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The coarse geometry of the Kakimizu complex

JESSE JOHNSON
ROBERTO PELAYO
ROBIN WILSON

We show that the Kakimizu complex of minimal genus Seifert surfaces for a knot in the 3–sphere is quasi-isometric to a Euclidean integer lattice $\mathbb{Z}^n$ for some $n \geq 0$.

57M25; 57N10

1 Introduction

In general, a knot $K \subset S^3$ may have multiple nonisotopic minimal genus Seifert surfaces. To understand all these possibilities, Kakimizu [6] defined a simplicial complex $\mathcal{MS}(K)$, later referred to as the *Kakimizu complex*. Each vertex σ of $\mathcal{MS}(K)$ is an isotopy class of minimal genus Seifert surfaces for K, and n–simplices are spanned by isotopy classes with pairwise disjoint Seifert surface representatives. The metric on $\mathcal{MS}(K)$ is defined by the minimal lengths of edge paths between vertices. Kakimizu [6] defined an alternative metric on the complex using the infinite cyclic cover of K and showed that this metric is equal to the edge path metric.

Recently, $\mathcal{MS}(K)$ has been shown to be connected (Scharlemann and Thompson [12]), simply connected, and contractible (Przytycki and Schultens [9]). In fact, $\mathcal{MS}(K)$ for several classes of knots has been computed, including special arborescent knots (Sakuma [10]) and certain composite knots [6]. The Kakimizu complex has also been computed for all prime knots up to 10 crossings (Kakimizu [7]).

For hyperbolic knots, Pelayo [8] and Sakuma and Shackleton [11] give a bound on the diameter of the Kakimizu complex that is quadratic in the genus of the knot, and Wilson [13] shows that $\mathcal{MS}(K)$ is finite. For satellite knots, however, $\mathcal{MS}(K)$ may be infinite, and may even be locally infinite (Banks [1]).

The goal of this paper is to describe the coarse geometry of the Kakimizu complex. Recall that a *quasi-isometry* is a map $f\colon X \to Y$ between metric spaces X, Y such that $\frac{1}{L} d_Y(f(x), f(y)) - L \leq d_X(x, y) \leq L d_Y(f(x), f(y)) + L$ for some constant L and every point of Y is within an L–neighborhood of the image $f(X)$. For torus and hyperbolic knots, $\mathcal{MS}(K)$ is finite and therefore quasi-isometric to a single point. For satellite knots, the large-scale structure may be more exciting.

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Theorem 1 *For any knot $K \subset S^3$, the Kakimizu complex $\mathcal{MS}(K)$ is quasi-isometric to $\mathbb{Z}^n$ for some $n \geq 0$.*

An upper bound on the dimension n can be calculated in a relatively straightforward fashion. Below, we define a subset of the complementary pieces in the JSJ decomposition for the complement of K called the core. It follows from the proof that the value of n is less than or equal to the number of JSJ tori in the interior of the core minus the number of fibered complementary components in the core. We believe this is also a lower bound on the rank, but do not include a proof of this in the present paper.

The outline of the paper is as follows: In Section 2, we examine how Seifert surfaces for K interact with the incompressible tori in a JSJ decomposition for the knot complement. In Section 3, we define a group action on $\mathcal{MS}(K)$ by an abelian group, generated by twisting around the tori in the JSJ decomposition, then in Section 4 we prove that this action induces a quasi-isometry from $\mathcal{MS}(K)$ to $\mathbb{Z}^n$.

Note that the proof here is for knots in the 3–sphere, rather than links. Przytycki and Schultens [9] discuss ways of generalizing the Kakimizu complex to manifolds with multiple boundary components, but our proof relies on certain properties that are unique to knots in S^3, particularly the classification (proved by Ryan Budney [3]) of Seifert fibered components of the complement of a JSJ decomposition. However, there are no known counterexamples to the obvious generalization of our theorem to links.

