^q)^{\# g} \setminus \mathrm{Int}(D^{p+q})$. We will write $V_{p,q} := V^1_{p,q}$ and $W_{p,q} := W^1_{p,q}$.

Definition 3.1 For an integer m let D^m_+ denote the m–dimensional half-disk, ie the subspace given by the set $\{\bar{x} \in \mathbb{R}^m \mid |\bar{x}| \leq 1,\ x_1 \geq 0\}$. The boundary of D^m_+ has the decomposition $\partial D^m_+ = \partial_0 D^m_+ \cup \partial_1 D^m_+$, where $\partial_0 D^m_+$ and $\partial_1 D^m_+$ are given by $\partial_0 D^m_+ = \{\bar{x} \in D^m_+ \mid x_1 = 0\}$ and $\partial_1 D^m_+ = \{\bar{x} \in D^m_+ \mid |\bar{x}| = 1\}$. In this way, $(D^m_+; \partial_0 D^m_+, \partial_1 D^m_+)$ forms a manifold triad.

We will construct a slight modification of the manifold $V_{p,q}$. Choose an embedding

$$\alpha\colon (D^{p+q}_+ \times \{1\}, \partial_0 D^{p+q}_+ \times \{1\}) \to (\partial_0 V_{p,q}, \partial_{0,1} V_{p,q}).$$

Let $\widehat{V}_{p,q}$ denote the manifold obtained by attaching $D^{p+q}_+ \times [0,1]$ to $V_{p,q}$ along the embedding α, ie

$$\widehat{V}_{p,q} = V_{p,q} \cup_\alpha (D^{p+q}_+ \times [0,1]).$$

We then denote by $\widehat{W}_{p,q}$ the manifold obtained by attaching $\partial_0 D^{p+q} \times [0,1]$ to $\partial_1 V_{p,q}$ along the restriction of α to $\partial_0 D^{p+q} \times \{1\}$.

Construction 3.2 We construct a subspace of $\widehat{V}_{p,q}$ as follows. Choose a basepoint $(a_0, b_0) \in \partial D^{p+1} \times S^q$ such that the pair $(a_0, -b_0)$ is contained in the face $\partial_1 V_{p,q}$. Choose an embedding $\gamma\colon [0,1] \to \widehat{W}_{p,q}$ that satisfies the following conditions:

(i) $\gamma(1) = (a_0, -b_0)$ and $\gamma(0) = (0,0) \in \partial_0 D^{p+q}_+ \times \{0\}$.

(ii) There exists $\epsilon \in (0,1)$ such that $\gamma(t) = (0,t) \in \partial_0 D^{p+q}_{+} \times [0,1]$ whenever $t \in (0,\epsilon)$.

(iii) $\gamma((0,1)) \cap [(D^{p+1} \times \{a_0\}) \cup (\{b_0\} \times S^q)] = \varnothing$.

We define $(B_{p,q}, C_{p,q}) \subset (\widehat{V}_{p,q}, \widehat{W}_{p,q})$ to be the pair of subspaces given by

$$\big(\gamma([0,1]) \cup (D^{p+1} \times \{b_0\}) \cup (\{a_0\} \times S^q), \gamma([0,1]) \cup (S^p \times \{b_0\}) \cup (\{a_0\} \times S^q)\big).$$

The inclusion $(B_{p,q}, C_{p,q}) \hookrightarrow (\widehat{V}_{p,q}, \widehat{W}_{p,q})$ is clearly a homotopy equivalence of pairs. We will refer to the pair $(B_{p,q}, C_{p,q})$ as the *core* of $(\widehat{V}_{p,q}, \widehat{W}_{p,q})$.

We now use this construction to construct a simplicial complex. Let $(M; \partial_0 M, \partial_1 M)$ be a compact manifold triad of dimension m. Let $\mathbb{R}^{m-1}_{+}$ denote $[0,\infty) \times \mathbb{R}^{m-2}$ and let $\partial\mathbb{R}^{m-1}_{+}$ denote $\{0\} \times \mathbb{R}^{m-2}$. Choose an embedding a: $[0,1) \times \mathbb{R}^{m-1}_{+} \to M$ with $a^{-1}(\partial_0 M) = \{0\} \times \mathbb{R}^{m-1}_{+}$ and $a^{-1}(\partial_1 M) = [0,1) \times \partial\mathbb{R}^{m-1}_{+}$. For each pair of positive integers p and q with $p+q+1 = m$, we define a simplicial complex $K^{\partial}(M,a)_{p,q}$.

Definition 3.3 The simplicial complex $K^{\partial}(M,a)_{p,q}$ is defined as follows:

(i) A vertex in $K^{\partial}(M,a)_{p,q}$ is defined to be a pair (t,ϕ), where $t \in (0,\infty)$ and ϕ: $\widehat{V}_{p,q} \to M$ is an embedding that satisfies:

  (a) $\phi^{-1}(\partial_0 M) = D^{p+q}_{+} \times \{0\}$ and $\phi^{-1}(\partial_1 M) = \widehat{W}_{p,q}$.

  (b) There exists $\epsilon > 0$ such that for $(s,z) \in [0,\epsilon) \times D^{m-1}_{+} \subset \widehat{V}_{p,q}$, the equality $\phi(s,z) = a(s, z + te_2)$ is satisfied, where $e_2 \in \mathbb{R}^{m-1}_{+}$ denotes the second basis vector.

(ii) A set of vertices $\{(\phi_0,t_0),\dots,(\phi_l,t_l)\}$ forms an l–simplex if $t_i \neq t_j$ and

$$\phi_i(B_{p,q}) \cap \phi_j(B_{p,q}) = \varnothing$$

whenever $i \neq j$, where recall that $B_{p,q} \subset \widehat{V}_{p,q}$ is the core from Construction 3.2.

Remark 3.4 The embedding a was necessary in order to define $K^{\partial}(M,a)_{p,q}$; however, it will not play a serious role in any of our proofs later on in the paper. For this reason we will drop the embedding a from the notation and will write $K^{\partial}(M)_{p,q} := K^{\partial}(M,a)_{p,q}$.

The majority of the technical work of this paper is devoted to proving Theorem 3.6 stated below. The statement of this theorem will require the use of a definition from [4], which we recall below:

Definition 3.5 A simplicial complex X is said to be *weakly Cohen–Macaulay* of dimension n if it is $(n-1)$–connected and the link of any p–simplex is $(n-p-2)$–connected. In this case we write $\mathrm{wCM}(X) \geq n$. The complex X is said to be *locally weakly Cohen–Macaulay* of dimension n if the link of any simplex is $(n-p-2)$–connected but no connectivity condition is imposed on X itself. In this case we shall write $\mathrm{lCM}(X) \geq n$.

Theorem 3.6 *Let $(M; \partial_0 M, \partial_1 M)$ be an m–dimensional manifold triad with $\partial_0 M$ nonempty. Let $p, q \in \mathbb{N}$ be such that $p+q+1 = m$ and suppose that the inequalities*

$$(8) \quad |q-p| < \min\{p-2, q-3\}, \quad |q-p| < \min\{\kappa(\partial_1 M) - 1, \kappa(M, \partial_1 M) - 2\}$$

are satisfied. Let $d = d(\pi_q(S^p))$ be the generating set rank and suppose that $r_{p,q}(M) \geq g$. Then the geometric realization $|K^\partial(M)_{p,q}|$ is $\frac{1}{2}(g-4-d)$–connected and $\mathrm{lCM}(K^\partial(M)_{p,q}) \geq \frac{1}{2}(g-1-d)$.

The proof of the above theorem takes place over the course of the next three sections of the paper. In addition to the simplicial complex defined above, we will need to work with two related semisimplicial spaces. Let $(M; \partial_0 M, \partial_1 M)$, p and q be as above and let a: $[0,1) \times \mathbb{R}^{m-1}_+ \to M$ be the same embedding as used in Definition 3.3. We define two semisimplicial spaces $K^\partial_\bullet(M)_{p,q}$ and $\overline{K}^\partial_\bullet(M,a)_{p,q}$ below:

Definition 3.7 The semisimplicial space $K^\partial_\bullet(M,a)_{p,q}$ is defined as follows:

(i) The space of 0–simplices $K^\partial_0(M,a)_{p,q}$ is defined to have the same underlying set as the set of vertices of the simplicial complex $K^\partial(M,a)_{p,q}$. That is, $K^\partial_0(M,a)_{p,q}$ is the space of pairs (t, ϕ), where $t \in \mathbb{R}$ and ϕ: $\widehat{V}_{p,q} \to M$ is an embedding which satisfies the same condition as in part (i) of Definition 3.3.

(ii) The space of l–simplices, $K^\partial_l(M,a) \subset (K^\partial_0(M,a))^{l+1}$, consists of ordered $(l+1)$–tuples,
$$((t_0, \phi_0), \dots, (t_l, \phi_l)),$$
such that $t_0 < \cdots < t_l$ and $\phi_i(B_{p,q}) \cap \phi_j(B_{p,q}) = \varnothing$ when $i \neq j$.

The spaces $K^\partial_l(M,a) \subset (\mathbb{R} \times \mathrm{Emb}(\widehat{V}_{p,q}, M))^{l+1}$ are topologized using the C^∞–topology on the space of embeddings. The assignments $[l] \mapsto K^\partial_l(M,a)$ define a semisimplicial space, denoted by $K^\partial_\bullet(M,a)_{p,q}$. The i^{th} face map d_i: $K^\partial_l(M,a) \to K^\partial_{l-1}(M,a)$ is defined by forgetting the i^{th} entry in the l–tuple $((t_0, \phi_0), \dots, (t_l, \phi_l))$.

Finally, $\overline{K}^{\partial}_{\bullet}(M,a)_{p,q} \subset K^{\partial}_{\bullet}(M,a)_{p,q}$ is defined to be the subsemisimplicial space of all simplices $((\phi_0,t_0),\dots,(\phi_l,t_l)) \in K^{\partial}_{l}(M,a)$ such that $\phi_i(\widehat{V}_{p,q}) \cap \phi_j(\widehat{V}_{p,q}) = \varnothing$ whenever $i \neq j$.

As in Remark 3.4, when working with $K^{\partial}_{\bullet}(M,a)_{p,q}$ and $\overline{K}^{\partial}_{\bullet}(M,a)_{p,q}$ we will drop the embedding a from the notation and write

$$\overline{K}^{\partial}_{\bullet}(M)_{p,q} := \overline{K}^{\partial}_{\bullet}(M,a)_{p,q} \quad \text{and} \quad K^{\partial}_{\bullet}(M)_{p,q} := K^{\partial}_{\bullet}(M,a)_{p,q}.$$

We will ultimately need to use the fact that the geometric realizations $|\overline{K}^{\partial}_{\bullet}(M,a)_{p,q}|$ and $|K^{\partial}_{\bullet}(M,a)_{p,q}|$ are highly connected. We prove that $|\overline{K}^{\partial}_{\bullet}(M,a)_{p,q}|$ and $|K^{\partial}_{\bullet}(M,a)_{p,q}|$ are highly connected by comparing them to the simplicial complex $K^{\partial}(M)_{p,q}$.

Corollary 3.8 *Let $(M;\partial_0 M,\partial_1 M)$ be an m–dimensional manifold triad with $\partial_0 M$ nonempty. Let $p,q \in \mathbb{N}$ be such that $p+q+1=m$ and suppose that the inequalities (8) are satisfied. Let $d = d(\pi_q(S^p))$ be the generating set rank and suppose that $r_{p,q}(M) \geq g$. Then the geometric realization $|\overline{K}^{\partial}_{\bullet}(M)_{p,q}|$ is $\frac{1}{2}(g-4-d)$–connected.*

Proof of Corollary 3.8 The result is obtained from Theorem 3.6 by assembling several results from [4]. Let $g \in \mathbb{Z}_{\geq 0}$ be given. Consider the discretization $K^{\partial}_{\bullet}(M)^{\delta}_{p,q}$. This is the semisimplicial space obtained by defining each $K^{\partial}_{l}(M)^{\delta}_{p,q}$ to have the same underlying set as $K^{\partial}_{l}(M)_{p,q}$, but topologized in the discrete topology. The correspondence $((\phi_0,t_0),\dots,(\phi_l,t_l)) \mapsto \{(\phi_0,t_0),\dots,(\phi_l,t_l)\}$ induces a map $|K^{\partial}_{\bullet}(M)^{\delta}_{p,q}| \to |K^{\partial}(M)_{p,q}|$, which is easily seen to be a homeomorphism. Indeed, if

$$\sigma = \{(\phi_0,t_0),\dots,(\phi_l,t_l)\} \leq K^{\partial}(M)_{p,q}$$

is a simplex, then $t_i \neq t_j$ if $i \neq j$, and so there is exactly one l–simplex in $K^{\partial}_{\bullet}(M)^{\delta}_{p,q}$ with underlying set equal to σ. It follows from this that $|K^{\partial}_{\bullet}(M)^{\delta}_{p,q}|$ is $\frac{1}{2}(g-4-d)$–connected.

Next, we compare the connectivity of $|K^{\partial}_{\bullet}(M)^{\delta}_{p,q}|$ to $|K^{\partial}_{\bullet}(M)_{p,q}|$. By replicating the same argument used in the proof of [4, Theorem 5.5] it follows that the connectivity of $|K^{\partial}_{\bullet}(M)_{p,q}|$ is bounded below by the connectivity of $|K^{\partial}_{\bullet}(M)^{\delta}_{p,q}|$, and thus $|K^{\partial}_{\bullet}(M)_{p,q}|$ is $\frac{1}{2}(g-4-d)$–connected as well. We remark that this proof from [4] uses the fact that $\mathrm{lCM}(K^{\partial}(M)_{p,q}) \geq \frac{1}{2}(g-1-d)$ (and this is established in Theorem 3.6). Finally, to finish the proof of the lemma we observe that the inclusion $\overline{K}^{\partial}_{\bullet}(M)_{p,q} \hookrightarrow K^{\partial}_{\bullet}(M)_{p,q}$ is a levelwise weak homotopy equivalence. This fact is proven by employing the same argument used in the proof of [4, Corollary 5.8]. We then apply [18, Proposition A.1] (or [2, Theorem 2.2]) which implies that any levelwise weak homotopy equivalence between semisimplicial spaces geometrically realizes to a weak homotopy equivalence between the geometric realizations. It follows that the induced map $|\bar{K}^{\partial}_{\bullet}(M)_{p,q}| \hookrightarrow |K^{\partial}_{\bullet}(M)_{p,q}|$ is a weak homotopy equivalence and thus completes the proof of the lemma. □

High-connectivity of the space $|\bar{K}^{\partial}_{\bullet}(M)_{p,q}|$ is the main ingredient needed for the proof of Theorem 2.7, which we carry out in Section 7. The main ingredient in proving Corollary 3.8 was Theorem 3.6. The next three sections then are geared toward developing all of the technical tools needed to prove Theorem 3.6.

4 The algebraic invariants

4.1 Invariants of manifold pairs

For what follows, let M be a compact, oriented manifold of dimension m with nonempty boundary. Let $A \subset \partial M$ be an oriented submanifold of dimension $m-1$. We will keep M and A fixed throughout the entire section. Let p and q be positive integers with $p+q+1 = m$ and suppose further that

(9) $$|q-p| < \min\{p-2, q-3\}, \quad |q-p| < \min\{\kappa(A)-1, \kappa(M,A)-2\},$$

where recall $\kappa(A)$ and $\kappa(M,A)$ denote the degrees of connectivity of A and (M,A), respectively. We will need to work with the homotopy groups $\pi_q(A)$ and $\pi_{p+1}(M,A)$. The above inequalities imply that A and (M,A) are both at least 1–connected and thus the groups $\pi_q(A)$ and $\pi_{p+1}(M,A)$ do not depend on a choice of basepoint (and thus we are justified in excluding a basepoint from the notation). Furthermore, (9) also implies that $p+1 > 2$, and thus the group $\pi_{p+1}(M,A)$ is abelian. Now, we will need to be able to represent elements of these homotopy groups $\pi_q(A)$ and $\pi_{p+1}(M,A)$ by smooth embeddings. The next lemma follows by assembling several results from [10; 19].

Lemma 4.1 *Let M be a manifold of dimension m with nonempty boundary and let $A \subset \partial M$ be a submanifold of dimension $m-1$. Let p and q be positive integers such that $p+q+1 = m$ and suppose that (9) holds. We may then draw the following conclusions about the homotopy groups $\pi_{p+1}(M,A)$ and $\pi_q(A)$:*

(i) *Any element of $\pi_{p+1}(M,A)$ can be represented by an embedding*
$$(D^{p+1}, S^p) \to (M,A),$$
unique up to isotopy.

(ii) *Any element of* $\pi_q(A)$ *can be represented by an embedding* $S^q \to A$*, which is unique up to regular homotopy.*

Proof As mentioned above, this lemma follows by applying results from [10; 19]. Our proof consists of showing how to apply these results and verifying that all of the necessary dimensional and connectivity conditions are satisfied. Let us start with statement (i). The key result to apply here is [10, Corollaries 1.1 and 2.1]. The combination of these results from [10] implies the following:

- Let X be an m–dimensional manifold and $Y \subset \partial X$ be an $(m-1)$–dimensional submanifold. Then every element of $\pi_k(X, Y)$ can be represented by an embedding, $(D^k, \partial D^k) \to (X, Y)$, uniquely up to isotopy, if the following conditions are satisfied:
  (a) (X, Y) is $(2k-m+2)$–connected.
  (b) $k \leq m-3$.

