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A bound for orderings of Reidemeister moves

JULIAN GOLD

We provide an upper bound on the number of ordered Reidemeister moves required to pass between two diagrams of the same link. This bound is in terms of the number of unordered Reidemeister moves required.

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In 1927 Kurt Reidemeister proved that any two link diagrams representing the same link may be joined by a finite sequence of Reidemeister moves. The importance of this theorem to knot theory cannot be overstated. Mathematicians like Alexander Coward [1; 2], Marc Lackenby [2], Bruce Trace [4], and Joel Hass and Jeffery Lagarias [3] have all explored properties of sequences of Reidemeister moves.

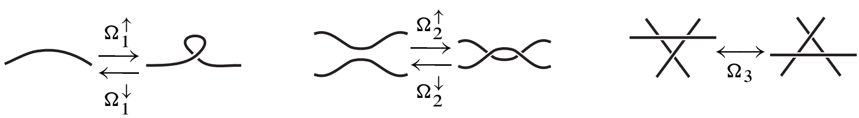

Figure 1: Reidemeister moves

In 2006, Alexander Coward showed in [1] that given any sequence of Reidemeister moves between link diagrams D_1 and D_2, it is possible to construct a new sequence ordered in the following way: first $\Omega_1^\uparrow$ moves, then $\Omega_2^\uparrow$ moves, then Ω_3 moves, finally $\Omega_2^\downarrow$ moves. We present, via the following theorem, an upper bound on the number of moves required for an ordered sequence in terms of the number of moves present in any sequence of Reidemeister moves.

Theorem 1 *Let D_1 and D_2 be diagrams for the same link that are joined by a sequence of M Reidemeister moves. Let $N = 6^{M+1}M$. Then there exists a sequence of no more than $\exp^{(N)}(N)$ moves from D_1 to D_2 ordered in the following way: first $\Omega_1^\uparrow$, then $\Omega_2^\uparrow$, then Ω_3, then $\Omega_2^\downarrow$ and finally $\Omega_1^\downarrow$.*

Here the function exp is defined as $\exp(x) = 2^x$ and $\exp^{(r)}(x)$ is the function exp iterated r times on input x.

We define a link diagram to be a 4–valent graph embedded in $\mathbb{R}^2$ with crossing information recorded at each vertex. All diagrams will be oriented, so that they

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represent oriented links. We regard two diagrams as the same if there is an ambient isotopy of $\mathbb{R}^2$ taking one diagram to the other, preserving crossing information and the orientation of each link component. To prove Theorem 1, we will adapt the methods Alexander Coward uses in [1] and borrow the following terminology.

Definition Let D be a link diagram and suppose $c\colon [0,1] \to \mathbb{R}^2$ is an embedded path whose image C intersects D transversely at finitely many points, where $c(0) \in D$ and $c(1) \notin D$. We stipulate that no point of intersection of D and C is a vertex of D. At each such point, apart from $c(0)$, we designate whether C passes over or under D.

Let $C \times [-\epsilon, \epsilon]$ be a small neighborhood of C such that

$$(C \times [-\epsilon, \epsilon]) \cap D = (C \cap D) \times [-\epsilon, \epsilon].$$

Then define the diagram D' as the 4–valent graph

$$D \cup \partial(C \times [-\epsilon, \epsilon]) \setminus (c(0) \times (-\epsilon, \epsilon))$$

with crossing information induced by the path c. We write $D \rightsquigarrow D'$ and say that D' is obtained from D by *adding a tail along C*. Additionally, we will call C the *core of this tail*. We require that adding a tail to a diagram D produces a diagram D' where $c(D') > c(D)$. Figure 2 illustrates the construction of a tail.

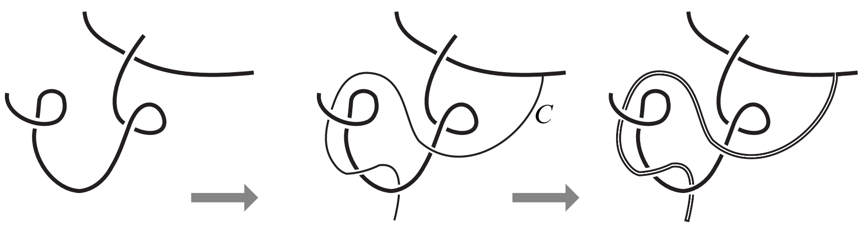


Figure 2: Adding a tail

Definition Suppose $D_1 \rightsquigarrow D_2$ via some path $c\colon [0,1] \to \mathbb{R}^2$. Suppose additionally that $c(1)$ lies in a small neighborhood of some crossing χ of D_1. Let D_3 be as in Figure 3, a diagram obtained from D_2 by performing two $\Omega_2^{\uparrow}$ moves followed by one Ω_3 move.

