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Conservative subgroup separability for surfaces with boundary

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If F is a compact surface with boundary, then a finitely generated subgroup without peripheral elements of $G=\pi_1(F)$ can be separated from finitely many other elements of G by a finite index subgroup of G corresponding to a finite cover $\widetilde{F}$ with the same number of boundary components as F.

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Suppose F is a compact surface with nonempty boundary. A nontrivial element of $\pi_1(F)$ is *peripheral* if it is represented by a loop freely homotopic into ∂F. A covering space $p\colon \widetilde{F} \to F$ is called *conservative* if F and $\widetilde{F}$ have the same number of boundary components: $|\partial F| = |\partial \widetilde{F}|$.

Theorem 0.1 (Main Theorem) *Let F be a compact, connected surface with $\partial F \neq \phi$ and $H \subset \pi_1(F)$ a finitely generated subgroup. Assume that no element of H is peripheral. Given a (possibly empty) finite subset $B \subset \pi_1(F) \setminus H$, there exists a finite-sheeted cover $p\colon \widetilde{F} \to F$ such that:*

(i) *There is a compact, connected, π_1–injective subsurface $S \subset \widetilde{F}$ such that $p_*(\pi_1(S)) = H$.*

(ii) *$p_*(\pi_1(\widetilde{F}))$ contains no element of B.*

(iii) *$\widetilde{F} \setminus S$ is connected and $incl_*\colon H_1(S) \to H_1(\widetilde{F})$ is injective.*

(iv) *The covering is conservative.*

This theorem, for F orientable and without (iii), is due to Masters and Zhang [5] and is a key ingredient in their proof that cusped hyperbolic 3–manifolds contain quasi-Fuchsian surface groups [4; 5]. Without (iii) and (iv) the theorem is a special case of well-known theorems on subgroup separability of free groups (see Hall, Jr [1]) and surface groups (see Scott [6; 7]). For a discussion of subgroup separability and 3–manifolds, see Long and Reid [3].

The proof in [5] uses the folded graph techniques due to Stallings; see Kapovich and Myasnikov [2]. The shorter proof below uses cut and cross-join of surfaces. A cover is

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called *good* if properties (i)–(ii) hold and *very good* if (i)–(iii) hold. The idea is to start with a good cover and then pass to a second cover which is very good. Then *cross-join operations* (defined below) are used to reduce the number of boundary components of a very good cover until it is conservative.

1 Constructing a very good cover

We first explain a geometric condition on a cover of F which ensures it is good, and then use Theorem 1.3 to construct a very good cover.

Choose a basepoint x in the interior of F and suppose $p\colon F_H \to F$ is the cover corresponding to H. There is a compact, connected, incompressible subsurface S in the interior of F_H which is a retract of F_H and which contains a lift $\tilde{x}$ of x. Each element $g \in \pi_1(F,x)$ determines a unique lift $\tilde{x}(g) \in F_H$ of the basepoint x. The surface S can be chosen large enough to contain $\{\tilde{x}(b) : b \in B\}$. Then $p|_S\colon S \to F$ is a local homeomorphism.

If $\pi\colon F' \to F$ is any cover and there is a lift of $p|_S$ to $\theta\colon S \to F'$ (thus $\pi \circ \theta = p|_S$) which is injective, we say S *lifts to an embedding in the cover* F'. The work of M Hall [1] and P Scott [6] shows there is a finite cover $F' \to F$ such that S lifts to an embedding in F'.

Proposition 1.1 (Good cover) *Under the hypotheses of the main theorem, if $\pi\colon F' \to F$ is any cover and S lifts to an embedding in F', then the cover is good.*

Proof With the notation above, a based loop representing an element $b \in B$ lifts to a path in F' that starts at the basepoint $\tilde{x} \in L = \theta(S)$ but ends at some other point $\tilde{x}(b) \neq \tilde{x}$ in L. □

Addendum 1.2 (Very good cover) *There is a very good cover $\widetilde{F} \to F$ of finite degree with $|\partial \widetilde{F}|$ even.*

Proof We start with a good cover F' of F with finite degree and the subsurface $S \subset F'$ described above and then construct a cover of F' with the required properties. Let $p\colon \widetilde{F} \to F'$ be the regular cover given by the kernel of the map of $\pi_1(F')$ onto $H_1(F', S; \mathbb{Z}/2)$. There is a lift $\widetilde{S}$ of S to this cover by construction. The conclusions follow from Theorem 1.3 below. □

The following allows us to lift a π_1–injective subsurface to a regular cover where it is H_1–injective and nonseparating.

