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Algebraic & Geometric Topology 18 (2018) 1823–1856

The nonorientable 4–genus for knots with 8 or 9 crossings

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The nonorientable 4–genus of a knot in the 3–sphere is defined as the smallest first Betti number of any nonorientable surface smoothly and properly embedded in the 4–ball with boundary the given knot. We compute the nonorientable 4–genus for all knots with crossing number 8 or 9. As applications we prove a conjecture of Murakami and Yasuhara and compute the clasp and slicing number of a knot.

57M25, 57M27

1 Introduction

1.1 Background

Knots and the surfaces they bound have been intimately related from the origins of knot theory. The classification of surfaces has made it easy to impart a measure of complexity on the knots that bound them. For instance, the Seifert genus $g_3(K)$ of a knot K, defined as the minimal genus of any surface S in S^3 with $\partial S = K$, was defined by Seifert [28] already in 1935. There are several other natural choices of surfaces to consider, leading to several flavors of knot genera.

Work of Fox [8; 9] and Fox and Milnor [10] led to the definition of the (*smooth, oriented*) 4*–genus* (*or slice genus*) $g_4(K)$ of a knot K as the minimal genus of any smoothly and properly embedded surface S in the 4–ball D^4 with $\partial S = K$. The *topological* (*oriented*) 4*–genus* $g_4^{\text{top}}(K)$ is defined analogously by requiring that the embedding $S \hookrightarrow D^4$ be locally topologically flat instead of smooth. Note that $g_4^{\text{top}}(K) \le g_4(K) \le g_3(K)$.

In another direction, Clark [3] defined the *nonorientable* 3*–genus* or 3*–dimensional crosscap number* $\gamma_3(K)$ as the smallest first Betti number of any nonorientable surface $\Sigma \subset S^3$ with $\partial\Sigma = K$. The *nonorientable* (*smooth*) 4*–genus* or 4*–dimensional crosscap number* $\gamma_4(K)$ was defined by Murakami and Yasuhara [20] as the minimal first Betti number of any nonorientable surface Σ smoothly and properly embedded in D^4 and

Published: 3 April 2018 DOI: 10.2140/agt.2018.18.1823

with $\partial\Sigma = K$. Some authors additionally define $\gamma_4(K) = 0$ for any slice knot K, but in the interest of a more unifying treatment we adopt the definition from the previous sentence. Just as in the case of oriented surfaces, so too for nonorientable surfaces there is a topological version of this invariant denoted by $\gamma_4^{\mathrm{top}}(K)$. The inequalities $\gamma_4^{\mathrm{top}}(K) \le \gamma_4(K) \le \gamma_3(K)$ again hold in the nonorientable setting. The oriented and nonorientable genera are easily seen to satisfy

$$\gamma_i(K) \le 2g_i(K) + 1 \quad \text{for } i = 3, 4 \tag{1-1}$$

with an analogous inequality holding for the topological 4–genera. Indeed if K bounds a properly embedded, smooth, genus-g surface $S \subset D^4$, then the surface Σ obtained from S by removing a disk neighborhood of an interior point and replacing it by a Möbius band has $\partial\Sigma = K$ and $b_1(\Sigma) = 2g + 1$, demonstrating $\gamma_4(K) \le 2g_4(K) + 1$.

The subject of study in this present work is the smooth nonorientable 4–genus γ_4. Having been introduced relatively recently, the literature available on γ_4 is relatively sparse. First results go back to the work of Viro [30] who uses Witt classes of intersection forms of 4–manifolds to obstruct a knot K from bounding a smoothly and properly embedded Möbius band in D^4. He uses his findings to demonstrate that $\gamma_4(4_1) > 1$.

Let $\sigma(K)$ and $\mathrm{Arf}(K)$ denote the signature and Arf invariant of K. Yasuhara [31] proves that if a knot K bounds a Möbius band in D^4, then there exists an integer x such that

$$|8x + 4\cdot\mathrm{Arf}(K) - \sigma(K)| \le 2.$$

This proves that $\gamma_4(K) > 1$ for any knot K with $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 4 \pmod 8$, the knot $K = 4_1$ being one example.

Gilmer and Livingston [12] use linking forms on the 2–fold branched cover of K, Heegaard Floer homology and Casson–Gordon invariants to show, for instance, that $\gamma_4(4_1 \# 5_1) = 3$, the largest known value for γ_4 at that time and still the largest known value for γ_4 among alternating knots (see however Theorem 1.1 below). They also prove the congruence relation

$$\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv \sigma(W(\Sigma)) - \beta(D^4, \Sigma) \pmod 8 \tag{1-2}$$

valid for any knot K that bounds a nonorientable surface Σ smoothly and properly embedded in D^4. Here $\sigma(K)$ and $\mathrm{Arf}(K)$ are as above, while $W(\Sigma)$ denotes the 2–fold cover of D^4 branched along Σ, and $\sigma(W(\Sigma))$ denotes its signature. Lastly, $\beta(D^4, \Sigma)$ is the Brown invariant (see [12] and Kirby and Melvin [17]) of the pair (D^4, Σ). It is easy to show that $\operatorname{rk} H_2(W(\Sigma);\mathbb{Z}) = \operatorname{rk} H_1(\Sigma;\mathbb{Z})$ implying the bound $|\sigma(W(\Sigma))| \le \operatorname{rk} H_1(\Sigma;\mathbb{Z})$, while work in [17] shows the same bound to hold for the Brown invariant (see also Corollary 2.2). The congruence (1-2), along with the discussion of this paragraph, implies again that if K is a knot with $\sigma(K) + 4\cdot \operatorname{Arf}(K) \equiv 4 \pmod 8$, then K cannot bound an embedded Möbius band in D^4. Relation (1-2) makes frequent appearances throughout this work, and we shall refer to it as the *Gilmer–Livingston* (*congruence*) *relation*.

Using tools from Heegaard Floer homology, Batson [1] is able to show that γ_4 is an unbounded function. He does so by proving the bound

$$\tfrac{1}{2}\sigma(K) - d(S^3_{-1}(K)) \le \gamma_4(K), \tag{1-3}$$

whose notation we now explain. For a rational number r we let $S^3_r(K)$ be the manifold resulting from r–framed Dehn surgery on K, and for an integral homology 3–sphere Y we use $d(Y)$ to denote its Heegaard Floer correction term; see Ozsváth and Szabó [26]. Batson shows that the left-hand side of (1-3) equals $k-1$ for the torus knot $K = T_{(2k,2k-1)}$, $k \in \mathbb{N}$, demonstrating the unboundedness of γ_4.

Ozsváth, Stipsicz and Szabó [25] in 2015 define a concordance invariant $\upsilon(K)$, derived from their family of concordance invariants $\Upsilon_K(t)$ [24]. They prove the lower bound

$$\left|\tfrac{1}{2}\sigma(K) - \upsilon(K)\right| \le \gamma_4(K) \tag{1-4}$$

and use it to provide another proof of the unboundedness of γ_4 by demonstrating that $\gamma_4\big(\#^n\, T_{(3,4)}\big) \ge n$, where $\#^n\, T_{(3,4)}$ is the n–fold connected sum of the $(3,4)$ torus knot $T_{(3,4)}$ with itself. The converse inequality $\gamma_4\big(\#^n\, T_{(3,4)}\big) \le n$ is easy to verify by finding an explicit Möbius band bounded by $T_{(3,4)}$, leading to $\gamma_4\big(\#^n\, T_{(3,4)}\big) = n$.

1.2 Results and applications

As of this writing, the KnotInfo knot tables of Cha and Livingston [2] only contain values for γ_4 for knots with 7 or fewer crossings. Our goal and the main result of this work is to extend this tabulation to include all 70 knots with 8 and 9 crossings.

Theorem 1.1 *The values of γ_4 for the 21 knots with crossing number 8 are*

$\gamma_4(K) = 1$ *for* $K = 8_3, 8_4, 8_5, 8_6, 8_7, 8_8, 8_9, 8_{10}, 8_{11}, 8_{14}, 8_{16}, 8_{19}, 8_{20}, 8_{21}$;

$\gamma_4(K) = 2$ *for* $K = 8_1, 8_2, 8_{12}, 8_{13}, 8_{15}, 8_{17}$;

$\gamma_4(K) = 3$ *for* $K = 8_{18}$.

Theorem 1.2 *The values of γ_4 for the 49 knots with crossing number 9 are*

$$\gamma_4(K) = 1 \quad \text{for } K = 9_1, 9_3, 9_4, 9_5, 9_6, 9_7, 9_8, 9_9, 9_{13}, 9_{15}, 9_{17}, 9_{19}, 9_{21}, 9_{22}, 9_{23}, 9_{25}, 9_{26}, 9_{27}, 9_{28}, 9_{29}, 9_{31}, 9_{32}, 9_{35}, 9_{36}, 9_{41}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{46}, 9_{47}, 9_{48};$$

$$\gamma_4(K) = 2 \quad \text{for } K = 9_2, 9_{10}, 9_{11}, 9_{12}, 9_{14}, 9_{16}, 9_{18}, 9_{20}, 9_{24}, 9_{30}, 9_{33}, 9_{34}, 9_{37}, 9_{38}, 9_{39}, 9_{40}, 9_{49}.$$

As already mentioned in Section 1.1, the nonorientable slice genus $\gamma_4(K)$ was introduced by Murakami and Yasuhara in their work [20] with the difference that they defined γ_4 of a slice knot K to be zero. Murakami and Yasuhara observed the inequality $\gamma_4(K) \le 2g_4(K) + 1$, the $i = 4$ version of (1-1). In the following conjecture they ask whether this inequality is the best possible bound relating $\gamma_4(K)$ and $g_4(K)$.

Conjecture [20, Conjecture 2.10] *There exists a nonslice knot K such that $\gamma_4(K) = 2g_4(K) + 1$.*

Theorem 1.1 verifies the Murakami–Yasuhara conjecture.

Corollary 1.3 *There exist nonslice knots K, for instance $K = 8_{18}$, such that $\gamma_4(K) = 2g_4(K) + 1$. Accordingly, the inequality $\gamma_4(K) \le 2g_4(K) + 1$ is sharp for some knots and cannot be improved upon.*

Recall that the *unknotting number* $u(K)$ of a knot K is the minimum number of crossing changes in any diagram of K that renders K unknotted. Similarly, the *slicing number* $u_s(K)$ of a knot K is defined as the minimum number of crossing changes in any diagram of K that transforms K into a slice knot. These two quantities fit into the double inequality

$$g_4(K) \le u_s(K) \le u(K). \tag{1-5}$$

Of these, a proof of the left inequality can be found in Scharlemann [27], while the right inequality is obvious since the unknot is slice. The inequality $u_s(K) \le u(K)$ is a strict inequality for many knots K, for instance for any nontrivial slice knot K. To show that the inequality $g_4(K) \le u_s(K)$ may also be strict is rather more difficult. The first example of a knot K where this occurs, namely $K = 7_4$, was discovered by Livingston [18]. Owens [21] and Owens and Strle [23], by relying on gauge-theoretic techniques, are able to calculate $u_s(K)$ for all knots K with 10 or fewer crossings and find many more examples with $g_4(K) < u_s(K)$. In general however, both $u(K)$ and $u_s(K)$ remain difficult knot invariants to compute.

The 4*–dimensional clasp number* $c_4(K)$ of a knot K is the smallest number of double points of any immersed disk in the 4–ball D^4, with boundary K. The clasp number also fits into a double inequality, namely

$$g_4(K) \le c_4(K) \le u_s(K), \tag{1-6}$$

of which the left one is proved in Shibuya [29], while the right one is obvious. The inequality $g_4(K) \le c_4(K)$ may be strict — an example is given in [20] — but we are not aware of a knot K with $c_4(K) < u_s(K)$. The relation between the nonorientable 4–genus $\gamma_4(K)$ and the clasp number $c_4(K)$ was worked out by Murakami and Yasuhara:

Proposition 1.4 [20, Proposition 2.3] *For any knot K,*

$$\gamma_4(K) \le \begin{cases} c_4(K) & \text{if } c_4(K) \text{ is even and } c_4(K) \neq 2, \\ c_4(K)+1 & \text{otherwise.} \end{cases} \tag{1-7}$$

This inequality and its proof were independently communicated to us by Chuck Livingston, whose input we gratefully acknowledge. The reason for this detour into exploring $c_4(K)$ and $u_s(K)$ is to demonstrate in the next example that our computation of $\gamma_4(8_{18})$ in conjunction with Proposition 1.4 can be used to obtain a proof of the strict inequalities $g_4(8_{18}) < c_4(8_{18})$ and $g_4(8_{18}) < u_s(8_{18})$, facts that were first obtained by Owens and Strle [23] using rather different techniques.