We say D_3 is obtained from D_1 by *adding a lollipop* and write $D_1 \circ\!\!\!\rightarrow D_3$. The *lollipop* itself is defined as $\overline{D_3 \setminus D_1}$. The *tail part of the lollipop* is $\overline{(D_3 \cap D_2) \setminus D_1}$, and the closure of the rest of the lollipop is the *circle part of the lollipop*. We say that the lollipop is *centered at* χ.

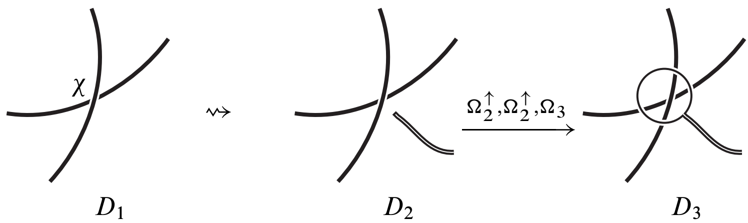


Figure 3: Adding a lollipop

We think of a sequence $\mathcal{S}$ of Reidemeister moves, tails and lollipops between link diagrams L_1 and L_2 in the following way:

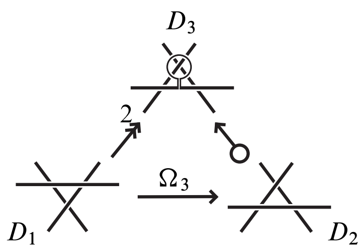
$$\mathcal{S}: L_1 = D_0 \xrightarrow{a_1} D_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} D_n = L_2$$

Here each a_i is a Reidemeister move, a tail or a lollipop. A tail or lollipop may be added from D_i to D_{i+1} (eg $D_i \rightsquigarrow D_{i+1}$) or from D_{i+1} to D_i (eg $D_i \leftsquigarrow D_{i+1}$). We say the *length* of $\mathcal{S}$ is n. The intermediate link diagrams D_i are often omitted from the figures in this paper for clarity, but are implicit in any sequence.

If a link diagram D_2 is reached from D_1 by a sequence of $\Omega_2^{\uparrow}$ moves of length n, we write $D_1 \twoheadrightarrow^n D_2$. The following lemma allows us to take a sequence $\mathcal{S}$ and produce a sequence $\mathcal{S}'$ with one less Ω_3 move.

Lemma 2 *Let D_1 and D_2 be link diagrams such that $D_1 \xrightarrow{\Omega_3} D_2$. Then there exists a diagram D_3 such that $D_1 \twoheadrightarrow^2 D_3$ and $D_2 \multimap D_3$.*

Proof The diagram



satisfies the required conditions. □

If an Ω_3 move occurs in a sequence of Reidemeister moves, tails and lollipops

$$\mathcal{S}: A \to \cdots \to B \xrightarrow{\Omega_3} C \to \cdots \to D,$$

we may apply Lemma 2 to $\mathcal{S}$ to get a new sequence

$$\mathcal{S}' : A \to \cdots \to B \xrightarrow{\Omega_2^{\uparrow}} B' \xrightarrow{\Omega_2^{\uparrow}} B'' \leftarrow\!\!\circ\, C \to \cdots \to D.$$

When we apply Lemma 2 to construct $\mathcal{S}'$ from $\mathcal{S}$, we call this *capping* the Ω_3 move from B to C. The following proposition and its corollary will also allow us to build new sequences from old ones in a useful way.