2 The Kakimizu complex and the JSJ decomposition

In [6], Kakimizu computes the Kakimizu complex for the connected sum of two nonfibered knots K_1 and K_2 with unique incompressible Seifert surfaces. In this case, $\mathcal{MS}(K_1 \# K_2)$ is isometric to $\mathbb{Z} \subset \mathbb{R}$. These Seifert surfaces come from taking the canonical Seifert surface obtained by forming the connected sum of the minimal genus Seifert surfaces for each knot and spinning it around the incompressible swallow-follow torus in the complement of the composite knot. When a knot has more than two factors, more incompressible tori would mean more ways to potentially create new Seifert surfaces by spinning.

To understand this structure, let M_K be the knot complement, and consider the JSJ decomposition of M_K: Let $T_1, \dots, T_n$ be a minimal collection of pairwise disjoint, incompressible tori such that the complement of $\bigcup T_i$ consists of Seifert fibered pieces and atoroidal (hyperbolic) pieces. Each T_i bounds a solid torus in S^3 containing K on one side; we will transversely orient each T_i so that the knot is on the negative side. If we consider a neighborhood $\mathcal{N}(T_i)$ of each torus and take the complement of the interior of these neighborhoods in M_K, then $M_K - \bigcup_{i=1}^n \mathrm{int}(\mathcal{N}(T_i))$ is a collection of compact connected components that we will call *blocks*.

Let B be a block not containing the knot and let M be the component of the complement $M_K \setminus B$ that contains K. If K is not nullhomologous in M, then we will say that B is a *core block*. This implies that every minimal genus Seifert surface for K must intersect the torus ∂M and therefore the interior of the core block B. So in particular, every minimal genus Seifert surface for K must intersect every core block. If the block B contains the knot, then we also define it to be a core block. We refer to the union of all the core blocks as the *core* of the JSJ decomposition of M_K. The reader can check that the core is a connected subset of M_K.

One consequence of the definition of the core is that minimal genus Seifert surfaces must intersect tori in the interior of the core in a very controlled manner.

Proposition 2 *Let T be a JSJ torus in the interior of the core. There is a fixed slope α of T such that every Seifert surface S for the knot must intersect T in one or more parallel loops with exactly this slope.*

Proof Since T is a JSJ torus, it separates S^3 into two components M and M', where M is a solid torus containing the knot and M' is on the opposite side. The component M' is homeomorphic to the complement of a (different) knot in S^3 so the inclusion map $H_1(T) \to H_1(M')$ has infinite cyclic image and infinite cyclic kernel. Let $\alpha \subset T$ be a loop representing a generator of this kernel.

The intersection $S \cap T$ is a collection of parallel loops with orientations induced from S and T. Since T is in the interior of the core, this intersection must be homologically nontrivial in M. (Otherwise, S would represent a 2–cycle in M with boundary K.) Thus the loops $S \cap T$ define a nontrivial element of the first homology group of T. On the other side of T, the surface $S' = S \cap M'$ implies that the loops $S \cap T$ determine a trivial element of the first homology of M'. Since these loops define a nontrivial element of $H_1(T)$, they must represent a power of the generator of the kernel map, ie $S \cap T$ is a collection of parallel copies of the longitude α. □

Lemma 3 *Let S be a minimal genus Seifert surface for K and let $T = \bigcup_{i=1}^n T_i$ be the collection of all JSJ tori. If S is isotoped to intersect T minimally, then $|S \cap T| \leq 6g - 4$, where g is the genus of S.*

Proof Let $B_1, \ldots, B_k$ denote the blocks of the JSJ decomposition. Each B_i is a submanifold of M_K that is either a hyperbolic link complement and hence is atoroidal, or Seifert fibered. Notice that S meets each B_i in a collection of orientable, disjoint, essential surfaces that are properly embedded in B_i.

Since S is a once-punctured surface of genus g, there are at most $3g-2$ pairwise disjoint isotopy classes of essential loops in S. Consider the components of $S \cap T$, each of which is a simple closed curve. Since both S and T are incompressible and the complement of K is irreducible, we can assume that all intersection curves are essential in both surfaces. (Otherwise we could reduce the number of intersections, contradicting the minimality of $S \cap T$.) Therefore, there are no more than $3g-2$ isotopy classes in S of curves $S \cap T$.