Example 1.5 The knot $K = 8_{18}$ has $\gamma_4(K) = 3$ by Theorem 1.1. Proposition 1.4 implies that $2 \le c_4(K)$, and thus $2 \le u_s(K)$ by (1-6). Since $u(K) = 2$ we obtain $c_4(K) = u_s(K) = 2$ by (1-5) and (1-6), while $g_4(K) = 1$.

Organization

In Section 2 we provide needed background material. We remind the reader of the definition of the Brown invariant, introduce nonoriented band moves on knot diagrams, and review the Goeritz form of a knot and Donaldson's diagonalization theorem. The main results of this section are the obstruction theorems (Theorems 2.10–2.12). Section 3 looks at all 70 knots with crossing number 8 or 9 and employs the techniques from Section 2 to compute their values of γ_4, which proves Theorems 1.1 and 1.2. Section 4 concludes with some observations and open questions.

Acknowledgements

We wish to thank Pat Gilmer and Chuck Livingston for helpful comments. Jabuka gratefully acknowledges support from the Simons Foundation, Grant #246123.

2 Background

This section describes the techniques used to determined the values of γ_4 for knots with 8 or 9 crossings. The techniques come in two flavors: constructive and obstructive. The former takes the form of a nonoriented band move on knot diagrams (described in Section 2.2) in such a way that if two knots are related by such a move, their γ_4–values differ by at most 1; see Proposition 2.4. The obstructive techniques use Donaldson's celebrated diagonalization theorem for definite 4–manifolds, in combination with a construction of Goeritz. These are described in Section 2.3.

2.1 The Brown invariant

This section recalls the definition of the Brown invariant $\beta(D^4,\Sigma)$ of a smoothly and properly embedded nonorientable surface $\Sigma \hookrightarrow D^4$. Our exposition follows that of [17].

Let V be a finite-dimensional $\mathbb{Z}_2$–vector space equipped with a nonsingular inner product $\cdot\colon V\times V\to\mathbb{Z}_2$, that is, an inner product for which $x\cdot y=0$ for all $y\in V$ implies $x=0$. We call $(V,\cdot)$ *even* if $x\cdot x=0$ for all $x\in V$; otherwise we say $(V,\cdot)$ is *odd*. Every such inner product space $(V,\cdot)$ can be decomposed as a direct sum of orthogonal subspaces isomorphic to

$$P=\mathbb{Z}_2x \quad\text{and}\quad T=\mathbb{Z}_2y\oplus\mathbb{Z}_2z$$

with $x\cdot x=1=y\cdot z$ and $y\cdot y=0=z\cdot z$. These two irreducible spaces satisfy the isomorphism $P\oplus T\cong P\oplus P\oplus P$, and there are no other relations among them. Accordingly, every inner product space $(V,\cdot)$ is isomorphic to either mP (the m–fold orthogonal sum of P) or nT (the n–fold orthogonal sum of T). The former are the odd inner product spaces, the latter the even ones.

A *quadratic form on* $(V,\cdot)$ is a function $q\colon V\to\mathbb{Z}_4$ with $q(x+y)=q(x)+q(y)+2x\cdot y$ for all $x,y\in V$. Here $\cdot 2\colon \mathbb{Z}_2\to\mathbb{Z}_4$ is the unique homomorphism sending 1 to 2. Restricting q to the irreducible summands of $(V,\cdot)$ gives a decomposition of q as a sum of quadratic forms on P or T.

The space $P=\mathbb{Z}_2x$ admits two quadratic forms q_{-1} and q_1, defined by $q_i(x)=i$. Similarly, the space $T=\mathbb{Z}_2y\oplus\mathbb{Z}_2z$ admits exactly four quadratic forms $q_{0,0}$, $q_{0,2}$, $q_{2,0}$ and $q_{2,2}$, given by $q_{i,j}(y)=i$ and $q_{i,j}(z)=j$. Of these the first three are mutually isomorphic, but are not isomorphic to the fourth one, giving precisely two isomorphism classes of quadratic forms on T.

The relation $P\oplus T\cong 3P$ of inner product spaces induces relations among the quadratic forms $q_{i,j}$ and q_k as $q_{\pm1}\oplus q_{0,0}\cong q_{\pm1}\oplus q_{-1}\oplus q_1$ and $q_{\pm1}\oplus q_{2,2}\cong 3q_{\mp1}$. Of course since $q_{0,0}$ is isomorphic to both $q_{0,2}$ and $q_{2,0}$, we may replace $q_{0,0}$ in the first relation above by either of $q_{0,2}$ or $q_{2,0}$. These relations further imply the relations

$$2q_{0,0}\cong 2q_{2,2} \quad\text{and}\quad 4q_{-1}\cong 4q_1, \tag{2-1}$$

which lead to the following unique decomposition of a quadratic form $(V,\cdot,q)$:

$$q\cong\begin{cases}\text{direct sums of copies of } q_{0,0} \text{ and at most one copy of } q_{2,2} & \text{if } (V,\cdot) \text{ is even,}\\ \text{direct sums of copies of } q_1 \text{ and at most three copies of } q_{-1} & \text{if } (V,\cdot) \text{ is odd.}\end{cases}$$

We define the *Brown invariant* $\beta(q)\in\mathbb{Z}_8$ *of* $(V,\cdot,q)$ by setting

$$\beta(q_{0,0})=\beta(q_{0,2})=\beta(q_{2,0})=0,\quad \beta(q_{2,2})=4,\quad \beta(q_{-1})=-1,\quad \beta(q_1)=1,$$

and by imposing additivity $\beta(q'\oplus q'')=\beta(q')+\beta(q'')$ under the direct sum of the quadratic forms q' and q''. The relations (2-1) show that the Brown invariant is well defined modulo 8.

For the next lemma we define the norm $|x|$ for $x\in\mathbb{Z}_8$ as the smallest absolute value $|y|$ with $x\equiv y \pmod 8$. For example $|7|=1$.

Lemma 2.1 *For an odd quadratic inner product space $(V,\cdot,q)$, we have*

$$|\beta(q)|\le \dim_{\mathbb{Z}_2} V.$$

Proof Since $(V,\cdot)$ is odd we can write $(V,\cdot)\cong nP$ with $n=\dim_{\mathbb{Z}_2}V$. Then q is isomorphic to an n–fold direct sum of copies of P_{-1} and P_1, and its Brown invariant $\beta(q)$ is therefore an n–fold sum whose summands are either -1 or 1. It follows that $|\beta(q)|\le n$ as claimed. □

Given a nonorientable surface $\Sigma\subset D^4$, smoothly and properly embedded, Guillou and Marin [16] define an odd form $q_\Sigma\colon H_1(\Sigma;\mathbb{Z}_2)\to\mathbb{Z}_2$ that is quadratic with respect to the linking pairing $\cdot$ on $H_1(\Sigma;\mathbb{Z}_2)$. We omit the details of the definition of q_Σ as they are not relevant to our subsequent discussion. The *Brown invariant* $\beta(\Sigma,D^4)\in\mathbb{Z}_8$

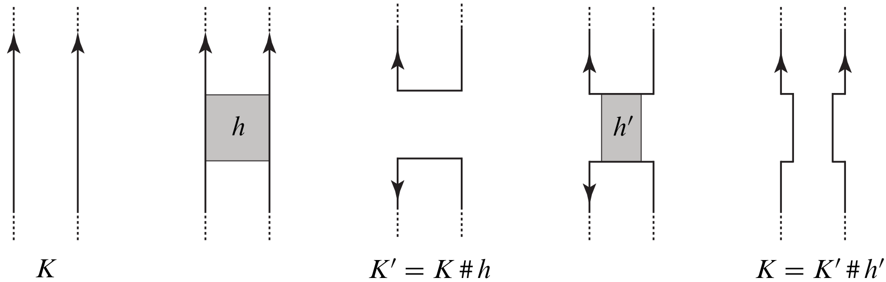


Figure 1: Nonorientable band moves are symmetric: if a knot K' is obtained from a knot K by a nonorientable band move using the band h, then K is also obtained from K' by a nonorientable band move using the "dual band" h' of h.

of the embedding $\Sigma \subset D^4$ is defined as the Brown invariant $\beta(q_\Sigma)$ of the quadratic inner product space $(H_1(\Sigma;\mathbb{Z}_2), \cdot, q_\Sigma)$. The following is a direct consequence of Lemma 2.1.

Corollary 2.2 *For a nonorientable surface $\Sigma \subset D^4$, smoothly and properly embedded, the inequality $|\beta(\Sigma, D^4)| \le b_1(\Sigma)$ holds.*

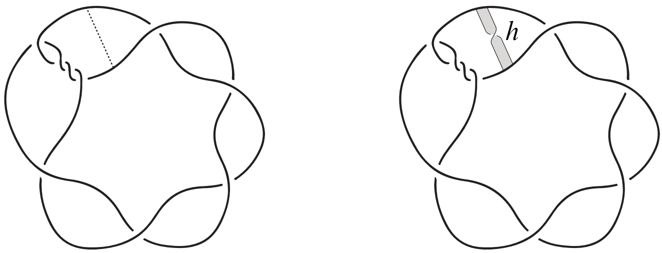
2.2 Nonoriented band moves

We describe here a move on knots that will be one of our fundamental tools in seeking concrete nonorientable surfaces Σ, smoothly and properly embedded in D^4, with boundary a given knot K.

Definition 2.3 A *nonoriented band move* on an oriented knot K is the operation of attaching an oriented band $h = [0,1] \times [0,1]$ to K along $[0,1] \times \partial[0,1]$ in such a way that the orientation of the knot agrees with that of $[0,1] \times \{0\}$ and disagrees with that of $[0,1] \times \{1\}$ (or vice versa), and then performing surgery on h, that is replacing the arcs $[0,1] \times \partial[0,1] \subseteq K$ by the arcs $\partial[0,1] \times [0,1]$.

The resulting knot K' shall be said to have been *obtained from K by a nonoriented band move*, and we write $K' = K \# h$ to indicate this operation.

Note that if K' was obtained from K by a nonoriented band move and $K' = K \# h$, then the knot K is also obtained from K' by a nonoriented band move and $K = K' \# h'$ where h' is the "dual band" of h; see Figure 1.



$9_4 \xrightarrow{1} 10_3$ $\qquad$ $10_3 = 9_4 \# h$

Figure 2: Our convention for labeling nonoriented bands is illustrated at left, where the handle h is represented by a dotted line, and its "framing" of 1 is indicated in the labeled arrow. The band is fully drawn, including its right-handed half-twist, at right. The caption on the left is shorthand notation for the caption on the right. As convention dictates, "positive framings" correspond to right-handed half-twists, and "negative framings" to left-handed half-twists.

Remark 2.1 (on band-move notation) Before proceeding, we pause to introduce some pictorial notation for band moves. We shall represent a band h in a knot diagram of K by drawing a dotted line representing the core $\{\frac{1}{2}\} \times [0,1]$ of h. We shall then use an integer n to indicate the number of half-twists to be introduced into h with respect to the blackboard (or paper) framing, where as is usual $n > 0$ corresponds to n right-handed half-twists and $n < 0$ corresponds to $|n|$ left-handed half-twists. This framing shall appear in the caption of the figure, where we write $K \xrightarrow{n} K'$ to indicate that $K' = K \# h$ and h is the band obtained from its core $\{\frac{1}{2}\} \times [0,1]$ by adding the framing n. Figure 2 illustrates this convention.

Remark 2.2 We note that in writing $K \xrightarrow{n} K'$ we mean that the knot K under the indicated nonoriented band move transforms into either the knot K' or its reverse mirror knot $-K'$. In all of our computations we determined K' from its crossing number and its Alexander polynomial, two data points which do not differentiate between K' and $-K'$. Since $\gamma_4(K') = \gamma_4(-K')$ this does not affect our claims.

The following proposition is an easy but very useful observation.

Proposition 2.4 *If the knots K and K' are related by a nonoriented band move, then*

$$\gamma_4(K) \leq \gamma_4(K') + 1.$$

If a knot K is related to a slice knot K' by a nonoriented band move, then $\gamma_4(K) = 1$.

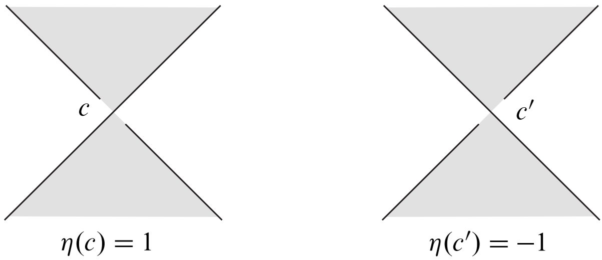


Figure 3: The weights ± 1 associated to the two different types of crossings c and c'

Proof Let Σ' be a nonorientable smoothly embedded surface in D^4 with $\partial\Sigma' = K'$ and with $b_1(\Sigma') = \gamma_4(K')$. Let h be a band such that K is obtained from K' by a nonoriented band move on h, and let Σ be the surface in D^4 obtained by attaching the band h to Σ' along $[0,1] \times \partial[0,1] \subseteq K'$, and pushing the interior of h into D^4 so as to make Σ properly (and smoothly) embedded in D^4. Then Σ is a nonorientable surface with $\partial\Sigma = K$ and with $b_1(\Sigma) = b_1(\Sigma') + 1$, and so

$$\gamma_4(K) \leq b_1(\Sigma) = b_1(\Sigma') + 1 = \gamma_4(K') + 1,$$

as needed. If K' is slice, the above construction can be repeated by using a slice disk for Σ', rendering Σ a Möbius band. □

2.3 Goeritz forms and Donaldson's diagonalization theorem

Associated to a projection D of knot K are two "black-and-white" checkerboard colorings. Each is a coloring of the regions of the knot projection with black and white colors such that no two regions sharing an edge receive the same color. There are exactly two such colorings: one in which the unbounded region is colored white and the other in which it is black.