To apply this to our situation we set $(M, A) = (X, Y)$, $m = p+q+1$ and $k = p+1$. The inequality $|p-q| < \kappa(M, A) - 2$ implies that

$$\kappa(M, A) \geq 2(p+1) - (p+q+1) + 2 = 2k - m + 2.$$

The inequality $|q-p| < q-3$ implies that

$$p+1 = k \leq (p+q+1) - 3 = m-3.$$

Thus, conditions (a) and (b) are satisfied, and [10, Corollaries 1.1 and 2.1] implies that any element of $\pi_{p+1}(M, A)$ can be represented by an embedding, uniquely up to isotopy.

For statement (ii) of the lemma we use [19, Proposition 1 and Lemma 1]. The combination of these results implies the following:

- Let X be an m–dimensional manifold. All elements of $\pi_k(X)$ are represented by an embedding, uniquely up to regular homotopy, if the following conditions are satisfied:
  (a) X is $(2k-m+2)$–connected.
  (b) $2m \geq 3k+3$.

To apply this to our situation we set $X = A$, $m = p+q$, and $k = q$. The inequality $|q-p| < \kappa(A) - 1$ implies condition (a) and the inequality $|q-p| < p-2$ implies condition (b). This establishes statement (ii) of the lemma. □

Remark 4.2 The statement from part (i) of Lemma 4.1 actually holds true with a weakened version of condition (9). Indeed, to apply [10, Corollaries 1.1 and 2.1] in our situation, it is only required that $\dim(M) - (p+1) \geq 3$ (and thus $q \geq 3$) and that the connectivity condition from (9) hold as well. However, the full strength of (9) will be needed later on in the paper. We will have to assume (9) (exactly as it is written) in order apply Theorem 4.10, which is based on the work carried out in the appendix.

We now proceed to define some invariants associated to the pair (M, A). We first define a bilinear map

$$\tau^{\partial}_{p,q}\colon \pi_{p+1}(M,A) \otimes \pi_q(S^p) \to \pi_q(A), \quad ([f],[\phi]) \mapsto [(f|_{\partial D^{p+1}}) \circ \phi], \tag{10}$$

where $f\colon (D^{p+1}, \partial D^{p+1}) \to (M, A)$ represents a class in $\pi_{p+1}(M,A)$ and $\phi\colon S^q \to S^p$ represents a class in $\pi_q(S^p)$. By the inequalities of (9), it follows from the *Freudenthal suspension theorem* that the suspension homomorphism $\Sigma\colon \pi_{q-1}(S^{p-1}) \to \pi_q(S^p)$ is surjective. For this reason the formula in (10) is indeed bilinear and thus $\tau^{\partial}_{p,q}$ is well defined.

We have a bilinear intersection pairing

$$\lambda^{\partial}_{p,q}\colon \pi_{p+1}(M,A) \otimes \pi_q(A) \to \mathbb{Z}, \tag{11}$$

which is defined by sending a pair $([f],[g]) \in \pi_{p+1}(M,A) \otimes \pi_q(A)$ to the oriented algebraic intersection number associated to the maps $f|_{\partial D^{p+1}}\colon S^p \to A$ and $g\colon S^q \to A$, which we may assume are embeddings by Lemma 4.1. Note that this oriented algebraic intersection number is well defined because the manifold A is assumed to be oriented.

We have a $(-1)^q$–symmetric bilinear pairing

$$\mu_q\colon \pi_q(A) \otimes \pi_q(A) \to \pi_q(S^p) \tag{12}$$

defined in the same way as in [15, Construction 3.1]. We refer the reader there for the definition.

The next proposition shows how the maps $\tau^{\partial}_{p,q}$, $\lambda^{\partial}_{p,q}$ and μ_q are related to each other. The proof follows from [15, Proposition 3.4].

Proposition 4.3 *Let $x \in \pi_p(M,A)$, $y \in \pi_q(A)$ and $z \in \pi_q(S^p)$. Then the equation*

$$\mu_q(\tau^{\partial}_{p,q}(y,z), x) = \lambda^{\partial}_{p,q}(y,x) \cdot z$$

is satisfied.

Remark 4.4 The formula in Proposition 4.3 makes sense even in the case when $q < p$ and thus $\pi_q(S^p) = 0$.

We will also need to consider a function

$$\alpha_q\colon \pi_q(A) \to \pi_{q-1}(\mathrm{SO}_p), \tag{13}$$

defined by sending $x \in \pi_q(A)$ to the element in $\pi_{q-1}(\mathrm{SO}_p)$ which classifies the normal bundle associated to an embedding $S^q \to A$ which represents x.

The map α_q is not in general a homomorphism. As will be seen in Proposition 4.5, the bilinear form μ_q measures the failure of α_q to preserve additivity. In order to describe the relationship between α_q and μ_q, we must define some auxiliary homomorphisms. Let $\overline{\pi}_q\colon \pi_{q-1}(\mathrm{SO}_p) \to \pi_q(S^p)$ be the map given by the composition $\pi_{q-1}(\mathrm{SO}_p) \to \pi_{q-1}(S^{p-1}) \to \pi_q(S^p)$, where the first map is induced by the bundle projection $\mathrm{SO}_p \to \mathrm{SO}_p/\mathrm{SO}_{p-1} \cong S^{p-1}$ and the second is the suspension homomorphism. Let $d_q\colon \pi_q(S^p) \to \pi_{q-1}(\mathrm{SO}_p)$ be the boundary homomorphism associated to the fiber sequence $\mathrm{SO}_p \to \mathrm{SO}_{p+1} \to S^p$. The proposition below follows directly from [19, Theorem 1].

Proposition 4.5 *The following equations are satisfied for all $x, y \in \pi_q(A)$:*

$$\alpha_q(x+y) = \alpha_q(x) + \alpha_q(y) + d_q(\mu_q(x,y)), \quad \mu_q(x,x) = \overline{\pi}_q(\alpha_q(x)).$$

Proof The above equations follow from Wall's theorem [19, Theorem 1]. To apply Wall's theorem one simply has to check that the manifold involved satisfies the appropriate dimensional and connectivity conditions with respect to the integer q. Wall's theorem requires that $2\dim(A) \geq 3q+3$, $q \geq 2$ and that A be $(2q - \dim(A) + 2)$–connected. These conditions all follow from our initial assumption of (9). $\square$

We will also need the following proposition, which describes how α_q and $\tau^{\partial}_{p,q}$ are related.

Proposition 4.6 *For all $(x,z) \in \pi_{p+1}(M,A) \times \pi_q(S^p)$, we have $\alpha_q(\tau^{\partial}_{p,q}(x,z)) = 0$.*

Proof Let $\partial_{p+1}\colon \pi_{p+1}(M,A) \to \pi_p(A)$ denote the boundary map. In [15, page 9] a bilinear map

$$F_{p,q}\colon \pi_{p-1}(\mathrm{SO}_q) \otimes \pi_q(S^p) \to \pi_{q-1}(\mathrm{SO}_p)$$

is defined. By [15, Proposition 3.8] (combined with the definition of the map $\tau^{\partial}_{p,q}$), it follows that

$$\alpha_q(\tau^{\partial}_{p,q}(x,z)) = F_{p,q}\big(\alpha_p(\partial_{p+1}(x)), z\big)$$

for all $(x,z) \in \pi_{p+1}(M,A) \times \pi_q(S^p)$. Let $f\colon (D^{p+1}, \partial D^{p+1}) \to (M,A)$ be an embedding that represents the class $x \in \pi_{p+1}(M,A)$. Notice that the normal bundle of $f(D^{p+1})$ is automatically trivial since the disk is contractible. It follows that $f|_{\partial D^{p+1}}(\partial D^{p+1}) \subset A$ has trivial normal bundle as well, and thus it follows that $\alpha_p(\partial_{p+1}(x)) = 0$ since $f|_{\partial D^{p+1}}$ represents the class $\partial_{p+1}x \in \pi_p(A)$. Since $F_{p,q}$ is bilinear, it follows that

$$\alpha_q(\tau^{\partial}_{p,q}(x,z)) = F_{p,q}\big(\alpha_p(\partial_{p+1}(x)), z\big) = F_{p,q}(\alpha_p(0), z) = 0$$

for all x and z. This concludes the proof of the proposition. □

Using the maps defined above we will work with the algebraic structure defined by the 6–tuple

$$(\pi_{p+1}(M,A), \pi_q(A), \tau^{\partial}_{p,q}, \lambda^{\partial}_{p,q}, \mu_q, \alpha_q). \tag{14}$$

We refer to this structure as the *Wall form* associated to the pair (M,A). We summarize the salient properties of (14) in the following lemma. This lemma should be compared to [15, Lemma 3.9].

Lemma 4.7 *Let (M,A), p, and q be exactly as above. The maps*

- $\tau^{\partial}_{p,q}\colon \pi_{p+1}(M,A) \otimes \pi_q(S^p) \to \pi_q(A)$,
- $\lambda^{\partial}_{p,q}\colon \pi_{p+1}(M,A) \otimes \pi_q(A) \to \mathbb{Z}$,
- $\mu_q\colon \pi_q(A) \otimes \pi_q(A) \to \pi_q(S^p)$,
- $\alpha_q\colon \pi_q(A) \to \pi_{q-1}(\mathrm{SO}_p)$,

satisfy the following conditions. For all $x, x' \in \pi_{p+1}(M,A)$, $y, y' \in \pi_q(A)$ and $z \in \pi_q(S^p)$ we have

(i) $\lambda^{\partial}_{p,q}(x, \tau^{\partial}_{p,q}(x', z)) = 0$,

(ii) $\mu_q(\tau^{\partial}_{p,q}(x,z), y) = \lambda^{\partial}_{p,q}(x,y)\cdot z$,

(iii) $\alpha_q(y+y') = \alpha_q(y) + \alpha_q(y') + d_q(\mu_q(y,y'))$,

(iv) $\mu_q(y,y) = \overline{\pi}_q(\alpha_q(y))$,

(v) $\alpha_q(\tau^{\partial}_{p,q}(x,z)) = 0$.

4.2 Modifying intersections

We will ultimately need to use $\lambda^{\partial}_{p,q}$ and μ_q to study intersections of embedded submanifolds. Let (M,A), p and q be exactly as in the previous section. For embeddings $f\colon (D^{p+1},S^p)\to(M,A)$ and $g\colon S^q\to A$, the integer $\lambda_{p,q}([f],[g])$ is equal to the signed intersection number of $f(S^p)$ and $g(S^q)$ in A. By our assumption of (9), A is simply connected and $p,q\geq 3$, and so, by application of the *Whitney trick* [12, Theorem 6.6], one can deform f through a smooth isotopy to a new embedding $f'\colon (D^{p+1},S^p)\to(M,A)$ such that $f'(S^p)$ and $g(S^q)$ intersect transversally in A at exactly $|\lambda^{\partial}_{p,q}(x,y)|$–many points, all with positive orientation. We now consider embeddings $f,g\colon S^q\to A$ whose images intersect transversally. The intersection $f(S^q)\cap g(S^q)$ is generically a $(q-p)$–dimensional closed manifold. We will need a higher-dimensional analogue of the Whitney trick that applies to the intersection of such embeddings. The first proposition below follows from [20; 7].

Theorem 4.8 (Wells [20]) *Let $f,g\colon S^q\to A$ be embeddings. Then there exists an isotopy $\Psi_t\colon S^q\to A$ with $\Psi_0=g$ and $\Psi_1(S^q)\cap f(S^q)=\varnothing$ if and only if $\mu_q([g],[f])=0$.*

Remark 4.9 The above theorem follows directly from the main theorem of [20]. To apply this result from [20] one has to verify that our dimensional/connectivity conditions from (9) imply the dimensional/connectivity conditions assumed in [20]. This verification is simple arithmetic, which we leave to the reader. One also has to verify that our invariant $\mu_q([g],[f])$ agrees with the invariant $\alpha(M,N;X)$, defined by Wells in [20, page 390]. This follows by comparing Wells's construction of $\alpha(M,N;X)$ with our construction of $\alpha_q([f],[g])$ carried out in [15, Construction 3.1]. It is straightforward to verify that the two constructions agree in the case that N and M are both spheres of the same dimension.

We will also need a technique for manipulating the intersections of embeddings $(D^{p+1},\partial D^{p+1})\to(M,A)$. The following theorem is a special case of Theorem A.1 from the appendix.

Theorem 4.10 *Let $f,g\colon (D^{p+1},S^p)\to(M,A)$ be embeddings. Then there is an isotopy of embeddings*

$$\Psi_t\colon (D^{p+1},S^p)\to(M,A)$$

such that $\Psi_0=f$ and $\Psi_1(D^{p+1})\cap g(D^{p+1})=\varnothing$.

Remark 4.11 The above theorem follows directly from Theorem A.1. One just has to verify that our dimensional/connectivity conditions (9) imply the dimensional/connectivity conditions in the statement of Theorem A.1. This is arithmetic, which is left to the reader.

Remark 4.12 We emphasize that in Theorem 4.10 the embeddings f and g are completely arbitrary; the theorem holds for any two such embeddings so long as (M, A), p and q satisfy (9). Furthermore, we emphasize that the restriction $\Psi_t|_{S^p}$ is not in general the constant isotopy. The theorem would not be true if we insisted on keeping the restriction $\Psi_t|_{S^p}$ fixed for all t.

We will need to apply the above theorems inductively. For the statement of the next two results, let (M, A) and p and q be exactly as in the statements of the previous two theorems. The following corollary is proven in a way similar to [16, Corollary 7.5].

Corollary 4.13 *Let $f_0, \dots, f_n\colon S^q \to A$ be a collection of embeddings such that:*

(i) $\mu_q(f_0, f_i) = 0$ *for all* $i = 1, \dots, n$.

(ii) *The collection of embeddings $f_1, \dots, f_n$ is pairwise transverse.*

Then there exists an isotopy $\Psi_t\colon S^q \to A$ with $t \in [0, 1]$ and $\Psi_0 = f_0$ such that $\Psi_1(S^q) \cap f_i(S^q) = \varnothing$ for $i = 1, \dots, n$.

Proof Recall that $\dim(A) = p+q$, where p and q satisfy (9). In the case that $p > q$ the statement of this corollary is trivial: the invariant $\mu_q(-,-)$ is identically zero and the result follows by general position. So let us assume that $q \geq p$. We prove the corollary by induction on the integer n. The base case where $n = 1$ follows from Theorem 4.8. For the inductive step let the embeddings $f_0, \dots, f_n\colon S^q \to A$ be given. By the inductive assumption we may assume that

$$f_0(S^q) \cap f_i(S^q) = \varnothing \quad \text{for } i = 1, \dots, n-1.$$

To complete the proof we need to find an isotopy of f_0 to a new embedding whose image is disjoint from $f_i(S^q)$ for all $i = 1, \dots, n$.

First we observe that since the embeddings $f_1, \dots, f_n$ were assumed to be pairwise transverse, it follows that for each $i = 1, \dots, n-1$, the intersection $f_i(S^q) \cap f_n(S^q)$ is a submanifold of dimension

$$2q - \dim(A) = 2q - (p+q) = q - p.$$

Let $U \subset A$ be an open regular neighborhood of $\bigcup_{i=1}^{n-1} f_i(S^q) \subset A$ that is disjoint from $f_0(S^q)$. Let X denote the complement $A \setminus U$ and let V denote the complement $f_n(S^q) \setminus U$. Defined in this way, X is a $(p+q)$–dimensional compact manifold, V is a q–dimensional compact manifold and $(V, \partial V) \subset (X, \partial X)$ is a neatly embedded submanifold. Since U was chosen to be disjoint from $f_0(S^q)$ we have $f_0(S^q) \subset \mathrm{Int}(X)$. We wish to apply Wells's theorem from [20] to the submanifolds $f_0(S^q), V \subset X$ to obtain an isotopy of $f_0(S^q) \subset X$ (through embeddings in X) that makes $f_0(S^q)$ disjoint from V. The assumption that $\mu_q([f_0],[f_n]) = 0$ implies that Wells's invariant $\alpha(f_0(S^q), V; X)$ (see Remark 4.9 and [20, Section 2]) vanishes as well. To apply [20] we need to verify that the manifolds V and X satisfy the connectivity conditions from the statement of Wells's theorem. For this we need to use the transversality assumption about the embeddings f_i. We make the following claim about the connectivity of X and V:

Claim 4.14 *The manifolds X and V satisfy the following connectivity conditions:*

(i) *V is $(q-p)$–connected.*

(ii) *X is $(q-p+1)$–connected.*

Proof Let us first prove part (i). Let $g\colon S^{q-p} \to V$ be a map. Since S^q is $(q-1)$–connected, the composite map $S^{q-p} \xrightarrow{\alpha} V \hookrightarrow f_n(S^q)$ extends to a map $\widehat{g}\colon D^{q-p+1} \to f_n(S^q)$. Condition (9) implies $p \leq q < p-2$, and from this it follows that

$$\dim(D^{p-q+1}) + \dim(f_i(S^q) \cap f_n(S^q)) < q \quad \text{for all } i = 1, \dots, n-1.$$

By general position the map $\widehat{g}$ may be deformed by a homotopy, relative ∂D^{p+q+1}, to a new map $\widetilde{g}$ with image disjoint from $f_i(S^q)$ for all $i = 1, \dots, n-1$. Since U is a regular neighborhood of $\bigcup_{i=1}^{n-1} f_i(S^q)$, the map may be deformed further so that its image is disjoint from U and thus is contained in V. This proves that $\pi_{q-p}(V) = 0$. The proof that $\pi_k(V) = 0$ for all $k \leq q-p$ is similar.