Proposition 3 *Suppose $D_1 \rightsquigarrow D_1'$ (or $D_1 \circ\!\!\rightarrow D_1'$) and also that $D_1 \twoheadrightarrow^1 D_2$. Then there exists a diagram D_2' such that $D_2 \rightsquigarrow D_2'$ ($D_2 \circ\!\!\rightarrow D_2'$ respectively) and $D_1' \twoheadrightarrow^a D_2'$, where*

(a) $$c(D_2') - c(D_2) \le 2(c(D_1') - c(D_1))$$

(b) $$a \le c(D_1') - c(D_1).$$

$$\begin{tikzcd} D_1' & \\ D_1 \arrow[u, rightsquigarrow] \arrow[r, twoheadrightarrow, "1"] & D_2 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} D_1' \arrow[r, twoheadrightarrow, "a"] & D_2' \\ D_1 \arrow[u, rightsquigarrow] \arrow[r, twoheadrightarrow, "1"] & D_2 \arrow[u, rightsquigarrow] \end{tikzcd}$$

Proof The diagram D_2 is obtained from D_1 by a single $\Omega_2^{\uparrow}$ move which takes place over two (possibly non-distinct) edges e_1 and e_2 of D_1. Pick points p_1 and p_2 on e_1 and e_2 respectively, so that p_1 and p_2 lie outside a small neighborhood of the tail $D_1 \rightsquigarrow D_1'$. We can perform the $\Omega_2^{\uparrow}$ move from D_1 to D_2 by adding a tail along a path γ, which starts at p_1 and ends slightly beyond p_2.

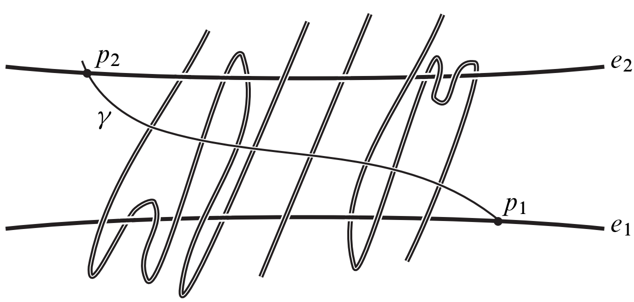


Figure 4: Constructing D_2' by adding a tail along γ

Diagram D_1' contains the points p_1 and p_2. We may arrange that the intersection of γ with the tail $D_1 \rightsquigarrow D_1'$ contains at most $2\lfloor (c(D_1') - c(D_1))/4 \rfloor$ points. Figure 4 depicts such an arrangement. Adding a tail along γ, we construct a diagram D_2' with

$$c(D_2') - c(D_1') \le 4\left\lfloor \frac{c(D_1') - c(D_1)}{4} \right\rfloor + 2.$$

Hence

$$c(D_2') - c(D_1') \le c(D_1') - c(D_1) + 2.$$

We note that $c(D_1') - c(D_1) + 2 \le 2(c(D_1') - c(D_1))$, because adding a tail to a diagram must raise its crossing number by at least two. This implies the desired bound on a. Also

$$c(D_2') - c(D_1') \le c(D_1') - c(D_1) + 2$$

implies, by adding $c(D_1')$ to both sides and subtracting $c(D_2)$, that

$$c(D_2') - c(D_2) \le 2c(D_1') - c(D_1) + 2 - c(D_2).$$

Using $c(D_2) = c(D_1) + 2$ we get

$$c(D_2') - c(D_2) \le 2c(D_1') - 2c(D_1).$$

In the case that $D_1 \mathrel{\circ\!\!\rightarrow} D_1'$, choose p_1 and p_2 to be outside the circle part of the lollipop, and the above considerations go through. $\square$

Corollary 4 is a natural generalization of Proposition 3.

Corollary 4 *Suppose $D_1 \rightsquigarrow D_1'$ (or $D_1 \mathrel{\circ\!\!\rightarrow} D_1'$) and also that $D_1 \twoheadrightarrow^n D_2$. Then there exists a diagram D_2' such that $D_2 \rightsquigarrow D_2'$ ($D_1 \mathrel{\circ\!\!\rightarrow} D_1'$ respectively) and $D_1' \twoheadrightarrow^b D_2'$, where*

$$b \le 2^n (c(D_1') - c(D_1)).$$

$$\begin{tikzcd} D_1' & \\ & \\ D_1 \arrow[uu, rightsquigarrow] \arrow[r, two heads, "n"] & D_2 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} D_1' \arrow[r, two heads, "b"] & D_2' \\ & \\ D_1 \arrow[uu, rightsquigarrow] \arrow[r, two heads, "n"] & D_2 \arrow[uu, rightsquigarrow] \end{tikzcd}$$