Theorem 1.3 *Suppose F is a compact, connected surface, possibly with boundary, which contains a compact, connected subsurface $S \neq F$. Assume that $S \cap \partial F$ is a (possibly empty) union of components of ∂F and no component of $\mathrm{cl}(F \setminus S)$ is a disc or a boundary parallel annulus. Let $p\colon \widetilde{F} \to F$ be the cover corresponding to the kernel of the natural homomorphism of $\pi_1(F)$ onto $G = H_1(F, S; \mathbb{Z}/2)$. If $\widetilde{S}_0$ is a connected component of $p^{-1}(S)$ then $X = \mathrm{cl}(\widetilde{F} \setminus \widetilde{S}_0)$ is connected and the map $i_*\colon H_1(\widetilde{S}_0) \to H_1(\widetilde{F})$ induced by inclusion is injective. Moreover $|\partial \widetilde{F}|$ is even.*

Proof The hypotheses imply $G \neq 0$. Let Y be a connected component of X. Then $\partial Y = (Y \cap \partial \widetilde{F}) \sqcup (Y \cap \widetilde{S}_0)$. We claim that $p(Y) \supset S$. Otherwise $p|_Y\colon Y \to \mathrm{cl}(F \setminus S)$ is a covering map which is injective since $p|(Y \cap \widetilde{S}_0)$ is injective. Thus Y is a lift of a component Z of $\mathrm{cl}(F \setminus S)$.

If $Z \cap S$ is connected, then since Z is not a disc or boundary parallel annulus, the image of $H_1(Z; \mathbb{Z}/2)$ in G is not trivial. Thus Z does not lift to the G–cover, a contradiction.

Hence $Z \cap S$ contains at least two distinct circle components B_1, B_2. There is a loop $\alpha = \beta \cdot \gamma \subset F$ which is the union of two arcs connecting B_1 and B_2: one arc $\beta \subset Z$ and one arc $\gamma \subset S$. Since α has mod 2 algebraic intersection number 1 with the boundary component B_1 of S it is a nonzero element of G. It follows that the lift $\widetilde{\beta} \subset Y$ of β has endpoints in different components of $p^{-1}(S)$, since otherwise α would lift to a loop. But $\partial \widetilde{\beta} \subset \partial Y \subset \partial \widetilde{S}_0$ which is a contradiction. Thus $p(Y) \supset S$.

It follows that Y contains some component $\widetilde{S}_1 \neq \widetilde{S}_0$ of $p^{-1}(S)$ in its interior. However the cover is regular so there is a covering transformation τ taking $\widetilde{S}_0$ to $\widetilde{S}_1$. Thus if $\widetilde{S}_0$ is not orientable then Y is not orientable and if $\widetilde{S}_0$ contains a component of $\partial \widetilde{F}$ then so does Y.

Choose some Riemannian metric on F. This metric pulls back to one on $\widetilde{F}$ which is preserved by covering transformations. If X is not connected, let Y be a component of X with smallest area.

As shown above, Y contains an $\widetilde{S}_1 \neq \widetilde{S}_0$ in its interior. The covering transformation τ taking $\widetilde{S}_0$ to $\widetilde{S}_1$ takes each component of $\widetilde{F} \setminus \widetilde{S}_0$ to a component of $\widetilde{F} \setminus \widetilde{S}_1$ with the same area. One of the components of $\widetilde{F} \setminus \widetilde{S}_1$ contains $\widetilde{S}_0$, so all the others must be strictly contained in Y, which contradicts that Y has minimal area. Hence $X = Y$ is connected.