Part (ii) follows by a similar general position argument. Let $g\colon S^{q-p+1} \to X$ be a map. Condition (9) implies that A is at least $(q-p+1)$–connected and so $S^{q-p+1} \xrightarrow{g} X \hookrightarrow A$ extends to a map $\widehat{g}\colon D^{q-p+2} \to A$. Since

$$p+q > \dim(D^{q-p+2}) + \dim(S^q) = 2q-p+2,$$

by general position there exists a homotopy of $\widehat{g}$, relative ∂D^{q-p+2}, to a new map whose image is disjoint from $\bigcup_{i=1}^{n-1} f_i(S^q)$. Since U is a regular neighborhood of

$\bigcup_{i=1}^{n-1} f_i(S^q)$, the map may be deformed further so that its image is disjoint from U and thus is contained in X. This proves that $\pi_{q-p+1}(X) = 0$. The argument that $\pi_k(X) = 0$ for all $k \leq q-p+1$ is similar. □

By the above claim, it follows that the connectivity and dimensional conditions from the statement of the main theorem of [20] are satisfied. This together with the fact that $\alpha(f_0(S^q), V; X) = 0$ implies that there is an isotopy $\Psi_t\colon S^q \to X$, $t \in [0,1]$, with $\Psi_0 = f_0$ and $\Psi_1(S^q) \cap V = \varnothing$. This concludes the proof of the corollary. □

The next corollary is proven in the same way as Corollary 4.13 but using Theorem 4.10 instead of Theorem 4.8. Since the argument is the same we omit the proof.

Corollary 4.15 *Let $g_0, \dots, g_k\colon (D^{p+1}, \partial D^{p+1}) \to (M, A)$ be a collection of embeddings such that the collection of submanifolds $g_1(D^{p+1}), \dots, g_k(D^{p+1}) \subset M$ is pairwise transverse. Then there exists an isotopy*

$$\Psi_t\colon (D^{p+1}, \partial D^{p+1}) \to (M, A), \quad t \in [0,1],$$

with $\Psi_0 = g_0$, such that $\Psi_1(D^{p+1}) \cap g_i(D^{p+1}) = \varnothing$ for $i = 1, \dots, k$.

5 Wall forms

We now formalize the algebraic structure studied in Section 4. Much of this section is a recollection of definitions and results from [15, Section 5]. We begin by introducing the category underlying our main construction.

Definition 5.1 Fix a finitely generated abelian group H. An object $\boldsymbol{M}$ in the category Ab^2_H is defined to be a pair of abelian groups $(\boldsymbol{M}_-, \boldsymbol{M}_+)$ equipped with a bilinear map $\tau\colon \boldsymbol{M}_- \otimes H \to \boldsymbol{M}_+$. Objects of Ab^2_H are referred to as H–pairs. A morphism $f\colon \boldsymbol{M} \to \boldsymbol{N}$ of H–pairs is defined to be a pair of group homomorphisms $f_-\colon \boldsymbol{M}_- \to \boldsymbol{N}_-$ and $f_+\colon \boldsymbol{M}_+ \to \boldsymbol{N}_+$ that satisfy $f_+ \circ \tau_{\boldsymbol{M}} = \tau_{\boldsymbol{N}} \circ (f_- \otimes \mathrm{Id}_{\boldsymbol{M}})$. We will refer to morphisms in Ab^2_H as *H–maps*.

We build on the above definition as follows. Fix once and for all a finitely generated abelian group H. All of our constructions will take place in the category Ab^2_H. Let $\boldsymbol{G}$ be an abelian H–pair, equipped with homomorphisms $\partial\colon H \to \boldsymbol{G}_+$ and $\pi\colon \boldsymbol{G}_+ \to H$. Then, let $\epsilon = \pm 1$. We call such a 4–tuple $(\boldsymbol{G}, \partial, \pi, \epsilon)$ a *form-parameter*. Fix a form-parameter $(\boldsymbol{G}, \partial, \pi, \epsilon)$ and let $\boldsymbol{M}$ be a finitely generated H–pair. Consider the following data:

- A bilinear map $\lambda\colon \boldsymbol{M}_- \otimes \boldsymbol{M}_+ \to \mathbb{Z}$.
- An ϵ–symmetric bilinear form $\mu\colon \boldsymbol{M}_+ \otimes \boldsymbol{M}_+ \to H$.
- Functions $\alpha_\pm\colon \boldsymbol{M}_\pm \to \boldsymbol{G}_\pm$.

Our main definition is given below:

Definition 5.2 The 5–tuple $(\boldsymbol{M}, \lambda, \mu, \alpha)$ is said to be a *Wall form* with parameters $(\boldsymbol{G}, \partial, \pi, \epsilon)$ if the following conditions are satisfied for all $x, x' \in \boldsymbol{M}_-$, $y, y' \in \boldsymbol{M}_+$ and $h \in H$:

(i) $\lambda(x, \tau_{\boldsymbol{M}}(x', h)) = 0$.

(ii) $\mu(\tau_{\boldsymbol{M}}(x, h), y) = \lambda(x, y) \cdot h$.

(iii) $\alpha_-(x + x') = \alpha_-(x) + \alpha_-(x')$.

(iv) $\alpha_+(y + y') = \alpha_+(y) + \alpha_+(y') + \partial(\mu(y, y'))$.

(v) $\mu(y, y) = \pi(\alpha_+(y))$.

(vi) $\alpha_+(\tau_{\boldsymbol{M}}(x, h)) = \tau_{\boldsymbol{G}}(\alpha_-(x), h)$.

The Wall form $(\boldsymbol{M}, \lambda, \mu, \alpha)$ is said to be *reduced* if α_- is identically zero. In the case of a reduced Wall form, condition (vi) then translates to $\alpha_+(\tau_{\boldsymbol{M}}(x, h)) = 0$ for all $x \in \boldsymbol{M}_-$ and $h \in H$. A *morphism* between Wall forms (with the same form-parameter) is an H–map $f\colon \boldsymbol{M} \to \boldsymbol{N}$ that preserves all values of λ, μ and α.

We will often denote a Wall form by its underlying H–pair, ie $\boldsymbol{M} := (\boldsymbol{M}, \lambda, \mu, \alpha)$. We will need notation for orthogonal complements.

Definition 5.3 Let $\boldsymbol{N} \le \boldsymbol{M}$ be Wall forms. We define a new sub-Wall form $\boldsymbol{N}^\perp \le \boldsymbol{M}$ by setting

$$\boldsymbol{N}_-^\perp := \{x \in \boldsymbol{M}_- \mid \lambda(x, w) = 0 \text{ for all } w \in \boldsymbol{N}_+\},$$
$$\boldsymbol{N}_+^\perp := \{y \in \boldsymbol{M}_+ \mid \lambda(v, y) = 0 \text{ and } \mu(y, w) = 0 \text{ for all } v \in \boldsymbol{N}_-,\ w \in \boldsymbol{N}_+\}.$$

It can be easily checked that $\tau(\boldsymbol{N}_-^\perp \otimes H) \le \boldsymbol{N}_+^\perp$ and thus $\boldsymbol{N}^\perp$ actually is a sub-H–pair of $\boldsymbol{M}$. We call $\boldsymbol{N}^\perp$ the *orthogonal complement* to $\boldsymbol{N}$ in $\boldsymbol{M}$. Two sub-Wall forms $\boldsymbol{N}, \boldsymbol{N}' \le \boldsymbol{M}$ are said to be *orthogonal* if $\boldsymbol{N} \cap \boldsymbol{N}' = \boldsymbol{0}$, $\boldsymbol{N} \le (\boldsymbol{N}')^\perp$ and $\boldsymbol{N}' \le \boldsymbol{N}^\perp$.

We will need to use the simplicial complex from [15, Definition 4.13]. For this we must recall the definition of the *standard Wall form*. This requires a few steps. Fix a finitely generated abelian group H. We define an H–pair $\boldsymbol{W} \in \mathrm{Ob}(\mathrm{Ab}_H^2)$ by setting $\boldsymbol{W}_- = \mathbb{Z}$ and $\boldsymbol{W}_+ = \mathbb{Z} \oplus H$. The map $\tau\colon \boldsymbol{W}_- \otimes H \to \boldsymbol{W}_+$ is defined by the formula

$$\tau(t \otimes h) = (0, t \cdot h) \in \mathbb{Z} \oplus H = \boldsymbol{W}_+.$$

For $g \in \mathbb{N}$, we denote by $\boldsymbol{W}^g$ the g–fold direct sum $\boldsymbol{W}^{\oplus g}$. We let $\boldsymbol{W}$ denote the H–pair $\boldsymbol{W}^1$, and $\boldsymbol{W}^0$ is understood to be the trivial H–pair. Fix elements $a \in \boldsymbol{W}_-$ and $b \in \boldsymbol{W}_+$ which correspond to $1 \in \mathbb{Z}$ and $(1,0) \in \mathbb{Z} \oplus H$, respectively. For $g \in \mathbb{N}$, we denote by $a_i \in \boldsymbol{W}_-^g$ and $b_i \in \boldsymbol{W}_+^g$ for $i = 1, \dots, g$ the elements that correspond to the elements a and b coming from the i^{th} direct summand of $\boldsymbol{W}$ in $\boldsymbol{W}^g$. Now fix a form-parameter $(\boldsymbol{G}, \partial, \pi, \epsilon)$. We endow $\boldsymbol{W}^g$ with the structure of a Wall form with parameter $(\boldsymbol{G}, \partial, \pi, \epsilon)$ by setting

$$(15)\quad \lambda(a_i, b_j) = \delta_{i,j}, \quad \mu(b_i, b_j) = 0, \quad \alpha_+(b_i) = 0, \quad \alpha_-(a_i) = 0 \quad \text{for } i, j = 1, \dots g.$$

These values together with the conditions imposed from Definition 5.2 determine the maps λ, μ and α completely. The vanishing of $\alpha_-(a_i)$ for all i implies that α_- is identically zero, thus $(\boldsymbol{W}^g, \lambda, \mu, \alpha)$ is a reduced Wall form with parameter $(\boldsymbol{G}, \partial, \pi, \epsilon)$ (the fact that $\alpha_+(b_i) = 0$ for all i does not imply that $\alpha_+ = 0$, however). We call this the *standard Wall form of rank* g with parameter $(\boldsymbol{G}, \pi, \partial, \epsilon)$.

We will use the standard Wall form $\boldsymbol{W}$ to probe other Wall forms. The simplicial complex defined in the next definition is an algebraic analogue of the simplicial complex defined in Section 3.

Definition 5.4 For a Wall form $\boldsymbol{M}$ let $L(\boldsymbol{M})$ be the simplicial complex whose vertices are given by morphisms $f\colon \boldsymbol{W} \to \boldsymbol{M}$. A set of vertices $\{f_0, \dots, f_l\}$ is an l–simplex if the sub-Wall forms $f_0(\boldsymbol{W}), \dots, f_l(\boldsymbol{W}) \le \boldsymbol{M}$ are pairwise orthogonal.

To state the main theorem regarding the simplicial complex $L(\boldsymbol{M})$ we need to introduce a notion of rank for a Wall form. The definition below is analogous to the rank $r_{p,q}(-)$ associated to a manifold triad $(M; \partial_0 M, \partial_1 M)$, defined back in Section 2.

Definition 5.5 For a Wall form $\boldsymbol{M}$, the rank of $\boldsymbol{M}$ is defined to be the nonnegative integer

$$r(\boldsymbol{M}) := \max\{g \in \mathbb{N} \mid \text{there exists a morphism } \boldsymbol{W}^g \to \boldsymbol{M}\}.$$

One of the key technical results proven in [15] (see [15, Theorem 5.1]) is the theorem stated below. For this, let d denote the generating set rank $d(H)$, which recall is the quantity $d(H) = \min\{k \in \mathbb{N} \mid \text{there exists an epimorphism } \mathbb{Z}^{\oplus k} \to H\}$.

Theorem 5.6 *Suppose that $r(\boldsymbol{M}) \geq g$. Then* $\mathrm{lCM}(L(\boldsymbol{M})) \geq \frac{1}{2}(g-1-d)$ *and the geometric realization* $|L(\boldsymbol{M})|$ *is* $\frac{1}{2}(g-4-d)$ *connected.*

We now show how to use the constructions from Section 4 to associate a Wall form to a pair of manifolds. Let M be a compact, oriented manifold of dimension m with nonempty boundary. Let $A \subset \partial M$ be a submanifold of dimension $m-1$. Let p and q be positive integers with $p+q+1=m$. Suppose that p and q satisfy the inequalities from (9). We set $H = \pi_q(S^p)$ and denote by $\mathcal{W}^{\partial}_{p,q}(M,A)$ the H–pair given by setting

$$\mathcal{W}^{\partial}_{p,q}(M,A)_- := \pi_{p+1}(M,A), \quad \mathcal{W}^{\partial}_{p,q}(M,A)_+ := \pi_q(A),$$

and then by setting τ equal to the bilinear map

$$\tau := \tau^{\partial}_{p,q}\colon \pi_{p+1}(M,A) \otimes \pi_q(S^p) \to \pi_q(A)$$

from (10). We need to define a suitable form-parameter. Let $\mathbf{G}_{p,q}$ denote the abelian group $\pi_{p-1}(\mathrm{SO}_q)$. The group $\mathbf{G}_{p,q}$ together with the maps from Proposition 4.5, $d_q\colon \pi_q(S^p) \to \pi_{q-1}(\mathrm{SO}_p)$ and $\overline{\pi}_q\colon \pi_{q-1}(\mathrm{SO}_p) \to \pi_q(S^p)$, make the 4–tuple

$$(\boldsymbol{G}_{p,q}, d_q, \pi_q, (-1)^q)$$

into a form parameter. It follows then directly from Lemma 4.7 that the 4–tuple

$$(\mathcal{W}^{\partial}_{p,q}(M,A), \lambda^{\partial}_{p,q}, \mu_q, \alpha_q) \tag{16}$$

is a reduced Wall form with form-parameter $(\boldsymbol{G}_{p,q}, d_q, \pi_q, (-1)^q)$. We call the Wall form of (16) the *Wall form of degree* (p,q) *associated to* (M,A). This construction should be compared to [15, Section 4.3].

We now state a basic proposition, which follows directly from the definitions of $\tau^{\partial}_{p,q}$, $\lambda^{\partial}_{p,q}$, μ_q and α_q.

Proposition 5.7 *Let M and N be m–dimensional manifolds with nonempty boundary. Let $A \subset \partial M$ and $B \subset \partial N$ be submanifolds of dimension $m-1$. Let $p,q \in \mathbb{Z}_{\geq 0}$ be chosen with $p+q+1=m$ so that the Wall forms $\mathcal{W}^{\partial}_{p,q}(N,B)$ and $\mathcal{W}^{\partial}_{p,q}(M,A)$ are defined. Then any embedding $\varphi\colon (N,B) \to (M,A)$ induces a unique morphism of Wall forms $\varphi_*\colon \mathcal{W}^{\partial}_{p,q}(N,B) \to \mathcal{W}^{\partial}_{p,q}(M,A)$.*

6 High connectivity of $K^{\partial}(M)_{p,q}$

Let $(M;\partial_0 M,\partial_1 M)$ be an m–dimensional manifold triad with $\partial_0 M\neq\varnothing$. Let p and q be positive integers with $p+q+1=m$ and suppose that the inequalities of (9) are satisfied. In this section we will prove Theorem 3.6, which asserts that $|K^{\partial}(M)_{p,q}|$ is $\frac{1}{2}(r_{p,q}(M)-4-d)$–connected and that $\mathrm{lCM}(K^{\partial}(M)_{p,q})\geq\frac{1}{2}(r_{p,q}(M)-1-d)$, where $d=d(\pi_q(S^p))$ is the generating set rank. Our strategy is to compare $K^{\partial}(M)_{p,q}$ to the complex $L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))$ from Definition 5.4 associated to the Wall form $\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M)$. In view of Theorem 5.6 we will need to construct a simplicial map $K^{\partial}(M)_{p,q}\to L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))$ and then prove that it is highly connected. The construction of this map and proof of its high-connectivity is carried out over the course of this section, which contains the technical core of the paper.

6.1 A simplicial map

Let $(M;\partial_0 M,\partial_1 M)$ be an m–dimensional manifold triad with $\partial_0 M\neq\varnothing$ and let p and q be positive integers with $p+q+1=m$. We will construct a simplicial map

$$H_{p,q}\colon K^{\partial}(M)_{p,q}\to L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M)). \tag{17}$$

Recall the pair $(\widehat{V}_{p,q},\widehat{W}_{p,q})$. Recall from Construction 3.2 the core, $(B_{p,q},C_{p,q})\xhookrightarrow{\simeq}(\widehat{V}_{p,q},\widehat{W}_{p,q})$. Let $(a_0,b_0)\in S^p\times S^q$ be the basepoint used in the construction of $(B_{p,q},C_{p,q})$ from Construction 3.2. Let $\sigma\in\pi_{p+1}(\widehat{V}_{p,q},\widehat{W}_{p,q})$ be the class represented by the embedding

$$(D^{p+1}\times\{b_0\},S^p\times\{b_0\})\hookrightarrow(B_{p,q},C_{p,q})\hookrightarrow(\widehat{V}_{p,q},\widehat{W}_{p,q})$$

and let $\rho\in\pi_q(\widehat{W}_{p,q})$ be the class represented by the embedding

$$\{a_0\}\times S^q\hookrightarrow C_{p,q}\hookrightarrow\widehat{W}_{p,q}.$$

It follows directly from the construction of $\widehat{V}_{p,q}$ that $\lambda_{p,q}(\sigma,\rho)=1$ and $\alpha_q(\rho)=0$. Using this observation we may define a morphism of Wall forms,

$$T_{p,q}\colon \boldsymbol{W}\to\mathcal{W}^{\partial}_{p,q}(\widehat{V}_{p,q},\widehat{W}_{p,q}),\qquad a\mapsto\sigma,\quad b\mapsto\rho, \tag{18}$$

where $a\in\boldsymbol{W}_-$ and $b\in\boldsymbol{W}_+$ are the standard generators used in the construction of $\boldsymbol{W}$. We are now ready to define the simplicial map $H_{p,q}$ from (17). Let $\phi\in K^{\partial}(M)_{p,q}$ be a vertex. We define

$$H_{p,q}(\phi)\in L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))$$

to be the morphism of Wall forms given by the composite

$$\boldsymbol{W} \xrightarrow{T_{p,q}} \mathcal{W}^{\partial}_{p,q}(\widehat{V}_{p,q}, \widehat{W}_{p,q}) \xrightarrow{\phi_*} \mathcal{W}^{\partial}_{p,q}(M, \partial_1 M),$$

where the second map ϕ_* is the morphism of Wall forms induced by the embedding ϕ. It easy to see that this map $H_{p,q}$ is a simplicial map. Indeed, if

$$\phi_1, \phi_2\colon (\widehat{V}_{p,q}, \widehat{W}_{p,q}) \to (M, \partial_1 M)$$

are embeddings with disjoint images, then the sub-Wall forms of $\mathcal{W}^{\partial}_{p,q}(\widehat{V}_{p,q}, \widehat{W}_{p,q})$ given by the images of $(\phi_1)_*$ and $(\phi_2)_*$, respectively, are orthogonal. This follows immediately from the definitions of $\lambda^{\partial}_{p,q}$ and μ_q.