In order to bound the number of components in $S \cap T$, it suffices to bound the number of parallel pairwise disjoint curves of $S \cap T$. Disjoint curves in $S \cap T$ that are parallel in S cobound an annulus A in S, and this annulus must be incompressible and properly embedded in some block B_i. If A were boundary parallel then we could reduce $S \cap T$. Thus A is an essential annulus so B_i must be Seifert fibered with A isotopic to a union of fibers.

Assume for contradiction there are three adjacent pairwise disjoint curves in $S \cap T$ that are parallel in S. Then the three curves correspond to two adjacent essential annuli A_1 and A_2 contained in adjacent Seifert fibered blocks. Without loss of generality, we can assume that A_1 is properly embedded in block B_1 and A_2 is properly embedded in block B_2. Because each A_i is a union of fibers, the two fiberings of the common boundary torus T_j induced from the Seifert fiberings of B_1 and B_2 have the same slope. Therefore these two fiberings can be isotoped to agree on T_j (see [5]), contradicting the minimality of the JSJ decomposition since T_j can be removed from the collection of JSJ tori. Therefore, there can be at most two adjacent curves of $S \cap T$ that are parallel in S in each isotopy class of curves. Hence $|S \cap T| \le 2(3g-2) = 6g-4$. □

The following is a slight generalization of the main result in [13]. The proof can be found in [13], however the statement is for manifolds with one toroidal boundary component. It is not difficult to modify the proof to also hold for manifolds with a finite number of toroidal boundary components, by a minor modification of the normal surface equations.

Theorem 4 [13] *Let M_L be a link complement. Let $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes for the link L. If M_L contains an infinite collection of essential surfaces S_i of the same Euler characteristic such that ∂S_i is isotopic to a subcollection of the α_i and there exists an integer N such that $|\partial S_i| \le N$ for each i, then M_L contains a closed incompressible torus.*

The following corollary follows immediately from Theorem 4.

Corollary 5 *Let M_L be a link complement and $N \in \mathbb{N}$. Suppose that M_L contains no closed essential tori. Let $\alpha_1, \dots, \alpha_k$ be a set of preferred longitudes for the link L. Then M_L contains at most finitely many essential surfaces S_i of maximal Euler characteristic such that ∂S is isotopic to a subcollection of the α_i and $|\partial S_i| \leq N$ for each i.*

A Seifert fibered block of a JSJ decomposition may be toroidal, but the Seifert fibered blocks of a knot complement in S^3 are much more restricted. Budney [3, Proposition 3.2] gives the following classification of Seifert fibered submanifolds of S^3.

Lemma 6 [3] *Let $V \neq S^3$ be a Seifert fibered submanifold of S^3, then V is diffeomorphic to one of the following:*

- *A Seifert fibered space over an n–times punctured sphere with two exceptional fibers, appearing as the complement of n regular fibers in a Seifert fibering of S^3.*
- *A Seifert fibered space over an n–times punctured sphere with 1 exceptional fiber, appearing as the complement of $n-1$ regular fibers in a Seifert fibering of an embedded solid torus in S^3.*
- *A Seifert fibered space over an n–times punctured sphere with no exceptional fibers.*

We will use the lemma above to show that there are only finitely many incompressible surfaces in a Seifert fibered block for a knot complement.

Theorem 7 *Let K be a knot and B a core block of the JSJ decomposition for M_K. There exist finitely many essential surfaces $S_1, S_2, \dots, S_m$ such that for any minimal genus Seifert surface S for K in the core of M_K, every component of $S \cap B$ is isotopic to one of the S_i.*