To show the injectivity of i_*, note the long exact homology sequence of the pair $(\widetilde{F}, \widetilde{S}_0)$ yields

$$0 \longrightarrow H_2(\widetilde{F}) \xrightarrow{j_*} H_2(\widetilde{F}, \widetilde{S}_0) \xrightarrow{\delta} H_1(\widetilde{S}_0) \xrightarrow{i_*} H_1(\widetilde{F}),$$

so that we have the following equivalences: $\ker i_* = 0$ if and only if $\operatorname{Image}\delta = 0$ if and only if j_* is an isomorphism. By excision $H_2(\widetilde{F}, \widetilde{S}_0) \cong H_2(X, X \cap \widetilde{S}_0) \cong H_2(X, \partial X \cap \partial \widetilde{S}_0)$.

Suppose $\partial F \neq \phi$. Then $X \cap \partial \widetilde{F} \neq \phi$, since otherwise $\partial \widetilde{F} \subset \widetilde{S}_0$, but we have shown $\tau(\widetilde{S}_0) \subset X$, which is a contradiction. Now $X \cap \partial \widetilde{F} \neq \phi$ implies $H_2(X, \partial X \cap \partial \widetilde{S}_0) = 0$, so that $\operatorname{Image}\delta = 0$ hence $\ker i_* = 0$.

The remaining case is $\partial F = \phi$. Here $X \cap \widetilde{S}_0 = \partial \widetilde{S}_0 = \partial X$. If $\widetilde{F}$ is orientable, then so is X, and it follows that j_* is an isomorphism, hence $\ker i_* = 0$.

If $\widetilde{F}$ is nonorientable, we claim X must also be nonorientable; hence $H_2(X, \partial X) = 0$ so that $0 = \operatorname{Image}\delta = \ker i_*$.

Indeed, if X is orientable then $\widetilde{F}$ is orientable. This is because $\tau(\widetilde{S}_0) \subset X$ so $\tau(\widetilde{S}_0)$ orientable. This is a lift of S so S is orientable. Thus the homomorphism $\pi_1(F) \to \mathbb{Z}_2$ that sends a loop to 0 if and only if it is orientation preserving vanishes on $\pi_1(S)$ and so factors through G. It follows that every orientation reversing loop in F has nonzero image in G so $\widetilde{F}$ is orientable.

It remains to show $|\partial \widetilde{F}|$ is even. The action of G on $\widetilde{F}$ is free. Since $\mathbb{Z}_2^2$ does not act freely on S^1 it follows that if $|\partial \widetilde{F}|$ is odd then $G \cong \mathbb{Z}_2$ and is generated by some component C of ∂F. Let Z be the component of $\operatorname{cl}(F \setminus S)$ that contains C. By excision $\mathbb{Z}_2 \cong G \cong H_1(Z, Z \cap S)$. Since Z is not a disc or an annulus with C one of the boundary components the only other possibility is that $Z = \operatorname{cl}(F \setminus S)$ is a pair of pants with only one boundary component in S. But then $|\partial F|$ is even hence so is $|\partial \widetilde{F}|$. □

The following is easily deduced from the proof of Theorem 1.3 and will be used in the next two sections of the paper.

Remark 1.4 If F is a surface and $S \subset F$ is a subsurface and $X = \operatorname{cl}(F \setminus S)$ is connected and $X \cap \partial F \neq \phi$ then $i_*\colon H_1(S) \to H_1(F)$ is injective.

2 Cross-joining covers

Suppose F is a surface and α_1 and α_2 are disjoint arcs properly embedded in F. Let $N(\alpha_i) \equiv \alpha_i \times [-1,1]$ be disjoint regular neighborhoods of the arcs α_i in F such that $\alpha_i \equiv \alpha_i \times 0$ and $N(\alpha_i) \cap \partial F = (\partial \alpha_i) \times [-1,1]$. The sets $\alpha_i \times (0, \pm 1] \subset F$ are called the $\pm$ *sides* of α_i.