Later on we will need to use the following proposition:

Proposition 6.1 *Let $(M; \partial_0 M, \partial_1 M)$ be an m–dimensional manifold triad with $\partial_0 M \neq \varnothing$ and let p and q be positive integers with $p+q+1 = m$. Then*

$$r(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)) \geq r_{p,q}(M).$$

Proof Let g denote the rank $r_{p,q}(M)$. By the definition of $r_{p,q}(-)$ there exists an embedding

$$f\colon (V^g_{p,q}, W^g_{p,q}) \to (M, \partial_1 M).$$

For $i = 1, \dots, g$, let $\iota_i\colon (V_{p,q}, W_{p,q}) \hookrightarrow (V^g_{p,q}, W^g_{p,q})$ be the inclusion of $V_{p,q}$ into the i^{th} boundary connect-sum factor. Let

$$j\colon \mathcal{W}^{\partial}_{p,q}(V_{p,q}, W_{p,q}) \to \mathcal{W}^{\partial}_{p,q}(\widehat{V}_{p,q}, \widehat{W}_{p,q})$$

be the morphism of Wall forms induced by the inclusion $(V_{p,q}, W_{p,q}) \hookrightarrow (\widehat{V}_{p,q}, \widehat{W}_{p,q})$. Since this inclusion is a homotopy equivalence it follows that j is an isomorphism of Wall forms. Summing the maps $(\iota_i)_*$ yields the morphism of Wall-forms

$$\begin{tikzcd} \boldsymbol{W}^g \arrow[r, "(T_{p,q} \circ j^{-1})^{\oplus g}"] & {[\mathcal{W}^{\partial}_{p,q}(V_{p,q}, W_{p,q})]^{\oplus g}} \arrow[r, "\bigoplus_{i=1}^g (\iota_i)_*"] & \mathcal{W}^{\partial}_{p,q}(V^g_{p,q}, W^g_{p,q}) \arrow[d, "f_*"] \\ & & \mathcal{W}^{\partial}_{p,q}(M, \partial_1 M). \end{tikzcd}$$

The existence of this morphism implies that $r(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)) \geq g$. This concludes the proof of the proposition. $\square$

6.2 Cohen–Macaulay complexes and the link lifting property

Our next step is to prove that the simplicial map $H_{p,q}$ from (17) is highly connected. Let us first introduce some terminology and notation. Recall that for any set K, a *symmetric relation* is a subset $\mathcal{R} \subset K \times K$ that is invariant under the "coordinate permutation" map $(x, y) \mapsto (y, x)$. A subset $C \subset K$ is said to be *$\mathcal{R}$–related* if $(x, y) \in \mathcal{R}$ for any two elements $x, y \in C$. We will need to consider symmetric relations defined on the set of vertices of a simplicial complex. Let X be a simplicial complex and let $\mathcal{R} \subset X \times X$ be a symmetric relation on the vertices of X. The relation $\mathcal{R}$ is said to be *edge-compatible* if for any 1–simplex $\{x, y\} < X$, the pair (x, y) is an element of $\mathcal{R}$. For any simplex $\sigma < X$ we let $\mathrm{lk}_X(\sigma)$ denote the *link* of the simplex σ.

Definition 6.2 Let $f\colon X \to Y$ be a simplicial map between two simplicial complexes. Let $\mathcal{R} \subset X \times X$ be a symmetric relation on the set of vertices of the complex X. The map f is said to have the *link lifting property with respect to* $\mathcal{R}$ if the following condition holds:

- Let $y \in Y$ be any vertex and let $A \subset X$ be a finite $\mathcal{R}$–related set such that $f(a) \in \mathrm{lk}_Y(y)$ for all $a \in A$. Then, given another finite set of vertices $B \subset X$ (not necessarily $\mathcal{R}$–related), there exists a vertex $x \in X$ with $f(x) = y$ such that $a \in \mathrm{lk}_X(x)$ for all $a \in A$ and $(b, x) \in \mathcal{R}$ for all $b \in B$.

The following lemma below is a restatement of [16, Lemma 2.3]. Its proof in [16] abstracts and formalizes the argument used in the proof of [4, Lemma 5.4].

Lemma 6.3 *Let X and Y be simplicial complexes and let $f\colon X \to Y$ be a simplicial map. Let $\mathcal{R} \subset X \times X$ be an edge-compatible symmetric relation. Suppose that the following conditions are satisfied:*

(i) *f has the link lifting property with respect to $\mathcal{R}$;*

(ii) $\mathrm{lCM}(Y) \geq n$.

Then the induced map $|f|_*\colon \pi_j(|X|) \to \pi_j(|Y|)$ is injective for all $j \leq n-1$. Furthermore, suppose that in addition to properties (i) *and* (ii) *the map f satisfies:*

(iii) *$f(\mathrm{lk}_X(\zeta)) \leq \mathrm{lk}_Y(f(\zeta))$ for all simplices $\zeta < X$.*

Then it follows that $\mathrm{lCM}(X) \geq n$.

6.3 Proof of Theorem 3.6

We now give the proof of Theorem 3.6. We do this by applying Lemma 6.3 to the simplicial map $H_{p,q}\colon K^{\partial}(M)_{p,q} \to L(\mathcal{W}_{p,q}(M, \partial_1 M))$. In order to apply this lemma to $H_{p,q}$, we will need to define a suitable symmetric relation on the vertices of the complex $K^{\partial}(M)_{p,q}$.

Definition 6.4 We define $\mathcal{T} \subset K^{\partial}(M)_{p,q} \times K^{\partial}(M)_{p,q}$ to be the subset consisting of those pairs (ϕ_1, ϕ_2) such that $\phi_1(B_{p,q})$ and $\phi_2(B_{p,q})$ are transverse in M.

Clearly the subset $\mathcal{T} \subset K^{\partial}(M)_{p,q} \times K^{\partial}(M)_{p,q}$ is a symmetric relation on the vertices. Furthermore, this relation is edge-compatible, ie if the set $\{\phi_1, \phi_2\} \leq K^{\partial}(M)_{p,q}$ is a 1–simplex then the pair (ϕ_1, ϕ_2) is contained in $\mathcal{T}$. Before finally giving the proof of Theorem 3.6 we make the following remark:

Remark 6.5 The proof of Theorem 3.6 given below is analogous to [15, Proof of Theorem 2.3, page 26] but with some new technical ingredients added. The above-mentioned proof in [15] does not make explicit use of the link lifting property with respect to the relation $\mathcal{T}$, and for this reason the proof of [15, Theorem 2.3] has a gap. The argument provided below is the corrected version of the proof filling in that gap.

A similar gap exists in the preprint of Galatius and Randal-Williams [3, Proof of Lemma 4.3, page 11], which was filled by the authors in the published version [4, Theorem 2.4]. This technical theorem is a generalization of the *coloring lemma* of Hatcher and Wahl [8].

The terminology "link lifting property with respect to a relation" is not used by Galatius and Randal-Williams in their paper. This is terminology of our own and it was introduced with the purpose of extracting the main idea from the proof of their theorem so as to break the argument down into steps. Similar terminology and tools are also used in our other homological stability paper [16].

Proof of Theorem 3.6 Let $r_{p,q}(M) \geq g$ and let $d = d(\pi_q(S^p))$ be the generating set rank of $\pi_q(S^p)$. We will show that $|K(M)_{p,q}|$ is $\frac{1}{2}(g-4-d)$–connected and $\mathrm{lCM}(K^{\partial}(M)_{p,q}) \geq \frac{1}{2}(g-1-d)$. By Proposition 6.1,

$$r(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)) \geq r_{p,q}(M) \geq g,$$

and thus Theorem 5.6 implies that the space $|L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))|$ is $\frac{1}{2}(g-4-d)$–connected and that $\mathrm{lCM}(L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))) \geq \frac{1}{2}(g-1-d)$. The proof of the theorem will follow from Lemma 6.3 once we verify the following two properties:

(i) The map $H_{p,q}$ from (17) has the link lifting property with respect to $\mathcal{T}$ (see Definition 6.2).

(ii) $H_{p,q}(\mathrm{lk}_{K^{\partial}(M)_{p,q}}(\zeta)) \leq \mathrm{lk}_{L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))}(H_{p,q}(\zeta))$ for any $\zeta \in K^{\partial}(M)_{p,q}$.

We begin by verifying property (i). Let $f\colon \boldsymbol{W} \to \mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)$ be a morphism of Wall forms, which we consider to be a vertex of $L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))$. Let $\phi_1, \dots, \phi_k \in K^{\partial}(M)_{p,q}$ be a collection of $\mathcal{T}$–related vertices such that

$$H_{p,q}(\phi_i) \in \mathrm{lk}_{L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))}(f)$$

for $i = 1, \dots, k$. Let $\psi_1, \dots, \psi_m \in K^{\partial}(M)_{p,q}$ be another arbitrary collection of vertices (that is not necessarily $\mathcal{T}$–related). To show that $H_{p,q}$ has the link lifting property with respect to $\mathcal{T}$, we will construct a vertex $\phi \in K^{\partial}(M)_{p,q}$ with $H_{p,q}(\phi) = f$ such that $\phi_i \in \mathrm{lk}_{K^{\partial}(M)_{p,q}}(\phi)$ for $i = 1, \dots, k$ and $(\phi_j, \phi) \in \mathcal{T}$ for $j = 1, \dots, m$.

Let $\zeta\colon (D^{p+1}, S^p) \to (M, \partial_1 M)$ and $\xi\colon S^q \to \partial_1 M$ be embeddings that represent the classes

$$f_-(a) \in \pi_{p+1}(M, \partial_1 M) \quad \text{and} \quad f_+(b) \in \pi_q(\partial_1 M),$$

respectively (it follows from Lemma 4.1 that these classes may be represented by embeddings). Since the cores $\phi_1(B_{p,q}), \dots, \phi_k(B_{p,q})$ are transverse and $H_{p,q}(\phi_i) \in \mathrm{lk}_{K^{\partial}(M)_{p,q}}(f)$ for all $i = 1, \dots, k$, we may apply Corollaries 4.13 and 4.15 to deform the embeddings ζ and ξ through isotopies to new embeddings ζ' and ξ' such that the images $\zeta'(D^{p+1})$ and $\xi'(S^q)$ are disjoint from the cores $\phi_i(B_{p,q})$ for $i = 1, \dots, k$. Let $\partial_1 M'$ denote the complement $\partial_1 M \setminus \big(\bigcup_{i=1}^k \phi_i(C_{p,q})\big)$. Since $\lambda_{p,q}(f_-(a), f_+(b)) = 1$, we may apply the *Whitney trick* (see [12, Theorem 6.6]) to deform $\xi'\colon S^q \to \partial_1 M'$, through an isotopy of embeddings into $\partial_1 M'$, to a new embedding $\xi''\colon S^q \to \partial_1 M'$ with the property that $\zeta'(\partial D^{p+1})$ and $\xi''(S^q)$ intersect transversally at exactly one point in $\partial_1 M'$. By *Thom's transversality theorem* [9, Theorem 2.1] we may further arrange $\zeta'(D^{p+1})$ and $\xi''(S^q)$ to be transverse to each of the cores $\psi_1(B_{p,q}), \dots, \psi_m(B_{p,q})$ while keeping them disjoint from $\phi_1(B_{p,q}), \dots, \phi_k(B_{p,q})$.

It follows by the above construction that the pair of subspaces

$$(\zeta'(D^{p+1}) \cup \xi''(S^q), \zeta'(\partial D^{p+1}) \cup \xi''(S^q))$$

is homeomorphic to the pair $(D^{p+1} \vee S^q, S^p \vee S^q)$. Now, both $\zeta'(S^p)$ and $\xi''(S^q)$ have trivial normal bundles in $\partial_1 M$; the normal bundle of $\xi''(S^q)$ is trivial because $\alpha_q(f_+(b)) = 0$ and the normal bundle of $\zeta'(S^p)$ is trivial because it bounds the disk $(\zeta'(D^{p+1}), \zeta'(S^p)) \subset (M, \partial_1 M)$, which must have trivial normal bundle since the disk is contractible. Let $U \subset \partial_1 M$ be a regular neighborhood of $\zeta'(S^p) \cup \xi''(S^q)$

(which is a wedge of a p–sphere with a q–sphere). The manifold U is diffeomorphic to the manifold obtained by forming the push-out of the diagram

$$S^p \times D^q \xleftarrow{i_p \times \mathrm{Id}_{D^q}} D^p \times D^q \xrightarrow{\mathrm{Id}_{D^p} \times i_q} D^p \times S^q,$$

where $i_p\colon D^p \hookrightarrow S^p$ and $i_q\colon D^q \hookrightarrow S^q$ are embeddings. It is easily seen that this push-out is diffeomorphic to $W_{p,q} = S^p \times S^q \setminus \mathrm{Int}(D^{p+q})$ after smoothing corners, thus we have a diffeomorphism $U \cong W_{p,q}$. By shrinking U down arbitrarily close to its "core" $\zeta'(S^p) \cup \xi'(S^q) \cong S^p \vee S^p$, we may assume that U is disjoint from $\phi_i(C_{p,q})$ for all $i = 1, \dots, k$.

Let $\overline{U} \subset M$ be the submanifold diffeomorphic to $U \times [0,1]$ obtained by adding a collar to $U \subset \partial_1 M$ in M. We extend the embedding ζ' to an embedding

$$\bar{\zeta}\colon (D^{p+1} \times D^q, S^p \times D^q) \to (M, \partial_1 M)$$

such that $\bar{\zeta}(S^p \times D^q) \subset U$ and $\bar{\zeta}|_{S^p \times \{0\}} = \zeta'$. We let $V \subset M$ be the subspace obtained by forming the union of $\overline{U}$ with $\bar{\zeta}(D^{p+1} \times D^q)$. By shrinking $\bar{\zeta}(D^{p+1} \times D^q)$ down to $\bar{\zeta}(D^{p+1} \times \{0\})$, we may assume that V is again disjoint from $\phi_i(B_{p,q})$ for all $i = 1, \dots, k$. By Proposition 6.6 (proven below) there is a diffeomorphism $V \cong V_{p,q} = D^{p+1} \times S^q$. Furthermore, the boundary of V has the decomposition $\partial V = \partial_0 V \cup \partial_1 V$, with $\partial_1 V = \partial V \cap \partial_1 M$ and $\partial_0 V = \partial V \setminus \mathrm{Int}(\partial_1 V)$. We have diffeomorphisms $\partial_1 V \cong W_{p,q}$ and $\partial_0 V \cong D^{p+q}$. Using the identifications

$$V \cong V_{p,q}, \quad \partial_0 V \cong D^{p+q} \quad \text{and} \quad \partial_1 V \cong W_{p,q},$$

we obtain an embedding $(V_{p,q}, W_{p,q}) \hookrightarrow (M, \partial_1 M)$ with image equal to $(V, \partial_1 V) \subset (M, \partial_1 M)$.

We then choose an embedding $\gamma\colon [0,1] \hookrightarrow \partial_1 M$, disjoint from $\phi_i(B_{p,q})$ for all $i = 1, \dots, k$, and with $\gamma(0) \in \partial_{0,1} V$ and $\gamma(1) \in \partial_0 M$. Taking the union of a thickening of this arc with V yields an embedding $(\widehat{V}_{p,q}, \widehat{W}_{p,q}) \to (M, \partial_1 M)$ that satisfies condition (i) of Definition 3.3. This in turn yields a vertex $\phi \in K^{\partial}(M)_{p,q}$ with $H_{p,q}(\phi) = f$ such that $\phi(B_{p,q}) \cap \phi_i(B_{p,q}) = \varnothing$ for all $i = 1, \dots, k$. It follows that ϕ_i is contained in the link of ϕ for $i = 1, \dots, k$. By construction, $\phi(B_{p,q})$ is transverse to $\psi_j(B_{p,q})$ for all $j = 1, \dots, m$. This proves that the map $H_{p,q}$ has the link lifting property with respect to $\mathcal{T}$.