Proof If B is a hyperbolic block, then it is atoroidal and $\partial(S \cap B)$ is fixed and bounded by Proposition 2 and Lemma 3. In this case, the conclusion follows directly from Corollary 5. If B is a Seifert fibered block, then Lemma 6 gives us three possibilities. In the first case, B is Seifert fibered over an n–times punctured sphere with two singular fibers, arising as the complement of n solid tori. Since M_K is a knot complement, there can only be one such solid torus in the complement of B. To see this, note that any solid torus contains a compressing disk for its boundary. Since the boundary tori of B are incompressible in the knot complement, the knot must intersect this compressing disk. However, since there is only one knot, there exists only one solid torus in the complement of the block. Thus $n = 1$ and B is Seifert fibered over a once-punctured sphere with two critical fibers. Such Seifert fibered spaces are known to contain no closed essential surfaces and thus are atoroidal. Applying Corollary 5, the conclusion again follows.

In the second case, B is Seifert fibered over an n–punctured sphere with one critical fiber, arising as the complement of $n-1$ solid tori in a Seifert fibered solid torus. Once again, as B is a subset of a knot complement, there can be at most 1 solid torus, so $n \leq 2$. These Seifert fibered spaces are also atoroidal and thus applying Corollary 5, the conclusion follows. In the last case, B is Seifert fibered over an n–times punctured sphere with no exceptional fibers and is thus a product. Since B is a core block, by Proposition 2, the boundary of $S \cap B$ is specified. Because S^2 contains only separating loops, there is a unique incompressible surface boundary of a specified homotopy class in this product space. □

3 Group actions on the Kakimizu complex

Every automorphism of the knot complement induces an automorphism of the Kakimizu complex. For a given T_i, let U be a closed regular neighborhood homeomorphic to $I \times T_i$. We will define an automorphism of the knot complement that is the identity outside of U and spins around the JSJ torus in a given direction. Consider the universal cover of T_i, which is homeomorphic to the plane. Choose a coordinate system on this plane. For every integer vector $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, there is a family of automorphisms ϕ_t of T_i for $t \in I$ that lift to translations of the plane taking $(0,0)$ to (m,n) and such that ϕ_0 and ϕ_1 are the identity on T_i. We obtain an automorphism of the knot complement as follows:

$$\Phi(x) = \begin{cases} (t, \phi_t(z)) & \text{if } x = (t,z) \in I \times T_i \\ x & \text{else} \end{cases}$$

For a fixed T_i, each choice of integer vector $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ gives one of these automorphisms of the knot complement. Furthermore, composition of these automorphisms corresponds to integer vector addition in $\mathbb{Z} \times \mathbb{Z}$, which forms an abelian group of rank 2. For each torus T_i in the interior of the core, let α be a loop in T_i defining the slope on T given by Proposition 2. Then spinning parallel to α takes every Seifert surface onto itself, and thus acts by the identity on $\mathcal{MS}(K)$. The quotient of $\mathbb{Z}^2$ by the infinite cyclic subgroup generated in this way is an infinite cyclic group, and we will let Φ_i be a representative for the generator of this quotient group, ie an element of the corresponding coset in $\mathbb{Z}^2$.

To prove Theorem 1, we will first prove that the result holds for a particular subcomplex of $\mathcal{MS}(K)$ that is locally finite ($\mathcal{MS}(K)$ is not locally finite in general [1]). We define the *core Kakimizu complex* $\mathcal{MS}(C)$ to be the subcomplex spanned by Seifert surfaces that can be isotoped into the core C.

Consider G', the group of all automorphisms of the knot complement generated by the homeomorphisms Φ_i defined above. Because each Φ_i takes each block onto itself, the action of G' restricts to an action on $\mathcal{MS}(C)$. For $i \neq j$, the support of Φ_i is disjoint from that of Φ_j, so such homeomorphisms commute and G' is abelian. Let N be the (normal) subgroup of G' that acts trivially on the core Kakimizu complex $\mathcal{MS}(C)$. Then $G = G'/N$ is also a finitely generated abelian group. In order to understand the action of G' on $\mathcal{MS}(C)$, we will first describe some key properties of $\mathcal{MS}(C)$.

