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Link invariants derived from multiplexing of crossings

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We introduce the multiplexing of a crossing, replacing a classical crossing of a virtual link diagram with a mixture of classical and virtual crossings. For integers m_i $(i=1,\dots,n)$ and an ordered n–component virtual link diagram D, a new virtual link diagram $D(m_1,\dots,m_n)$ is obtained from D by the multiplexing of all crossings. For welded isotopic virtual link diagrams D and D', the virtual link diagrams $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are welded isotopic. From the point of view of classical link theory, it seems very interesting that new classical link invariants are obtained from welded link invariants via the multiplexing of crossings.

57M25, 57M27

1 Introduction

An n–component *virtual link diagram* is an immersion of n circles in the plane whose singularities are transverse double points, which are labeled either as a *classical crossing* or as a *virtual crossing* as illustrated in the figure below:

classical crossing   virtual crossing

Note that we do not use here the usual drawing convention for virtual crossings, which is a small circle around the corresponding double point.

Virtual isotopy is an equivalence relation on virtual link diagrams generated by classical Reidemeister moves R1–3 and virtual Reidemeister moves VR1–4 illustrated in Figure 1. We remark that VR1–4 imply a *detour move*, which replaces an arc passing through a number of virtual crossings with any other such arc, with the same endpoints. *Welded isotopy* is the extension of virtual isotopy which also allows the move OC (meaning *overcrossings commute*) illustrated in Figure 2. A *welded link* is an equivalence class of virtual link diagrams under welded isotopy. M Goussarov, M Polyak and O Viro [1]

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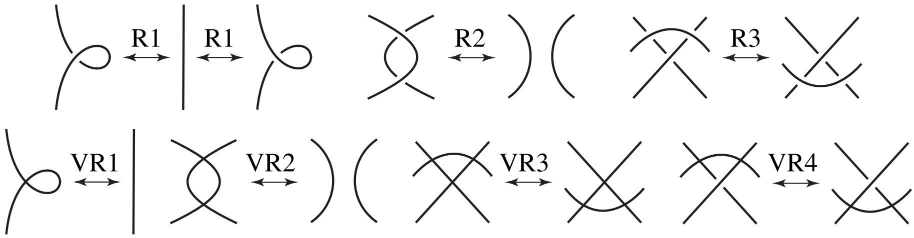

Figure 1: Classical and virtual Reidemeister moves

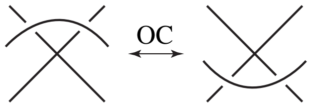

Figure 2: Move OC

essentially proved that welded isotopic classical link diagrams are equivalent, that is, they can be transformed into each other by a sequence of classical Reidemeister moves. Therefore, we can consider welded links as a natural generalization of classical links.

In this paper, we introduce the *multiplexing* of a crossing for a virtual link diagram as a local change on a classical crossing, as shown in Figure 3. Let m_i be integers $(i = 1, \dots, n)$ and D an ordered n–component virtual link diagram. By multiplexing all classical crossings of D, we obtain the virtual link diagram $D(m_1, \dots, m_n)$ of D associated with $(m_1, \dots, m_n)$; see Section 2 for the precise definition. We show that if virtual link diagrams D and D' are welded isotopic, then $D(m_1, \dots, m_n)$ and $D'(m_1, \dots, m_n)$ are welded isotopic for any $(m_1, \dots, m_n) \in \mathbb{Z}^n$ (Theorem 2.1).

The *group* of a virtual link diagram is known to be a welded link invariant [3, Section 4]. Hence, by Theorem 2.1, the group $G(D(m_1, \dots, m_n))$ of $D(m_1, \dots, m_n)$ is a link invariant of D. We remark that $G(D(m, \dots, m))$ is isomorphic to the *generalized link group* $G_m(D)$ defined independently by A J Kelly [4] and M Wada [8]. Therefore, $G(D(m_1, \dots, m_n))$ is a generalization of $G_m(D)$. As an application, we show that for a nonzero integer m and for *classical knot* diagrams D and D', D is equivalent to D' or its mirror image if and only if $D(m)$ is welded isotopic to $D'(m)$ or its mirror image (Theorem 3.2).

