The intersecting kernels of Heegaard splittings

FENGCHUN LEI
JIE WU

Let $V \cup_S W$ be a Heegaard splitting for a closed orientable 3–manifold M. The inclusion-induced homomorphisms $\pi_1(S) \to \pi_1(V)$ and $\pi_1(S) \to \pi_1(W)$ are both surjective. The paper is principally concerned with the kernels $K = \mathrm{Ker}(\pi_1(S) \to \pi_1(V))$, $L = \mathrm{Ker}(\pi_1(S) \to \pi_1(W))$, their intersection $K \cap L$ and the quotient $(K \cap L)/[K, L]$. The module $(K \cap L)/[K, L]$ is of special interest because it is isomorphic to the second homotopy module $\pi_2(M)$. There are two main results.

(1) We present an exact sequence of $\mathbb{Z}(\pi_1(M))$–modules of the form

$$(K \cap L)/[K, L] \hookrightarrow R\{x_1, \dots, x_g\}/J \xrightarrow{T^\phi} R\{y_1, \dots, y_g\} \xrightarrow{\theta} R \xrightarrow{\epsilon} \mathbb{Z},$$

where $R = \mathbb{Z}(\pi_1(M))$, J is a cyclic R–submodule of $R\{x_1, \dots, x_g\}$, T^ϕ and θ are explicitly described morphisms of R–modules and T^ϕ involves Fox derivatives related to the gluing data of the Heegaard splitting $M = V \cup_S W$.

(2) Let $\mathcal{K}$ be the intersection kernel for a Heegaard splitting of a connected sum, and $\mathcal{K}_1$, $\mathcal{K}_2$ the intersection kernels of the two summands. We show that there is a surjection $\mathcal{K} \to \mathcal{K}_1 * \mathcal{K}_2$ onto the free product with kernel being normally generated by a single geometrically described element.

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1 Introduction

Let $M = V \cup_S W$ be a Heegaard splitting, where V and W are handlebodies of genus g and S is a closed Riemann surface of genus g with $\partial V = S$ and $\partial W = S$. Consider an essential simple closed curve λ in S. If λ bounds a disk D_1 in the manifold V and a disk D_2 in W, then by gluing D_1 and D_2 together along λ, we obtain an embedding of the 2–sphere S^2 in the 3–manifold M and so the splitting is reducible. Motivated from these observations, the purpose of this article is to study the intersecting subgroup of the kernels of $\pi_1(S) \to \pi_1(V)$ and $\pi_1(S) \to \pi_1(W)$. In other words, we investigate the (possibly singular) curves on the Riemann surface S that can be extended to (possibly singular) disks in V and W, respectively. Note that the intersecting subgroup addressed

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here is the kernel of the splitting homomorphism introduced by Stallings [19] and studied by others, for example, Jaco [10] and Papkyriakopoulos [17].

The combinatorial problem of determining the intersecting subgroups is related to the classical Whitehead Asphericity Question in low dimensional topology; see Bogley [4]. Recent development on combinatorial determinations on the general homotopy groups of spheres in homotopy theory also concerns the intersecting subgroups of free groups or braid groups; see Berrick, Cohen, Wong and Wu [1], Cohen and Wu [7; 8], Li and Wu [13] and Wu [20]. In our cases, the intersection is given by an explicit subgroup of the fundamental group of the Riemann surface of genus g with its image under an automorphism. The investigation on intersecting subgroups in our special cases might help for searching the methods for attacking the Whitehead Asphericity Question and the homotopy groups of spheres.

We first consider the algebraic determination on the intersecting kernel of Heegaard splittings. Let V_g be the standard handlebody of genus g with $\partial V_g = S_g$ the Riemann surface of genus g. Given a diffeomorphism $\varphi\colon S_g \to S_g$, the resulting construction $M = V_g \cup_\varphi V_g$ with equivalence relation generated by $x \sim \varphi(x)$ for $x \in S_g$ gives a Heegaard splitting. Clearly any Heegaard splitting of 3–manifolds is given in such a way. (See Section 2 for details.) Recall that $\pi_1(S_g)$ admits the standard presentation with generators $a_1, \dots, a_g, b_1, \dots, b_g$ and a single relation $[a_1, b_1] \cdots [a_g, b_g] = 1$. The fundamental group $\pi_1(V_g)$ is then the free group F_g^a of rank g with free basis $a_1, \dots, a_g$. Let $i\colon S_g \to V_g$ be the canonical inclusion. Then $i_*\colon \pi_1(S_g) \to \pi_1(V_g)$ is the group homomorphism with $i_*(a_j) = a_j$ and $i_*(b_j) = 1$ for $1 \le j \le g$. Let $\mathrm{KB}_g = \langle b_1, \dots, b_g \rangle^N$ be the normal closure of $b_1, \dots, b_g$ in $\pi_1(S_g)$. Then $\mathrm{Ker}(i_*\colon \pi_1(S_g) \to \pi_1(V_g)) = \mathrm{KB}_g$. Observe that the inclusion of S_g to the second copy of V_g is given by the composite