By Lemma 6.3 it follows that the induced map

$$\pi_i(|K^{\partial}(M)_{p,q}|) \to \pi_i(|L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)|)$$

is injective for all $i < \frac{1}{2}(g-1-d)$ and thus $|K^{\partial}(M)_{p,q}|)$ is $\frac{1}{2}(g-4-d)$–connected. In order to conclude that $\mathrm{lCM}(K^{\partial}(M)_{p,q}) \geq \frac{1}{2}(g-1-d)$, we need to establish property (ii). We need to verify that

$$H_{p,q}(\mathrm{lk}_{K^{\partial}(M)_{p,q}}(\zeta)) \leq \mathrm{lk}_{L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))}(H_{p,q}(\zeta))$$

for any simplex $\zeta \in K^{\partial}(M)_{p,q}$. This property follows immediately from the fact that if $\phi_1, \phi_2 \in K^{\partial}(M)_{p,q}$ are such that $\phi_1(B_{p,q}) \cap \phi_2(B_{p,q}) = \varnothing$, then the morphisms of Wall forms $H_{p,q}(\phi_1)$ and $H_{p,q}(\phi_2)$ are orthogonal. This concludes the proof of the theorem. □

There is one claim in the above proof that still needs verification, namely that the manifold V that we constructed is diffeomorphic to $V_{p,q}$. We now prove that claim. Pick a basepoint $(a,b) \in S^p \times S^q$ such that $(S^p \times \{b\}) \cup (\{a\} \times S^q) \subset W_{p,q} = S^p \times S^q \setminus \mathrm{Int}(D^{p+q})$. Let $f\colon S^p \to W_{p,q}$ be the embedding given by the chain of inclusions $S^p \hookrightarrow S^p \times \{b\} \hookrightarrow (S^p \times \{b\}) \cup (\{a\} \times S^q) \hookrightarrow W_{p,q}$. This embedding has a trivial normal bundle. Let $f'\colon S^p \times D^q \to W_{p,q}$ be an embedding with $f'|_{S^p \times \{0\}} = f$. Finally, let

$$\bar{f}\colon S^p \times D^q \to W_{p,q} \times [0,1]$$

denote the embedding given by $S^p \times D^q \xrightarrow{f'} W_{p,q} \hookrightarrow W_{p,q} \times \{1\} \hookrightarrow W_{p,q} \times [0,1]$. Let V denote the manifold obtained by attaching the handle $D^{p+1} \times D^q$ to $W_{p,q} \times [0,1]$ along the embedding $\bar{f}$, ie

$$V = (W_{p,q} \times [0,1]) \cup_{\bar{f}} (D^{p+1} \times D^q).$$

The boundary of $W_{p,q}$ has the decomposition $\partial V = (W_{p,q} \times \{0\}) \cup (\partial W_{p,q} \times [0,1]) \cup W'$, where W' is the manifold obtained from $W_{p,q}$ by performing surgery along the embedding $\bar{f}$. Consequently, the manifold W' is diffeomorphic to a disk D^{p+q} and so ∂V is diffeomorphic to $W_{p,q}$. The following proposition was used in the above proof of Theorem 3.6.

Proposition 6.6 *Let $p, q \in \mathbb{N}$ satisfy the inequality $|q-p| < \min\{p-2, q-3\}$. Then the manifold V constructed above is diffeomorphic to $D^{p+1} \times S^q = V_{p,q}$.*

Proof Let $g\colon S^q \to V$ be the embedding given by the chain of inclusions

$$S^q \hookrightarrow \{a\} \times S^q \hookrightarrow W_{p,q} \hookrightarrow W_{p,q} \times \{\tfrac{1}{2}\} \hookrightarrow W_{p,q} \times [0,1] \hookrightarrow V.$$

This embedding has trivial normal bundle and so extends to an embedding

$$\bar{g}\colon D^{p+1} \times S^q \to \mathrm{Int}(V).$$

Furthermore, $\overline{g}$ induces an isomorphism on homology and thus is a homotopy equivalence since V is simply connected. Let X be the complement $V \setminus \mathrm{Int}(\overline{g}(D^{p+1} \times S^q))$. The boundary of X decomposes as the disjoint union $\partial X = \partial V \sqcup \partial \overline{g}(D^{p+1} \times S^q)$. By excision we have the isomorphism $0 = H_i(V, \overline{g}(D^{p+1} \times S^q)) \cong H_i(X, \partial \overline{g}(D^{p+1} \times S^q))$ for all $i \in \mathbb{Z}_{\geq 0}$, and then by *Lefschetz duality* we obtain

$$0 = H_i(X, \partial \overline{g}(D^{p+1} \times S^q)) \cong H^i(X, \partial V) = 0 \quad \text{for all } i \in \mathbb{Z}_{\geq 0}.$$

Since X and ∂V are simply connected, it follows that X is an h–cobordism between ∂V and the manifold $\partial \overline{g}(D^{p+1} \times S^q)$. Since $p + q + 1 \geq 6$, it follows by the *h–cobordism theorem* [12] that X is diffeomorphic to the cylinder $\partial V \times [0, 1]$. By shrinking down this cylinder, it follows that the embedding $\overline{g}\colon D^{p+1} \times S^q \to V$ is isotopic to a diffeomorphism $D^{p+1} \times S^q \cong V$. This completes the proof of the proposition. □

With the above proposition established, the proof of Theorem 3.6 is now complete. By Corollary 3.8, it now follows that the geometric realization $|\overline{K}^{\partial}(M)_{p,q}|$ is $\frac{1}{2}[g - 4 - d(\pi_q(S^p))]$–connected whenever $r_{p,q}(M) \geq g$.

7 Homological stability

With our main technical result Theorem 3.6 established, in this section we show how this theorem implies our homological stability theorem, Theorem 2.7. The constructions and arguments in this section are essentially the same as what was done in [4, Section 6] and so we merely provide an outline while referring the reader there for details.

7.1 A model for $\mathrm{BDiff}(M, \partial_0 M)$

Let $(M; \partial_0 M, \partial_1 M)$ be a compact manifold triad of dimension m with $\partial_0 M$ and $\partial_1 M$ nonempty. We construct a concrete model for $\mathrm{BDiff}(M, \partial_0 M)$. Fix once and for all an embedding $\theta\colon (\partial_0 M, \partial_{0,1} M) \to (\mathbb{R}^{\infty}_{+}, \partial \mathbb{R}^{\infty}_{+})$ and let $(S, \partial S)$ denote the submanifold pair $(\theta(\partial_0 M), \theta(\partial_{0,1} M)) \subset (\mathbb{R}^{\infty}_{+}, \partial \mathbb{R}^{\infty}_{+})$.

Definition 7.1 We define $\mathcal{M}(M)$ to be the set of compact m–dimensional submanifold triads

$$(M'; \partial_0 M', \partial_1 M') \subset ([0, \infty) \times \mathbb{R}^{\infty}_{+}; \{0\} \times \mathbb{R}^{\infty}_{+}, [0, \infty) \times \partial \mathbb{R}^{\infty}_{+})$$

such that:

(i) $(\partial_0 M', \partial_{0,1} M') = (S, \partial S)$.

(ii) $(M', \partial_1 M')$ contains $([0,\epsilon)\times S, [0,\epsilon)\times \partial S)$ for some $\epsilon > 0$.

(iii) $(M'; \partial_0 M', \partial_1 M')$ is diffeomorphic to $(M; \partial_0 M, \partial_1 M)$.

Denote by $\mathcal{E}(M)$ the space of smooth (neat) embeddings

$$(M; \partial_0 M, \partial_1 M) \to ([0,\infty)\times\mathbb{R}^{\infty}_{+}; \{0\}\times\mathbb{R}^{\infty}_{+}, [0,\infty)\times\partial\mathbb{R}^{\infty}_{+}),$$

topologized in the C^{∞}–topology. The space $\mathcal{M}(M)$ is topologized as a quotient of the space $\mathcal{E}(M)$, where two embeddings are identified if they have the same image.

It follows from Definition 7.1 that $\mathcal{M}(M)$ equals the orbit space $\mathcal{E}(M)/\mathrm{Diff}(M, \partial_0 M)$. By [14, Lemma A.1], it follows that the quotient map $\mathcal{E}(M) \to \mathcal{E}(M)/\mathrm{Diff}(M, \partial_0 M) = \mathcal{M}(M)$ is a locally trivial fiber bundle. In [5] it is proven that the space $\mathcal{E}(M)$ is weakly contractible. These two facts together imply that there is a weak homotopy equivalence $\mathcal{M}(M) \simeq \mathrm{BDiff}(M, \partial_0 M)$, and thus we may take the space $\mathcal{M}(M)$ to be a model for the classifying space of the diffeomorphism group $\mathrm{Diff}(M, \partial_0 M)$.

Now let p and q be positive integers with $p+q+1 = m$. Recall from Section 2 the relative cobordism

$$(K_{p,q}, \partial_1 K_{p,q})\colon (\partial_0 M\times\{0\}, \partial_{0,1} M\times\{0\}) \rightsquigarrow (\partial_0 M\times\{1\}, \partial_{0,1} M\times\{1\}).$$

Choose an embedding

$$\alpha\colon (K_{p,q}; \partial_0 K_{p,q}, \partial_1 K_{p,q}) \to ([0,1]\times\mathbb{R}^{\infty}_{+}; \{0,1\}\times\mathbb{R}^{\infty}_{+}, [0,1]\times\partial\mathbb{R}^{\infty}_{+})$$

that satisfies $\alpha(i,x) = (i,\theta(x))$ for all $(i,x)\in\{0,1\}\times\partial_0 M = \partial_0 K_{p,q}$. For a submanifold $M' \subset [0,\infty)\times\mathbb{R}^{\infty}_{+}$, denote by $M' + e_1 \subset [1,\infty)\times\mathbb{R}^{\infty}_{+}$ the submanifold obtained by translating M' over 1–unit in the first coordinate. For $M' \in \mathcal{M}(M)$, the submanifold $\alpha(K_{p,q}) \cup (M' \cup e_1) \subset [0,\infty)\times\mathbb{R}^{\infty}_{+}$ is an element of the space $\mathcal{M}(M\cup_{\partial_0} K_{p,q})$, and thus we have a continuous map

$$s_{p,q}\colon \mathcal{M}(M) \to \mathcal{M}(M\cup_{\partial_0} K_{p,q}), \quad M' \mapsto \alpha(K_{p,q}) \cup (M' + e_1). \tag{19}$$

The construction of $s_{p,q}$ depends on the choice of embedding α. Any two such embeddings are isotopic and thus it follows that the homotopy class of $s_{p,q}$ does not depend on any such choice.

7.2 A semisimplicial resolution

Let $(M; \partial_0 M, \partial_1 M)$ be an m–dimensional manifold triad as in Section 7.1. Fix $p, q \in \mathbb{N}$ such that $p + q + 1 = m$. We now construct a semisimplicial resolution of the moduli space $\mathcal{M}(M)$. Choose once and for all a coordinate patch $c\colon (\mathbb{R}^{m-1}_{+}, \partial\mathbb{R}^{m-1}_{+}) \hookrightarrow (S, \partial S)$. Such a coordinate patch induces for each $M' \in \mathcal{M}(M)$ a germ of an embedding $[0,1) \times \mathbb{R}^{m-1}_{+} \to M'$ as in the definition of $\overline{K}^{\partial}_{\bullet}(M')_{p,q}$ (see Definition 3.7). For each nonnegative integer l, we define $X^{\partial}_{l}(M)_{p,q}$ to be the set of pairs (M', ϕ) where $M' \in \mathcal{M}(M)$ and $\phi \in \overline{K}^{\partial}_{l}(M')_{p,q}$. We topologize $X^{\partial}_{l}(M)_{p,q}$ by identifying it with the quotient space

$$(\mathcal{E}(M) \times \overline{K}^{\partial}_{l}(M)_{p,q})/\mathrm{Diff}(M, \partial_0 M), \tag{20}$$

where $\mathrm{Diff}(M, \partial_0 M)$ acts on $\mathcal{E}(M) \times \overline{K}^{\partial}_{l}(M)_{p,q}$ by the diagonal action. The identification of $X^{\partial}_{l}(M)_{p,q}$ with (20) is given by the map

$$X^{\partial}_{l}(M)_{p,q} \to (\mathcal{E}(M) \times \overline{K}^{\partial}_{l}(M)_{p,q})/\mathrm{Diff}(M, \partial_0 M), \quad (M', \phi) \mapsto [\Psi, \phi],$$

where $\Psi \in \mathcal{E}(M)$ is some choice of embedding with $\Psi(M) = M'$.

The assignments $[l] \mapsto X^{\partial}_{l}(M)_{p,q}$ make $X^{\partial}_{\bullet}(M)_{p,q}$ into a semisimplicial space, where the face maps are induced by the face maps in $\overline{K}^{\partial}_{\bullet}(M)_{p,q}$. The forgetful maps $X^{\partial}_{l}(M)_{p,q} \to \mathcal{M}(M)$, $(M', \phi) \mapsto M'$, assemble to yield the augmented semisimplicial space $X^{\partial}_{\bullet}(M)_{p,q} \to X^{\partial}_{-1}(M)_{p,q}$, where the space $X^{\partial}_{-1}(M)_{p,q}$ is set equal to $\mathcal{M}(M)$. We have the following proposition:

Proposition 7.2 *The augmentation map*

$$|X^{\partial}_{\bullet}(M)_{p,q}| \to X^{\partial}_{-1}(M)_{p,q}$$

is $\frac{1}{2}(r_{p,q}(M) - 2 - d)$–connected, where $d = d(\pi_p(S^q))$ is the generating set length.

Proof For each $l \in \mathbb{Z}_{\geq 0}$ the forgetful map $X^{\partial}_{l}(M)_{p,q} \to \mathcal{M}(M)$ is a locally trivial fiber bundle with fiber given by the space $\overline{K}^{\partial}_{l}(M)_{p,q}$. By [2, Lemma 2.14] it follows that the sequence

$$|\overline{K}^{\partial}_{\bullet}(M)_{p,q}| \to |X^{\partial}_{\bullet}(M)_{p,q}| \to X^{\partial}_{-1}(M)_{p,q}$$

is a homotopy fiber sequence. The proposition then follows from Corollary 3.8 using the long exact sequence on homotopy groups associated to a fiber sequence. □

7.3 Proof of Theorem 2.7

We now will show how to use the augmented semisimplicial space $X^{\partial}_{\bullet}(M)_{p,q} \to X^{\partial}_{-1}(M)_{p,q}$ to complete the proof of Theorem 2.7. First, we fix some new notation which will make the steps of the proof easier to state. For what follows let $(M; \partial_0 M, \partial_1 M)$ be a compact m–dimensional manifold triad with nonempty boundary. As in the previous sections, choose positive integers p and q with $p+q+1 = m$ that satisfy the inequalities (7) with respect to $(M, \partial_1 M)$. We work with the same choice of p and q for the rest of the section. For each $g \in \mathbb{N}$ we denote by M_g the manifold obtained by forming the boundary connected sum of M with $(D^{p+1} \times S^q)^{\# g}$ along the face $\partial_1 M$. Clearly we have $r_{p,q}(M_g) \geq g$. We consider the spaces $\mathcal{M}(M_g)$. For each $g \in \mathbb{N}$ we have the stabilization map $s_{p,q}\colon \mathcal{M}(M_g) \to \mathcal{M}(M_{g+1})$. Using the weak homotopy equivalences $\mathcal{M}(M_g) \simeq \mathrm{BDiff}(M_g, \partial_0 M_g)$, Theorem 2.7 translates to the following statement:

Theorem 7.3 *The induced map $(s_{p,q})_*\colon H_k(\mathcal{M}(M_g); \mathbb{Z}) \to H_k(\mathcal{M}(M_{g+1}); \mathbb{Z})$ is an isomorphism when $k \leq \frac{1}{2}(g-3-d)$ and an epimorphism when $k \leq \frac{1}{2}(g-1-d)$.*

Since $r_{p,q}(M_g) \geq g$, Proposition 7.2 implies that the map $|X^{\partial}_{\bullet}(M_g)_{p,q}| \to X^{\partial}_{-1}(M_g)_{p,q}$ is $\frac{1}{2}(g-2-d)$–connected. For each pair of integers $g, k \in \mathbb{Z}_{\geq 0}$ with $k < g$, there is a map

$$F^g_k\colon \mathcal{M}(M_{g-k-1}) \to X^{\partial}_k(M_g)_{p,q} \tag{21}$$

defined as follows. Let $K^k_{p,q} \subset [0, k+1] \times \mathbb{R}^{\infty}_{+}$ denote the $(k+1)$–fold concatenation of the submanifold $\alpha(K_{p,q}) \subset [0,1] \times \mathbb{R}^{\infty}_{+}$ used in the construction of stabilization map (19), ie

$$K^k_{p,q} = \alpha(K_{p,q}) \cup [\alpha(K_{p,q}) + e_1] \cup \cdots \cup [\alpha(K_{p,q}) + k \cdot e_1].$$

For each $k \in \mathbb{Z}_{\geq 0}$ we fix a k–simplex $(\zeta_0, \ldots, \zeta_k) \in \overline{K}^{\partial}(K^k_{p,q})_{p,q}$. The map F^g_k from (21) is defined by the formula $F^g_k(M') = ((\zeta_0, \ldots, \zeta_k), K^k_{p,q} \cup [(k+1) \cdot e_1 + M'])$. It follows directly from the definition of F^g_k that for each pair $k < g$, the diagram

$$\begin{tikzcd} \mathcal{M}(M_{g-k-1}) \arrow[r, "s_{p,q}"] \arrow[d, "F^g_k"] & \mathcal{M}(M_{g-k}) \arrow[d, "F^g_k"] \\ X^{\partial}_k(M_g)_{p,q} \arrow[r, "d_k"] & X^{\partial}_{k-1}(M_g)_{p,q} \end{tikzcd} \tag{22}$$

is commutative. The following proposition is proven in the same way as [3, Propositions 5.3 and 5.5]. For this reason we omit the proof and refer the reader to these analogous results from [3; 4] for details.