Proof Let D_1, D_2 and D_1' be as in the statement of the theorem. We work in the case $D_1 \rightsquigarrow D_1'$, but the proof for lollipops is identical. Let $\mathcal{E}$ be the sequence of $\Omega_2^{\uparrow}$ moves of length n from D_1 to D_2,

$$\mathcal{E} : D_1 = E_0 \twoheadrightarrow^1 E_1 \twoheadrightarrow^1 \cdots \twoheadrightarrow^1 E_n = D_2,$$

and let $E_0' = D_1'$. We use Proposition 3 to construct a diagram E_1' such that $E_1 \rightsquigarrow E_1'$ and $E_0' \twoheadrightarrow^{b_0} E_1'$, where $b_0 \le c(E_0') - c(E_0)$. Apply Proposition 3 again to the triple (E_1, E_1', E_2) to build a diagram E_2'. Iterate this, constructing the diagrams E_2' through E_n', as below.

$$
\begin{tikzcd}
D'_1 = E'_0 \arrow[r, two heads, "b_0"] \arrow[rrrrr, no head, bend left, "b"] & E'_1 \arrow[r, two heads, "b_1"] & \cdots & E'_{n-2} \arrow[r, two heads, "b_{n-2}"] & E'_{n-1} \arrow[r, two heads, "b_{n-1}"] & E'_n = D'_2 \\
D_1 = E_0 \arrow[r, two heads, "1"] \arrow[u, rightsquigarrow] & E_1 \arrow[r, two heads, "1"] \arrow[u, rightsquigarrow] & \cdots & E_{n-2} \arrow[r, two heads, "1"] \arrow[u, rightsquigarrow] & E_{n-1} \arrow[r, two heads, "1"] \arrow[u, rightsquigarrow] & E_n = D_2 \arrow[u, rightsquigarrow]
\end{tikzcd}
$$

Proposition 3(b) gives us that $b_i \leq c(E'_i) - c(E_i)$, while Proposition 3(a) tells us $c(E'_i) - c(E_i) \leq 2^i (c(E'_0) - c(E_0))$. The sequence of Ω_2 moves from E'_0 to E'_n has length b, where $b = \sum_{i=0}^{n-1} b_i$. Hence,

$$b \leq (2^n - 1)(c(E'_0) - c(E_0)).$$

Take $D'_2 = E'_n$ to complete the proof. □

Theorem 5 below makes use of Lemma 2, Proposition 3 and Corollary 4 to begin building an ordered sequence from an unordered sequence.

Theorem 5 *Let D_2 be a link diagram obtained from D_1 via a sequence of Ω_2 and Ω_3 moves of length M. Then there exists a diagram D_3 such that $D_1 \twoheadrightarrow^c D_3$ with D_3 is obtained from D_2 by adding a sequence no more than M tails and lollipops. Further, $c \leq \exp^{(M)}(6M)$.*

Proof Consider a sequence $\mathcal{A}$ of Ω_2 and Ω_3 moves of length M from D_1 to D_2, of which N are Ω_3:

$$\mathcal{A} : D_1 = A_0 \to A_1 \to \cdots \to A_M = D_2$$

Using Lemma 2, cap every Ω_3 move to build a new sequence $\mathcal{E}_1$:

$$\mathcal{E}_1 : D_1 = E_0 \to E_1 \to \cdots \to E_{M+2N} = D_2,$$

where $\mathcal{E}_1$ contains no Ω_3 moves. This is depicted in Figure 5.

If $E_i \xrightarrow{\Omega_2^{\downarrow}} E_{i+1}$, we relabel this as $E_i \leftsquigarrow E_{i+1}$, because a $\Omega_2^{\uparrow}$ move may be performed by adding a tail. Define a *local minimum* of $\mathcal{E}_1$ to be a diagram E_i such that

$$E_{i-1} \leftsquigarrow E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1} \qquad \text{or} \qquad E_{i-1} \leftarrow\!\!\circ\; E_i \xrightarrow{\Omega_2^{\uparrow}} E_{i+1}.$$

Let $E_J \in \{E_1, \ldots, E_{M+2N-1}\}$ be the local minimum appearing in $\mathcal{E}_1$ with greatest index. Let r_1 be the number of consecutive $\Omega_2^{\uparrow}$ moves in $\mathcal{E}_1$ to the right of E_J. Let ℓ_1 be the number of consecutive $\Omega_2^{\uparrow}$ moves in $\mathcal{E}_1$ to the left of E_{J-1}.