Given a homeomorphism $h\colon N(\alpha_1) \to N(\alpha_2)$ taking the $+$ side of α_1 to the $+$ side of α_2, the *cross-join* of F along (α_1, α_2) is the surface K defined as follows. The surface $F^- = F \setminus (\alpha_1 \cup \alpha_2)$ contains four subsurfaces $\alpha_i \times (0, \pm 1]$. Let F^{cut} be the surface obtained by completing these subsurfaces to $\alpha_i \times [0, \pm 1]$. Thus F^{cut} has two copies α_i^+, α_i^- of α_i in $\partial F^{\mathrm{cut}}$ and identifying these copies suitably produces F. The surface K is the quotient of F^{cut} obtained by using h to identify α_1^- to α_2^+ and α_1^+ to α_2^-. Note that here we do not require F to be connected, so that α and β might be in different components of F.

There are two special cases of cross-join which will be used to change the number of boundary components of a surface:

Lemma 2.1 *Suppose the compact surface F contains two disjoint properly embedded arcs α and β. In addition suppose that*

(1) *either F is connected and the endpoints of α, β lie on four distinct components of ∂F,*

(2) *or F is the union of two connected components A and B and $\alpha \subset A$ has both endpoints on the same boundary component and $\beta \subset B$ has endpoints on distinct boundary components.*

Then a surface K obtained by cross-joining along these arcs has $|\partial K| = |\partial F| - 2$. Furthermore $\chi(K) = \chi(F)$ and K is connected.

Proof We verify that K is connected. In the first case this follows since the arcs do not disconnect the boundary components on which they have endpoints; therefore $F \setminus (\alpha \cup \beta)$ is connected. In the second case it follows because $B \setminus \beta$ is connected, and every point in K is connected to a point in this subset by an arc. □

Suppose $p\colon \widetilde{F} \to F$ is a (possibly not connected) covering of surfaces and α is an arc properly embedded in F. Suppose $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are two distinct lifts of α to $\widetilde{F}$; then they are disjoint. The map p provides a homeomorphism between small regular neighborhoods of these two arcs. Using this to cross-join produces a surface $\widetilde{F}'$ and since the identifications are compatible with p there is a covering map $p'\colon \widetilde{F}' \to F$.

An important special case is when $\widetilde{F}$ is a $(d+1)$–fold cover which is the disjoint union of a 1–fold cover $F_1 \to F$ and some connected d–fold cover $F_d \to F$. Then cross-joining an arc in F_1 with one in F_d produces a connected cover of degree $d+1$.

To produce a new cover F' of F by a cross-join along two arcs in some cover $\widetilde{F}$ requires the arcs to be disjoint from each other. If S is embedded in $\widetilde{F}$ and these arcs are also disjoint from S, then S lifts to an embedding in F', so the cover F' is good. We call the combination of these two properties the *disjointness condition*.

There is a metric condition, involving some arbitrary choice of Riemannian metric on F, that ensures the disjointness condition is satisfied and therefore that the new cover is good. The next lemma provides a uniform upper bound on the lengths of the arcs we will use to cross-join *in any cover* of F.

Lemma 2.2 (Short arcs) *Suppose F is a compact, connected surface with a Riemannian metric such that the diameter of F is ℓ. If $\widetilde{F}$ is a finite connected cover of F then:*

(1) *If A and B are distinct components of ∂F then there is an arc α in F connecting them and length$(\alpha) \leq \ell$.*

(2) *If some component A of ∂F has (at least) two preimages in $\partial \widetilde{F}$ then there is an embedded arc α in F of length at most 2ℓ which lifts to an arc with endpoints on distinct preimages of A.*

Proof The first claim is obvious. For the second claim, since every point in $\widetilde{F}$ is within a distance at most ℓ of some point in $p^{-1}(A)$ and $\widetilde{F}$ is connected, some point in $\widetilde{F}$ is within a distance at most ℓ of points in two distinct components of $p^{-1}(A)$. This gives an arc β in $\widetilde{F}$ of length at most 2ℓ which connects two distinct components of $p^{-1}(A)$.