Associated to either checkerboard coloring of the knot projection D is a bilinear form first described by Goeritz [13]. Our exposition follows that given by Gordon and Litherland [14].

Let $X_0, X_1, \dots, X_n$ denote the white regions in the checkerboard coloring of D. We associate to every crossing c in D a weight $\eta(c) = \pm 1$ as in Figure 3. Let $P_{i,j}$ be the set of double points in D that are incident to both X_i and X_j. For $i, j \in \{0, \dots, n\}$

let g_{ij} be the integer

$$g_{ij} = \begin{cases} -\sum\limits_{c \in P_{i,j}} \eta(c) & \text{if } i \neq j, \\ -\sum\limits_{k \neq i} g_{ik} & \text{if } i = j. \end{cases} \tag{2-2}$$

Let $G' = [g_{ij}]$ be the $(n+1) \times (n+1)$ matrix comprised of the coefficients g_{ij}, we refer to G' as the *pre-Goeritz matrix* associated to the above choice of checkerboard coloring of D. The *Goeritz matrix* $G = [g_{ij}]$ is the $n \times n$ matrix obtained from G' by deleting its 0^{th} row and column. The bilinear form $(\mathbb{Z}^n, G)$ is symmetric and nondegenerate, indeed $\det G = \det K$.

Let F be a smoothly and properly embedded surface in D^4 with $\partial F = K$, and let $W = W(F)$ be the twofold cover of D^4 with branching set F. The surface F may be chosen to be either oriented or nonorientable. Note that the boundary of W is $Y = Y(K)$, the twofold cover of S^3 with branching set K. We denote by Q_W the intersection form on $H_2(W;\mathbb{Z})/\mathrm{Tor}$.

Theorem 2.5 (Gordon and Litherland [14, Theorem 3]) *Let K be a knot and D a projection of K. Pick a checkerboard coloring of D, and let $n+1$ be the number of white regions. Let F' be the surface with boundary equal to K, obtained from the black regions (with twisted bands added to connect the black disks), and let F be obtained from F' by pushing its interior into D^4. With $W = W(F)$ described as above, there is an isomorphism*

$$(H_2(W;\mathbb{Z})/\mathrm{Tor}, Q_W) \cong (\mathbb{Z}^n, G)$$

of integral, symmetric, bilinear forms.

Corollary 2.6 *If a knot K has a projection D with a (positive or negative) definite Goeritz matrix G, then its twofold branched cover $Y(K)$ bounds a smooth, compact (positive or negative) definite 4–manifold W.*

If D is an alternating knot projection of a knot K then the Goeritz matrices $G_{\pm}$ associated to either of the two possible checkerboard colorings of D are definite, the subscript in $G_{\pm}$ indicating the type of definiteness for each case. Indeed, by a beautiful result of Greene's [15], this property characterizes alternating knots. We note that all knots with 8 or 9 crossings have alternating diagrams, with the exception of the 11 knots

$$8_{19},\ 8_{20},\ 8_{21},\ 9_{42},\ 9_{43},\ 9_{44},\ 9_{45},\ 9_{46},\ 9_{47},\ 9_{48},\ 9_{49}. \tag{2-3}$$

Theorem 2.5 points to the importance of understanding the 4–manifold $W(\Sigma)$, and we pause to elucidate some of its algebrotopological properties before continuing. It is not hard to show that $b_2(W(\Sigma)) = b_1(\Sigma)$; see Lemma 1 in [12]. More difficult is the proof that $b_1(W(\Sigma)) = 0$, which is given in Lemma 2 in [19]. The next proposition, whose proof can be found in [11; 22], describes other relevant aspects of the algebraic topology of $W(\Sigma)$.

Proposition 2.7 (Gilmer [11], Owens and Strle [22]) *Let W be a smooth, oriented, compact 4–manifold with $b_1(W) = 0$ and with $Y = \partial W$ a rational homology 3–sphere. Let ℓ denote the determinant of the intersection pairing*

$$Q_W\colon\ H_2(W;\mathbb{Z})/\mathrm{Tor} \otimes H_2(W;\mathbb{Z})/\mathrm{Tor} \to \mathbb{Z},$$

and let n be the order of

$$\mathrm{Im}(\mathrm{Tor}(H_2(W;\mathbb{Z}))) \to \mathrm{Tor}(H_2(W,Y;\mathbb{Z})).$$

Then $|H_1(Y)| = \ell \cdot n^2$.

Corollary 2.8 *Let K be a knot bounding a Möbius band $\Sigma \subset D^4$, and let $W = W(\Sigma)$. Then the absolute value of the square of the single generator of $H_2(W;\mathbb{Z})/\mathrm{Tor}$ equals a natural number ℓ that divides $\det K$ with quotient a square. In particular, if $\det K$ is square-free, then $\ell = \pm \det K$.*

We conclude this section by quoting a beautiful result of Donaldson's.

Theorem 2.9 [6] *Let X be a smooth, closed, oriented 4–manifold whose intersection form $(H_2(X;\mathbb{Z})/\mathrm{Tor}, Q_X)$ is definite. Then Q_X is diagonalizable over $\mathbb{Z}$.*

2.4 Lower bounds on $\gamma_4(K)$

A combination of the Gilmer–Livingston congruence relation (1-2) and Theorems 2.5 and 2.9 can be used to obtain lower bounds on $\gamma_4(K)$ for certain knots K. We distinguish three cases according to whether $\sigma(K) + 4\cdot \mathrm{Arf}(K)$ is congruent mod 8 to 2, 4 or 0. The corresponding lower bounds on $\gamma_4(K)$ are stated in Theorems 2.10, 2.11 and 2.12, respectively.

To begin, let K be a knot such that $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 2 \pmod 8$ and suppose that K bounds a smoothly and properly embedded Möbius band Σ in D^4. Let $W = W(\Sigma)$

be the twofold cover of D^4 with branching set Σ, and note that the Gilmer–Livingston congruence relation implies that W is positive definite. Corollary 2.8 dictates that the square of the sole generator of $H_2(W(\Sigma))/\mathrm{Tor}$ is some $\ell \in \mathbb{N}$ that is a divisor of $\det K$ such that $\det K/\ell$ is a square. We capture this statement by writing $Q_{W(\Sigma)} = [\ell]$.

Assume additionally that K has a checkerboard coloring whose associated Goeritz matrix G is negative definite, and let $W(F)$ be the 4–manifold as in Theorem 2.5. We can then create the smooth, closed, oriented 4–manifold X by gluing $W(\Sigma)$ to $W(F)$ along their boundaries:

$$X = W(F) \cup_{Y(K)} (-W(\Sigma)).$$

Then X is negative definite, and so by Theorem 2.9 its intersection form Q_X must be diagonalizable over $\mathbb{Z}$. The direct sum $Q_{W(F)} \oplus Q_{W(\Sigma)} = Q_{W(F)} \oplus [-\ell]$ of the intersection forms of $W(F)$ and $W(\Sigma)$ clearly embeds into Q_X, a condition that can be explicitly checked for a concrete knot K. Conversely, if $Q_{W(F)} \oplus Q_{W(\Sigma)}$ does not embed into a diagonal form of equal rank, then K cannot bound a Möbius band in D^4, and we conclude that $\gamma_4(K) \geq 2$. We summarize this conclusion in the next theorem where we use the term 1–*definite* as a synonym for positive definite, and similarly -1–*definite* as a substitute for negative definite.

Theorem 2.10 *Let K be a knot with $\sigma(K) + 4 \cdot \mathrm{Arf}(K) \equiv 2\epsilon \pmod 8$ for a choice of $\epsilon \in \{\pm 1\}$. Assume that K admits a checkerboard coloring for which the associated Goeritz form G is $-\epsilon$–definite.*

If there is no embedding of $G \oplus [-\ell]$ into the ϵ–definite diagonal form $(\mathbb{Z}^{\mathrm{rank}(G)+1}, \epsilon\,\mathrm{Id})$ for any divisor $\ell \in \mathbb{N}$ of $\det K$ with $\det K/\ell$ a square, then $\gamma_4(K) \geq 2$.

If K is a knot with $\sigma(K)+4\cdot\mathrm{Arf}(K) \equiv 4 \pmod 8$ then the Gilmer–Livingston relation (1-2) reduces to

$$\sigma(W(\Sigma)) - \beta(D^4, \Sigma) \equiv 4 \pmod 8$$

for any nonorientable surface $\Sigma \subset D^4$ with $\partial\Sigma = K$, and implies that $\gamma_4(K) \geq 2$. If a nonorientable surface Σ with $b_1(\Sigma) = 2$ existed, then the above relation would force $\sigma(W(\Sigma)) = \pm 2$ according to Corollary 2.2. Since $b_2(W(\Sigma)) = 2$, we conclude that $W(\Sigma)$ is either positive or negative definite. If K is an alternating knot such that both its Goeritz forms $G_\pm$ are definite, then one can again form a smooth, oriented, closed and definite 4–manifold X as

$$X = \begin{cases} W(F_-) \cup_{Y(K)} (-W(\Sigma)) & \text{if } \sigma(W(\Sigma)) > 0, \\ W(F_+) \cup_{Y(K)} (-W(\Sigma)) & \text{if } \sigma(W(\Sigma)) < 0. \end{cases} \tag{2-4}$$

The surfaces $F_\pm$ are the surfaces formed by the black regions of the checkerboard coloring used to create the Goeritz form $G_\pm$; see Theorem 2.5. In either case, Donaldson's Theorem 2.9 implies that the intersection form Q_X of X must be diagonalizable. The difficulty faced in this case is that it is generally hard to determine the intersection form $Q_{W(\Sigma)}$ of $W(\Sigma)$, given that Σ is a hypothetical surface. Nevertheless, if Σ existed, we would still obtain an embedding of $Q_{W(F_\pm)}$ into a diagonal definite form of rank two larger, since $b_2(X) = b_2(W(F_\pm)) + 2$. We summarize this in the next theorem.

Theorem 2.11 *Let K be a knot with $\sigma(K) + 4\cdot \mathrm{Arf}(K) \equiv 4 \pmod 8$ and assume that the Goeritz matrices $G_\pm$ of K, associated to the two possible checkerboard colorings of a knot projection D of K, are positive and negative definite, respectively* (with the subscript $\pm$ indicating the definiteness type of $G_\pm$).

If there is no embedding of G_+ into the positive-definite form $(\mathbb{Z}^{\mathrm{rank}(G_+)+2}, \mathrm{Id})$ nor of G_- into the negative-definite form $(\mathbb{Z}^{\mathrm{rank}(G_-)+2}, -\mathrm{Id})$, then $\gamma_4(K) \geq 3$.

Lastly, we turn to the case of a knot K with $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 0 \pmod 8$. For such knots the Gilmer–Livingston relation offers no information about $\gamma_4(K)$. Suppose that K is alternating such that its Goeritz forms $G_\pm$ are definite. If K bounded a Möbius band $\Sigma \subset D^4$, then $W(\Sigma)$ is either positive or negative definite with intersection form $[\ell]$ for some nonzero integer ℓ dividing $\det K$ and with $\det K/|\ell|$ a square, according to Corollary 2.8. We then form again the definite, smooth, compact 4–manifold X as in (2-4). By Donaldson's theorem, X must have a diagonalizable intersection form Q_X.

Theorem 2.12 *Let K be a knot with $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 0 \pmod 8$, and assume that the Goeritz matrices $G_\pm$ of K, associated to the two possible checkerboard colorings of a knot projection D of K, are positive and negative definite, respectively* (with the subscript $\pm$ indicating the definiteness type of $G_\pm$).

If there is no embedding of $G_+ \oplus [\ell]$ into the positive-definite form $(\mathbb{Z}^{\mathrm{rank}(G_+)+1}, \mathrm{Id})$ nor of $G_- \oplus [-\ell]$ into the negative-definite form $(\mathbb{Z}^{\mathrm{rank}(G_-)+1}, -\mathrm{Id})$ for any divisor $\ell \in \mathbb{N}$ of $\det K$ with $\det K/\ell$ a square, then $\gamma_4(K) \geq 2$.