Proposition 7.4 *Let $g \geq 4+d$. We have the following:*

(i) *For each $k < g$, the map $F^g_k\colon \mathcal{M}(M_{g-k-1}) \to X^{\partial}_k(M_g)_{p,q}$ is a weak homotopy equivalence.*

(ii) *The face maps $d_i\colon X^{\partial}_k(M_g)_{p,q} \to X^{\partial}_{k-1}(M_g)_{p,q}$ are weakly homotopic.*

To finish the proof of Theorem 7.3, consider the spectral sequence associated to the augmented semisimplicial space $X^{\partial}_{\bullet}(M_g)_{p,q} \to X^{\partial}_{-1}(M_g)_{p,q}$ with E^1–term given by $E^1_{j,l} = H_j(X^{\partial}_l(M_g)_{p,q};\mathbb{Z})$ for $l \geq -1$ and $j \geq 0$. The differential is given by $d^1 = \sum(-1)^i(d_i)_*$, where $(d_i)_*$ is the map on homology induced by the i^{th} face map in $X^{\partial}_{\bullet}(M_g)_{p,q}$. The group $E^{\infty}_{j,l}$ is a subquotient of the relative homology group

$$H_{j+l+1}(X^{\partial}_{-1}(M_g)_{p,q}, |X^{\partial}_{\bullet}(M_g)_{p,q}|;\mathbb{Z}).$$

Proposition 7.4 together with Proposition 7.2 and commutativity of diagram (22) imply the following facts:

(a) For $g \geq 4+d$, there are isomorphisms $E^1_{j,l} \cong H_l(\mathcal{M}(M_{g-j-1});\mathbb{Z})$.

(b) The differential

$$d^1\colon H_l(\mathcal{M}(M_{g-j-1});\mathbb{Z}) \cong E^1_{j,l} \to E^1_{j-1,l} \cong H_l(\mathcal{M}(M_{g-j});\mathbb{Z})$$

is equal to $(s_{p,q})_*$ when j is even and is equal to zero when j is odd.

(c) The term $E^{\infty}_{j,l}$ is equal to 0 when $j+l \leq \frac{1}{2}(g-2-d)$.

To complete the proof one uses (c) to prove that the differential $d^1\colon E^1_{2j,l} \to E^1_{2j-1,l}$ is an isomorphism when $0 < j \leq \frac{1}{2}(g-3-d)$ and an epimorphism when $0 < j \leq \frac{1}{2}(g-1-d)$. This is done by carrying out the exact same inductive argument given in [3, Section 5.2, Proof of Theorem 1.2]. This establishes Theorem 7.3 and the main result of this paper, Theorem 2.7.

8 Tangential structures

In this section we prove an analogue of Theorem 1.1 for the moduli spaces of manifolds equipped with tangential structures. Recall that a tangential structure is a map $\theta\colon B \to$

$BO(d)$. A θ–structure on a d–dimensional manifold M is a bundle map (fiberwise linear isomorphism) $\ell\colon TM \to \theta^*\gamma^d$. More generally, if M is an l–dimensional manifold with $l \leq d$, then a θ–structure on M is a bundle map $TM \oplus \epsilon^{d-l} \to \theta^*\gamma^d$.

Fix a tangential structure $\theta\colon B \to BO(d)$. Let M be a d–dimensional manifold with boundary. Let $P \subset \partial M$ be a codimension-0 submanifold and let $\ell_P\colon TP \oplus \epsilon^1 \to \theta^*\gamma^d$ be a θ–structure. We define

$$\mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P) \subset \mathrm{Bun}(TM, \theta^*\gamma^d)$$

to be the subspace consisting of those θ–structures on M that agree with ℓ_P when restricted to P. The formula $\mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P) \times \mathrm{Diff}(M, P) \to \mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P)$, $(\ell, f) \mapsto \ell \circ Df$, defines a continuous action of the topological group $\mathrm{Diff}(M, P)$ on the space $\mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P)$. We define $\mathrm{BDiff}_\theta(M, \ell_P)$ to be the *homotopy quotient* $\mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P)/\!\!/\mathrm{Diff}(M, P)$.

We proceed to construct stabilization maps analogous to those defined in Section 2. This will require us to make some choices. Let $p, q \in \mathbb{Z}_{\geq 0}$ be integers such that $p + q + 1 = d$.

Definition 8.1 Fix once and for all a bundle map $\tau\colon \mathbb{R}^d \to \theta^*\gamma^d$. This choice determines a canonical θ–structure on any framed d–dimensional manifold. If X is any such framed d–dimensional manifold, we denote this canonical θ–structure on X by ℓ^τ_X.

Notice that since the manifold $V^1_{p,q} \cong D^{p+1} \times S^q$ admits an embedding into $\mathbb{R}^{p+q+1}$, $V^1_{p,q}$ is parallelizable. Choose once and for all a framing, $TV^1_{p,q} \cong V^1_{p,q} \times \mathbb{R}^{p+q+1}$, and consider the canonical θ–structure $\ell^\tau_{V^1_{p,q}}$ induced by this chosen framing. We call an arbitrary θ–structure $\ell\colon TV^1_{p,q} \to \theta^*\gamma^d$ *standard* if it is homotopic to the canonical θ–structure $\ell^\tau_{V^1_{p,q}}$ defined above.

Let $(M; \partial_0 M, \partial_1 M)$ be a d–dimensional manifold triad with $\partial_0 M$ and $\partial_1 M$ nonempty. Let $(K_{p,q}; \partial_1 K_{p,q})$ be the relative cobordism between

$$(\partial_0 M \times \{0\}, \partial_{0,1} M \times \{0\}) \quad \text{and} \quad (\partial_0 M \times \{1\}, \partial_{0,1} M \times \{1\})$$

introduced in Section 2. Let us write $P := \partial_0 M$ and fix a θ–structure $\ell_P\colon TP \oplus \epsilon^1 \to \theta^*\gamma^d$. Choose a θ–structure $\ell_{K_{p,q}}\colon TK_{p,q} \to \theta^*\gamma^d$ that agrees with ℓ_P on both components of

$$\partial_0 K_{p,q} = \partial_0 M \times \{0, 1\},$$

and that is standard (in the sense of Definition 8.1) when restricted to $V^1_{p,q}$ (where $V^1_{p,q}$ is considered as a submanifold of $K_{p,q} = (\partial_0 M \times [0,1]) \natural V^1_{p,q}$). With this choice of θ–structure we obtain a map

$$\mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P) \to \mathrm{Bun}(T(M \cup_P K_{p,q}), \theta^*\gamma^d; \ell_P), \quad \ell \mapsto \ell \cup \ell_{K_{p,q}}.$$

This map is $\mathrm{Diff}(M, P)$–equivariant, and thus it induces a map

$$s^{\theta}_{p,q}\colon \mathrm{BDiff}_{\theta}(M, \ell_P) \to \mathrm{BDiff}_{\theta}(M \cup K_{p,q}, \ell_P). \tag{23}$$

In addition to the inequalities imposed on p and q in the statement of Theorem 2.7, the following theorem will require us to also impose the further condition $q \leq p$, and to assume that $\theta\colon B \to BO(d)$ is such that the space B is q–connected.

Theorem 8.2 *Let p and q be positive integers with $p + q + 1 = d = \dim(M)$ and suppose that the inequalities of (7) are satisfied. Suppose further that $q \leq p$ and that $\theta\colon B \to BO(d)$ is such that B is q–connected. Suppose that $r_{p,q}(M) \geq g$. Then the homomorphism*

$$(s^{\theta}_{p,q})_*\colon H_k(\mathrm{BDiff}_{\theta}(M, \ell_P); \mathbb{Z}) \to H_k(\mathrm{BDiff}_{\theta}(M \cup_P K_{p,q}, \ell_P); \mathbb{Z})$$

is an isomorphism when $k \leq \frac{1}{2}(g-4)$ and an epimorphism when $k \leq \frac{1}{2}(g-2)$.

The proof of the above theorem is similar to Theorem 2.7 and requires only slight modifications. The main ingredient of the proof is to show that the tangentially structured analogue of the simplicial complex $K^{\partial}(M)_{p,q}$ (see Definition 8.3 below) is highly connected relative to the rank $r_{p,q}(M)$. With this high-connectivity established, the proof of Theorem 8.2 follows in exactly the same way as the proof of Theorem 2.7, as outlined in Section 7. We will show explicitly how to prove high-connectivity of the complex (Proposition 8.5), and refer the reader to [4, Section 7] for the rest of the argument, which by this point is standard.

We proceed to construct a simplicial complex (and related semisimplicial spaces) analogous to the one constructed in Section 3. Let P and $\ell_P\colon TP \oplus \epsilon^1 \to \theta^*\gamma^d$ be as in the statement of Theorem 8.2. Let $(M; \partial_0 M, \partial_1 M)$ be a d–dimensional compact manifold triad, with $\partial_0 M = P$. Choose a θ–structure $\ell_M \in \mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P)$. The definition below should be compared to [4, Definition 7.10].

Definition 8.3 Let $a\colon [0,1) \times \mathbb{R}^{d-1}_+ \to M$ be an embedding with $a^{-1}(\partial_0 M) = \{0\} \times \mathbb{R}^{d-1}_+$ and $a^{-1}(\partial_1 M) = [0,1) \times \partial\mathbb{R}^{d-1}_+$. For each pair of positive integers p

and q with $p+q+1=d$, we define a simplicial complex $K^{\partial}(M,\ell_M,a)_{p,q}$ as follows:

(i) A vertex in $K^{\partial}(M,\ell_M,a)_{p,q}$ is defined to be a triple (t,ϕ,γ), where (t,ϕ) is an element of $K^{\partial}(M,a)_{p,q}$ and γ is a path in $\mathrm{Bun}(\hat{V}_{p,q},\theta^*\gamma^d)$, starting at $\phi^*\ell_M$, ending at $\ell_{\hat{V}_{p,q}}$ and constant on the subset $D^{p+q}_+\times\{0\}\subset\hat{V}_{p,q}$.

(ii) A set of vertices $\{(t_0,\phi_0,\gamma_0),\dots,(t_l,\phi_l,\gamma_l)\}$ forms an l–simplex if $t_i\neq t_j$ and $\phi_i(B_{p,q})\cap\phi_j(B_{p,q})=\varnothing$ whenever $i\neq j$, just as in Definition 3.3.

As in Section 3 we will also need to work with a semisimplicial space $K^{\partial}_{\bullet}(M,\ell_M,a)_{p,q}$ analogous to the simplicial complex defined above.

(i) The space of 0–simplices $K^{\partial}_0(M,\ell_M,a)_{p,q}$ is defined to have the same underlying set as the set of vertices of the simplicial complex $K^{\partial}(M,\ell_M,a)_{p,q}$.

(ii) The space of l–simplices $K^{\partial}_l(M,\ell_M,a)\subset(K^{\partial}_0(M,\ell_M,a))^{l+1}$ consists of the ordered $(l+1)$–tuples

$$((t_0,\phi_0,\gamma_0),\dots,(t_l,\phi_l,\gamma_l))$$

such that $t_0<\dots<t_l$ and $\phi_i(B_{p,q})\cap\phi_j(B_{p,q})=\varnothing$ when $i\neq j$.

The spaces $K^{\partial}_l(M,\ell_M,a)\subset(\mathbb{R}\times\mathrm{Emb}(\hat{V}_{p,q},M)\times\mathrm{Bun}(T\hat{V}_{p,q},\theta^*\gamma^d))^{l+1}$ are topologized using the C^∞–topology on the space of embeddings and the compact–open topology on the space of bundle maps. The assignments $[l]\mapsto K^{\partial}_l(M,\ell_M,a)$ define a semisimplicial space, denoted by $K^{\partial}_{\bullet}(M,\ell_M,a)_{p,q}$, with face maps defined the same way as in Definition 3.7.

Finally, the subsemisimplicial space $\overline{K}^{\partial}_{\bullet}(M,\ell_M,a)_{p,q}\subset K^{\partial}_{\bullet}(M,\ell_M,a)_{p,q}$ is defined to be the subsemisimplicial space consisting of all simplices

$$((t_0,\phi_0,\gamma_0),\dots,(t_l,\phi_l,\gamma_l))\in K^{\partial}_l(M,\ell_M,a)$$

such that $\phi_i(\hat{V}_{p,q})\cap\phi_j(\hat{V}_{p,q})=\varnothing$ whenever $i\neq j$.

The key technical result that we will need is the lemma stated below. This lemma is the source of the requirement that $q\leq p$ and that the space B be q–connected. Fix a θ–structure ℓ_D on the disk D^{d-1} and fix an embedding $D^{d-1}\hookrightarrow\partial V_{p,q}$. We consider the space $\mathrm{Bun}(TV_{p,q},\theta^*\gamma^d;\ell_D)$.

Lemma 8.4 *Suppose that $q\leq p$ and that $\theta\colon B\to BO(d)$ is chosen so that B is q–connected. Then, given any two elements $\ell_1,\ell_2\in\mathrm{Bun}(TV_{p,q},\theta^*\gamma^d;\ell_D)$, there exists a diffeomorphism $f\in\mathrm{Diff}(V_{p,q},D^{p+q})$ such that ℓ_1 is homotopic to $\ell_2\circ Df$.*

Proof Since the space B is q–connected and $V_{p,q}$ is homotopy equivalent to S^q, it follows that the underlying map $V_{p,q} \to B$ of any θ–structure on $V_{p,q}$ is null-homotopic. It follows that every θ–structure on $V_{p,q}$ is homotopic to one that is induced by a framing of the tangent bundle. That is, every $\ell \in \mathrm{Bun}(TV_{p,q}, \theta^*\gamma^d; \ell_D)$ is homotopic to a θ–structure of the form

$$TV_{p,q} \xrightarrow{\cong} V_{p,q} \times \mathbb{R}^d \xrightarrow{\mathrm{pr}} \mathbb{R}^d \xrightarrow{\tau} \theta^*\gamma^d,$$

where the first arrow is a framing of the tangent bundle. Fix a framing

$$\phi_D\colon TD^{d-1} \oplus \epsilon^1 \xrightarrow{\cong} D^{d-1} \times \mathbb{R}^d.$$

Let $\mathrm{Fr}(TV_{p,q}, \phi_D)$ denote the space of framings of $TV_{p,q}$ that agree with ℓ_D when restricted to the disk $D^{d-1} \subset \partial V_{p,q}$. From the observation made above, to prove the lemma it will suffice to prove the following statement: given any two framings $\varphi_1, \varphi_2 \in \mathrm{Fr}(TV_{p,q}, \phi_D)$, there exists $f \in \mathrm{Diff}(V_{p,q}, D^d)$ such that $\varphi_1 \circ Df$ is homotopic (through framings) to φ_2. Let $\varphi_1, \varphi_2 \in \mathrm{Fr}(TV_{p,q}, \phi_D)$. The tangent bundle $TV_{p,q}$ has a natural splitting $E \oplus N \cong TV_{p,q}$. The bundle E is the pull-back of the tangent bundle TS^q over the projection $V_{p,q} \to S^q$ and N is the pull-back of the normal bundle of $S^q \hookrightarrow V_{p,q}$ over the projection $V_{p,q} \to S^q$. The bundle $N \to V_{p,q}$ is trivial and has fibers of dimension $p+1$. Pick once and for all a standard framing

$$\phi\colon TV_{p,q} \xrightarrow{\cong} V_{p,q} \times \mathbb{R}^d.$$

Since $p \geq q$ by assumption, the stabilization map $\pi_q(\mathrm{SO}_{p+1}) \to \pi_q(\mathrm{SO})$ is surjective. From this it follows that φ_i (for $i = 1, 2$) is homotopic to a framing of the form

$$TV_{p,q} \xrightarrow{\cong} E \oplus N \xrightarrow{\mathrm{Id}_E \oplus \widehat{\varphi}_i} E \oplus N \xrightarrow{\cong} TV_{p,q} \xrightarrow{\phi} V_{p,q} \times \mathbb{R}^d \tag{24}$$

for some bundle isomorphism $\widehat{\varphi}_i\colon N \xrightarrow{\cong} N$. So, let us assume that φ_1 and φ_2 are of this form, with $\widehat{\varphi}_1, \widehat{\varphi}_2\colon N \xrightarrow{\cong} N$ defined as in the above composition. Choose once and for all a trivialization

$$\psi\colon N \xrightarrow{\cong} V_{p,q} \times \mathbb{R}^{p+1}$$

of the bundle N. There exists some map $g\colon S^q \to \mathrm{SO}(p+1)$ such that the bundle map

$$\psi \circ \widehat{\varphi}_2^{-1} \circ \widehat{\varphi}_1 \circ \psi^{-1}\colon V_{p,q} \times \mathbb{R}^{p+1} \xrightarrow{\cong} V_{p,q} \times \mathbb{R}^{p+1}$$

is given by the formula

$$(x, y) \mapsto (x, g(x) \cdot y) \quad \text{for all } (x, y) \in V_{p,q} \times \mathbb{R}^{p+1}.$$

We may assume that the map g is smooth and that $g(x) = \mathrm{Id}_{\mathbb{R}^{p+1}}$ for all $x \in D \subset V_{p,q}$. Let

$$G\colon S^q \times D^{p+1} \to S^q \times D^{p+1}$$

be the diffeomorphism given by the formula

$$(v, w) \mapsto (v, g(v) \cdot w) \quad \text{for all } (v, w) \in S^q \times D^{p+1}.$$

With G constructed in this way, it follows that $\varphi_1 \circ DG = \varphi_2$. This completes the proof of the lemma. $\square$

The main technical ingredient in the proof of Theorem 8.2 is the following proposition:

Proposition 8.5 *Let $(M; \partial_0 M, \partial_1 M)$ and $p+q+1 = d$ satisfy the inequalities (2). Suppose further that $q \le p$ and that $\theta\colon B \to BO(d)$ is chosen so that B is q–connected. Let $r_{p,q}(M) \ge g$. Then, for any $\ell_M \in \mathrm{Bun}(TM, \theta^*\gamma^d; \ell_P)$, the geometric realization $|\bar{K}^{\partial}_{\bullet}(M, \ell_M, a)_{p,q}|$ is $\frac{1}{2}(g-4)$–connected.*

Proof As before, the degree of connectivity of $|\bar{K}^{\partial}_{\bullet}(M, \ell_M, a)_{p,q}|$ is bounded below by the degree of connectivity of $|K^{\partial}(M, \ell_M, a)_{p,q}|$. To prove the theorem it will suffice to show that $|K^{\partial}(M, \ell_M, a)_{p,q}|$ is $\frac{1}{2}(g-4)$–connected. Let

$$\mathcal{T} \subset K^{\partial}(M, \ell_M, a)_{p,q} \times K^{\partial}(M, \ell_M, a)_{p,q}$$

be the symmetric relation from Definition 6.4. As in Section 6 we have a simplicial map

$$H_{p,q}\colon K^{\partial}(M, \ell_M, a)_{p,q} \to L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)),$$

defined by sending (t, ϕ, γ) to the morphism of Wall forms $W_{p,q} \to \mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)$ induced by ϕ. We will need to show that the map $H_{p,q}$ has the following properties:

(i) The map $H_{p,q}$ has the link lifting property with respect to $\mathcal{T}$ (see Definition 6.2).