It seems very interesting from the viewpoint of classical link theory that $D(m_1, \dots, m_n)$ might not be welded isotopic to a classical link diagram even if D is a classical one, and new classical link invariants are expected from known welded link invariants via the multiplexing of crossings. For example, there is a 3–component *classical* link diagram D with trivial Alexander polynomial such that for $m_1 \neq m_2$ and $m_3 \neq 0$, the Alexander polynomial of $D(m_1, m_2, m_3)$ is nontrivial and that $D(m_1, m_2, m_3)$ is not welded isotopic to a classical link diagram (Example 5.1).

2 Multiplexing of crossings

Let $(m_1, \ldots, m_n)$ be an ordered set of integers and $D = D_1 \cup \cdots \cup D_n$ an ordered n–component virtual link diagram. For a classical crossing of D whose overpass belongs to D_j, we define the *multiplexing* of the crossing associated with m_j as a local change shown in Figure 3. When $m_j = 0$, the multiplexing of the crossing is the virtualization of it. The number of classical crossings that appear in the multiplexing of the crossing is the absolute value of m_j. Let $D(m_1, \ldots, m_n)$ denote the virtual link diagram obtained from D by the multiplexing of all classical crossings of D associated with $(m_1, \ldots, m_n)$. Then we have the following theorem.

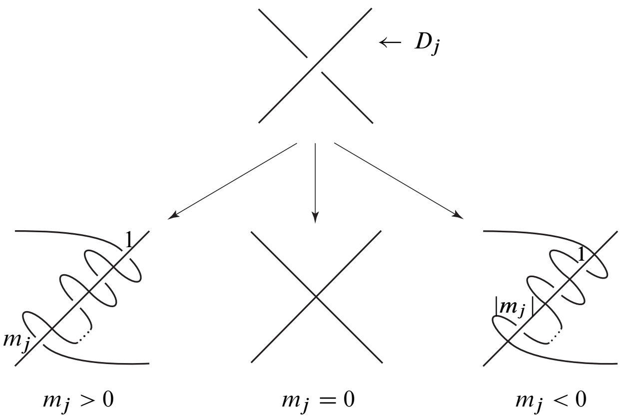

Figure 3: Multiplexing of a crossing

Theorem 2.1 *If two ordered n–component virtual link diagrams D and D' are welded isotopic, then $D(m_1, \ldots, m_n)$ and $D'(m_1, \ldots, m_n)$ are welded isotopic for any $(m_1, \ldots, m_n) \in \mathbb{Z}^n$.*

Remark 2.2 There exist equivalent classical link diagrams D and D' such that $D(m_1, \ldots, m_n)$ and $D'(m_1, \ldots, m_n)$ are not *virtually* isotopic for some $(m_1, \ldots, m_n)$. For example, let D be the classical knot diagram illustrated in the left-hand side of Figure 4. Then the virtual knot diagram $D(2)$ is not virtually isotopic to the trivial one [3]. Let D' be the trivial knot diagram without crossings, then $D'(2) = D'$. Therefore, D and D' are equivalent, but $D(2)$ and $D'(2)$ are not virtually isotopic.

3 Generalized link groups

Kelly [4] and Wada [8], independently, introduced a family of link invariants G_m $(m \in \mathbb{Z})$ which are groups generalizing the fundamental group of the complement of a

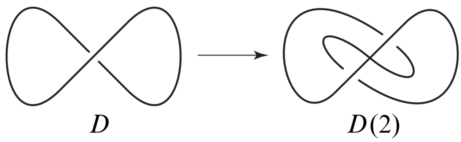

Figure 4: The diagram $D(2)$, obtained from D by the multiplexing of the crossing, is not virtually isotopic to the trivial knot diagram.

classical link in the 3–sphere S^3. Let D be an oriented classical link diagram of a classical link L. The *generalized link group* $G_m(D)$ of D is defined as follows: each arc of D yields a generator, and each crossing of D gives a relation as shown in Figure 5. Note that $G_1(D) \cong \pi_1(S^3 \setminus L)$. In [4; 8], they proved that $G_m(D)$ is a classical link invariant. As we mentioned in the introduction, $G(D(m,\dots,m))$ is isomorphic to $G_m(D)$. Hence, $D(m,\dots,m)$ gives us a geometric interpretation of $G_m(D)$. (We remark that N Kamada and S Kamada [2] gave a geometric interpretation of the groups of virtual link diagrams.) Moreover, Theorem 2.1 implies that G_m can be defined for not only classical link diagrams but also virtual link diagrams, and it is a welded link invariant.