Claim 8 *$\mathcal{MS}(C)$ is nonempty.*

Proof Let S be a Seifert surface for the knot K such that $S \cap (\bigcup T_i)$ is minimal over all Seifert surfaces for K. If S stays inside the core, then $\mathcal{MS}(C)$ is nonempty. If S exits the core, then it must do so by intersecting some JSJ torus T that separates a core block from a non-core block. Assume that S intersects T minimally. Since T is not in the interior of the core, the knot is homologically trivial in the component X of the complement of T that contains the knot. As noted above, X is a solid torus. Because S is orientable, the intersection curves of S with T define the trivial element of the first homology group of X.

If these loops are nonmeridional then each individual loop defines a nontrivial element of the homology of X, so there must be an equal number with each orientation. If the loops are meridional then the complement $S \setminus X$ defines a meridional surface for the knot complement $S^3 \setminus X$. Since the meridian is homologically nontrivial in $S^3 \setminus X$, this implies that there are again an equal number of loops with each orientation. So in either case, the loops $S \cap T$ occur in pairs with opposite orientations.

Choose a pair of adjacent curves of intersection β and γ with opposite orientations. Then β and γ cobound an annulus $A \subset T$ with interior disjoint from S. Cut the Seifert surface S along the curves β and γ and attach the resulting boundary components to ∂A, then push the resulting surface slightly into the interior of the block, reducing the number of intersections of S with T and thus contradicting the assumption that $S \cap (\bigcup T_i)$ is minimal. Therefore the vertex representing S is in $\mathcal{MS}(C)$, so $\mathcal{MS}(C) \neq \varnothing$. □

Claim 9 *$\mathcal{MS}(C)$ is connected.*

Proof To show that the subcomplex $\mathcal{MS}(C)$ is connected, we examine the construction used by Scharlemann and Thompson [12] to find a path in the Kakimizu complex between any vertices v_1 and v_2. Let S and S' be Seifert surfaces representing the vertices v_1 and v_2, respectively. By taking double curve sums, Scharlemann and Thompson create a sequence of minimal genus Seifert surfaces S_i for $0 \leq i \leq k$ such that $S_i \cap S_{i+1} = \varnothing$ for $0 \leq i < k$, with S_0 isotopic to S and S_k isotopic to S'. If S and S' are both in C then so are all the double curve sums. Thus, $S_i \subset C$ for each i and the path is contained in $\mathcal{MS}(C) \subset \mathcal{MS}(K)$. □

Claim 10 *$\mathcal{MS}(C)$ is locally finite.*

Proof Let S be a minimal genus Seifert surface representing a vertex $v \in \mathcal{MS}(C)$. For any minimal genus Seifert surface $S' \subset C$ disjoint from S, there are finitely many possibilities for the intersection of S' with each core block of the JSJ decomposition by Theorem 7. The surface S' is determined by these intersections and the annuli that connect these subsurfaces inside the regular neighborhoods of the JSJ tori in the interior of the core. Because S intersects every component of T in the interior of the core and S' is disjoint from S, there are finitely many ways the subsurfaces can be connected together (in particular, no annulus can spin all the way around such a torus without crashing through S) so there are finitely many minimal genus Seifert surfaces (up to isotopy) disjoint from S. □

In order to prove that our main theorem holds for the subcomplex $\mathcal{MS}(C)$, we will need Theorem 25 from [4], which is stated below.

Theorem 11 [4] *Let X be a metric space that is geodesic and proper, let G be a group and let $G \times X \to X$ be an action by isometries. Assume that the action is proper and that the quotient X/G is compact.*

Then the group G is finitely generated and quasi-isometric to X. More precisely, for any $x_0 \in X$, the mapping $G \to X$ given by $g \mapsto gx_0$ is a quasi-isometry.

We will first show that $\mathcal{MS}(C)/G$ is finite.