$$s_g \xrightarrow{\varphi} \cong S_g \xhookrightarrow{i} V_g.$$

Thus the intersecting kernel is given by

$$\mathrm{KB}_g \cap \varphi_*^{-1}(\mathrm{KB}_g)$$

and so the algebraic problem is how to determine the intersecting kernel

$$\mathrm{KB}_g \cap \phi^{-1}(\mathrm{KB}_g) = \langle b_1, \dots, b_g \rangle^N \cap \langle (\phi^{-1}(b_1), \dots, \phi^{-1}(b_g) \rangle^N$$

for any automorphism ϕ of $\pi = \pi_1(S_g)$. Since both KB_g and $\phi^{-1}(\mathrm{KB}_g)$ are normal subgroups of π, the commutator subgroup $[\mathrm{KB}_g, \phi^{-1}(\mathrm{KB}_g)]$ is contained in the subgroup $\mathrm{KB}_g \cap \phi^{-1}(\mathrm{KB}_g)$. Assume that the words $\phi^{-1}(b_1), \dots, \phi^{-1}(b_g)$ are given. Then a set of generators for the commutator subgroup $[\mathrm{KB}_g, \phi^{-1}(\mathrm{KB}_g)]$ can be listed and so the algebraic problem is reduced to how to determine the quotient group

$$(\mathrm{KB}_g \cap \phi^{-1}(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi^{-1}(\mathrm{KB}_g)] \tag{1-1}$$

which measures how far the intersecting subgroup is from the commutator subgroup. Observe that the above group is abelian because the commutator subgroup

$$[\mathrm{KB}_g \cap \phi^{-1}(\mathrm{KB}_g), \mathrm{KB}_g \cap \phi^{-1}(\mathrm{KB}_g)] \subseteq [\mathrm{KB}_g, \phi^{-1}(\mathrm{KB}_g)].$$

Our determination of the group given in Equation (1-1) is as follows. For a ring R, let

$$R\{x_1, \dots, x_n\} = \bigoplus_{j=1}^{n} Rx_j = R^{\oplus n}$$

be the direct sum, where Rx_j is a copy of R labeled by x_j. For a group G, let $\mathbb{Z}(G)$ be the group ring of G and let $\epsilon\colon \mathbb{Z}(G) \to \mathbb{Z}$ be the augmentation. Recall that a derivation $\partial\colon \mathbb{Z}(G) \to \mathbb{Z}(G)$ means a linear map such that $\partial(vw) = \partial(v)\epsilon(w) + v\partial(w)$. Let F_n be the free group of rank n with a basis $a_1, \dots, a_n$. Then there is a unique derivation $\partial_j = \partial/\partial a_j\colon \mathbb{Z}(F_n) \to \mathbb{Z}(F_n)$ such that $\partial_j(a_i) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker δ. For a homomorphism $\theta\colon \mathbb{Z}(F_n) \to \mathbb{Z}(G)$ and an element $w \in \mathbb{Z}(F_n)$, let $\partial_j^{\theta}(w) = \theta(\partial_j(w))$ be the image of $\partial_j(w)$ in the group ring $\mathbb{Z}(G)$. If the homomorphism θ is clear, we simply write $\partial_j(w)$ for $\partial_j^{\theta}(w)$ as an element in $\mathbb{Z}(G)$.

Theorem 1.1 *Let $M = V_g \cup_\phi V_g$ be a Heegaard splitting given by a diffeomorphism $\phi\colon S_g \to S_g$. Let*

$$\pi_1(S_g) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

be the standard presentation of $\pi_1(S_g)$ and let $q\colon \pi_1(S_g) \to \pi_1(V_g) = F_g^a$ be the canonical quotient homomorphism. Then the following hold:

(1) *The group $\pi_1(M)$ admits a presentation with generators $a_1, \dots, a_g$ and the relations given by $q(\phi_*(b_j)) = 1$ for $1 \le j \le g$.*

(2) *Let $R = \mathbb{Z}(\pi_1(M))$. Then there is an exact sequence of R–modules*

$$(\mathrm{KB}_g \cap \phi_*^{-1}(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi_*^{-1}(\mathrm{KB}_g)] \hookrightarrow R\{x_1, \dots, x_g\}/J \xrightarrow{T^\phi} R\{y_1, \dots, y_g\} \xrightarrow{\theta} R \xrightarrow{\epsilon} \mathbb{Z},$$

where J is the R–submodule of $R\{x_1, \dots, x_g\}$ generated by $\sum_{j=1}^{g}(a_j - 1)x_j$, T^ϕ is a morphism of R–modules with $T^\phi(x_i) = \sum_{j=1}^{g} \partial_j(\phi_*(b_i))y_j$, and θ is a morphism of R–modules with $\theta(y_i) = a_i - 1$.

By this result, the computation of the group $(\mathrm{KB}_g \cap \phi_*^{-1}(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi_*^{-1}(\mathrm{KB}_g)]$ depends on the presentation of the fundamental group $\pi_1(M)$ induced by the automorphism ϕ_* on $\pi_1(S_g)$. Consider the Jacobian of the automorphism ϕ_*:

$$\begin{pmatrix} \left(\partial(\phi_*(a_i))/\partial a_j\right)_{g\times g} & \left(\partial(\phi_*(b_i))/\partial a_j\right)_{g\times g} \\ \left(\partial(\phi_*(a_i))/\partial b_j\right)_{g\times g} & \left(\partial(\phi_*(b_i))/\partial b_j\right)_{g\times g} \end{pmatrix}_{2g\times 2g}$$

over $\mathbb{Z}(\pi_1(M))$. Then T^{ϕ} is determined by the $\mathbb{Z}(\pi_1(M))$–linear transformation

$$R\{x_1,\dots,x_g\} \longrightarrow R\{y_1,\dots,y_g\}$$

given by the matrix

$$\left(\frac{\partial(\phi_*(b_i))}{\partial a_j}\right)_{g\times g}.$$

A direct consequence of Theorem 1.1 is to give an algebraic criterion for testing the irreducibility of Heegaard splittings.