This last theorem is stated only for completeness and possible future applications. By happenstance, all knots K with 8 or 9 crossings with $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 0 \pmod 8$ admit nonoriented band moves to slice knots, and accordingly all such knots have γ_4 equal to 1; see Section 3.4.

3 Computations of γ_4

This section computes the value $\gamma_4(K)$ for all knots K with crossing number equal to 8 or 9, thereby proving Theorems 1.1 and 1.2. The computations are organized into four subsections: Section 3.1 considers those 8– and 9–crossing knots that are slice (and hence have γ_4 equal to 1), are concordant to a knot with a known value of γ_4, or admit a single nonorientable band move resulting in a slice knot. Sections 3.2, 3.3 and 3.4 consider knots K with $\sigma(K)+4\cdot\mathrm{Arf}(K)$ congruent to 4, 2 and 0 modulo 8, respectively, and rely on Theorems 2.10 and 2.11 to work out the their value of γ_4. We note that while these theorems only apply to alternating knots, by happenstance most of the nonalternating knots (2-3) with 8 or 9 crossings are already addressed in Section 3.1 as they are all either slice (in the case of 8_{20} and 9_{46}), concordant to a knot with $\gamma_4=1$ (in the case of 8_{21}), or admit a single nonoriented band move to a slice knot (in the case of 8_{19}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{47} and 9_{48}). The only remaining nonalternating knot 9_{49} is addressed in Section 3.2 by solely relying on Proposition 2.4, which in turn does not use the assumption of the knot being alternating.

3.1 Slice knots, concordant knots, and band moves to slice knots

Among knots with crossing number 8 or 9, the smoothly slice knots are precisely [2]

$$8_8,\ 8_9,\ 8_{20} \quad \text{and} \quad 9_{27},\ 9_{41},\ 9_{46}. \tag{3-1}$$

For each of these six knots, γ_4 equals 1.

Additionally, there are smooth concordances [5; 4] between the knots 8_{10} and -3_1 (with $-K$ referring to the reverse mirror of K) as well as between the knots 8_{21} and 3_1. Since $\gamma_4(3_1)=1$, and since γ_4 is a invariant of smooth concordance, we obtain

$$\gamma_4(8_{10})=1 \quad \text{and} \quad \gamma_4(8_{21})=1. \tag{3-2}$$

Among knots with 8 or 9 crossings, the 38 knots

$$\begin{gathered} 8_3,\ 8_4,\ 8_5,\ 8_6,\ 8_7,\ 8_{11},\ 8_{14},\ 8_{16},\ 8_{19}, \\ 9_1,\ 9_3,\ 9_4,\ 9_5,\ 9_6,\ 9_7,\ 9_8,\ 9_9,\ 9_{13},\ 9_{15},\ 9_{17},\ 9_{19},\ 9_{21},\ 9_{22},\ 9_{23}, \\ 9_{25},\ 9_{26},\ 9_{28},\ 9_{29},\ 9_{31},\ 9_{32},\ 9_{35},\ 9_{36},\ 9_{42},\ 9_{43},\ 9_{44},\ 9_{45},\ 9_{47},\ 9_{48} \end{gathered} \tag{3-3}$$

bound smoothly and properly embedded Möbius bands in D^4. This is seen in Figures 10–13, where we exhibit band moves from each of the knots in (3-3) to a slice knot. The claim about bounding Möbius bands then follows from Proposition 2.4. The only knot from the list (3-3) not found in the aforementioned figures is the knot $K = 9_4$, which was addressed in Figure 2.

3.2 Knots with $\sigma(K) + 4\cdot \mathrm{Arf}(K) \equiv 4 \pmod 8$

Among knots K with crossing number 8 or 9, those that satisfy the congruence $\sigma(K)+4\cdot \mathrm{Arf}(K) \equiv 4 \pmod 8$ are precisely the 19 knots

$$\begin{gathered} 8_1,\ 8_2,\ 8_{12},\ 8_{13},\ 8_{15},\ 8_{17},\ 8_{18}, \\ 9_{10},\ 9_{11},\ 9_{14},\ 9_{18},\ 9_{20},\ 9_{24},\ 9_{30},\ 9_{33},\ 9_{34},\ 9_{37},\ 9_{38},\ 9_{49}. \end{gathered} \tag{3-4}$$

The Gilmer–Livingston relation (1-2) implies that $\gamma_4(K) \geq 2$ for any such knot. We verify that $\gamma_4(K) = 2$ for all knots K in (3-4) with the exception of $K = 8_{18}$ by constructing an explicit nonorientable smooth and properly embedded surface Σ in D^4 with $b_1(\Sigma) = 2$ and $\partial\Sigma = K$. The existence of such a surface follows from Proposition 2.4 by finding a nonorientable band move from K to a knot K' with $\gamma_4(K') = 1$. Such band moves are described in Figures 14–15. We address the exceptional knot 8_{18} next:

Proposition 3.1 $\gamma_4(8_{18}) = 3$.

Proof Theorem 2.11 implies that $\gamma_4(8_{18}) \geq 3$ provided we can prove that neither of the two Goeritz matrices $G_\pm$ of 8_{18} embed into a diagonal form of equal definiteness (positive or negative) and of rank two larger.

We start by considering G_-, the negative definite Goeritz matrix associated to the checkerboard coloring of 8_{18} as given in Figure 4. In that figure, G_- is the incidence matrix of the given graph, whereby each of the vertices $e_1,\dots,e_4$ (corresponding to generators of $\mathbb{Z}^4$) has square -3 (that is, $G_-(e_i,e_i) = -3$), and any pair of vertices sharing an edge pairs to 1, while vertices not sharing an edge pair to 0.

An embedding

$$\varphi\colon (\mathbb{Z}^4, G_-) \hookrightarrow (\mathbb{Z}^6, -\mathrm{Id})$$

is a monomorphism $\varphi\colon \mathbb{Z}^4 \to \mathbb{Z}^6$ such that $\varphi(a)\cdot\varphi(b) = G_-(a,b)$ for any pair $a, b \in \mathbb{Z}^4$, where the dot product refers to the product $-\mathrm{Id}$ on $\mathbb{Z}^6$. Let $\{e_i\}_{i=1}^4$ be the basis for $(\mathbb{Z}^4, G_-)$ as described by Figure 4 (right), and let $\{f_i\}_{i=1}^6$ be the standard basis for $(\mathbb{Z}^6, -\mathrm{Id})$, that is, the basis with $f_i \cdot f_j = -\delta_{ij}$. In a further simplification of notation we shall also write $e_i \cdot e_j$ to mean $G_-(e_i, e_j)$; the nature of the vectors engaging in the dot product determines the particular dot product being used.

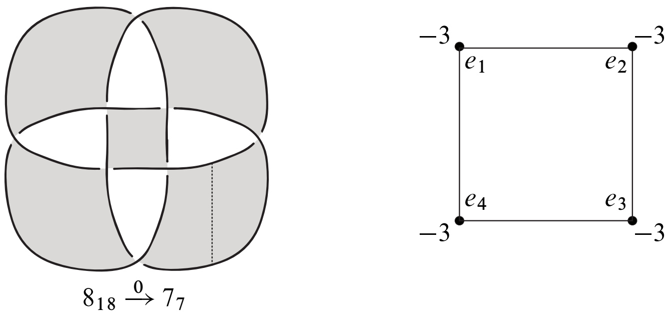


Figure 4: $K = 8_{18}$ (left) and its negative-definite Goeritz form G_- (right)

If an embedding φ existed, it would have to send e_1 (up to a change of basis of $(\mathbb{Z}^6, -\mathrm{Id})$) to

$$\varphi(e_1) = f_1 + f_2 + f_3.$$

Since $e_1 \cdot e_3 = 0$, we have that $\varphi(e_3)$ must share an even number of basis elements $\{f_i\}_{i=1}^6$ with the basis elements f_1, f_2, f_3 occurring in the formula for $\varphi(e_1)$. Thus that shared number is either 0 or 2. If it is 0, then $\varphi(e_3) = f_4 + f_5 + f_6$. However, since $e_1 \cdot e_2 = e_3 \cdot e_2 = 1$, the expression $\varphi(e_2)$ must share an odd number of basis elements $\{f_i\}_{i=1}^6$ with those occurring in each of the formulas for $\varphi(e_1)$ and $\varphi(e_3)$. That odd number cannot be 3 and thus must be 1, which is impossible as all six basis elements $\{f_i\}_{i=1}^6$ occur in $\varphi(e_1)$ and $\varphi(e_3)$.

It follows that $\varphi(e_3)$ must have two basis elements in common with $\varphi(e_1)$, and so, again up to a change of basis of $(\mathbb{Z}^6, -\mathrm{Id})$, it must be that

$$\varphi(e_3) = f_1 - f_2 + f_4.$$

Suppose that $\varphi(e_2) = \sum_{i=1}^6 \lambda_i f_i$. Then the values of $e_2 \cdot e_i$ for $i = 1, 2, 3$ lead to these equations in the integer coefficients $\lambda_1, \dots, \lambda_6$:

$$\begin{aligned} -\lambda_1 - \lambda_2 - \lambda_3 &= 1, \\ -\lambda_1 + \lambda_2 - \lambda_4 &= 1, \\ \lambda_1^2 + \cdots + \lambda_6^2 &= 3. \end{aligned}$$

Writing $\lambda_3 = -\lambda_1 - \lambda_2 - 1$ and $\lambda_4 = -\lambda_1 + \lambda_2 - 1$ by using the first two equations and plugging these into the third equation yields

$$\lambda_1^2 + 3\lambda_2^2 + 2(\lambda_1 + 1)^2 + \lambda_5^2 + \lambda_6^2 = 3.$$

It follows that $\lambda_2 = 0$ and that either $\lambda_1 = 0$ or $\lambda_1 = -1$, leading to two possibilities for $\varphi(e_2)$:

$$\varphi(e_2) = -f_3 - f_4 + f_5 \quad \text{or} \quad \varphi(e_2) = -f_1 + f_5 + f_6.$$

We suppose first that $\varphi(e_2) = -f_3 - f_4 + f_5$ and write $\varphi(e_4) = \sum_{i=1}^{6} \mu_i f_i$ for some integers coefficients $\mu_1, \dots, \mu_6$ subject to the equations

$$\begin{aligned} -\mu_1 - \mu_2 - \mu_3 &= 1, \\ \mu_3 + \mu_4 - \mu_5 &= 0, \\ -\mu_1 + \mu_2 - \mu_4 &= 1, \\ \mu_1^2 + \cdots + \mu_6^2 &= 3. \end{aligned}$$

The first three of these equations lead to $\mu_3 = -\mu_1 - \mu_2 - 1$, $\mu_4 = -\mu_1 + \mu_2 - 1$ and $\mu_5 = -2(\mu_1 + 1)$, which when plugged into the fourth equation yield

$$\mu_1^2 + 3\mu_2^2 + 6(\mu_1 + 1)^2 + \mu_6^2 = 3.$$

It follows immediately that $\mu_1 = -1$, $\mu_2 = 0$ and $\mu_6^2 = 2$, and the latter equation of course has no integral solution μ_6. Therefore the choice of $\varphi(e_2) = -f_3 - f_4 + f_5$ does not lead to an embedding φ.

Secondly, suppose that $\varphi(e_2) = -f_1 + f_5 + f_6$, the only remaining possibility for $\varphi(e_2)$, and write again $\varphi(e_4) = \sum_{i=1}^{6} \eta_i f_i$ for integers $\eta_1, \dots, \eta_6$, this time subject to

$$\begin{aligned} -\eta_1 - \eta_2 - \eta_3 &= 1, \\ \eta_1 - \eta_5 - \eta_6 &= 0, \\ -\eta_1 + \eta_2 - \eta_4 &= 1, \\ \eta_1^2 + \cdots + \eta_6^2 &= 3. \end{aligned}$$

The first three of these equations imply $\eta_3 = -\eta_1 - \eta_2 - 1$, $\eta_4 = -\eta_1 + \eta_2 - 1$ and $\eta_6 = \eta_1 - \eta_5$, which when inserted into the fourth equation lead to

$$\eta_1^2 + 3\eta_2^2 + 2(\eta_1 + 1)^2 + \eta_5^2 + (\eta_1 - \eta_5)^2 = 3.$$

We are forced to conclude that $\eta_2 = 0$ and that either $\eta_1 = 0$ or $\eta_1 = -1$. The case of $\eta_1 = 0$ forces the equation $2\eta_5^2 = 1$, while the case of $\eta_1 = -1$ leads to $\eta_5 + (1 + \eta_5)^2 = 2$, neither of which has integral solutions.

We find that both possibilities for $\varphi(e_2)$ lead to equations for the coefficients of $\varphi(e_4)$ that have no integral solutions, and thus the embedding $\varphi\colon (\mathbb{Z}^4, G_-) \hookrightarrow (\mathbb{Z}^6, -\mathrm{Id})$ cannot exist.