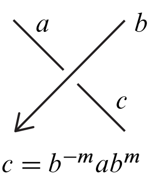

Figure 5: A relation of the generalized link group $G_m(D)$

It is well known that the square knot SK and the granny knot GK are a pair of distinct knots with isomorphic fundamental groups. C Tuffley [7] proved that $G_m(\mathrm{SK})$ and $G_m(\mathrm{GK})$ are not isomorphic for $m \geq 2$. Moreover, S Nelson and W D Neumann proved the following theorem:

Theorem 3.1 [6, Theorem 1.1] *Let m be an integer with $m \geq 2$, and let D, D' be classical knot diagrams. Then D is equivalent to D' or D'_* if and only if $G_m(D) \cong G_m(D')$, where D'_* is the mirror image of D'.*

This theorem together with Theorem 2.1 implies the following.

Theorem 3.2 *Let m be a nonzero integer, and let D, D' be classical knot diagrams. Then D is equivalent to D' or D'_* if and only if $D(m)$ is welded isotopic to $D'(m)$ or $(D'(m))_*$.*

Proof Since we have that $D'_*(m) = (D'(m))_*$, the "only if" part immediately holds by Theorem 2.1. Thus, let us prove the if part.

For $m = 1$, it is trivial [1, Theorem 1.B].

Suppose that $m \geq 2$. If $D(m)$ is welded isotopic to $D'(m)$, then $G(D(m)) \cong G(D'(m))$ [3, Section 4]. Therefore, $G_m(D) \cong G_m(D')$.

If $D(m)$ is welded isotopic to $(D'(m))_* = D'_*(m)$, then $G(D(m)) \cong G(D'_*(m))$, and hence $G_m(D) \cong G_m(D'_*)$. By Theorem 3.1, D is equivalent to D' or D'_*.

If $m \leq -1$, then it is not hard to see that $D(|m|)$ and $(D(m))(-1)$ are welded isotopic. Hence, Theorem 2.1 implies that if $D(m)$ and $D'(m)$ are welded isotopic, then $D(|m|)$ and $D'(|m|)$ are welded isotopic. Therefore, the proof follows from the case when $m \geq 1$. □

4 Proof of Theorem 2.1

In this section we will prove Theorem 2.1. Let us first prove the following lemma.

Lemma 4.1 *The local moves A, B, C^+ and C^- illustrated in Figure 6 are realized by welded isotopy. Here, the square bounded by dashed lines in the move B may contain virtual crossings but not classical crossings.*

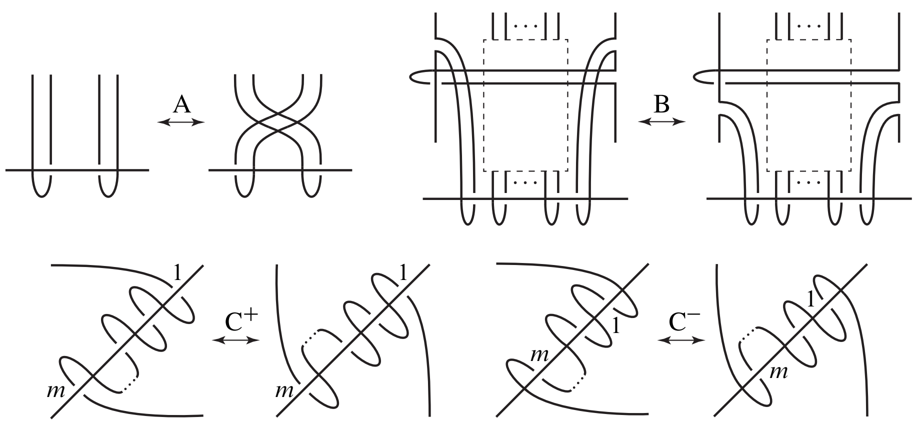

Figure 6: Local moves A, B, C^+ and C^- are realized by welded isotopy.

Proof **Move A** See Figure 7.

Move B See Figure 8, where V denotes virtual isotopy.