Corollary 1.2 (Irreducibility criterion of Heegaard splittings) *Let $M = V_g \cup_{\phi} V_g$ be a Heegaard splitting given by a diffeomorphism $\phi\colon S_g \to S_g$. Let*

$$\pi_1(S_g) = \langle a_1,\dots,a_g, b_1,\dots,b_g \mid [a_1,b_1]\cdots[a_g,b_g] = 1\rangle$$

be the standard presentation of $\pi_1(S_g)$. Let

$$T^{\phi}\colon R\{x_1,\dots,x_g\}/J \longrightarrow R\{y_1,\dots,y_g\}$$

be the linear transformation defined as above. Then M is irreducible if and only if T^{ϕ} is a monomorphism.

Proof By Brown and Loday [6, Corollary 3.4], we have

$$\pi_2(M) \cong (\mathrm{KB}_g \cap \phi_*^{-1}(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi_*^{-1}(\mathrm{KB}_g)].$$

According to Milnor [15, Theorem 2] together with the positive solution to Poincaré conjecture (see Morgan and Tian [16]), M is irreducible if and only if $\pi_2(M) = 0$. The assertion follows directly from Theorem 1.1. □

Our next result concerns the intersecting subgroups of the connected sums of the Heegaard splittings. For our convenience, we also use $(M; V, W; S)$ to denote a Heegaard splitting $V \cup_S W$ for M. For a Heegaard splitting $\mathcal{M} = (M; V, W; S)$, let $K(\mathcal{M}) = \mathrm{Ker}(i_*\colon \pi_1(S) \to \pi_1(V)) \cap \mathrm{Ker}(j_*\colon \pi_1(S) \to \pi_1(W))$, where $i\colon S \to V$ and $j\colon S \to W$ are the inclusions. Let $\mathcal{M}_i = (M_i; V_i, W_i; S_i)$ with $i = 1, 2$ be two Heegaard splittings. Then there is a natural way to define the connected sum

$\mathcal{M} = \mathcal{M}_1 \#_{S^2} \mathcal{M}_2 = (M; V, W; S)$ of the splittings $\mathcal{M}_1$ and $\mathcal{M}_2$ (See Section 2 for details). The simple close curve $C = S^2 \cap S$ determines an element $[C]$ in $\pi_1(S)$. Let $G_1 * G_2$ denote the free product of the groups G_1 and G_2. For a subset $A = \{g_\alpha\}$ of a group G, let $\langle A \rangle^N$ or $\langle g_\alpha \rangle^N$ denote the normal closure generated by the elements g_α in A.

Theorem 1.3 *Let $\mathcal{M}_1 = (M_1; V_1, W_1; S_1)$, $\mathcal{M}_2 = (M_2; V_2, W_2; S_2)$ be two Heegaard splittings, and $\mathcal{M} = \mathcal{M}_1 \#_{S^2} \mathcal{M}_2 = (M; V, W; S)$ the connected sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. Then there is a short exact sequence of groups*

$$\{1\} \longrightarrow \langle [C] \rangle^N \longrightarrow K(\mathcal{M}) \longrightarrow K(\mathcal{M}_1) * K(\mathcal{M}_2) \longrightarrow \{1\},$$

where C is the intersecting curve of the 2–sphere S^2 and the Heegaard surface S.

The article is organized as follows. In Section 2, we review some basic properties of Heegaard splittings. The proof of Theorem 1.1 is given in Section 3. In Section 4, we give the proof of Theorem 1.3.

2 Preliminaries

In this section we review some of the definitions and results which will be used in the paper, and fix some notation.

2.1 Fundamental facts on Heegaard splittings – brief review

Let S_g be a closed, connected, oriented surface, and let $\mathrm{Diff}^\pm S_g$ ($\mathrm{Diff}^+ S_g$, resp.) be the groups of diffeomorphisms (orientation-preserving diffeomorphisms, resp.) of S_g. The mapping class group Γ_g (extended mapping class group $\Gamma_g^\pm$, resp.) of S_g is the group $\mathrm{Diff}^+ S_g$ ($\mathrm{Diff}^\pm S_g$, resp.) modulo those diffeomorphisms which are isotopic to the identity.

Let H_g be a handlebody of genus g, and S_g the boundary of H_g with induced orientation. The handlebody subgroup $\mathcal{H}_g^\pm \subset \Gamma_g^\pm$ is the (nonnormal) subgroup of all mapping classes that have representatives that extend to diffeomorphisms of H_g.