Let $\gamma\colon [0, 2R] \to \widetilde{F}$ be a shortest arc connecting two distinct components of $p^{-1}(A)$ and parametrized by arc length. Then $R \leq \ell$. To complete the proof we show that γ projects to an embedded arc in F. Observe that

$$d_{\widetilde{F}}(\gamma(t), p^{-1}(A)) = \min(t, 2R - t)$$

otherwise there is a shorter arc connecting two distinct components of $p^{-1}(A)$. It follows that

$$d_F(p(\gamma(t)), A) = \min(t, 2R - t)$$

This means that the distance in F of a point on $p \circ \gamma$ from A is given by arc length along $p \circ \gamma$. It follows that $\alpha = p \circ \gamma$ is the required embedded arc. □

An arc of length at most 2ℓ is called *short*. The next lemma provides a conservative cyclic cover with large diameter of a surface F. If a short arc in F connects two distinct boundary components, then so does every covering translate of it. If S lifts to the cover then there are many different translates of the short arc that are far from each other and far from the lift of S. In particular the disjointness condition is satisfied by suitable translates of a lifted short arc in this cover.

Lemma 2.3 (Big covers) *Suppose F is a compact connected surface with $k \geq 2$ boundary components and which contains a compact, connected, incompressible subsurface $S \subset \text{interior}(F)$ with $F \setminus S$ connected. Given $n > 0$ there is a conservative finite cyclic cover $\widetilde{F} \to F$ of degree bigger than n and a lift, $\widetilde{S}$, of S to $\widetilde{F}$. Furthermore $\widetilde{F} \setminus \widetilde{S}$ is connected and the map $i_*\colon H_1(\widetilde{S}) \to H_1(\widetilde{F})$ induced by inclusion is injective.*

Proof Let Y be the surface obtained from $F \setminus \text{interior}(S)$ by gluing a disc onto each component of ∂S. Then Y is a connected surface with k boundary components and there is a natural isomorphism of $H_1(F)/H_1(S)$ onto $H_1(Y)$. Choose a prime $p > \max(k, n)$. Because Y is connected, there is an epimorphism from $H_1(Y)$ onto $\mathbb{Z}/p$ which sends one component of ∂Y to $k-1$ and all the other $(k-1)$ components of ∂Y to -1. Now $(k-1)$ is coprime to p because $2 \leq k < p$. Therefore this defines a *conservative* cyclic p–fold cover $\widetilde{Y}$ of Y. It also determines a conservative cyclic p–fold cover $\widetilde{F}$ of F such that S lifts to $\widetilde{S}$. Since $\widetilde{Y}$ is connected, it follows that $X = \widetilde{F} \setminus \widetilde{S}$ is connected. Also $X \cap \partial\widetilde{F} = \partial\widetilde{F}$ is not empty. Hence i_* is injective by Remark 1.4. □

3 Proof of main theorem

In this section all covers are of finite degree. Given a cover $p\colon \widetilde{F} \to F$, the *excess* number of boundary components $E(p)$ for this cover is defined as $E(p) = |\partial\widetilde{F}| - |\partial F|$. By Addendum 1.2 there is a very good cover $p\colon \widetilde{F} \to F$ with $|\partial\widetilde{F}|$ even. If $E(p) = 0$ the theorem is proved; otherwise we construct another very good cover with smaller excess, and repeating this process a finite number of times yields a very good cover with zero excess.

These constructions use various very good covers of F. We will choose a lift of S to each cover and *identify* this lift with S and refer to the lift as S. This should not cause confusion.

We will also use the big cover Lemma 2.3 to replace a very good cover $\widetilde{F} \to F$ by another very good cover $F' \to F$ with the same excess and very large diameter. This process is called *taking a big cover*. We will rename F' as $\widetilde{F}$.

Given $\widetilde{F}$ very good, we will use (except in one case) one of the two cross-joins described in Lemma 2.1 to produce a new *connected* very good cover F' of F with smaller excess. We first take a big cover so that there are lifts of a short arc that are far apart and disjoint from S in $\widetilde{F}$. Then we change $\widetilde{F}$ with a cross-join to produce a good connected cover F', which has smaller excess by Lemma 2.1.

To verify F' is very good we check below that $F' \setminus S$ is *connected*, then by Remark 1.4 this implies $incl_* f\colon H_1(S) \to H_1(F')$ is injective, so F' is very good.