Moves C^+ and C^- Let F be the local move illustrated in Figure 9 which is realized by a detour move. Figure 10 (resp. Figure 11) indicates the proof for move C^+ (resp. C^-). While the proof is described only when $m = 4$ in Figures 10 and 11, it is essentially the same in all other cases. □

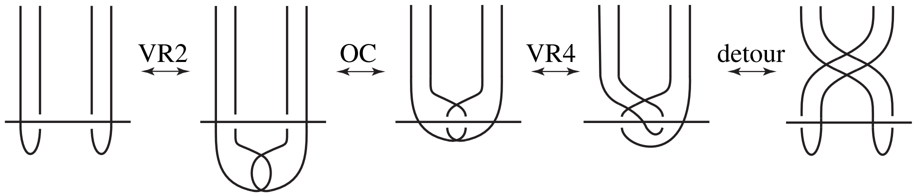

Figure 7: Proof for move A

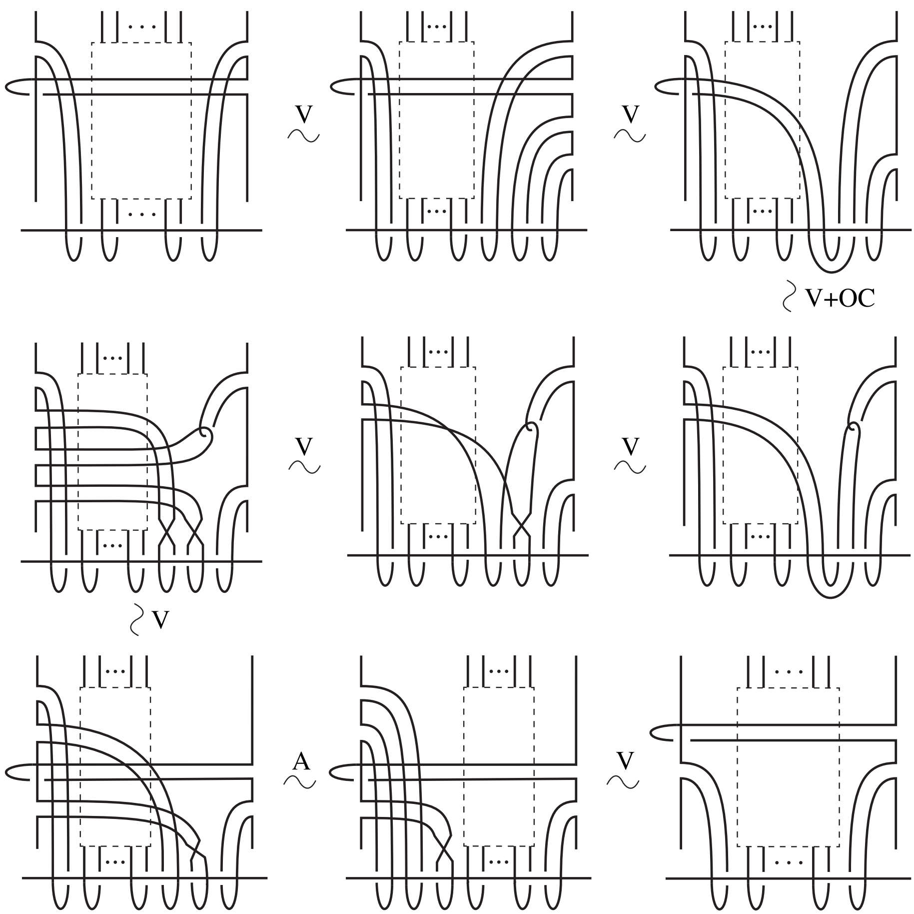

Figure 8: Proof for move B

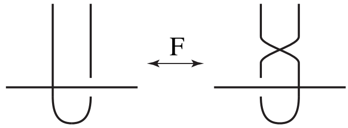

Figure 9: Move F

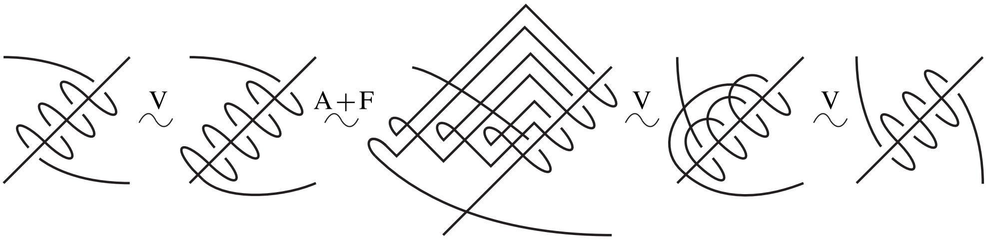

Figure 10: Proof for move C^+

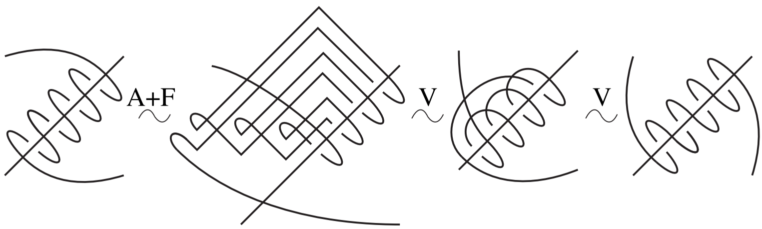

Figure 11: Proof for move C^-

Proof of Theorem 2.1 It suffices to show that if D and D' are related by one of R1, R2, R3, VR4 and OC, then $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are welded isotopic.