Let $H'_g = \tau(H_g)$ be a diffeomorphic image of H_g with $\tau(x) = x$ for all $x \in S_g = \partial H_g$. For an element $\phi \in \mathrm{Diff}^\pm S_g$, ϕ defines a 3–manifold M in the following way: $M = H_g \cup_\phi H'_g = H_g \cup H'_g / x \sim \phi(x)$, for all $x \in S_g$, ie M is obtained by gluing H_g and H'_g together via a diffeomorphism ϕ: $\partial H_g \to \partial H'_g$. The surface $S_g = \partial H_g = \partial H'_g$ embedded in M is called a *Heegaard surface*, and $H_g \cup_{S_g} H'_g$ is called a *Heegaard* *splitting* for M. The Heegaard splitting is also denoted by $\mathcal{M} = (M; H_g, H_g'; S_g; \phi)$. Clearly, the topological type of M depends only on the mapping class Φ of ϕ, so we sometimes use $H_g \cup_\Phi H_g'$ to denote the Heegaard splitting.

It is a well-known fact that every closed, connected, orientable 3–manifold can be obtained from a Heegaard splitting (see for example Scharlemann [18] for a proof).

Heegaard splittings are not unique in general. Suppose M admits Heegaard splittings $H_g \cup_{\Phi_1} H_g'$ and $H_g \cup_{\Phi_2} H_g'$ with defining maps $\phi_1, \phi_2 \in \mathrm{Diff}^{\pm} S_g$. We say the two splittings are *equivalent* if the two splitting surfaces are isotopic, or equivalently, there exists a diffeomorphism $M \to M$ which takes H_g to H_g, H_g' to H_g', and so S_g to S_g.

Proposition 2.1 *The Heegaard splittings* $(M; H_g, H_g'; S_g; \phi)$ *and* $(M; V_g, V_g'; F_g; \varphi)$ *for* M *defined by* $\phi, \varphi \in \mathrm{Diff}^{\pm} S_g$ *are equivalent if and only if* φ *is in the double coset* $\mathcal{H}_g^{\pm} \phi \mathcal{H}_g^{\pm} \subset \mathrm{Diff}^{\pm} S_g$.

Proof It is essentially due to Birman [2]. Suppose there exists a diffeomorphism $h\colon M \to M$ with $h(H_g) = H_g$, $h(H_g') = H_g'$. Let $h_0 = h|H_g$, $h_0' = h|H_g'$, $h_1 = h_0|\partial H_g$, and $h_1' = h_0'|\partial H_g'$. In order for h to be well-defined on $\partial H_g = \partial H_g'$, we have the following commutative diagram:

$$\begin{tikzcd}
\partial H_g \arrow[r, "\phi"] \arrow[d, "h_1"'] & \partial H_g' \arrow[d, "h_1'"] \\
\partial H_g \arrow[r, "\varphi"] & \partial H_g'
\end{tikzcd}$$

Thus $\phi \circ h_1' = h_1 \circ \varphi$, so $\varphi = (h_1)^{-1} \circ \phi \circ h_1'$, where $(h_1)^{-1}, h_1' \in \mathcal{H}_g^{\pm}$, as required.

Conversely, if φ is in the double coset $\mathcal{H}_g^{\pm} \phi \mathcal{H}_g^{\pm}$, we can construct a diffeomorphism from M to M which takes H_g to H_g and H_g' to H_g'. □

In the category of oriented manifolds and orientation-preserving diffeomorphisms, we have an analogue of the correspondent description (see Birman [2; 3]).

2.2 Intersecting kernels of Heegaard splittings

From now on, when we do not need to stress the genus of a surface or a handlebody, we will omit the symbol g.

Definition 2.2 Let $\mathcal{M} = (M; H, H'; S; \phi)$ be a Heegaard splitting for a closed orientable 3–manifold M. Let $i\colon S \hookrightarrow H$ and $i'\colon S \hookrightarrow H'$ be the inclusions, and $i_*\colon \pi_1(S) \to \pi_1(H)$, $i'_*\colon \pi_1(S) \to \pi_1(H')$ the induced homomorphisms. Then $\operatorname{Ker} i_* \cap \operatorname{Ker} i'_* = \operatorname{Ker} i_* \cap \phi_*^{-1}(\operatorname{Ker} i_*)$ is called the *intersecting kernel* of the Heegaard splitting $\mathcal{M}$, and is denoted by $K(\mathcal{M})$.

Clearly $K(\mathcal{M})$ is a (normal) subgroup of $\pi_1(S)$. It is a well-known fact that every subgroup of $\pi_1(S)$ with finite index is an $Fuchsian$–group, and every subgroup of $\pi_1(S)$ with infinite index is free (refer to [14, Proposition 7.4]).

Example 2.3 Let $\mathcal{M} = (S^3; H_1, H'_1; T)$ be a genus 1 Heegaard splitting for S^3. Let a, b be two essential simple closed curves on the torus T such that a bounds a disk in V, b bounds a disk in W, and a and b intersect in a single point P, which we choose as a base point. Then $\{[a], [b]|\}$ is a basis for the free abelian group $\pi_1(T)$. Clearly,

$$\operatorname{Ker}(i_*\colon \pi_1(T) \to \pi_1(V)) = \{\, n[a] : n \in \mathbb{Z}\},$$
$$\operatorname{Ker}(j_*\colon \pi_1(T) \to \pi_1(W)) = \{\, n[b] : n \in \mathbb{Z}\}.$$

Thus $K(\mathcal{M}) = \{0\}$.