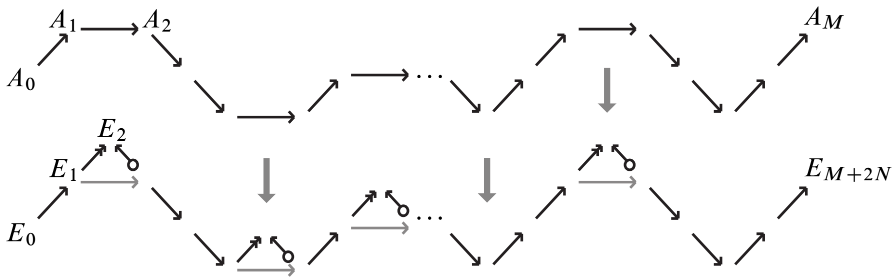


Figure 5: Constructing $\mathcal{E}_1$ from $\mathcal{A}$

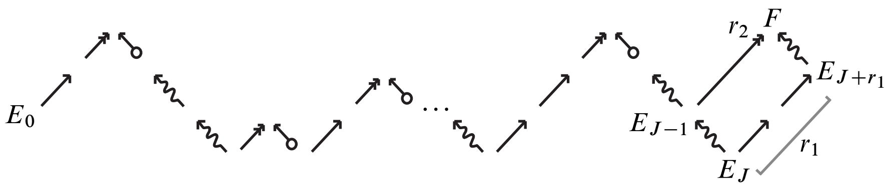


Figure 6: Constructing F (in this case, $E_{J+r_1} = E_{M+2N}$)

Apply Corollary 4 to the triple $(E_{J-1}, E_J, E_{J+r_1})$ to build a diagram F, where $E_{J-1} \twoheadrightarrow^{r_2'} F$ and where $E_{J+r_1} \circlearrowright\!\!\rightarrow F$ if $E_J \circlearrowright\!\!\rightarrow E_{J-1}$ or $E_{J+r_1} \rightsquigarrow F$ if $E_J \rightsquigarrow E_{J-1}$. Corollary 4 tells us $r_2' \le 4 \cdot 2^{r_1}$, in the worst case that $E_J \circlearrowright\!\!\rightarrow E_{J-1}$. Figure 6 depicts the construction of F. Define $\mathcal{E}_2$ to be the following sequence:

$$\mathcal{E}_2 : D_1 = E_0 \to E_1 \to \cdots \to E_{J-1} \to \cdots \to F \to E_{J+r_1} \to \cdots \to E_{M+2N}$$

Then $\mathcal{E}_2$ is a sequence of diagrams with r_2 consecutive $\Omega_2^{\uparrow}$ moves to the right of its last local minimum, where we have the following bound on r_2:

$$r_2 \le 4 \cdot 2^{r_1} + \ell_1 \le 2^{r_1+2+\ell_1}.$$

Let $\mathcal{E}_k$ be the sequence obtained by $k-1$ applications of Corollary 4, with r_k the number of $\Omega_2^{\uparrow}$ moves to the right of the last local minimum of $\mathcal{E}_k$. Let ℓ_k be the number of consecutive $\Omega_2^{\uparrow}$ moves preceding the diagram to the immediate left of the last local minimum of $\mathcal{E}_k$. Given the $\mathcal{E}_k$ and r_k, we may apply Corollary 4 as above to produce a sequence $\mathcal{E}_{k+1}$ and corresponding r_{k+1} with $r_{k+1} \le 2^{r_k+2+\ell_k}$, and hence

$$r_{k+1} \le \exp^{(k)}\left(r_1 + 2k + \sum_{i=1}^{k} \ell_i\right). \tag{1}$$

Iterate the constructions of the $(\mathcal{E}_k, r_k)$ until we produce a sequence $\mathcal{E}_K$ with no local minima and with r_K consecutive $\Omega_2^{\uparrow}$ moves following E_0. The number of times we