First observe that the cyclic cover produced by Lemma 2.3 leaves $\widetilde{F} \setminus S$ connected. Since the cross-join arcs are disjoint from S they also determine a cross-join of $\widetilde{F} \setminus S$. This cross-joined subsurface is $F' \setminus S$ which is connected by Lemma 2.1 as required.

Case when $|\partial F| = 1$ By Lemma 2.2 there is a properly embedded, short arc, α, in F which is covered by an arc β with endpoints on two distinct boundary circles of $\partial\widetilde{F}$. After taking a big cover we may assume the diameter of $\widetilde{F}$ is much larger than the length of α and the diameter of S; thus β can be chosen to be disjoint from S in $\widetilde{F}$.

Cross-join $(\widetilde{F}, \beta)$ with (F, α) to obtain a cover F' with one fewer boundary circle than $\widetilde{F}$. There is a lift of S to F' and $F' \setminus S$ is connected. Repeat the process until we obtain a cover with only one boundary component. This completes the proof when $|\partial F| = 1$.

Case when $|\partial F| \geq 2$ First we show how to make $E(p)$ even by performing a cross-join if needed. This first step will increase the number of boundary components.

Suppose $E(p)$ is odd. By Addendum 1.2 $|\partial\widetilde{F}|$ is even, so $|\partial F|$ is odd. We can make $E(p)$ even by cross-joining $(\widetilde{F}, \widetilde{\alpha})$ and (F, α) to obtain a cover $p'\colon F' \to F$. To perform the cross-join, choose a short embedded arc $\alpha \subset F$ with endpoints on two distinct circles C, C' of ∂F and a lift $\widetilde{\alpha} \subset \widetilde{F}$, with endpoints on two preimages $\widetilde{C}$, $\widetilde{C}'$. After taking a big cover we can assume that $\widetilde{\alpha}$ is disjoint from S in $\widetilde{F}$. Then F' is the cross-join of (F, α) and $(\widetilde{F}, \widetilde{\alpha})$. The surface S lifts to F' and $F' \setminus S$ is connected by Lemma 2.1.

Here is the outline of the rest of the proof. If $E(p) \neq 0$ then it is even. We proceed as follows using suitable cross-joins to construct new coverings. If there are two different components $C, C' \subset \partial F$ which both have more than one preimage in $\partial\widetilde{F}$ then we find a short arc α in F connecting C and C' and cross-join $\widetilde{F}$ to itself along two suitable lifts of α in $\widetilde{F}$. This reduces the excess by 2. After finitely many steps we obtain a cover so that at most one component $C \subset \partial F$ has more than one preimage. A single *cyclic cross-join* (defined below) is done simultaneously to reduce the excess to zero. Here are the details.

Suppose A and B are distinct circles in ∂F which both have (at least) two distinct preimages $\widetilde{A}_i$, $\widetilde{B}_i$ for $i = 1, 2$ in $\partial\widetilde{F}$. Choose a short arc γ in F with endpoints on A and B. Let α_i be a lift of γ with one endpoint on $\widetilde{A}_i$ and β_i a lift with an endpoint on $\widetilde{B}_i$. Inductively we assume that $\widetilde{F} \setminus S$ is connected. After taking a big cover we may assume that these arcs are all far apart and far from S. Thus there is a cover, obtained by cross-joining along any pair of distinct arcs chosen from this set of four, and S lifts to this cover.

We claim that there is a pair of these arcs which have endpoints on four distinct boundary components of $\widetilde{F}$. It follows from Lemma 2.1 that cross-joining along this pair reduces the excess by 2 and S lifts to the cover F' so produced. Furthermore, since $\widetilde{F} \setminus S$ is connected it follows that $F' \setminus S$ is connected by Lemma 2.1.

If α_1 and α_2 do not both have endpoints on the same lift $\widetilde{B}$ of B the pair (α_1, α_2) works. Similarly if β_1 and β_2 do not both have endpoints on the same lift $\widetilde{A}$ of A the pair (β_1, β_2) works. The remaining case is (after relabeling) α_1 and α_2 both have endpoints on a component $\widetilde{B} \neq \widetilde{B}_2$ which covers B and β_1, β_2 both have endpoints on some component $\widetilde{A} \neq \widetilde{A}_2$ which covers A. Then α_2 connects A_2 to $\widetilde{B} \neq \widetilde{B}_2$ and β_2 connects B_2 to $\widetilde{A} \neq \widetilde{A}_2$. Thus the pair (α_2, β_2) works.