Similarly, for a genus 1 Heegaard splitting $\mathcal{M}_1$ for a lens space $L(p,q)$ and $\mathcal{M}_2$ for $S^2 \times S^1$, we have $K(\mathcal{M}_1) = \{0\}$ and $K(\mathcal{M}_2) \cong \mathbb{Z}$.

Let V be a handlebody of genus $n \geq 2$, $\partial V = S$, $i\colon S \hookrightarrow V$ the inclusion, and $i_*\colon \pi_1(S) \to \pi_1(V)$ the induced homomorphism. Let $\{a_i, b_i, 1 \leq i \leq n\}$ be a canonical system of oriented simple closed curves on S, that is, $\{b_i, 1 \leq i \leq n\}$ is a collection of pairwise disjoint curves which bound a collection of n pairwise essential disks in V, the manifold obtained by cutting V open along the disks is a 3–ball, and $\{a_i, 1 \leq i \leq n\}$ is a collection of pairwise disjoint curves with $a_i \cap b_j = \varnothing$ if $i \neq j$ and $a_i \cap b_j$ a single point if $i = j$, for each pair of i, j. Choose a base point P in $S - \{a_i, b_i, 1 \leq i \leq n\}$, and by ambiguity still use $[a_i], [b_i]$ to denote the path class of a_i, b_i in $\pi_1(S, P) = \pi_1(P)$, $1 \leq i \leq n$. Then $\operatorname{Ker} i_* = \langle [b_i], 1 \leq i \leq n\rangle^N$, the normal closure of $\{[b_i], 1 \leq i \leq n\}$ in $\pi_1(S)$, and the quotient group $\pi_1(S)/\operatorname{Ker} i_*$ is a free group of rank n with a basis $\{[a_i], 1 \leq i \leq n\}$.

The next proposition shows that for a Heegaard splitting $\mathcal{M}$ of genus ≥ 2, $K(\mathcal{M})$ is never trivial.

Proposition 2.4 *Let $V \cup_S W$ be a Heegaard splitting of genus ≥ 2 for M. Let $i\colon S \hookrightarrow V$, $j\colon S \hookrightarrow W$ be the inclusions and $i_*\colon \pi_1(S) \to \pi_1(V)$, $j_*\colon \pi_1(S) \to \pi_1(W)$*

the induced homomorphisms. Then for any $\alpha \in \operatorname{Ker} i_*$*,* $\beta \in \operatorname{Ker} j_*$*, we have* $[\alpha, \beta] \in K(V \cup_S W)$*, where* $[\alpha, \beta]$ *is the commutator of* α *and* β *in* $\pi_1(S)$*. In the other words,* $[\operatorname{Ker} i_*, \operatorname{Ker} j_*] \lhd K(V \cup_S W)$*.*

Proof By $i_*(\alpha) = 1$, $i_*([\alpha, \beta]) = i_*(\alpha\beta\alpha^{-1}\beta^{-1}) = i_*(\alpha)i_*(\beta)i_*(\alpha^{-1})i_*(\beta^{-1}) = i_*(\beta)i_*(\beta)^{-1} = 1$. Similarly, $j_*([\alpha, \beta]) = 1$. Therefore, $[\alpha, \beta] \in \operatorname{Ker} i_* \cap \operatorname{Ker} j_* = K(V \cup_S W)$. □

Proposition 2.5 *Suppose that two Heegaard splittings* $\mathcal{M}_1 = (M; H_g, H'_g; S_g; \phi)$ *and* $\mathcal{M}_2 = (M; V_g, V'_g; F_g; \varphi)$ *for* M *defined by* $\phi, \varphi \in \operatorname{Diff}^{\pm} S_g$ *are equivalent. Then there exists a* $f \in \mathcal{H}_g^{\pm}$ *such that* $f_*(K(\mathcal{M}_1)) = K(\mathcal{M}_2)$*. In particular, the intersecting kernel is an invariant of Heegaard splittings.*

Proof Use the notation as before. By assumption, there exist $h, h' \in \mathcal{H}_g^{\pm}$ such that $\varphi = h \circ \phi \circ h'$. By definition,

$$\begin{aligned} K(\mathcal{M}_1) &= \operatorname{Ker} i_* \cap \phi_*^{-1}(\operatorname{Ker} i_*), \\ K(\mathcal{M}_2) &= \operatorname{Ker} i_* \cap \varphi_*^{-1}(\operatorname{Ker} i_*). \end{aligned}$$

Thus

$$\begin{aligned} K(\mathcal{M}_2) &= \operatorname{Ker} i_* \cap \varphi_*^{-1}(\operatorname{Ker} i_*) \\ &= \operatorname{Ker} i_* \cap (h \circ \phi \circ h')_*^{-1}(\operatorname{Ker} i_*) \\ &= \operatorname{Ker} i_* \cap h'^{-1}_* \circ \phi_*^{-1} \circ h_*^{-1}(\operatorname{Ker} i_*). \end{aligned}$$