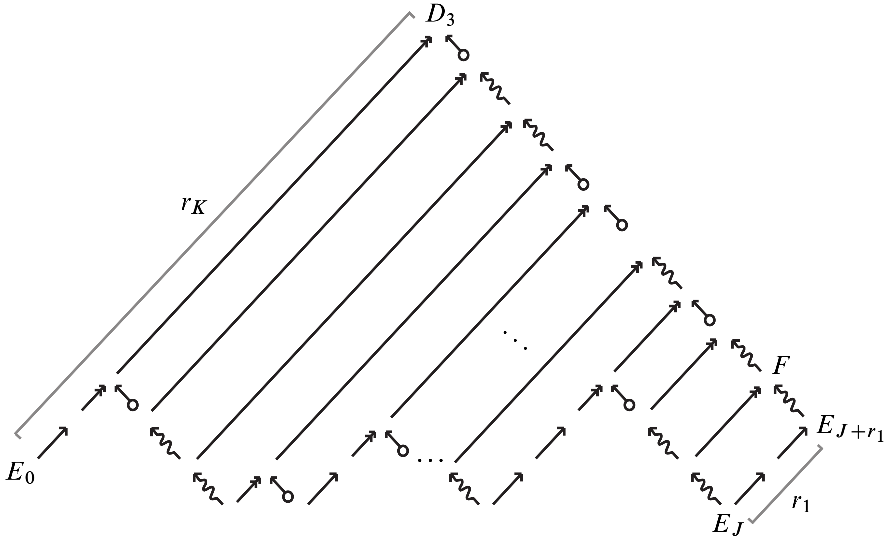


Figure 7: Repeatedly applying Corollary 4 to build D_3

apply Corollary 4 to construct $\mathcal{E}_K$ from $\mathcal{E}_1$ is exactly the number of tails and lollipops in $\mathcal{E}_1$, which is less than or equal to M. So $K \leq M+1$, and via (1) above,

$$r_K \leq \exp^{(M)}(6M),$$

where we've used that $r_1 \leq M$ and $\sum_{i=1}^{K-1} \ell_i \leq M + 2N \leq 3M$.

There are r_K moves of type $\Omega_2^{\uparrow}$ following $D_1 = E_0$ in $\mathcal{E}_K$, so let D_3 be the diagram obtained by performing these moves on D_1. Because D_3 is obtained from $E_{M+2N} = D_2$ by at most M tails and lollipops, Theorem 5 holds. □

The following theorem allows us to construct an ordered sequence of Ω_2 and Ω_3 moves from the tails and lollipops arising in Theorem 5.

Theorem 6 *Suppose D_2 is obtained from D_1 by a sequence $\mathcal{T}$ of tails and lollipops of length M:*

$$\mathcal{T} : D_1 = T_0 \xrightarrow{a_1} T_1 \xrightarrow{a_2} \cdots \xrightarrow{a_M} T_M = D_2$$

where either $T_i \rightsquigarrow T_{i+1}$ or $T_i \circ\!\!\rightarrow T_{i+1}$. Then there exists a diagram D_3 obtained from D_2 by a sequence of $\Omega_2^{\uparrow}$ moves of length no more than $\frac{M}{2}(c(D_2) - c(D_1)) + 2M$, followed by a sequence of Ω_3 moves of length no more than M. Additionally D_1 is obtained from D_3 by a sequence of $\Omega_2^{\downarrow}$ moves of length at most $\frac{M+1}{2}(c(D_2) - c(D_1)) + 2M$.

Proof Consider a crossing χ of the diagram D_2 about which the circle part of any lollipop in $\mathcal{T}$ is centered. There may be multiple lollipops (suppose there are k) centered at χ, so consider a point p_k on the outermost one. Let q be a point in a small enough neighborhood of χ such that a straight line segment from q to χ does not intersect D_2 except at χ.

Consider a path $c\colon [0,1] \to \mathbb{R}^2$ such that $c(0) = p_k$ and $c(1) = q$. Choose c in such a way that its image C intersects each concentric lollipop at only one point. The point of intersection of C and the ith concentric lollipop is denoted p_i. Let $\delta_k = 0$ and let $\delta_{k-1} < \delta_{k-2} < \cdots < \delta_1 \in (0,1)$ such that $c(\delta_i) = p_i$.

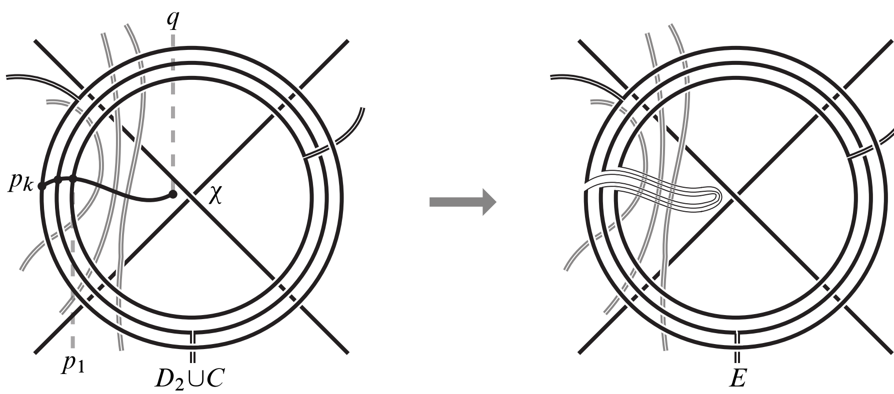


Figure 8: Adding concentric tails at the crossing χ

As in the proof of Proposition 3, we also choose c so that $C \cap D_2$ consists of no more than $2\left\lfloor \frac{c(D_2)-c(D_1)}{4} \right\rfloor$ points, excluding the points p_1 through p_k.