Lemma 12 *There are a finite number of minimal genus Seifert surfaces in the core, called* fundamental surfaces*, such that every Seifert surface for K in the core is either fundamental or can be obtained by spinning a fundamental surface around some number of JSJ tori in the interior of the core. In other words, there are a finite number of isotopy classes of minimal genus Seifert surfaces $\sigma_1, \dots, \sigma_j$ in the core such that any other isotopy class of minimal genus Seifert surfaces $\sigma' \in \mathcal{MS}(C)$ can be written as $\sigma' = g\sigma_k$ for some $g \in G$ and some fundamental surface σ_k.*

Proof Let S be a minimal genus Seifert surface for the knot K and assume that S meets the JSJ tori $T_1, \dots, T_n$ transversally and minimally. Let $\mathcal{N}(T_i)$ be a neighborhood of T_i. Recall that the blocks B_k are the components of the complement of the JSJ tori.

For each k, each component of $S \cap B_k$ is isotopic to one of the finitely many possible properly embedded surfaces as given in Theorem 7. Inside each neighborhood $\mathcal{N}(T_i)$, each component of $S \cap \mathcal{N}(T_i)$ is an incompressible annulus.

Therefore, the Seifert surface S is obtained from some finite collection of the incompressible surfaces in each block by connecting these pieces with annuli across T. Up to spinning around the torus, there are finitely many ways to connect the incompressible surfaces on either side of each torus. Therefore, every Seifert surface S is in the orbit of one of finitely many isotopy classes of Seifert surfaces coming from the finite number of ways of putting together the finite components in each block. □

Next, we will show that the action is proper.

Lemma 13 *The action of the group G on $\mathcal{MS}(C)$ is proper. That is, the stabilizer $G_v = \{g \in G \mid g(v) = v\}$ is finite for every vertex v in $\mathcal{MS}(C)$.*

Proof Let $v \in \mathcal{MS}(C)$ be a vertex of the core Kakimizu complex. To prove that G_v is finite, we will show that there is a monomorphism from G_v to a finite group.

By Lemma 12, there are finitely many orbits $O_1, O_2, \dots, O_r$ of the group G. Choose a representative of each orbit $v_i \in O_i$. Let $V = \{v, v_1, \dots, v_r\}$, and let d be the diameter of V. Let B be a ball in $\mathcal{MS}(C)$ of diameter d centered at the vertex v. By construction, $V \subset B$. Notice that each automorphism of G_v preserves distance between vertices, so the ball B is fixed setwise. Since $\mathcal{MS}(C)$ is locally finite by Claim 10, there are finitely many vertices in B. This induces a homomorphism from the stabilizer G_v to the permutation group of the (finitely many) vertices of B.

To see that this homomorphism is injective, we note that the kernel consists of all elements of G_v that fix B pointwise. Let g be such an automorphism in the kernel. Since g fixes B pointwise, then $g(v_i) = v_i$ for all i. For any $x \in \mathcal{MS}(C)$, $x = h(v_i)$ for some i, where h is some element of G. Since G is abelian, $gh = hg$, so $g(x) = g(h(v_i)) = h(g(v_i)) = h(v_i) = x$. Thus, g fixes every $x \in \mathcal{MS}(C)$. Because we quotiented out by the elements of G' that act trivially, g is the identity element in G and in G_v. Thus, the homomorphism from G_v to the finite permutation group is injective, and thus G_v is finite. □

We can now combine these results to prove the following:

Lemma 14 *$\mathcal{MS}(C)$ is quasi-isometric to a finitely generated abelian group.*

Proof As noted above, the metric on $\mathcal{MS}(C)$ is the path metric, so the complex is properly geodesic. Since each automorphism of G takes disjoint surfaces to disjoint surfaces, it preserves distances between vertices and thus acts isometrically on $\mathcal{MS}(C)$. By Lemma 12, $\mathcal{MS}(C)/G$ is finite and hence compact, and by Lemma 13, the action of G on $\mathcal{MS}(C)$ via left multiplication is proper. Thus Theorem 11 implies that $\mathcal{MS}(C)$ is quasi-isometric to G, a finitely generated abelian group. □

4 Proof of the main theorem

In the previous section, we proved that the core Kakimizu complex $\mathcal{MS}(C)$ is quasi-isometric to a finitely generated abelian group. To prove our main result, it remains to show that for any knot K, $\mathcal{MS}(K)$ is quasi-isometric to $\mathcal{MS}(C)$.