Repeating this process a finite number of times reduces the excess by an even number until either $|\partial\widetilde{F}| = |\partial F|$ or else there is a unique component C of ∂F with more than one preimage. In the latter case the excess is even so there is an odd number of preimages $p^{-1}(C) = \{C_0, \ldots, C_{2k}\}$.

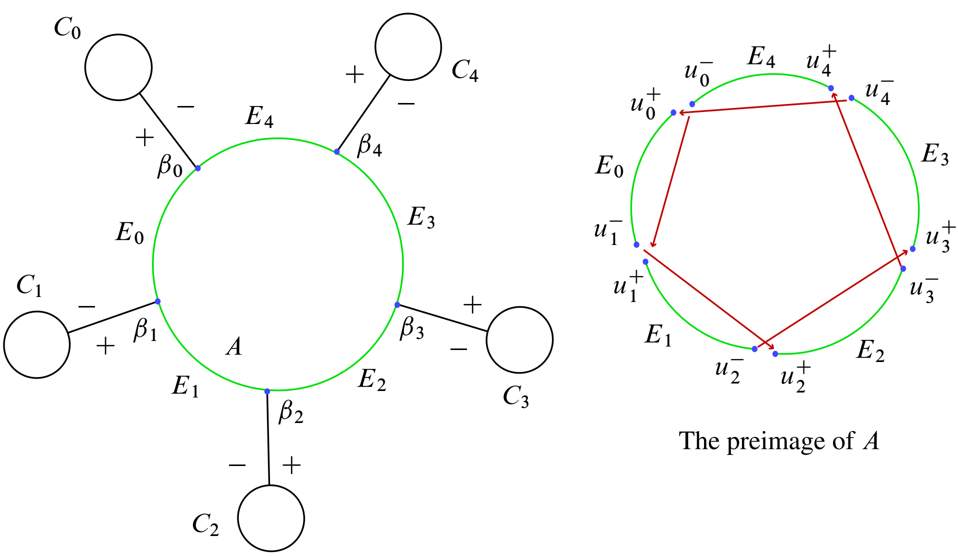

Figure 1: Cyclic cross-joining, $2k+1 = 5$ illustrated

Refer to Figure 1. Choose a component A of $\partial\widetilde{F}$ that does not cover C. This is possible because $|\partial F| \geq 2$. Let β be a short arc in F with endpoints on $p(A)$ and C. For each i there is a lift β_i of β with one endpoint on C_i and the other on A. After taking a big cover we may assume all these lifts are far apart and far from S. Orient each arc β_i so it points from A to C_i and call the left side $+$ and the right side $-$. Now cross-join cyclically as follows. Cut $\widetilde{F}$ along the union of these arcs and join the $-$ side of β_i to the $+$ side of β_{i+1}, with all integer subscripts taken mod $2k+1$.

The resulting cover has a single preimage of C. Indeed, each C_i has been cut at one point to give an interval $D_i = [t_i^+, t_i^-]$ where the label i denotes an endpoint of β_i and $t_i^{\pm}$ is on the $\pm$ side of β_i. These intervals are then glued by identifying t_i^- in D_i to t_{i+1}^+ in D_{i+1}. The result is obviously connected; it is a single circle.

To analyse the preimage of $p(A)$ the circle A was cut at $2k+1$ points to produce $2k+1$ subarcs $E_i = [u_i^+, u_{i+1}^-]$ where $u_i^{\pm}$ is on the $\pm$ side of β_i. Then E_i is glued to E_{i+2} by identifying u_{i+1}^- with u_{i+2}^+ (see figure 1). Since there are $2k+1$ intervals and the i–th one is glued to the $(i+2)$–th one the result is connected because 2 is coprime to $2k+1$. This gives the required conservative cover completing the proof of the main theorem. □

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