Add a tail along the path $c|_{[\delta_1,1]}$ to construct a diagram E_1 from D_2, where $c(E_1) - c(D_2) \leq c(D_2) - c(D_1)$. Perturb this tail slightly, so that it is closer to the crossing χ, and now add a second tail disjoint from the first tail along the path $c|_{[\delta_2,1]}$. This second tail introduces no more than $c(D_2) - c(D_1)$ crossings.

Repeating this process of perturbing and adding tails along $c|_{[\delta_i,1]}$ for all $i \in \{1,\ldots,k\}$, we produce a diagram E_k where $c(E_k) - c(D_2) \leq k(c(D_2) - c(D_1))$. We then build the diagram E by adding nested tails in the same way for every crossing of D_2 that is the center of some lollipop, so that $c(E) - c(D_2) \leq M(c(D_2) - c(D_1))$. The construction of E is depicted in Figure 8. The diagram E may be obtained from D_2 by a sequence of $\Omega_2^{\uparrow}$ moves of length at most $\frac{M}{2}(c(D_2) - c(D_1))$.

Now construct the diagram E' from E by performing the following at each crossing: if there are k concentric circles centered at a crossing χ, perform $2k$ type $\Omega_2^{\uparrow}$ moves,

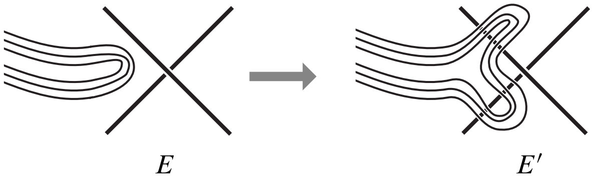


Figure 9: Perform $2k$ type $\Omega_2^{\uparrow}$ moves, so that each tail ‘forks’ over the crossing

forking the previously constructed tails over the edges of the crossing χ, as Figure 9 illustrates.

The diagram E' may be reached from D_2 via a sequence of $\Omega_2^{\uparrow}$ moves with length at most $\frac{M}{2}(c(D_2)-c(D_1))+2M$. Finally, construct the diagram D_3 by performing at most M moves of type Ω_3, as in Figure 10.

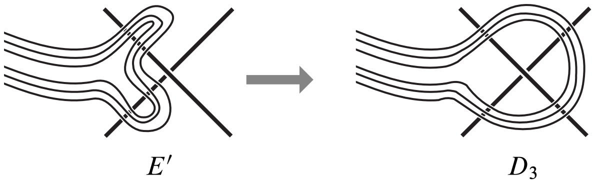


Figure 10: Performing Ω_3 moves to pass from E' to D_3

We may now pass from D_3 to D_1 by performing $\Omega_2^{\downarrow}$ moves as follows. Each tail and lollipop of $\mathcal{T}$ in D_2 is still present in D_3, with the circle parts of each lollipop modified. We remove them one at a time starting with the last tail or lollipop a_M in the sequence. If a_M is a lollipop, it now has the form depicted by Figure 11 in D_3, and may be removed by $\Omega_2^{\downarrow}$ moves. If a_M is a tail, it may likewise be removed by $\Omega_2^{\downarrow}$ moves. We continue to remove tails and lollipops in the reverse order they are added in $\mathcal{T}$ until we obtain D_1.

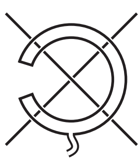


Figure 11

Because $c(D_3)=c(E')$, we know $c(D_3)-c(D_1)$ is exactly $c(E')-c(D_2)+c(D_2)-c(D_1)$, which is at most $M(c(D_2)-c(D_1))+4M+c(D_2)-c(D_1)$. Halving this gives us a bound on the number of $\Omega_2^{\downarrow}$ moves from D_3 to D_1. □

We consolidate previous results into Theorem 7, a special case of Theorem 1.