It is easy to see that $G_+ = -G_-$, and so an embedding $(\mathbb{Z}^4, G_+) \hookrightarrow (\mathbb{Z}^6, \mathrm{Id})$ would lead to an embedding $(\mathbb{Z}^4, G_-) \hookrightarrow (\mathbb{Z}^6, -\mathrm{Id})$ which was already shown not to exist. Therefore the conditions of Theorem 2.11 are met and it follows that $\gamma_4(8_{18}) \geq 3$. The equality $\gamma_4(8_{18}) = 3$ follows from the nonoriented band move indicated in Figure 4 (left) which transforms 8_{18} into the knot 7_7, and the fact that $\gamma_4(7_7) = 2$; see [2]. $\square$

3.3 Knots with $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 2 \pmod 8$

In this section we consider the 8– and 9–crossing knots K that satisfy the congruence relation $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv 2 \pmod 8$. These are precisely the 34 knots

$$\begin{gathered} -\underline{8_4},\ -\underline{8_6},\ \underline{8_7},\ -\underline{8_{10}},\ \underline{8_{11}},\ -\underline{8_{14}},\ -\underline{8_{16}},\ -\underline{8_{19}},\ -\underline{8_{21}}, \\ -9_2,\ -\underline{9_3},\ -\underline{9_5},\ -\underline{9_6},\ -\underline{9_8},\ \underline{9_9},\ 9_{12},\ \underline{9_{15}},\ 9_{16},\ -\underline{9_{17}},\ -\underline{9_{21}},\ \underline{9_{22}},\ -\underline{9_{25}}, \\ \underline{9_{26}},\ \underline{9_{28}},\ -\underline{9_{29}},\ -\underline{9_{31}},\ -\underline{9_{32}},\ \underline{9_{35}},\ 9_{39},\ 9_{40},\ \underline{9_{42}},\ \underline{9_{45}},\ \underline{9_{47}},\ -\underline{9_{48}}. \end{gathered} \tag{3-5}$$

The 8– and 9–crossing knots K that satisfy $\sigma(K) + 4\cdot\mathrm{Arf}(K) \equiv -2 \pmod 8$, the opposite congruence relation, are the mirror knots of those appearing in (3-5). The 29 underlined knots (or their mirror knots) in this list have been shown in Section 3.1 to have γ_4 equal to 1, leaving us only to deal with the remaining five knots.

We will show that each nonunderlined knot K from (3-5) meets the assumptions of Theorem 2.10, thereby proving that $\gamma_4(K) \geq 2$. We will then show that $\gamma_4(K) = 2$ by finding a nonoriented band move from K to a knot K' with $\gamma_4(K') = 1$.

Notational convention We will represent the Goeritz forms $(\mathbb{Z}^n, G)$ of the various nonunderlined knots from (3-5) as incidence matrices of weighted graphs. Recall that in such a presentation the generators $e_1, \dots, e_n$ of $\mathbb{Z}^n$ correspond to the n vertices of the weighted graph, which in turn correspond to the white regions in the checkerboard coloring of the diagram of K. Moreover, $G(e_i, e_i)$ is given by the weight of the vertex e_i for each $i = 1, \dots, n$, and if $n_{i,j}$ is the number of edges between the vertices e_i and e_j, then $G(e_i, e_j) = n_{ij}$. For simplicity of notation we shall write $e_i \cdot e_j$ to mean $G(e_i, e_j)$. This weighted graph approach to describing $(\mathbb{Z}^n, G)$ is merely a graphical tool that encodes the form (2-2).

An embedding

$$\varphi\colon (\mathbb{Z}^{n+1}, G \oplus [-d]) \hookrightarrow (\mathbb{Z}^{n+1}, -\mathrm{Id}) \tag{3-6}$$

is a monomorphism $\varphi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ with the property that $-\mathrm{Id}(\varphi(e_i), \varphi(e_j)) = e_i \cdot e_j$ for $i, j = 1, \dots, n+1$. Here e_{n+1} is the basis element of $\mathbb{Z}^n \oplus \mathbb{Z}$ corresponding to the last summand, and it has the properties

$$e_{n+1} \cdot e_{n+1} = -d \qquad \text{and} \qquad e_{n+1} \cdot e_i = 0 \quad \text{for } i = 1, \dots, n.$$

Recall that $d \in \mathbb{N}$ is a divisor of $\det K$ with $\det K/d$ a square; see Corollary 2.8. With the exception of $K = 9_{40}$, all nonunderlined knots K from (3-5) have square-free determinant, forcing $d = \det K$.

Let $\{f_i\}_{i=1}^{n+1}$ be the standard basis for $(\mathbb{Z}^{n+1}, -\mathrm{Id})$, that is, $-\mathrm{Id}(f_i, f_j) = -\delta_{ij}$ for all $1 \le i, j \le n+1$, and also write $f_i \cdot f_j$ to mean $-\mathrm{Id}(f_i, f_j)$. While the dot notation is used for both G and $-\mathrm{Id}$, the nature of the vectors involved in the dot product makes clear which form is meant.

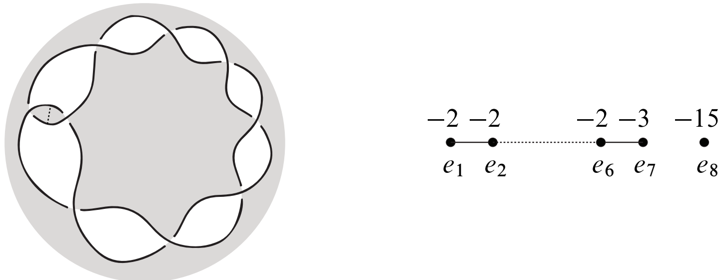
Our approach to showing that the embedding φ does not exist is to take advantage of the "rigidities" presented by vertices e_i with square -2 or -3. Any such vertex e_i under φ maps to either $f_1 - f_2$ or $f_1 + f_2 + f_3$, up to a change of basis of $(\mathbb{Z}^{n+1}, -\mathrm{Id})$. Moreover since each basis element f_i has square -1, then a pair of vertices e_i and e_j with $e_i \cdot e_j = 1$ must have the property $\varphi(e_i)$ and $\varphi(e_j)$ share an odd number of basis elements $\{f_i\}_{i=1}^{n+1}$, and similarly if $e_i \cdot e_j = 0$ then $\varphi(e_i)$ and $\varphi(e_j)$ must share an even number of basis elements $\{f_i\}_{i=1}^{n+1}$. These requirements are restrictive enough to show that φ cannot exist for the Goeritz forms of the nonunderlined knots in (3-5).

The nonexistence of the embedding (3-6) shows that $\gamma_4(K) \ge 2$ for the corresponding knot K. The equality $\gamma_4(K) = 2$ is derived by finding a nonoriented band move from K to a knot K' with $\gamma_4(K') = 1$.

Remark 3.1 We would like to emphasize that in each of the following arguments, our computations are valid up to a change of basis of $(\mathbb{Z}^{n+1}, -\mathrm{Id})$ and we will usually take this fact for granted to simplify the exposition.

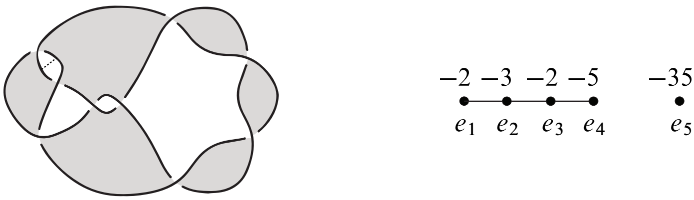
$\boldsymbol{K = -9_2}$ The negative definite Goeritz matrix G associated to the checkerboard coloring of the knot $K = -9_2$ from Figure 5 (left) is given as the incidence matrix of the weighted graph in Figure 5 (right), where all the missing vertices, indicated by the dotted line, have weight -2. Since $\det 9_2 = 15$ is square-free, we are seeking to obstruct the existence of an embedding

$$\varphi\colon (\mathbb{Z}^8, G \oplus [-15]) \hookrightarrow (\mathbb{Z}^8, -\mathrm{Id}).$$



$-9_2 \xrightarrow{0} 7_1$

Figure 5: $K = -9_2$ (left) and its Goeritz form (right)



$9_{12} \xrightarrow{0} 7_3$

Figure 6: $K = 9_{12}$ (left) and its Goeritz form (right)

If φ existed, we would have to have

$$\varphi(e_i) = f_i - f_{i+1} \quad \text{for } i = 1, \dots, 6.$$

Let $\varphi(e_7) = \sum_{i=1}^{8} \mu_i f_i$. Then since $e_7 \cdot e_6 = 1$ and $e_7 \cdot e_j = 0$ for $j = 1, \dots, 5$, it follows that

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 \quad \text{and} \quad -\mu_6 + \mu_7 = 1.$$

Since at most three of the coefficients μ_j are nonzero, we conclude that $\mu_j = 0$ for $j = 1, \dots, 6$ and $\mu_7 = 1$. It follows that $\varphi(e_7) = f_7 + \mu_8 f_8$, forcing the relation $1 + \mu_8^2 = 3$, which has no integral solution. Thus φ cannot exist, leading to $\gamma_4(-9_2) \geq 2$. The equality $\gamma_4(-9_2) = 2$ follows from the nonorientable band move in Figure 5 (left) which transforms -9_2 to 7_1 and the fact that $\gamma_4(7_1) = 1$.

$K = 9_{12}$ The negative definite Goeritz matrix G associated to the checkerboard coloring of the knot $K = 9_{12}$ from Figure 6 (left) is given by the incidence matrix in Figure 6 (right). Since $\det 9_{12} = 35$ is square-free, we seek to obstruct an embedding

$$\varphi\colon (\mathbb{Z}^5, G \oplus [-35]) \hookrightarrow (\mathbb{Z}^5, -\mathrm{Id}).$$

If φ existed, we would have

$$\varphi(e_4) = f_1 + f_2 + f_3 + f_4 + f_5 \quad \text{or} \quad \varphi(e_4) = f_1 + 2f_2.$$

Case 1 $(\varphi(e_4) = f_1 + f_2 + f_3 + f_4 + f_5)$ Write $\varphi(e_5) = \sum_{i=1}^5 \lambda_i f_i$ for integers $\lambda_1, \dots, \lambda_5$ to be determined. Since $e_5 \cdot e_5 = -35$ it follows that $35 = \sum_{i=1}^5 \lambda_i^2$, and $e_4 \cdot e_5 = 0$ implies that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$. These two relations are in contradiction with one another because

$$35 = \lambda_1^2 + \cdots + \lambda_5^2 \equiv (\lambda_1 + \cdots + \lambda_5)^2 \pmod 2 \equiv 0 \pmod 2,$$

showing that the choice of $\varphi(e_4) = f_1 + f_2 + f_3 + f_4 + f_5$ does not extend to an embedding φ.

Case 2 $(\varphi(e_4) = f_1 + 2f_2)$ Since $e_3 \cdot e_4 = 1$ and $\varphi(e_3)$ is a sum of only two basis elements $\{f_i\}_{i=1}^5$, it must be that $\varphi(e_3) = -f_1 + f_3$ or $\varphi(e_3) = f_1 - f_2$.

We first pursue the case of $\varphi(e_3) = -f_1 + f_3$. Since $e_2 \cdot e_3 = 1$, $\varphi(e_2)$ must share exactly one basis element with $\varphi(e_3)$. This shared element cannot be f_1 since this would force $\varphi(e_2) \cdot \varphi(e_4) \neq 0$, showing that $\varphi(e_3)$ and $\varphi(e_2)$ must share f_3. Note that $\varphi(e_2)$ cannot contain f_2 either since this would lead yet again to $\varphi(e_2) \cdot \varphi(e_4) \neq 0$. We are thus forced to conclude that $\varphi(e_2) = -f_3 + f_4 + f_5$. Write $\varphi(e_1) = \sum_{i=1}^5 \mu_i f_i$. Then $\mu_1 + 2\mu_2 = 0$, $\mu_1 = \mu_3$, $1 = \mu_3 - \mu_4 - \mu_5$ and $\sum_{i=1}^5 \mu_i^2 = 2$. The three linear equations lead to $\mu_1 = \mu_3 = -2\mu_2$ and $\mu_5 = -2\mu_2 - \mu_4 - 1$, which when plugged into the quadratic equation give

$$9\mu_2^2 + \mu_4^2 + (2\mu_2 + \mu_4 + 1)^2 = 2.$$

This forces $\mu_2 = 0$ and $\mu_4^2 + (\mu_4 + 1)^2 = 2$, the latter of which has no solution $\mu_4 \in \mathbb{Z}$.