Note that $h, h' \in \mathcal{H}_g^{\pm}$, so $h_*(\operatorname{Ker} i_*) = \operatorname{Ker} i_*$, and $h'_*(\operatorname{Ker} i_*) = \operatorname{Ker} i_*$. Hence

$$\begin{aligned} K(\mathcal{M}_2) &= \operatorname{Ker} i_* \cap h'^{-1}_* \circ \phi_*^{-1} \circ h_*^{-1}(\operatorname{Ker} i_*) \\ &= \operatorname{Ker} i_* \cap h'^{-1}_* \circ \phi_*^{-1}(\operatorname{Ker} i_*) \\ &= h'^{-1}_*(h'_*(\operatorname{Ker} i_*) \cap \phi_*^{-1}(\operatorname{Ker} i_*)) \\ &= h'^{-1}_*(\operatorname{Ker} i_* \cap \phi_*^{-1}(\operatorname{Ker} i_*)) \end{aligned}$$

Set $f = h'^{-1}$. The conclusion follows. □

3 Algebraic determination on intersecting kernels and the proof of Theorem 1.1

Let $\pi = \pi_1(S_g)$. Recall π admits a presentation with generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and a single relation $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1$, where the commutator $[a, b] = aba^{-1}b^{-1}$. Let KB_g be the normal subgroup of π generated by $b_1, b_2, \ldots, b_g$

and let $\phi\colon \pi \to \pi$ be an automorphism. Then there is a commutative diagram of short exact sequences of groups

$$\begin{tikzcd}
q_\phi(\mathrm{KB}_g) \arrow[r, hook] & \pi/\phi(\mathrm{KB}_g) \arrow[r, two heads] & \widehat{\pi} \\
\mathrm{KB}_g \arrow[u, two heads] \arrow[r, hook] & \pi \arrow[u, two heads, "q_\phi"'] \arrow[r, two heads, "q"] \arrow[ur, phantom, "\text{push}"] & F(a_1, a_2, \dots, a_g) \arrow[u, two heads] \\
\mathrm{KB}_g \cap \phi(\mathrm{KB}_g) \arrow[u, hook] \arrow[r, hook] & \phi(\mathrm{KB}_g) \arrow[u, hook] \arrow[r, two heads] & q(\phi(\mathrm{KB}_g)), \arrow[u, hook]
\end{tikzcd} \tag{3-1}$$

where q and q_ϕ are quotient homomorphisms and the top-right square is a push-out diagram. Since KB_g and $\phi(\mathrm{KB}_g)$ are normal subgroups of π, the commutator subgroup $[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]$ is a normal subgroup of π with

$$[\mathrm{KB}_g, \phi(\mathrm{KB}_g)] \subseteq \mathrm{KB}_g \cap \phi(\mathrm{KB}_g).$$

Modulo the subgroup $[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]$, Diagram (3-1) induces the following commutative diagram

$$\begin{tikzcd}
q_\phi(\mathrm{KB}_g) \arrow[r, hook] & \pi/\phi(\mathrm{KB}_g) \arrow[r, two heads] & \widehat{\pi} \\
\dfrac{\mathrm{KB}_g}{[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]} \arrow[u, two heads] \arrow[r, hook] & \pi/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)] \arrow[u, two heads, "q_\phi"'] \arrow[r, two heads, "q"] \arrow[ur, phantom, "\text{push}"] & F(a_1, \dots, a_g) \arrow[u, two heads] \\
(\mathrm{KB}_g \cap \dfrac{\phi(\mathrm{KB}_g))}{[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]} \arrow[u, hook] \arrow[r, hook] & \dfrac{\phi(\mathrm{KB}_g)}{[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]} \arrow[u, hook] \arrow[r, two heads] & q(\phi(\mathrm{KB}_g)). \arrow[u, hook]
\end{tikzcd} \tag{3-2}$$

For any group G, let $G^{\mathrm{ab}} = H_1(G)$ denote the abelianization of G.

Proposition 3.1 *There is a short splitting exact sequence of groups*

$$(\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)] \hookrightarrow \big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}} \twoheadrightarrow q_\phi(\mathrm{KB}_g)^{\mathrm{ab}}.$$

Proof By applying Hopf Exact Sequence to the short exact sequence in the left column of Diagram (3-2), there is an exact sequence

$$\begin{aligned} &H_2(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]) \to H_2(q_\phi(\mathrm{KB}_g)) \to R \to \\ &H_1(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]) \to H_1(q_\phi(\mathrm{KB}_g)) \to 0, \end{aligned} \tag{3-3}$$

where R is the quotient group of $(\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]$ by the commutator subgroup

$$\big[\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)], (\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big].$$

Since $$[\mathrm{KB}_g, \mathrm{KB}_g \cap \phi(\mathrm{KB}_g))] \subseteq [\mathrm{KB}_g, \phi(\mathrm{KB}_g)],$$

the commutator subgroup $\big[\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)], (\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big]$ is trivial and so

$$R = (\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)].$$

From the commutative diagram of short exact sequences of groups

$$\begin{tikzcd}
\mathrm{KB}_g \arrow[r, hook] \arrow[d, "\cong"] & \pi \arrow[r, two heads] \arrow[d, "\phi"] & F(a_1, \dots, a_g) \arrow[d] \\
\phi(\mathrm{KB}_g) \arrow[r, hook] & \pi \arrow[r, two heads] & \pi/\phi(\mathrm{KB}_g),
\end{tikzcd}$$

the group $\pi/\phi(\mathrm{KB}_g)$ is isomorphic to $F(a_1, \dots, a_g)$ and so $\pi/\phi(\mathrm{KB}_g)$ is a free group. It follows that the subgroup $q_\phi(\mathrm{KB}_g)$ is a free group. Thus

$$H_2(q_\phi(\mathrm{KB}_g)) = 0$$

and so the exact sequence in Equation (3-3) induces a short exact sequence of abelian groups

$$(\mathrm{KB}_g \cap \phi(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)] \hookrightarrow H_1(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]) \twoheadrightarrow H_1(q_\phi(\mathrm{KB}_g)).$$