Theorem 7 *Let D_2 be a link diagram obtained from D_1 by a sequence of Ω_2 and Ω_3 moves of length M. Then there is a sequence of at most $\exp^{(2M)}(6M)$ Reidemeister moves from D_1 to D_2 ordered in the following way: first $\Omega_2^{\uparrow}$ moves, then Ω_3 moves and finally $\Omega_2^{\downarrow}$ moves.*

Proof Given D_1 and D_2, construct a diagram D_3 using Theorem 5, where D_3 is obtained from D_1 by no more than $\exp^{(M)}(6M)$ type $\Omega_2^{\uparrow}$ moves, and where D_3 is obtained from D_2 by no more than M tails and lollipops. Note that $c(D_3)-c(D_2) \leq 2\cdot\exp^{(M)}(6M)+2M$.

From D_2 and D_3, apply Theorem 6 to construct a diagram D_4 with the following properties: there is a sequence of $\Omega_2^{\uparrow}$ moves whose length is no more than $M\cdot\exp^{(M)}(6M)+M^2+2M$, followed by a sequence of Ω_3 moves of length no more than M from D_3 to D_4. There is also a sequence of $\Omega_2^{\downarrow}$ moves whose length is at most $(M+1)\cdot\exp^{(M)}(6M)+M^2+3M$ from D_4 to D_2.

Following the sequences of moves constructed from D_1 to D_3, then to D_4 and finally to D_2, we have a sequence of no more than $(2M+2)\cdot\exp^{(M)}(6M)+M(2M+6)$ Reidemeister moves ordered as desired. For $M\geq 1$, $\exp^{(2M)}(6M)\geq(2M+2)\cdot\exp^{(M)}(6M)+M(2M+6)$. □

Before considering the more general case of an arbitrary sequence of M Reidemeister moves, we need two lemmas relating to Ω_1 moves. These lemmas allow us to take a sequence of Reidemeister moves and build a new sequence in which the Ω_1 moves occur only at the beginning and end.

Lemma 8 *Let A, B and C be link diagrams such that*

$$A \xrightarrow{\Omega} B \xrightarrow{\Omega_1^{\uparrow}} C$$

where Ω is an arbitrary Ω_2 or Ω_3 move. Then there exists a diagram B' which may be obtained from A by a single $\Omega_1^{\uparrow}$ move, and where C is obtained from B' by no more than six Ω_2 or Ω_3 moves. Additionally, if instead $\Omega=\Omega_1^{\downarrow}$, there is a diagram B' such that

$$A \xrightarrow{\Omega_1^{\uparrow}} B' \xrightarrow{\Omega_1^{\downarrow}} C.$$

Lemma 9 *Let A, B and C be link diagrams such that $A\xrightarrow{\Omega_1^{\downarrow}}B\xrightarrow{\Omega}C$, where Ω is an Ω_2 or Ω_3 move. Then there exists a diagram B' such that B' is obtained from A by no more than six Ω_2 or Ω_3 moves and where C may be obtained from B' by a single $\Omega_1^{\downarrow}$ move.*

The proofs of Lemma 8 and Lemma 9 are left to be verified by the reader, and Corollary 10 is a rapid consequence of these lemmas:

Corollary 10 *Let D_2 be obtained from D_1 by an arbitrary sequence of M Reidemeister moves, α of which are $\Omega_1^{\uparrow}$ and β of which are $\Omega_1^{\downarrow}$. Then there exist diagrams D_1' and D_2' such that D_1' is obtained from D_1 by α type $\Omega_1^{\uparrow}$ moves and D_2 is obtained from D_2' by β type $\Omega_1^{\downarrow}$ moves. Additionally, D_2' is obtained from D_1' by no more than $6^M M$ Reidemeister moves of type Ω_2 and Ω_3.*

We conclude by proving Theorem 1.

Proof of Theorem 1 Begin with an arbitrary sequence of M Reidemeister moves from diagram D_1 to diagram D_2, α of which are $\Omega_1^{\uparrow}$ and β of which are $\Omega_1^{\downarrow}$. Construct D_1' and D_2' as in Corollary 10. Apply Theorem 7 to the sequence of Ω_2 and Ω_3 moves from D_1' to D_2' to obtain a sorted sequence of Reidemeister moves from D_1 to D_2 of length at most

$$\exp^{(2\cdot 6^M M)}(6\cdot 6^M M)+\alpha+\beta \le \exp^{(6^{M+1}M)}(6^{M+1}M). \qquad \square$$

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UCLA Mathematics Department
Box 951555, Los Angeles, CA 90095-1555, USA

jugold@ucla.edu

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