Next we turn to the only remaining possibility, namely $\varphi(e_3) = f_1 - f_2$. Since $e_2 \cdot e_3 = 1$, $\varphi(e_3)$ and $\varphi(e_2)$ must share exactly one basis element f_i, $i = 1, \dots, 5$. Accordingly, we must have $\varphi(e_2) = \pm f_i \pm f_j$ with $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$. However each of these cases leads to $\varphi(e_2) \cdot \varphi(e_4) \neq 0$, contradicting $e_2 \cdot e_4 = 0$.

It follows that the embedding φ cannot exist, implying that $\gamma_4(9_{12}) \geq 2$. The nonoriented band move from 9_{12} to 7_3 in Figure 6 (left) shows that $\gamma_4(9_{12}) = 2$ seeing as $\gamma_4(7_3) = 1$.

$\boldsymbol{K = 9_{16}}$ The negative definite Goeritz matrix G associated to the checkerboard coloring of the knot $K = 9_{16}$ from Figure 7 (left) is given in Figure 7 (right). Since

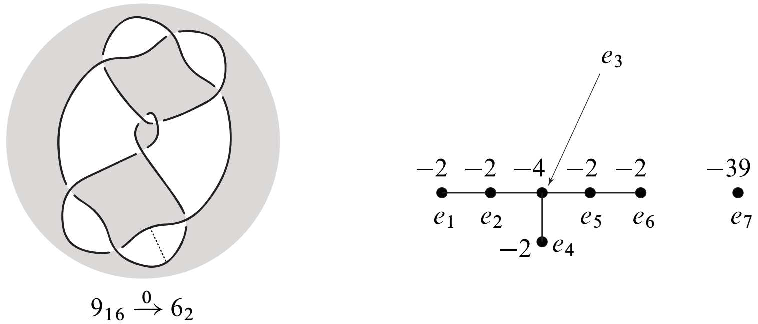


Figure 7: $K = 9_{16}$ (left) and its Goeritz form (right)

$\det 9_{16} = 39$ is square-free, we wish to obstruct the existence of an embedding

$$\varphi\colon (\mathbb{Z}^7, G \oplus [-39]) \hookrightarrow (\mathbb{Z}^7, -\mathrm{Id}).$$

Any such φ would have

$$\varphi(e_3) = f_1 + f_2 + f_3 + f_4.$$

Since $e_i \cdot e_4 = 1$ and $e_i \cdot e_i = -2$ for $i = 2, 4, 5$, each $\varphi(e_i)$ has exactly one basis element in common with $\varphi(e_3)$, and that common basis element is different for each $i = 2, 4, 5$. Indeed if we had for instance $-f_1$ common to $\varphi(e_2)$ and $\varphi(e_4)$, we would be forced to have $\varphi(e_2) = -f_1 + f_5$ and $\varphi(e_4) = -f_1 - f_5$, which would make it impossible to satisfy the relations $\varphi(e_1) \cdot \varphi(e_2) = 1$ and $\varphi(e_1) \cdot \varphi(e_4) = 0$ simultaneously. A similar argument shows that neither of the other two pairs $\{\varphi(e_2), \varphi(e_5)\}$ and $\{\varphi(e_4), \varphi(e_5)\}$ can share the same basis element with $\varphi(e_3)$. Thus we conclude that

$$\varphi(e_2) = -f_1 + f_5, \quad \varphi(e_4) = -f_2 + f_6, \quad \varphi(e_5) = -f_3 + f_7.$$

Since $e_1 \cdot e_2 = 1$, $\varphi(e_1)$ shares exactly one basis element with $\varphi(e_2)$. This shared element cannot be f_5 since the other basis element for $\varphi(e_1)$ would have to come from $\{f_2, f_3, f_4, f_6, f_7\}$, each choice of which would lead to $\varphi(e_1) \cdot \varphi(e_i) \neq 0$ for some $i \neq 1, 2$. This leaves $\varphi(e_1) = f_1 - f_4$ as the only possibility. Lastly, $e_6 \cdot e_5 = 1$ says that $\varphi(e_6)$ must contain one and only one of f_3 or f_7. However either choice for the other basis element in $\varphi(e_6)$ leads to one of $\varphi(e_6) \cdot \varphi(e_i)$, $i = 1, 2, 3, 4$ being nonzero, a contradiction. Thus φ cannot exist, and so $\gamma_4(9_{16}) \geq 2$, showing that $\gamma_4(9_{16}) = 2$ given the nonoriented band move from 9_{16} to 6_2 in Figure 7 (left) and seeing as $\gamma_4(6_2) = 1$.

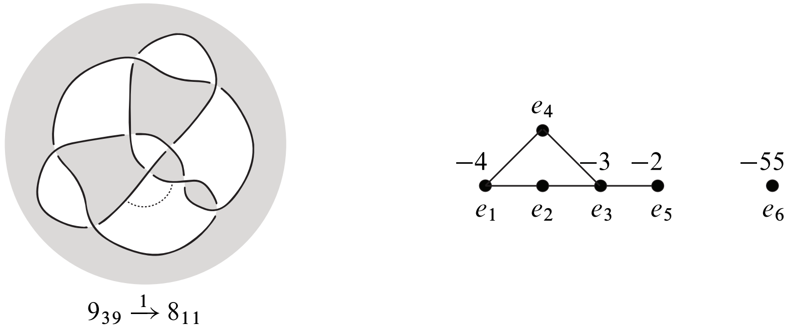


Figure 8: $K = 9_{39}$ (left) and its Goeritz form (right). Here e_2 has square -2 and e_4 has square -3.

$K = 9_{39}$ The negative definite Goeritz matrix G associated to the checkerboard coloring of the knot $K = 9_{39}$ in Figure 8 (left) is the incidence matrix of the weighted graph in Figure 8 (right). Since $\det 9_{39} = 55$ is square-free, we aim to show that no embedding

$$\varphi\colon (\mathbb{Z}^6, G \oplus [-55]) \hookrightarrow (\mathbb{Z}^6, -\mathrm{Id})$$

exists. Any such φ would have

$$\varphi(e_1) = f_1 + f_2 + f_3 + f_4 \quad \text{and} \quad \varphi(e_2) = -f_4 + f_5.$$

Note that $\varphi(e_1) = 2f_1$ is not possible because $e_1 \cdot e_2 = 1$. Since $e_2 \cdot e_3 = 1$ and $e_1 \cdot e_3 = 0$, we have that $\varphi(e_3)$ shares exactly one basis element with $\varphi(e_2)$ and an even number of basis elements with $\varphi(e_1)$. The shared element among $\varphi(e_3)$ and $\varphi(e_2)$ may be either f_4 or f_5, leading to the possibilities $\varphi(e_3) = f_4 - f_1 - f_6$ or $\varphi(e_3) = -f_5 - f_1 + f_2$.

Case 1 $(\varphi(e_3) = f_4 - f_1 - f_6)$ Since $e_3 \cdot e_4 = 1 = e_1 \cdot e_4$ then $\varphi(e_4)$ shares an odd number of basis elements with each of $\varphi(e_1)$ and $\varphi(e_3)$. This shared number of basis elements between $\varphi(e_4)$ and $\varphi(e_3)$ cannot be three since if it were then we would obtain $\varphi(e_5) \cdot \varphi(e_4) \equiv \varphi(e_5) \cdot \varphi(e_3) \pmod 2$ which is not a valid congruence. Thus $\varphi(e_4)$ shares one basis element with $\varphi(e_3)$. Note also that $\varphi(e_4)$ shares an even number of basis elements with $\varphi(e_2)$.

Case 1a ($\varphi(e_3)$ and $\varphi(e_4)$ share only f_4) In this case f_5 also appears as a summand in $\varphi(e_4)$ and we are led to $\varphi(e_4) = -f_4 - f_5 \pm f_i$ for some $i \in \{2, 3\}$. No matter which $i \in \{2, 3\}$ we pick, we arrive at an even number of shared basis elements between $\varphi(e_4)$ and $\varphi(e_1)$, a contradiction.

Case 1b ($\varphi(e_3)$ and $\varphi(e_4)$ share only f_1) In this case $\varphi(e_4)$ cannot contain f_4 or f_6, and therefore cannot contain f_5 either since $e_2 \cdot e_4 = 0$. Thus we are forced to conclude that in this case $\varphi(e_4) = f_1 - f_2 - f_3$. Moving on to $\varphi(e_5)$, the relation $e_5 \cdot e_3 = 1$ shows that $\varphi(e_5)$ shares with $\varphi(e_3)$ exactly one of f_1, f_4 or f_6.

(α) Suppose $\varphi(e_5)$ and $\varphi(e_3)$ share f_1. In this case we find that $\varphi(e_5) = f_1 \pm f_i$ for some $i \in \{2, 3, 5\}$. The relation $e_5 \cdot e_2 = 0$ shows that $i \neq 5$. Each of the possibilities $\varphi(e_5) = f_1 \pm f_2$ or $\varphi(e_5) = f_1 \pm f_3$ leads to one of $\varphi(e_5) \cdot \varphi(e_1)$ or $\varphi(e_5) \cdot \varphi(e_4)$ having the wrong value, a contradiction.

(β) Suppose $\varphi(e_5)$ and $\varphi(e_3)$ share f_4. Here $\varphi(e_5) = -f_4 \pm f_i$ for some $i \in \{2, 3, 5\}$. The relation $e_5 \cdot e_2 = 0$ forces $i = 5$ and $\varphi(e_5) = -f_4 - f_5$. However this leads to the incorrect value of 1 for $\varphi(e_5) \cdot \varphi(e_1)$, a contradiction.

(γ) Suppose $\varphi(e_5)$ and $\varphi(e_3)$ share f_6. Here $\varphi(e_5) = f_6 \pm f_i$ for some $i \in \{2, 3, 5\}$. The relation $e_5 \cdot e_2 = 0$ forces $i \neq 5$, and each of the remaining possibilities $\varphi(e_5) = f_6 \pm f_2$ and $\varphi(e_5) = f_6 \pm f_3$ leads to the incorrect value of ± 1 for $\varphi(e_5) \cdot \varphi(e_1)$, another contradiction.

Case 1c ($\varphi(e_3)$ and $\varphi(e_4)$ share only f_6) Since $e_4 \cdot e_2 = 0$, then $\varphi(e_4)$ cannot contain f_5 either, leaving us with the possibility of $\varphi(e_4) = f_6 \pm f_2 \pm f_3$. No matter the choice of signs, this leads to an even number of shared basis elements between $\varphi(e_4)$ and $\varphi(e_1)$, contradicting $e_4 \cdot e_1 = 1$.

We conclude that the case of $\varphi(e_3) = f_4 - f_1 - f_6$ does not lead to an embedding φ.

Case 2 ($\varphi(e_3) = -f_5 - f_1 + f_2$) As in the previous case, we find that $\varphi(e_4)$ shares one basis element with $\varphi(e_3)$, shares an even number of basis elements with $\varphi(e_2)$ and an odd number with $\varphi(e_1)$.

Case 2a ($\varphi(e_3)$ and $\varphi(e_4)$ only share f_1) In this case $\varphi(e_4)$ cannot contain f_2 or f_5 and thus also not f_4, since $\varphi(e_2) = -f_4 + f_5$. This leaves us with $\varphi(e_4) = f_1 \pm f_3 \pm f_6$, leading to an even number of shared basis elements between $\varphi(e_4)$ and $\varphi(e_1)$, a contradiction to the relation $e_4 \cdot e_1 = 1$.

Case 2b ($\varphi(e_3)$ and $\varphi(e_4)$ only share f_2) In this setup $\varphi(e_4)$ cannot contain f_1 or f_5 and thus also not f_4 (again since $e_2 \cdot e_4 = 0$). Similarly to the previous subcase we are left with $\varphi(e_4) = -f_2 \pm f_3 \pm f_6$ leading to the same contradiction as in the previous subcase.

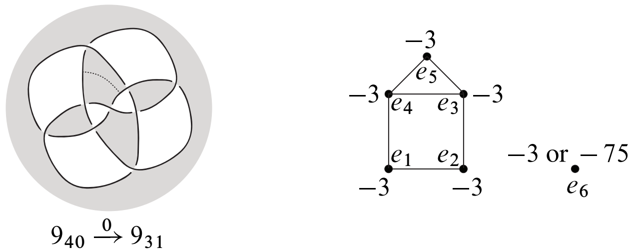


Figure 9: $K = 9_{40}$ (left) and its Goeritz form (right)

Case 2c ($\varphi(e_3)$ and $\varphi(e_4)$ only share f_5) Here $\varphi(e_4)$ cannot contain f_1 or f_2, while the relation $e_2 \cdot e_4 = 0$ implies that $\varphi(e_4)$ must contain f_4. This implies that $\varphi(e_4) = f_5 + f_4 \pm f_i$ for $i \in \{3, 6\}$. Since $\varphi(e_4)$ shares an odd number of elements with $\varphi(e_1)$ we conclude that $i = 6$ and that $\varphi(e_4) = f_5 + f_4 \pm f_6$. This, regardless of the sign choice, implies that $\varphi(e_4) \cdot \varphi(e_1) = -1$, a contradiction.