(ii) $H_{p,q}(\mathrm{lk}_{K^{\partial}(M,\ell_M,a)_{p,q}}(\zeta)) \le \mathrm{lk}_{L(\mathcal{W}^{\partial}_{p,q}(M,\partial_1 M))}(H_{p,q}(\zeta))$ for any simplex $\zeta \in K^{\partial}(M, \ell_M, a)_{p,q}$.

Condition (ii) is proven in the same way as in the proof of Theorem 3.6. The proof of condition (i) is similar to the proof of Theorem 3.6 but requires one extra step, which we describe below. This step will rely on Lemma 8.4 and thus requires the conditions that $q \leq p$ and B is q–connected. Let $f\colon W \to \mathcal{W}^{\partial}_{p,q}(M, \partial_1 M)$ be a morphism of Wall forms, ie a vertex of $L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))$. Let $(\phi_1, \gamma_1,), \dots, (\phi_k, \gamma_k) \in K^{\partial}(M, \ell_M, a)_{p,q}$ be a collection of $\mathcal{T}$–related vertices such that

$$H_{p,q}(\gamma_i, \phi_i) \in \mathrm{lk}(f) \quad \text{for } i = 1, \dots, k$$

(we have dropped the numbers $t_1, \dots, t_k$ from the notation to save space). Let $(\phi'_1, \gamma'_1), \dots, (\phi'_m, \gamma'_m) \in K^{\partial}(M, \ell_M, a)_{p,q}$ be another arbitrary collection of vertices (that is not necessarily $\mathcal{T}$–related). To show that $H_{p,q}$ has the link lifting property with respect to $\mathcal{T}$, we need to construct a vertex $(\phi, \gamma) \in K^{\partial}(M, \ell_M, a)_{p,q}$ with $H_{p,q}(\phi, \gamma) = f$ such that

$$(\phi_i, \gamma_i) \in \mathrm{lk}(\phi, \gamma) \quad \text{and} \quad ((\psi_j, \gamma_j), (\phi, \gamma)) \in \mathcal{T}$$

for all $i = 1, \dots, k$ and $j = 1, \dots, m$. By using the same procedure employed in the proof of Theorem 3.6, we may construct the embedding $\phi\colon \widehat{V}_{p,q} \to M$ in the same way that was done there. However, in order to obtain the path $\gamma\colon [0,1] \to \mathrm{Bun}(T\widehat{V}_{p,q}, \theta^*\gamma^d)$, we need to use Lemma 8.4. Since $q \leq p$ and the space B is q–connected, by Lemma 8.4 we may find a diffeomorphism $\varphi\colon \widehat{V}_{p,q} \to \widehat{V}_{p,q}$ that is the identity on the half-disk $D^{p+q}_{+} \subset \partial\widehat{V}_{p,q}$ such that the θ–structure $\varphi^*\phi^*\ell_M$ given by

$$T\widehat{V}_{p,q} \xrightarrow{D\varphi} T\widehat{V}_{p,q} \xrightarrow{D\phi} TM \xrightarrow{\ell_M} \theta^*\gamma^d \tag{25}$$

is on the same path component of $\mathrm{Bun}(T\widehat{V}_{p,q}, \theta^*\gamma^d)$ as the canonical θ–structure $\ell^{\tau}_{\widehat{V}_{p,q}}$. Letting $\gamma\colon [0,1] \to \mathrm{Bun}(T\widehat{V}_{p,q}, \theta^*\gamma^d)$ be a path from $\varphi^*\phi^*\ell_M$ to $\ell^{\tau}_{\widehat{V}_{p,q}}$ it follows that the pair $(\phi \circ \varphi, \gamma)$ is a vertex in the complex $K^{\partial}(M, \ell_M, a)_{p,q}$ that satisfies all of the desired conditions.

The construction of this vertex concludes our verification of the link lifting property. It follows from Lemma 6.3 that the degree of connectivity of $|K^{\partial}(M, \ell_M, a)_{p,q}|$ is bounded below by the degree of connectivity of $|L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))|$. Since $q \leq p$, $\pi_q(S^p)$ is either isomorphic to $\mathbb{Z}$ or is zero, thus the generating set length $d(\pi_q(S^p))$ is either equal to 1 or zero. It follows from Theorem 5.6 that $|L(\mathcal{W}^{\partial}_{p,q}(M, \partial_1 M))|$ is at least $\frac{1}{2}(g-4)$–connected. By what was proven above it follows that $|\overline{K}^{\partial}_{\bullet}(M, \ell_M, a)_{p,q}|$ is $\frac{1}{2}(g-4)$–connected as well. This concludes the proof of the proposition. □

With the above proposition established, the proof of Theorem 8.2 is obtained by implementing the same constructions from Section 7. We omit the rest of the proof and refer the reader to [4, Section 7] for details.

Appendix: Embeddings and disjunction

In this section we prove a disjunction result for embeddings of manifolds with boundary. This result implies Theorem 4.10, which is one of the main technical ingredients used to prove that the complex $K(M)_{p,q}$ is highly connected.

Theorem A.1 *Let $(M, \partial M)$ be a manifold pair of dimension m. Let $(P, \partial P)$ and $(Q, \partial Q)$ be manifold pairs of dimensions p and q, respectively, with $\partial P \neq \varnothing \neq \partial Q$. Let $f\colon (P, \partial P) \to (M, \partial M)$ and $g\colon (Q, \partial Q) \to (M, \partial M)$ be smooth embeddings and suppose that the following conditions are met:*

(i) $m > p + \frac{1}{2}q + 1$ *and* $m > q + \frac{1}{2}p + 1$.
(ii) $(P, \partial P)$, $(Q, \partial Q)$, P *and* Q *are* $(p+q-m)$*–connected.*
(iii) $(M, \partial M)$ *and* M *are* $(p+q-m+1)$*–connected.*

Then there exists an isotopy $\psi_s\colon (P, \partial P) \to (M, \partial M)$ for $s \in [0, 1]$ such that $\psi_0 = f$ and $\psi_1(P) \cap g(Q) = \varnothing$.

The proof of the above theorem is based on a technique developed by Hatcher and Quinn [7]. We recall the results of Hatcher and Quinn in the following section, develop some new techniques in the sections that follow and then finish the proof of Theorem A.1 in Section A.4.

Remark A.2 In view of the remarks in [7, page 333] we believe that a result similar to the one stated above may have been known to Hatcher and Quinn at time of writing their paper. However, to our knowledge no proof of this result exists in [7] (or in the literature) and so we provide the proof here in this appendix.

A.1 The Hatcher–Quinn invariant

We now review the construction of Hatcher and Quinn [7]. This construction involves the framed bordism groups of a space, twisted by a stable vector bundle.

Definition A.3 Let X be a space and let ζ be a stable vector bundle over X. For an integer n, $\Omega_n^{\mathrm{fr}}(X; \zeta)$ is defined to be the set of bordism classes of triples (M, f, F), where M is a closed n–dimensional smooth manifold, $f\colon M \to X$ is a map and $F\colon \nu_M \to f^*(\zeta)$ is an isomorphism of stable vector bundles covering the identity map on M, where ν_M denotes the stable normal bundle of M.

Let M, P and Q be smooth manifolds of dimensions m, p and q, respectively. Let t denote the integer $p+q-m$. Let $f\colon (P,\partial P)\to(M,\partial M)$ and $g\colon (Q,\partial Q)\to(M,\partial M)$ be smooth maps. We denote by $E(f,g)$ the *homotopy pull-back* of the maps f and g. Explicitly, $E(f,g)$ is the space defined by

$$E(f,g)=\{(x,y,\gamma)\in P\times Q\times \mathrm{Path}(M)\mid f(x)=\gamma(0),\ g(y)=\gamma(1)\},$$

where $\mathrm{Path}(M)$ is the space of continuous maps $[0,1]\to M$, topologized in the compact–open topology. Consider the diagram

$$\begin{tikzcd} E(f,g) \arrow[r, "\pi_P"] \arrow[d, "\pi_Q"] \arrow[dr, "\widehat{s}"] & P \arrow[d, "f"] \\ Q \arrow[r, "g"] & M \end{tikzcd} \tag{26}$$

where π_P and π_Q are the projections and $\widehat{s}$ is the map defined by $\widehat{s}(x,y,\gamma)=\gamma\left(\frac{1}{2}\right)$. Let ν_P and ν_Q denote the stable normal bundles associated to the manifolds P and Q, respectively. We denote by $\eta(f,g)$ the stable vector bundle over $E(f,g)$ given by the Whitney sum $\pi_P^*\nu_P\oplus\pi_Q^*\nu_Q\oplus\widehat{s}^*(TM)$. We will need to consider the bordism group $\Omega_t^{\mathrm{fr}}(E(f,g);\eta(f,g))$.

Suppose now that the maps $f\colon (P,\partial P)\to(M,\partial M)$ and $g\colon (Q,\partial Q)\to(M,\partial M)$ are transverse and that $f(\partial P)\cap g(\partial Q)=\varnothing$ (by f and g being transverse we mean that $f\times g\colon P\times Q\to M\times M$ is transverse to the diagonal $\Delta_M\subset M\times M$). It follows that the pull-back $f\pitchfork g:=(f\times g)^{-1}(\Delta_M)\subset P\times Q$ is a closed submanifold of dimension $p+q-m$. Let $\iota\colon f\pitchfork g\to E(f,g)$ denote the canonical embedding given by the formula $(x,y)\mapsto(x,y,c_{f(x)})$, where $c_{f(x)}\in\mathrm{Path}(M)$ is the constant path at the point $f(x)\in M$. The lemma below follows from [7, Proposition 2.1].

Lemma A.4 *Let $f\colon (P,\partial P)\to(M,\partial M)$ and $g\colon (Q,\partial Q)\to(M,\partial M)$ be transversal smooth maps such that $f(\partial P)\cap g(\partial Q)=\varnothing$. Then there is a natural bundle isomorphism $\hat{\iota}\colon \nu_{f\pitchfork g}\xrightarrow{\cong}\iota^*(\eta(f,g))$ such that the triple $(f\pitchfork g,\iota,\hat{\iota})$ determines a well-defined element of the bordism group $\Omega_t^{\mathrm{fr}}(E(f,g);\eta(f,g))$.*

Definition A.5 For transversal maps f and g with $f(\partial P)\cap g(\partial Q)=\varnothing$ as in the previous lemma, we will denote by $\alpha_t(f,g,M)\in\Omega_t^{\mathrm{fr}}(E(f,g);\eta(f,g))$ the element determined by the triple $(f\pitchfork g,\iota,\hat{\iota})$ given in Lemma A.4.

The main result from [7] is the following theorem:

Theorem A.6 *Let $f\colon (P,\partial P)\to(M,\partial M)$ and $g\colon (Q,\partial Q)\to(M,\partial M)$ be smooth embeddings such that $f(\partial P)\cap g(\partial Q)=\varnothing$. Suppose further that*

$$m>\max\{p+\tfrac{1}{2}q+1,\tfrac{1}{2}p+q+1\}.$$

If the class $\alpha_t(f,g,M)$ is equal to the zero element in $\Omega^{\mathrm{fr}}_t(E(f,g);\eta(f,g))$, then there exists an isotopy $\psi_s\colon (P,\partial P)\to(M,\partial M)$ such that $\psi_0=f$ and $\psi_s|_{\partial P}=f|_{\partial P}$ for all $s\in[0,1]$, and $\psi_1(P)\cap g(Q)=\varnothing$.

The bordism group $\Omega^{\mathrm{fr}}_t(E(f,g);\eta(f,g))$ in general can be quite difficult to compute. However, in the case where P, Q and M are all highly connected, the group reduces to a far simpler object.

Proposition A.7 *Let*

$$f\colon (P,\partial P)\to(M,\partial M)\quad\text{and}\quad g\colon (Q,\partial Q)\to(M,\partial M)$$

be smooth embeddings such that $f(\partial P)\cap g(\partial Q)=\varnothing$. Suppose that P and Q are both t–connected and that M is $(t+1)$–connected. Then the natural map $\Omega^{\mathrm{fr}}_t(\mathrm{pt})\to\Omega^{\mathrm{fr}}_t(E(f,g),\eta(f,g))$ is an isomorphism.

Proof Let P and Q be t–connected and let M be $(t+1)$–connected. There is a fiber sequence $\Omega M\to E(f,g)\to P\times Q$. The long exact sequence on homotopy groups implies that the space $E(f,g)$ is t–connected. The proof then follows by application of the *Atiyah–Hirzebruch spectral sequence*. □

Suppose that P and Q are t–connected and that M is $(t+1)$–connected. If

$$f\colon (P,\partial P)\to(M,\partial M)\quad\text{and}\quad g\colon (Q,\partial Q)\to(M,\partial M)$$

are smooth embeddings, as a consequence of the above proposition we may consider $\alpha_t(f,g;M)$ to be an element of the framed bordism group $\Omega^{\mathrm{fr}}_t(\mathrm{pt})$.

Remark A.8 The element $\alpha_t(f,g;M)$ is the obstruction to finding an isotopy, relative to the boundary ∂P, that pushes $f(P)$ off of $g(Q)$. This element $\alpha_t(f,g;M)$ very well may be nonzero for arbitrary f, g and M, and thus it may appear that Theorem A.1 is false. However, Theorem A.1 does not assert the existence of an isotopy that fixes the boundary of P. Indeed, as will be seen in the following sections, $\alpha_t(f,g;M)$ is not an obstruction to the existence of an isotopy that is nonconstant on the boundary of P.

A.2 Relative Hatcher–Quinn invariant

We will have to consider relative framed bordism groups.

Definition A.9 Let (X, A) be an excisive pair of spaces and let ζ be a stable vector bundle over X. For an integer n, $\Omega_n^{\mathrm{fr}}((X, A), \zeta)$ is defined to be the abelian group of bordism classes of triples (M, f, F), where $(M, \partial M)$ is an n–dimensional manifold pair, $f\colon (M, \partial M) \to (X, A)$ is a map and $F\colon \nu_M \to f^*(\zeta)$ is an equivalence class of stable bundle isomorphisms as before.

For any excisive space pair (X, A) and stable vector bundle ζ over X, there is a long exact sequence of bordism groups,

$$\cdots \to \Omega_n^{\mathrm{fr}}(A; \zeta|_A) \to \Omega_n^{\mathrm{fr}}(X; \zeta) \to \Omega_n^{\mathrm{fr}}((X, A); \zeta) \to \Omega_{n-1}^{\mathrm{fr}}(A; \zeta|_A) \to \cdots . \tag{27}$$

Remark A.10 This exact sequence follows as a consequence of [11, Theorem 3.1.5]. Indeed, this theorem identifies the bordism group $\Omega_n^{\mathrm{fr}}((X, A); \zeta)$ with the relative stable homotopy group $\pi_n(X^\zeta, A^\zeta)$, where X^ζ and A^ζ are the Thom spectra associated to the stable vector bundles $\zeta \to X$ and $\zeta|_A \to A$, respectively. From this isomorphism $\Omega_n^{\mathrm{fr}}((X, A); \zeta) \cong \pi_n(X^\zeta, A^\zeta)$, the long exact sequence (27) follows from the long exact sequence of homotopy groups associated to the pair (X^ζ, A^ζ). Alternatively, the exact sequence can be verified directly by hand. This is a simple exercise that we leave for the reader.

Using these relative bordism groups, we define a relative version of the Hatcher–Quinn invariant. Let $f\colon (P, \partial P) \to (M, \partial M)$ and $g\colon (Q, \partial Q) \to (M, \partial M)$ be embeddings. Unlike the case in the previous section, we now include the possibility that the intersection $f(\partial P) \cap g(\partial Q)$ be nonempty. If f and g are transversal (and by this we mean that both f and g and $f|_{\partial P}$ and $g|_{\partial Q}$ are transversal in the ordinary sense), then the pull-back $f \pitchfork g$ is a manifold with boundary given by $\partial(f \pitchfork g) = f|_{\partial P} \pitchfork g|_{\partial Q} \subset \partial P \times \partial Q$. We will need to construct a relative version of the bordism invariant that was defined in the previous section.

Let $\partial E(f, g)$ denote the homotopy pull-back $E(f|_{\partial P}, g|_{\partial Q})$. The space $\partial E(f, g)$ embeds naturally as a subspace of $E(f, g)$.