Since $q_\phi(\mathrm{KB}_g)$ is a free group, $H_1(q_\phi(\mathrm{KB}_g))$ is a free abelian group. Thus the above short exact sequence splits off and hence the result. □

Now we are going to determine $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ and $q_\phi(\mathrm{KB}_g)^{\mathrm{ab}}$. Let

$$N \hookrightarrow G \twoheadrightarrow G'$$

be a short exact sequence. Consider the (left) conjugation action of G on N given by $g \cdot x = gxg^{-1}$ for $g \in G$ and $x \in N$. Then N^{ab} is a (left) module over the group algebra $\mathbb{Z}(G)$. Observe that $g^{-1}xg \equiv x \mod [N, N]$ for $g, x \in N$. The $\mathbb{Z}(G)$–action on N^{ab} induces a $\mathbb{Z}(G')$–action on N^{ab}. From the short exact sequence $\mathrm{KB}_g \hookrightarrow \pi \twoheadrightarrow F(a_1, \dots, a_g)$, the abelian group $\mathrm{KB}_g^{\mathrm{ab}}$ is a (left) module over $\mathbb{Z}(\pi)$. In particular, $\mathrm{KB}_g^{\mathrm{ab}}$ is a (left) module over $\mathbb{Z}(\phi(\mathrm{KB}_g))$ because KB_g is a subgroup of π.

Proposition 3.2 *There is an isomorphism of abelian groups*

$$\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}} \cong \mathbb{Z} \otimes_{\mathbb{Z}(\phi(\mathrm{KB}_g))} \mathrm{KB}_g^{\mathrm{ab}}.$$

Proof Let $p\colon \mathrm{KB}_g \to \left(\mathrm{KB}_g / [\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\right)^{\mathrm{ab}}$ be the quotient map. Since

$$\left(\mathrm{KB}_g / [\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\right)^{\mathrm{ab}}$$

is abelian, the homomorphism p factors through the quotient group $\mathrm{KB}_g^{\mathrm{ab}}$. For $y \in \phi(\mathrm{KB}_g)$ and $x \in \mathrm{KB}_g$, the conjugation

$$yxy^{-1} \equiv x$$

in $\left(\mathrm{KB}_g / [\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\right)^{\mathrm{ab}}$ because $[y, x] = yxy^{-1}x^{-1} \in [\mathrm{KB}_g, \phi(\mathrm{KB}_g)]$. Thus the quotient homomorphism p factors through $\mathbb{Z} \otimes_{\mathbb{Z}(\phi(\mathrm{KB}_g)} \mathrm{KB}_g^{\mathrm{ab}}$. Similarly the quotient homomorphism

$$\mathrm{KB}_g \twoheadrightarrow \mathbb{Z} \otimes_{\mathbb{Z}(\phi(\mathrm{KB}_g))} \mathrm{KB}_g^{\mathrm{ab}}.$$

factors through the quotient $\left(\mathrm{KB}_g / [\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\right)^{\mathrm{ab}}$. The assertion follows. □

Let $F_{2g}^{a,b}$ be the free group of rank $2g$ generated by $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ and let F_g^a be the free group of rank g generated by $a_1, a_2, \dots, a_g$. Let

$$p\colon F_{2g}^{a,b} \longrightarrow F_g^a$$

be the group homomorphism such that $p(a_j) = a_j$ and $p(b_j) = 1$ for $1 \le j \le g$. Let $\widetilde{\mathrm{KB}}_g$ be the kernel of p. Clearly $\widetilde{\mathrm{KB}}_g$ is the normal closure of $b_1, \dots, b_g$ in $F_{2g}^{a,b}$.

Lemma 3.3 *For each g, $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}$ is a free module over $\mathbb{Z}(F_g^a)$ with a basis $\{b_1, b_2, \dots, b_g\}$.*

Proof From the short exact sequence,

$$\widetilde{\mathrm{KB}}_g \hookrightarrow F_{2g}^{a,b} \twoheadrightarrow F_g^a,$$

the free group $\widetilde{\mathrm{KB}}_g$ has a basis $\{w b_j w^{-1} \mid w \in F_g^a \ 1 \le j \le g\}$ and hence the result. □

Lemma 3.4 *Let J be the sub–$\mathbb{Z}(F_g^a)$–module of $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}$ generated by the element*

$$(a_1 - 1) \cdot b_1 + (a_2 - 1) \cdot b_2 + \cdots + (a_g - 1) \cdot b_g. \tag{3-4}$$