By using move C^+ or C^-, it is not hard to see that if D and D' are related by either R1 or R2, then $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are welded isotopic.

If D and D' are related by a single VR4, then $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are related by a detour move.

If D and D' are related by a single R3, then $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are related by virtual isotopy and moves A, B, $C^\pm$ and F. Figure 12 indicates the proof when $m_i = 3$ and $m_j = 2$. In the general case, the proof is essentially same, where move C^- is used instead of C^+ when m_i is negative.

If D and D' are related by a single OC, then by deformations similar to those in Figure 12, $D(m_1,\dots,m_n)$ and $D'(m_1,\dots,m_n)$ are related by virtual isotopy and moves A and $C^\pm$. $\square$

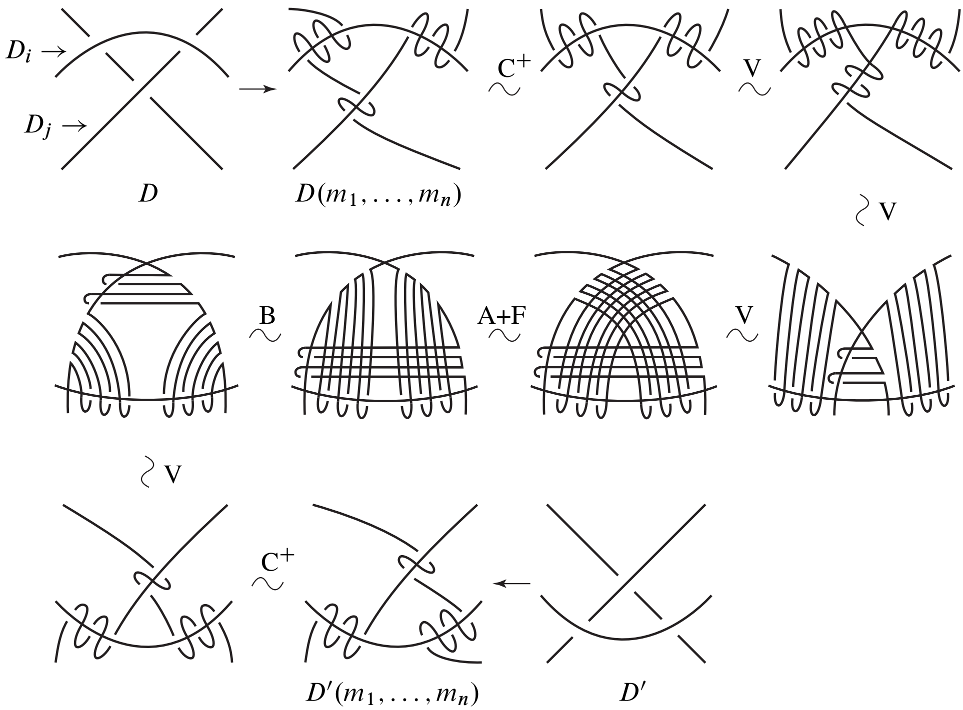

Figure 12: $D(m_1, \dots, m_n)$ and $D'(m_1, \dots, m_n)$ are related by virtual isotopy and moves A, B, C$^+$ and F when $m_i > 0$.

Remark 4.2 By using *arrow calculus*, introduced by J-B Meilhan and the third author in [5], we could prove Theorem 2.1 more simply. It might also be possible to show Theorem 2.1 by using Gauss diagrams. While our proof looks complicated, it is done by combining elementary deformations, and is, in particular, self-contained.

5 Examples

We are curious to have new *classical* link invariants from welded link invariants via the multiplexing of crossings. In fact, we have the following example.