Lemma A.11 *The pair $(E(f, g), \partial E(f, g))$ is excisive, ie $\partial E(f, g) \hookrightarrow E(f, g)$ is a cofibration.*

Proof It will suffice to show that there exists an open neighborhood $U \subset E(f,g)$ that contains the subspace $\partial E(f,g)$ as a deformation retract. Let $V_P \subset P$, $V_Q \subset Q$ and $V_M \subset M$ be open collar neighborhoods of the boundaries ∂P, ∂Q and ∂M, respectively, and let

$$\pi_P\colon V_P \to \partial P, \quad \pi_Q\colon V_Q \to \partial Q, \quad \pi_M\colon V_M \to \partial M$$

denote the standard deformation retractions. We define $U \subset E(f,g)$ to be the space of triples $(x,y,\gamma) \in V_P \times V_Q \times \mathrm{Path}(V_M)$ such that $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. Since V_P, V_Q and V_M are open collar neighborhoods of the boundaries ∂P, ∂Q and ∂M, respectively, it follows that U is an open neighborhood of $\partial E(f,g)$ (this uses the fact that $\mathrm{Path}(V_M)$ is an open subset of $\mathrm{Path}(M)$ when topologized in the compact–open topology). A deformation retraction $\pi\colon U \to \partial E(f,g)$ is then defined by the formula $(x,y,\gamma) \mapsto (\pi_P(x), \pi_Q(y), \pi_M \circ \gamma)$. The fact that π is a deformation retraction follows from the fact that π_P, π_Q and π_M are all deformation retractions. $\square$

We have a map of pairs

$$\iota\colon (f \pitchfork g, \partial(f \pitchfork g)) \to (E(f,g), \partial E(f,g)), \quad (x,y) \mapsto (x,y,c_{f(x)}).$$

The restriction of $\eta(f,g)$ to $\partial E(f,g)$ is equal to the stable bundle $\eta(f|_{\partial P}, g|_{\partial Q})$. To save space we will let $\hat{E}(f,g)$ denote the pair $(E(f,g), \partial E(f,g))$. We will need to consider the relative bordism group $\Omega_t^{\mathrm{fr}}(\hat{E}(f,g), \eta(f,g))$. Let

$$\hat{\partial}\colon \Omega_t^{\mathrm{fr}}(\hat{E}(f,g), \eta(f,g)) \to \Omega_{t-1}^{\mathrm{fr}}(\partial E(f,g), \eta(f,g)|_{\partial E(f,g)}) \tag{28}$$

be the boundary homomorphism in the long exact sequence associated to the pair $\hat{E}(f,g)$. Since the pair $\hat{E}(f,g) = (E(f,g), \partial E(f,g))$ is excisive it follows that the above map $\hat{\partial}$ fits into a long exact sequence as in (27). Using the same construction from Lemma A.4 we obtain:

Lemma A.12 *Let $f\colon (P,\partial P) \to (M,\partial M)$ and $g\colon (Q,\partial Q) \to (M,\partial M)$ be transversal maps. Then the pullback manifold $f \pitchfork g$ determines a class $\alpha_t^{\partial}(f,g,M) \in \Omega_t^{\mathrm{fr}}(\hat{E}(f,g), \eta(f,g))$. Furthermore, we have*

$$\hat{\partial}(\alpha_t^{\partial}(f,g,M)) = \alpha_{t-1}(f|_{\partial P}, g|_{\partial Q}, \partial M).$$

Lemma A.12 will be useful to us in order to prove the following result:

Proposition A.13 *Let $f\colon (P,\partial P)\to(M,\partial M)$ and $g\colon (Q,\partial Q)\to(M,\partial M)$ be embeddings. Suppose that $(P,\partial P)$ and $(Q,\partial Q)$ are $(p+q-m)$–connected and that $(M,\partial M)$ is $(p+q-m+1)$–connected. Then there exists an isotopy $\Psi_s\colon P\to M$ for $s\in[0,1]$ such that $\Psi_0=f$ and $\Psi_1(\partial P)\cap g(\partial Q)=\varnothing$.*

Proof Let t denote the integer $p+q-m$. The connectivity conditions in the statement of the proposition implies that the pair $(E(f,g),\partial E(f,g))$ is t–connected and, thus, the bordism group $\Omega_t^{\mathrm{fr}}(\widehat{E}(f,g),\eta(f,g))$ is trivial. It follows from this that $\widehat{\partial}(\widehat{\alpha}_t(f,g,M))=\alpha_{t-1}(f|_{\partial P},g|_{\partial Q},\partial M)=0$. We may then apply Theorem A.6 to obtain an isotopy $\psi_s\colon \partial P\to\partial M$ with $\psi_0=f|_{\partial P}$ such that $\psi_1(\partial P)\cap g(\partial Q)=\varnothing$. The proof of the proposition then follows by application of the *isotopy extension theorem* [9, Theorem 1.3]. □

A.3 Creating intersections

In this section we develop a technique for creating intersections with prescribed Hatcher–Quinn obstructions. Let M and Q be oriented, connected, compact manifolds of dimension m and q, respectively, and let $g\colon (Q,\partial Q)\to(M,\partial M)$ be an embedding. Let $r=m-q$ and let $f\colon (D^r,\partial D^r)\to(M,\partial M)$ be a smooth embedding transverse to g such that $f(\partial D^r)\cap g(\partial Q)=\varnothing$. Let $j\geq 0$ be an integer strictly less than r. With j chosen in this way it follows that $\pi_{r+j}(S^r)$ is in the stable range and thus we have an isomorphism $\pi_{r+j}(D^r,\partial D^r)\cong\pi_{r+j}(S^r)$. Let $\varphi\colon (D^{r+j},\partial D^{r+j})\to(D^r,\partial D^r)$ be a smooth map. Denote by

$$\mathcal{P}_j\colon \pi_{r+j}(D^r,\partial D^r)\cong\pi_{r+j}(S^r)\xrightarrow{\cong}\Omega_j^{\mathrm{fr}}(\mathrm{pt}) \tag{29}$$

the *Pontryagin–Thom* isomorphism for framed bordism (see [13]). The following lemma shows how to express $\alpha_j(f\circ\varphi,g,M)$ in terms of $\alpha_0(f,g,M)$ and the element $\mathcal{P}_j([\varphi])\in\Omega_j^{\mathrm{fr}}(\mathrm{pt})$.

Lemma A.14 *Let g, f and φ be as above and suppose that $f(\partial D^r)\cap g(\partial Q)=\varnothing$. Then*

$$\alpha_j(f\circ\varphi,g,M)=\alpha_0(f,g,M)\cdot\mathcal{P}_j([\varphi]),$$

where the product on the right-hand side is the product in the graded bordism ring $\Omega_*^{\mathrm{fr}}(\mathrm{pt})$.

Proof Let $\ell \in \mathbb{Z}$ denote the oriented, algebraic intersection number associated to the intersection of $f(D^r)$ and $g(Q)$. By application of the Whitney trick, we may deform f so that $f(D^r)$ is transverse to $g(Q)$ and

$$f(D^r) \cap g(Q) = \{x_1, \dots, x_\ell\}, \tag{30}$$

where the points x_i for $i = 1, \dots, \ell$ all have the same sign. It follows that

$$(f \circ \varphi)^{-1}(g(Q)) = \bigsqcup_{i=1}^{\ell} \varphi^{-1}(x_i).$$

For each $i \in \{1, \dots, \ell\}$, the normal framing at x_i (induced by the orientations of $f(D^r)$, $g(Q)$ and M) induces a framing on $\varphi^{-1}(x_i)$. We denote the element of $\Omega^{\mathrm{fr}}_j(\mathrm{pt})$ given by $\varphi^{-1}(x_i)$ with this induced framing by $[\varphi^{-1}(x_i)]$. By definition of the Pontryagin–Thom map $\mathcal{P}_j$, the element $[\varphi^{-1}(x_i)]$ is equal to $\mathcal{P}_j([\varphi])$ for $i = 1, \dots, \ell$. Using the equality (30), it follows that $\alpha_j(f \circ \varphi, g, M) = \ell \cdot \mathcal{P}_j([\varphi])$. The proof then follows from the fact that $\alpha_0(f, g, M)$ is identified with the algebraic intersection number associated to $f(D^r)$ and $g(Q)$. □

We apply the above lemma to the following proposition:

Proposition A.15 *Let Q and M have nonempty boundary and let $g\colon (Q, \partial Q) \to (M, \partial M)$ be a smooth embedding. Let r denote the integer $m - q$. There exists an embedding $f\colon (D^r, \partial D^r) \to (M, \partial M)$ that satisfies the following conditions:*

- $f(\partial D^r) \cap g(\partial Q) = \varnothing$.
- $f(D^r) \cap g(Q)$ *consists of a single point with positive orientation.*
- f *represents the trivial element in* $\pi_r(M, \partial M)$.

Proof We will prove the proposition by carrying out an explicit construction as follows:

(i) Choose a collar embedding $h\colon \partial Q \times [0, \infty) \to Q$ such that $h^{-1}(\partial Q) = \partial Q \times \{0\}$.

(ii) Choose a point $y \in \partial Q$, then define an embedding

$$\gamma\colon [0,1] \to g(Q), \quad \gamma(t) = g(h(y,t)).$$

We then let $x \in g(Q)$ denote the point $\gamma(1)$.

(iii) Choose an embedding $\alpha\colon (D^2_+, \partial_0 D^2_+) \to (M, \partial M)$ that satisfies the following conditions:

(a) $\alpha(D^2_+) \cap g(Q) = \gamma([0,1])$.

(b) $\alpha(\partial_1 D^2_+) \cap g(Q) = \{x\}$.

(c) $\alpha(D^2_+)$ intersects $g(Q)$ orthogonally (with respect to some metric on M).

(iv) Let r denote the integer $m-q$. Choose an $(r-1)$–frame of orthogonal vector fields $(v_1, \dots, v_{r-1})$ over the embedded half-disk $\alpha(D^2_+) \subset M$ with the property that v_i is orthogonal to $\alpha(D^2_+)$ and orthogonal to $g(Q)$ over the intersection $\alpha(D^2_+) \cap g(Q)$ for $i = 1, \dots, r-1$. Since the disk is contractible, there is no obstruction to the existence of such a frame.

The orthogonal $(r-1)$–frame chosen in step (iv) induces an embedding

$$\bar{f}\colon (D^{r+1}_+, \partial_0 D^{r+1}_+) \to (M, \partial M).$$

The orthogonality condition (condition (c)) in step (iii) of the above construction, together with the orthogonality condition on the frame chosen in step (iv), implies that $\bar{f}(D^{r+1}_+)$ is transverse to $g(Q)$. Furthermore, by choosing the vectors of the frame in step (iv) to have sufficiently small magnitude, condition (b) from step (iii) implies that $f(\partial_1 D^{r+1}) \cap g(\mathrm{Int}(Q)) = \{x\}$. We then set the map $f\colon (D^r, \partial D^{r-1}) \to (M, \partial M)$ equal to the embedding obtained by restricting $\bar{f}$ to $(\partial_1 D^{r+1}, \partial_{0,1} D^{r+1})$. This concludes the proof of the proposition. □

A.4 Proof of Theorem A.1

Let $f\colon (P, \partial P) \to (M, \partial M)$ and $g\colon (Q, \partial Q) \to (M, \partial M)$ be smooth embeddings exactly as in the statement of Theorem A.1. Suppose that conditions (i), (ii) and (iii) from the statement of Theorem A.1 are satisfied. We now have all of the necessary tools available to prove the theorem.

Proof of Theorem A.1 By Proposition A.13 we may assume that $f(\partial P) \cap g(\partial Q) = \varnothing$. Consider the element $\alpha_t(f, g, M) \in \Omega^{\mathrm{fr}}_t(\mathrm{pt})$, where as before $t = p+q-m$. Let $\varphi\colon (D^p, \partial D^p) \to (D^{m-q}, \partial D^{m-q})$ be a map such that $\mathcal{P}_t([\varphi]) = -\alpha_t(f, g, M)$ as elements of $\Omega^{\mathrm{fr}}_t(\mathrm{pt})$. By Proposition A.15 there exists a null-homotopic embedding $\phi\colon (D^{m-q}, \partial D^{m-q}) \to (M, \partial M)$ such that $\phi(D^{m-q})$ intersects the interior of $g(Q)$ at exactly one point. Furthermore, by general position we may assume that $\phi(\partial D^{m-q})$ is disjoint from $g(\partial Q)$. It follows from Lemma A.14 that

$$\alpha_t(\phi \circ \varphi, g, M) = \mathcal{P}_t([\varphi]) \cdot \alpha_0(\phi, g, M) = -\alpha_t(f, g, M). \tag{31}$$

Now, the map $\phi \circ \varphi$ has image disjoint from $g(\partial Q) \subset \partial M$. Let M' denote the complement $M \setminus g(\partial Q)$. By the connectivity and dimensional conditions from the statement of Theorem A.1, we may apply Lemma 4.1 (see also [10, Theorem 1]) to obtain a homotopy

$$\widehat{\varphi}_s\colon (D^p, \partial D^p) \to (M', \partial M'), \quad s \in [0,1], \tag{32}$$

with $\widehat{\varphi}_0 = \phi \circ \varphi$ and such that $\widehat{\varphi}_1$ is an embedding. Let us denote

$$\widehat{\varphi} := i_{M'} \circ \widehat{\varphi}_1\colon (D^p, \partial D^p) \to (M, \partial M),$$

where $i_{M'}\colon (M', \partial M') \hookrightarrow (M, \partial M)$ is the inclusion. Since the homotopy in (32) was through maps with image in $(M', \partial M')$, it follows that $\alpha_t(\widehat{\varphi}, g, M) = \alpha_t(\phi \circ \varphi, g, M)$, and then by the above calculation (31) we have $\alpha_t(\widehat{\varphi}, g, M) = -\alpha_t(f, g, M)$. Now, let $\hat{f}\colon (P, \partial P) \to (M, \partial M)$ be the embedding obtained by forming the boundary connected sum of the submanifolds

$$(f(P), f(\partial P)), (\widehat{\varphi}(D^p), \widehat{\varphi}(\partial D^p)) \subset (M, \partial M)$$

along an arc in ∂M that is disjoint from $g(\partial Q)$. Clearly this embedding is homotopic (as a map) to f. We emphasize that the homotopy taking $\hat{f}$ to f will not be constant on the boundary of P. We then have

$$\alpha_t(\hat{f}, g, M) = \alpha_t(f, g, M) + \alpha_t(\widehat{\varphi}, g, M) = \alpha_t(f, g, M) - \alpha_t(f, g, M) = 0.$$

By Theorem A.6 there is a diffeotopy (relative the boundary of M) that pushes $\hat{f}(P)$ off of $g(Q)$. Now since f is homotopic to $\hat{f}$, it follows that f is homotopic (through maps sending ∂P to ∂M) to an embedding with image disjoint from $g(\partial Q)$. We then apply [7, Theorem 1.1] to conclude that the embedding f is actually isotopic (rather than just homotopic) to such an embedding with image disjoint from $g(Q)$. Then Theorem A.1 follows by isotopy extension. □

References

[1] **B Botvinnik**, **N Perlmutter**, *Stable moduli spaces of high-dimensional handlebodies*, J. Topol. 10 (2017) 101–163 MR

[2] **J Ebert**, **O Randal-Williams**, *Semi-simplicial spaces*, preprint (2017) arXiv

[3] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds*, preprint (2012) arXiv

[4] **S Galatius**, **O Randal-Williams**, *Homological stability for moduli spaces of high dimensional manifolds, I*, J. Amer. Math. Soc. 31 (2018) 215–264 MR

[5] **J Genauer**, *Cobordism categories of manifolds with corners*, Trans. Amer. Math. Soc. 364 (2012) 519–550 MR

[6] **J L Harer**, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. 121 (1985) 215–249 MR

[7] **A Hatcher**, **F Quinn**, *Bordism invariants of intersections of submanifolds*, Trans. Amer. Math. Soc. 200 (1974) 327–344 MR

[8] **A Hatcher**, **N Wahl**, *Stabilization for mapping class groups of 3–manifolds*, Duke Math. J. 155 (2010) 205–269 MR

[9] **M W Hirsch**, *Differential topology*, Graduate Texts in Mathematics 33, Springer (1976) MR

[10] **J F P Hudson**, *Embeddings of bounded manifolds*, Proc. Cambridge Philos. Soc. 72 (1972) 11–20 MR

[11] **G Laures**, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. 352 (2000) 5667–5688 MR

[12] **J Milnor**, *Lectures on the h–cobordism theorem*, Princeton Univ. Press (1965) MR

[13] **J W Milnor**, *Topology from the differentiable viewpoint*, Univ. Press Virginia, Charlottesville, VA (1965) MR

[14] **N Perlmutter**, *Cobordism category of manifolds with Baas–Sullivan singularities*, Münster J. Math. 8 (2015) 119–167 MR

[15] **N Perlmutter**, *Homological stability for the moduli spaces of products of spheres*, Trans. Amer. Math. Soc. 368 (2016) 5197–5228 MR

[16] **N Perlmutter**, *Linking forms and stabilization of diffeomorphism groups of manifolds of dimension $4n+1$*, J. Topol. 9 (2016) 552–606 MR

[17] **O Randal-Williams**, **N Wahl**, *Homological stability for automorphism groups*, Adv. Math. 318 (2017) 534–626 MR

[18] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR

[19] **C T C Wall**, *Classification problems in differential topology, VI: Classification of $(s-1)$–connected $(2s+1)$–manifolds*, Topology 6 (1967) 273–296 MR

[20] **R Wells**, *Modifying intersections*, Illinois J. Math. 11 (1967) 389–403 MR

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