Having exhausted all possibilities and having been led to a contradiction in each, we conclude that the embedding φ cannot exist.

It follows that $\gamma_4(9_{39}) \geq 2$. Figure 8 (left) shows a band move from 9_{39} to 8_{11} and since $\gamma_4(8_{11}) = 1$, it follows that $\gamma_4(9_{39}) = 2$.

$\boldsymbol{K = 9_{40}}$ The negative definite Goeritz matrix G associated to the checkerboard coloring of the knot $K = 9_{40}$ in Figure 9 (left) is the incidence matrix of the weighted graph in Figure 9 (right). Since $\det 9_{40} = 75 = 3 \cdot 5^2$ we wish to obstruct the existence of an embedding

$$\varphi\colon (\mathbb{Z}^6, G \oplus [-d]) \hookrightarrow (\mathbb{Z}^6, -\mathrm{Id})$$

for $d = 3$ and for $d = 75$. If we assume that such a φ exists, then its restriction to $V := \mathrm{Span}(e_1, e_2, e_3, e_4)$ is an embedding of $(V, G|_{V\times V})$ into $(\mathbb{Z}^6, -\mathrm{Id})$. However the form $(V, G|_{V\times V})$ is isomorphic to the form considered in Proposition 3.1, where it was shown not to embed into $(\mathbb{Z}^6, -\mathrm{Id})$. It follows that $\gamma_4(9_{40}) \geq 2$, and since the nonoriented band move in Figure 9 (left) turns 9_{40} into 9_{31}, a knot with γ_4 equal to 1, we conclude that $\gamma_4(9_{40}) = 2$.

3.4 Knots with $\sigma(K) + 4 \cdot \mathrm{Arf}(K) \equiv 0 \pmod 8$

The 17 knots K with crossing number 8 or 9 that satisfy the congruence relation $\sigma(K) + 4 \cdot \mathrm{Arf}(K) \equiv 0 \pmod 8$ are, up to passing to mirrors, given by

$$\begin{gathered} \underline{8_3},\ \underline{8_5},\ \underline{8_8},\ \underline{8_9},\ \underline{8_{20}}, \\ \underline{9_1},\ \underline{9_4},\ \underline{9_7},\ \underline{9_{13}},\ \underline{9_{19}},\ \underline{9_{23}},\ \underline{9_{27}},\ \underline{9_{36}},\ \underline{9_{41}},\ \underline{9_{43}},\ \underline{9_{44}},\ \underline{9_{46}}. \end{gathered} \tag{3-7}$$

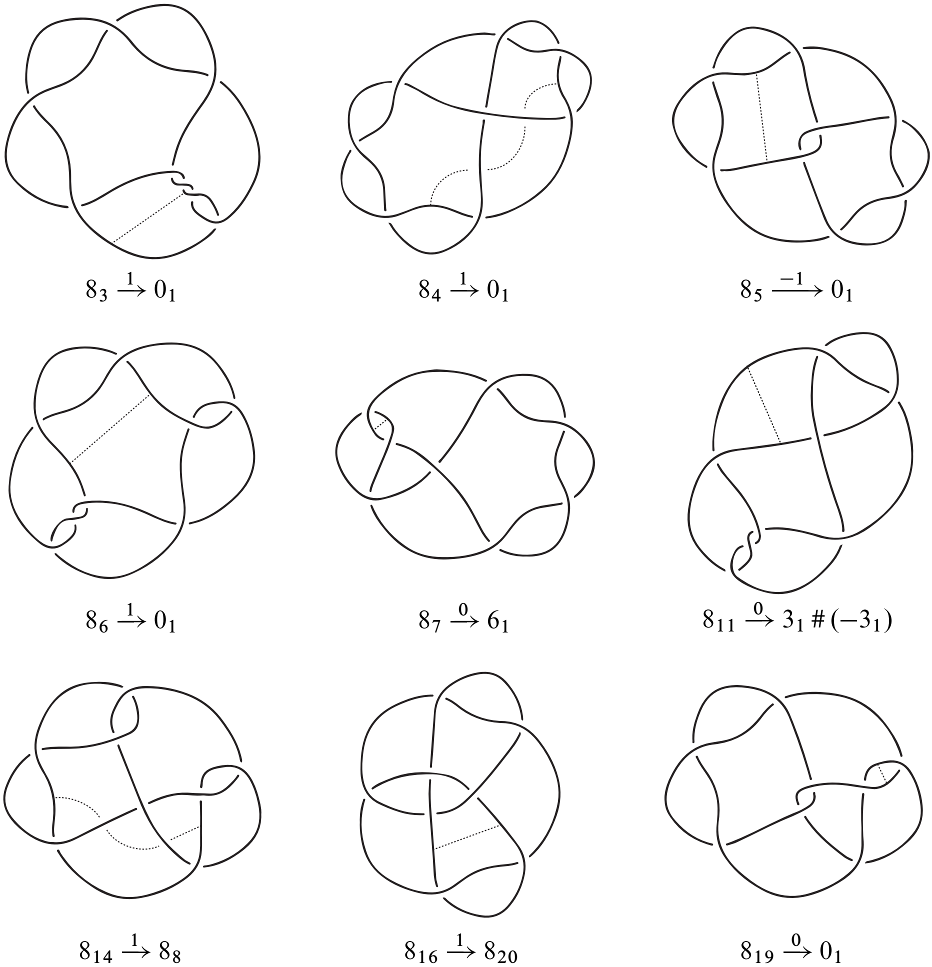


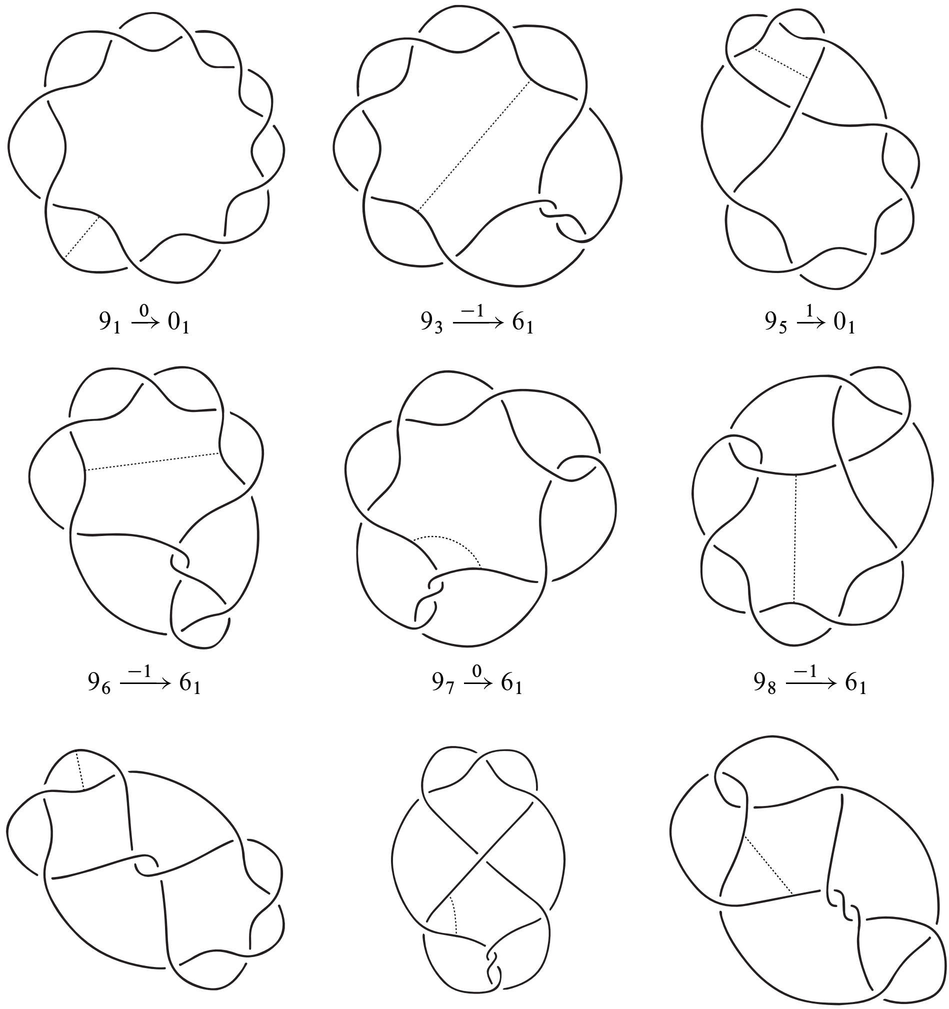
Figure 10: Nonoriented band moves from the knots 8_3, 8_4, 8_5, 8_6, 8_7, 8_{11}, 8_{14}, 8_{16}, 8_{19} to smoothly slice knots

All of these knots have already been considered in Section 3.1 with the exception of $K = 9_4$ for which Figure 2 shows a band move to the slice knot 10_3, demonstrating that $\gamma_4(9_4) = 1$.

4 Concluding remarks

Murakami and Yasuhara [20] proved that

$$\gamma_4(K) \leq \left\lfloor \tfrac{1}{2} c(K) \right\rfloor,$$



$9_1 \xrightarrow{0} 0_1$ $9_3 \xrightarrow{-1} 6_1$ $9_5 \xrightarrow{1} 0_1$

$9_6 \xrightarrow{-1} 6_1$ $9_7 \xrightarrow{0} 6_1$ $9_8 \xrightarrow{-1} 6_1$

$9_9 \xrightarrow{0} 6_1$ $9_{13} \xrightarrow{0} 8_9$ $9_{15} \xrightarrow{0} 8_8$

Figure 11: Nonoriented band moves from the knots 9_1, 9_3, 9_5, 9_6, 9_7, 9_8, 9_9, 9_{13}, 9_{15} to smoothly slice knots

where $c(K)$ is the crossing number of the knot K, and where $x \mapsto \lfloor x \rfloor$ is the floor function, giving the largest integer n less than or equal to the real number x. For the case of a knot K with $c(K) = 8$ or $c(K) = 9$ this inequality becomes $\gamma_4(K) \le 4$, an inequality which is strict for all such knots as demonstrated by Theorems 1.1 and 1.2. The known values of γ_4 from [2] show that this inequality remains strict for all knots K

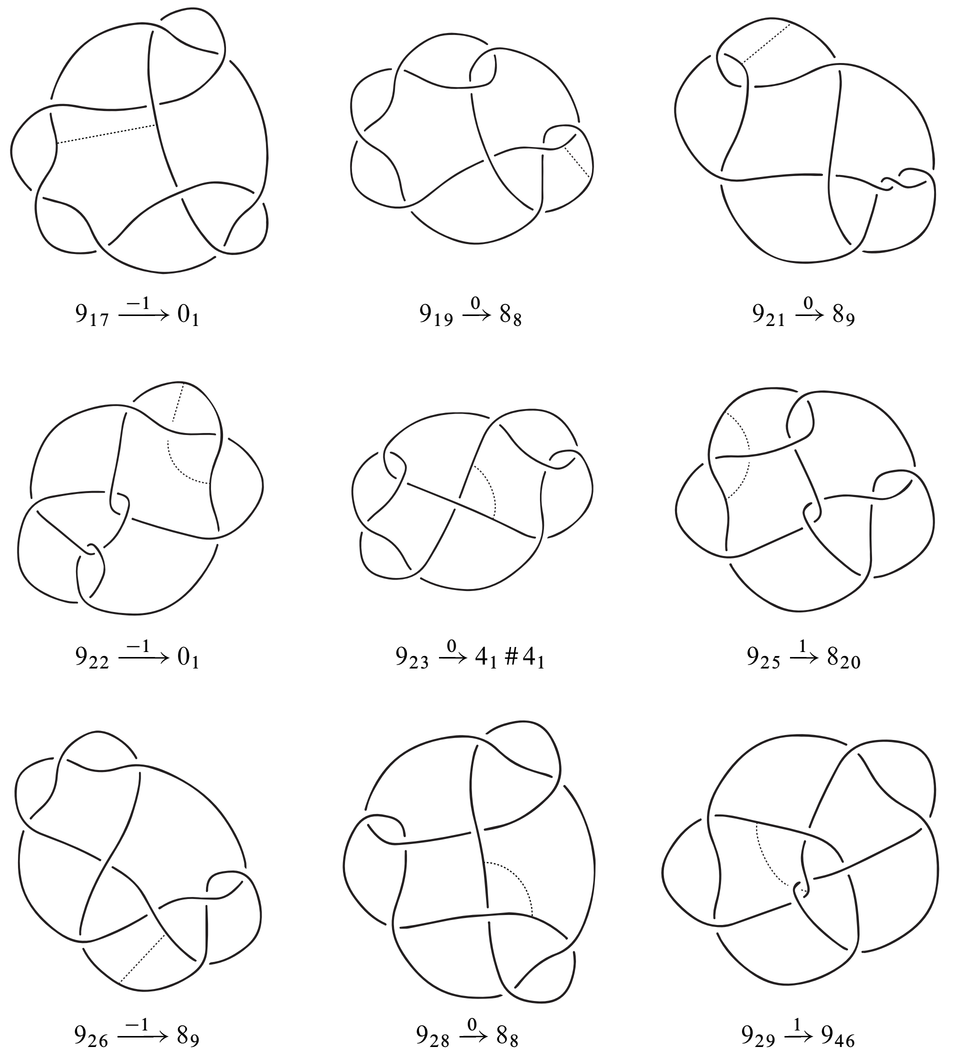


Figure 12: Nonoriented band moves from the knots 9_{17}, 9_{19}, 9_{21}, 9_{22}, 9_{23}, 9_{25}, 9_{26}, 9_{28}, 9_{29} to smoothly slice knots

with $c(K)$ equal to either 3, 5, 6 or 7. However the above inequality does become an equality for $K = 4_1$. These observations prompt the following question.