Lemma 15 *The core Kakimizu complex $\mathcal{MS}(C)$ is quasi-isometric to the entire Kakimizu complex $\mathcal{MS}(K)$.*

Proof We will show that $\mathcal{MS}(C)$ is quasi-isometric to $\mathcal{MS}(K)$ by showing that the inclusion map $\mathcal{MS}(C) \hookrightarrow \mathcal{MS}(K)$ preserves distances and every vertex σ in $\mathcal{MS}(K)$ is within a bounded distance from some vertex σ' in $\mathcal{MS}(C)$. First we note that the proof of Proposition 5 from [12] uses double curve sums to produce geodesics in $\mathcal{MS}(K)$ (as opposed to just paths). Since a double curve sum in C produces a new surface in C, this implies that the geodesics between vertices of $\mathcal{MS}(C)$ constructed in this way will be contained in $\mathcal{MS}(C)$. Thus, given two vertices in the core of the Kakimizu complex $\mathcal{MS}(C)$, measuring their distance in $\mathcal{MS}(C)$ is equivalent to measuring their distance in the entire Kakimizu complex $\mathcal{MS}(K)$. So, the inclusion map preserves distances.

Let S be a minimal genus Seifert surface for K in the isotopy class σ. If S is contained in the core, then $\sigma \in \mathcal{MS}(C)$. If not, then, S must intersect a JSJ torus T that bounds a block B inside the core and B' outside the core. The torus T separates the Seifert surface S into a compact surface S' inside the core and finitely many annuli A_i outside the core (because S has minimal genus). In fact, there are at most $3g-2$ annuli A_i since any minimal genus Seifert surface intersects the JSJ tori in at most $6g-4$ circles by Lemma 3. The boundaries $C_i^+ \cup C_i^-$ of each annulus A_i can be joined by annuli D_i that lie inside of B. Attaching these D_i to S' yields a minimal genus Seifert surface $\widetilde{S}$ that lies completely in the core C and is represented by an isotopy class $\sigma' \in \mathcal{MS}(C)$.

Each time we surger the surface at a single annulus in this way, the new surface is disjoint from the previous surface. Thus σ and σ' are connected by a path in $\mathcal{MS}(K)$ of distance at most the number of annuli outside C, ie at most $3g-2$. Thus, every $\sigma \in \mathcal{MS}(K)$ is within a bounded distance from $\mathcal{MS}(C)$, so the two complexes are quasi-isometric. □

Proof of Theorem 1 In Lemma 15, we showed that $\mathcal{MS}(K)$ is quasi-isometric to $\mathcal{MS}(C)$. Because quasi-isometry defines an equivalence relation, this implies, by Lemma 14, that $\mathcal{MS}(K)$ is quasi-isometric to a finitely generated abelian group, and therefore quasi-isometric to $\mathbb{Z}^n$ for some n. □

Note that the proof of Theorem 1 implies that the abelian group G that is quasi-isometric to $\mathcal{MS}(K)$ is generated by spinning around tori in the interior of the core, so its rank is at most the number of such tori. Moreover, for each block B that is homeomorphic to the complement of a fibered link, spinning around the torus closest to the knot K is equivalent (up to isotopy) to spinning around the remaining boundary tori. Thus the rank is at most the number of JSJ tori in the interior of the core minus the number of fibered core blocks.

We believe it should be possible to show that this is also a lower bound on the rank by generalizing the proof of Theorem 1.2 from [2]. However, the mechanics of such a proof would be mostly independent of the techniques above, and thus out of the scope of the current paper.

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Department of Mathematics, Oklahoma State University
Stillwater, OK 74078, USA

Department of Mathematics, University of Hawaii at Hilo
Hilo, HI 96720, USA

Department of Mathematics and Statistics, California State Polytechnic University
Pomona, CA 91768, USA

jjohnson@math.okstate.edu, robertop@hawaii.edu,
robinwilson@csupomona.edu

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