Example 5.1 Let $D = D_1 \cup D_2 \cup D_3$ be the ordered oriented 3–component classical link diagram illustrated in Figure 13. Then the Alexander polynomial $\Delta_D(t)$ of D is 0. On the other hand,

$$\Delta_{D(m_1,m_2,m_3)}(t) = g(t)(t^{m_1} - t^{m_2})^2(1 - t^{m_3}),$$

where

$$g(t) = \gcd\{1 - t^{m_1}, 1 - t^{m_2}, 1 - t^{m_3}\}.$$

Therefore, $\Delta_{D(m_1,m_2,m_3)}(t)$ is nontrivial for some (m_1, m_2, m_3), while $\Delta_D(t)$ vanishes. We remark that $D(m_1, m_2, m_3)$ is not welded isotopic to a classical link diagram when $m_1 \neq m_2$ since the intersection number of the first and second components of $D(m_1, m_2, m_3)$ is equal to $m_1 - m_2$ $(\neq 0)$.

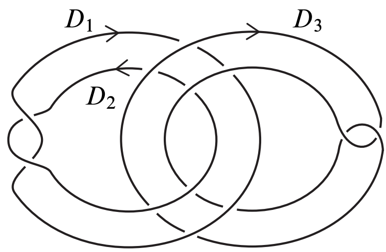

Figure 13: An ordered oriented 3–component classical link diagram with vanishing Alexander polynomial

In the example above, the 3–variable Alexander polynomial of D does not vanish. So far, we do not know if there is a classical link with vanishing multivariable Alexander polynomial such that our invariants via the multiplexing of crossings survive. But we have the following example.

Example 5.2 Let $D = D_1 \cup D_2 \cup D_3$ and $D' = D'_1 \cup D'_2 \cup D'_3$ be the ordered oriented 3–component virtual link diagrams illustrated on the left- and right-hand sides of Figure 14, respectively. Then, the 3–variable Alexander polynomials of D and D' are both equal to $(1-t_1)(1-t_2)(1-t_3)$. However,

$$\Delta_{D(m_1,m_2,m_3)}(t) = (1-t^{m_1})^2(1-t^{m_2})(1-t^{m_3})$$

and

$$\Delta_{D'(m_1,m_2,m_3)}(t) = (1-t^{m_1})(1-t^{m_2})^2(1-t^{m_3}).$$

Therefore, D and D' can be distinguished by the 1–variable Alexander polynomials of $D(m_1, m_2, m_3)$ and $D'(m_1, m_2, m_3)$, while the 3–variable Alexander polynomials of D and D' coincide.

Suppose that each m_i is equal to either 0 or 1. Then, by the definition of the multiplexing of crossings, an invariant of $D(m_1, \dots, m_n)$ might be weaker than that of D. (Note that $D(1, \dots, 1) = D$ and $D(0, \dots, 0)$ is a diagram of the n–component trivial link.) But even if some of the m_i are 0, it seems still interesting to consider $D(m_1, \dots, m_n)$, because it would give us useful invariants that are handled easily. For example, we have the following.

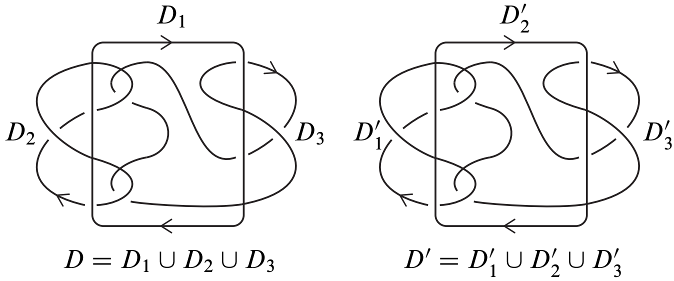

Figure 14: Two ordered oriented 3–component virtual link diagrams with the same 3–variable Alexander polynomial

Example 5.3 Let $D = D_1 \cup D_2 \cup D_3$ be the ordered oriented 3–component classical link diagram illustrated in the left-hand side of Figure 15. Then the second Alexander polynomial of $D(1,1,0)$ is equal to $(1-t)^2$. Hence, $D(1,1,0)$ provides a concise way to determine that D is nontrivial.

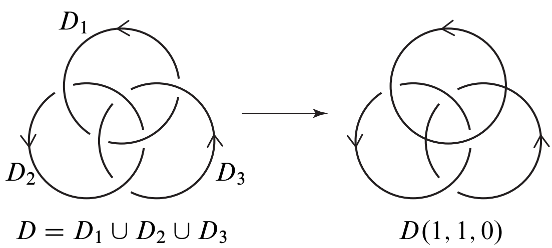

Figure 15

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