Question 4.1 *Does there exist a knot K with $c(K) > 4$ and with $\gamma_4(K) = \lfloor \frac{1}{2} c(K) \rfloor$?*

The knot $K = 8_{18}$ is special among 8– and 9–crossing knots, being the only knot that maximizes γ_4, with a maximal value of 3. We note that 8_{18} is also special among this

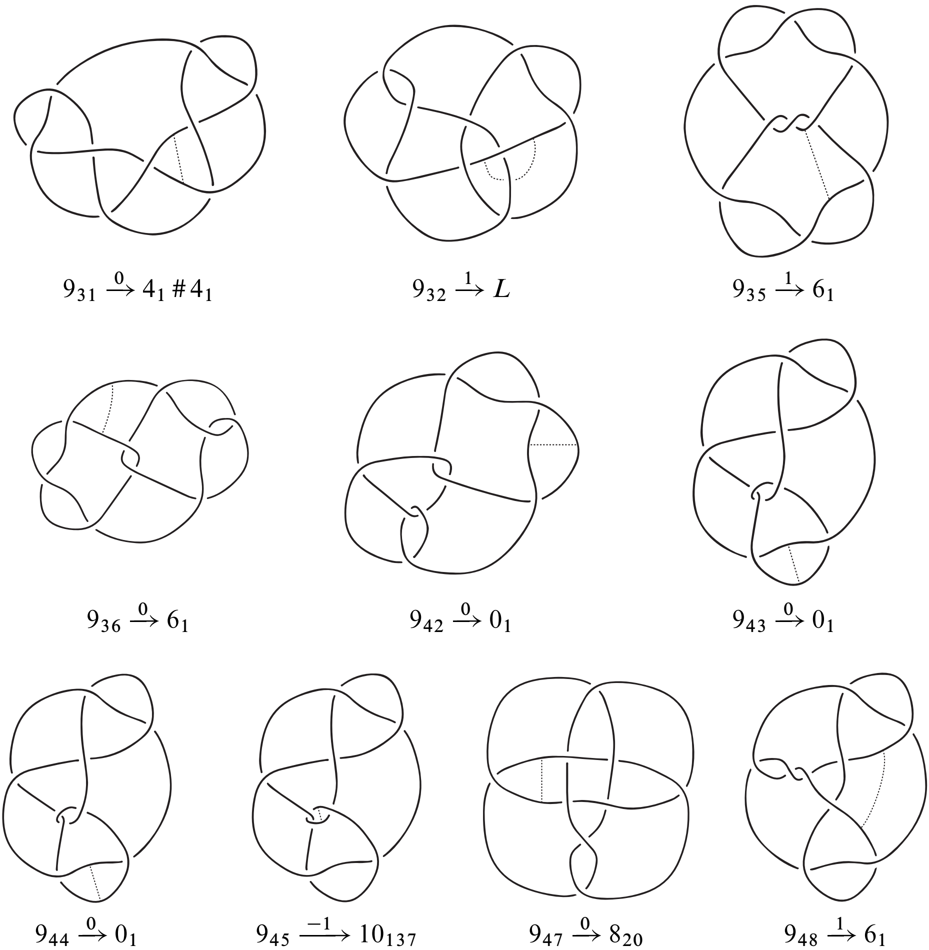


Figure 13: Nonoriented band moves from the knots 9_{31}, 9_{32}, 9_{35}, 9_{36}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{47}, 9_{48} to smoothly slice knots. In the top center, the knot L obtained by the indicated band move equals either the knot $11n_4$ or $11n_{21}$. We couldn't fully identify this knot, but both $11n_4$ or $11n_{21}$ are slice knots, which is sufficient for our purpose.

set of knots as it has the largest full symmetry group, namely the dihedral group D_8; see [2]. Other knots with 8 or 9 crossings have smaller full symmetry groups, given by $\mathbb{Z}_i$ and D_j with $i = 1, 2$ and $j = 1, 2, 3, 4, 6$. The group D_8 does not appear again as the full symmetry group for any knot K with $c(K) \leq 11$, and only the knot 10_{123} has larger full symmetry group, namely D_{10}. However 10_{123} is slice, and so $\gamma_4(10_{123}) = 1$.

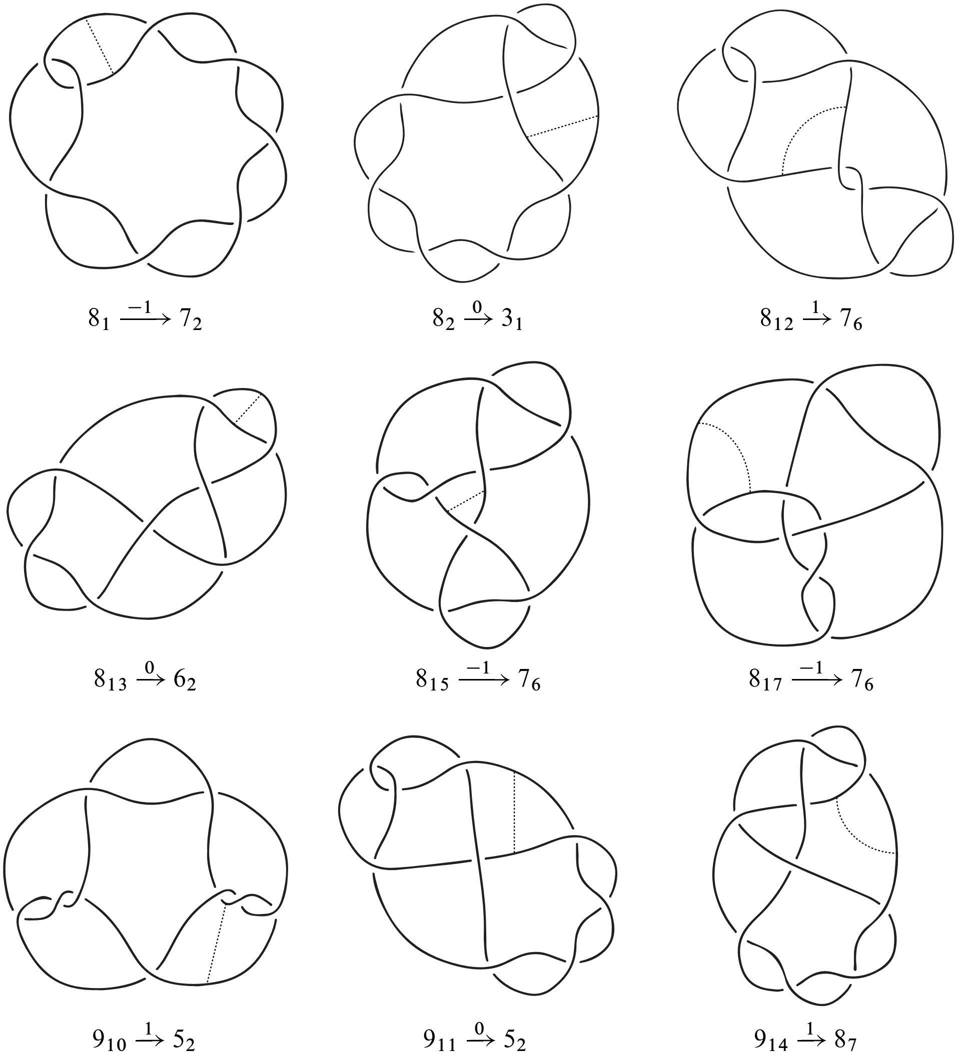
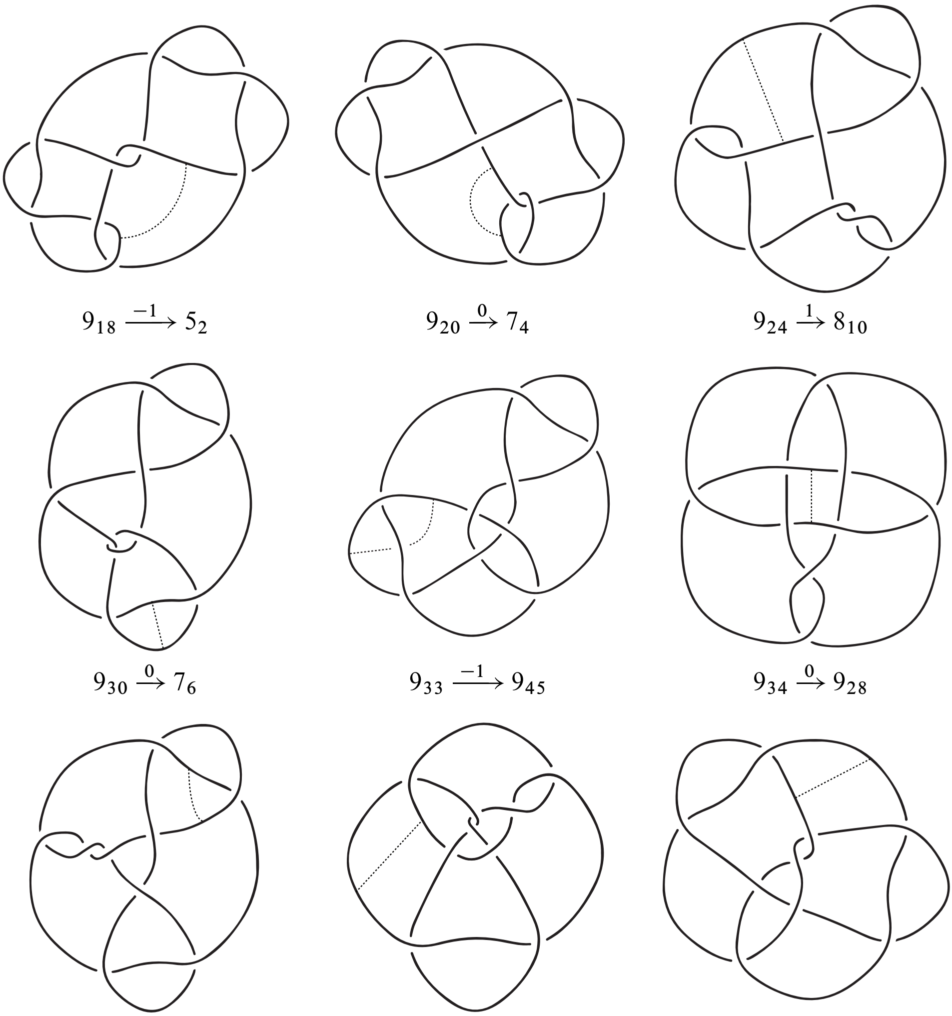


Figure 14: Nonoriented band moves from the knots 8_1, 8_2, 8_{12}, 8_{13}, 8_{15}, 8_{17}, 9_{10}, 9_{11}, 9_{14} to knots with γ_4 equal to 1

Question 4.2 *Is there a connection between $\gamma_4(K)$ and the full symmetry group of a nonslice knot K?*

A beautiful result of Edmonds [7] stipulates that a p–periodic knot K possesses a Seifert surface $S \subset S^3$ of genus $g_3(K)$ that is preserved under the $\mathbb{Z}_p$–action on S^3, making the connection between symmetries of a knot and its various genera plausible.



$9_{18} \xrightarrow{-1} 5_2$ $9_{20} \xrightarrow{0} 7_4$ $9_{24} \xrightarrow{1} 8_{10}$

$9_{30} \xrightarrow{0} 7_6$ $9_{33} \xrightarrow{-1} 9_{45}$ $9_{34} \xrightarrow{0} 9_{28}$

$9_{37} \xrightarrow{0} 8_{10}$ $9_{38} \xrightarrow{-1} 8_{14}$ $9_{49} \xrightarrow{-1} 8_{21}$

Figure 15: Nonoriented band moves from the knots 9_{18}, 9_{20}, 9_{24}, 9_{30}, 9_{33}, 9_{34}, 9_{37}, 9_{38}, 9_{49} to knots with γ_4 equal to 1

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Received: 29 August 2017 Revised: 30 November 2017