Then there is an isomorphism of $\mathbb{Z}(F_g^a)$–modules

$$\widetilde{\mathrm{KB}}_g^{\mathrm{ab}} / J \cong \mathrm{KB}_g^{\mathrm{ab}}.$$

Proof Consider the commutative diagram of short exact sequences of groups

$$\begin{tikzcd}
\widetilde{\mathrm{KB}}_g \arrow[r, hook] \arrow[d] & F_{2g}^{a,b} \arrow[r, two heads] \arrow[d] & F_g^a \arrow[d, equal] \\
\mathrm{KB}_g \arrow[r, hook] & \pi \arrow[r, two heads] & F_g^a .
\end{tikzcd}$$

The group KB_g is the quotient group of $\widetilde{\mathrm{KB}}_g$ by the normal closure generated by the element

$$C = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = (a_1 b_1 a_1^{-1}) b_1^{-1} (a_2 b_2 a_2^{-1}) b_2^{-1} \cdots (a_g b_g a_g^{-1}) b_g^{-1}.$$

Let C' be the image of C in $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}$. Then

$$\begin{aligned} C' &= (a_1 \cdot b_1 - b_1) + (a_2 \cdot b_2 - b_2) + \cdots + (a_g \cdot b_g - b_g) \\ &= (a_1 - 1) \cdot b_1 + (a_2 - 1) \cdot b_2 + \cdots + (a_g - 1) \cdot b_g \end{aligned}$$

in $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}$. Let $p\colon \widetilde{\mathrm{KB}}_g^{\mathrm{ab}} \to \mathrm{KB}_g^{\mathrm{ab}}$ be the quotient homomorphism. By the above commutative diagram of short exact sequences, p is a homomorphism of (right) $\mathbb{Z}(F_g^a)$–modules with

$$p(C') = 0.$$

It follows that the quotient homomorphism $p\colon \widetilde{\mathrm{KB}}_g^{\mathrm{ab}} \to \mathrm{KB}_g^{\mathrm{ab}}$ factors through $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}/J$. Let

$$\bar{p}\colon \widetilde{\mathrm{KB}}_g^{\mathrm{ab}}/J \twoheadrightarrow \mathrm{KB}_g^{\mathrm{ab}}$$

be the resulting homomorphism of (left) $\mathbb{Z}(F_g^a)$–modules.

Now consider the quotient homomorphism

$$p'\colon \widetilde{\mathrm{KB}}_g \twoheadrightarrow \widetilde{\mathrm{KB}}_g^{\mathrm{ab}} \twoheadrightarrow \widetilde{\mathrm{KB}}_g^{\mathrm{ab}}/J.$$

Since $p'(C) = 0$, the group homomorphism p' factors through the quotient group $\mathrm{KB}_g = \widetilde{\mathrm{KB}}_g/\langle C\rangle^N$. Moreover, since $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}/J$ is abelian, the resulting homomorphism $\mathrm{KB}_g \to \widetilde{\mathrm{KB}}_g^{\mathrm{ab}}/J$ factors through the quotient group $\mathrm{KB}_g^{\mathrm{ab}}$ which gives the inverse of $\bar{p}$. The proof is finished. □

Let $\hat{a}_j$ denote the image of a_j in $\hat{\pi}$ under the quotient homomorphism $F_g^a \to \hat{\pi}$. Since the conjugation action of the subgroup $\phi(\mathrm{KB}_g)$ of π on $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ is trivial, the conjugation action of π $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ induces an action of $\hat{\pi}$ and so $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ is a module over $\mathbb{Z}(\hat{\pi})$.

Proposition 3.5 *As a $\mathbb{Z}(\hat{\pi})$–module, $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ admits a presentation that is generated by letters $b_1, b_2, \dots, b_g$ with the single defining relations given by the equation*

$$\sum_{j=1}^{g} (\hat{a}_j - 1) \cdot b_j = 0.$$

Proof Consider the quotient homomorphism $F_{2g}^{a,b} \to \pi$. By Lemma 3.3,

$$\widetilde{\mathrm{KB}}_g^{\mathrm{ab}} = \mathbb{Z}(F_g^a) \otimes \mathbb{Z}\{b_1, b_2, \dots, b_g\}.$$

By Proposition 3.2,

$$\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}} \cong \mathbb{Z} \otimes_{\mathbb{Z}(\phi(\mathrm{KB}_g))} \mathrm{KB}_g^{\mathrm{ab}} = \mathbb{Z} \otimes_{\mathbb{Z}(q(\phi(\mathrm{KB}_g)))} \mathrm{KB}_g^{\mathrm{ab}}$$

as modules over $\mathbb{Z}(F_g^a)$. Together with Lemma 3.4, $\big(\mathrm{KB}_g/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\big)^{\mathrm{ab}}$ is the quotient left $\mathbb{Z}(F_g^a)$–module of $\widetilde{\mathrm{KB}}_g^{\mathrm{ab}}$ modulo the following relations:

1. The action of the subgroup $q(\phi(\mathrm{KB}_g))$ becomes trivial.
2. The element $C' = \sum_{j=1}^{g} (a_j - 1) \cdot b_j = 0$.

By taking the first type relations, we obtain

$$\begin{aligned}\mathbb{Z} \otimes_{\mathbb{Z}(q(\phi(\mathrm{KB}_g)))} \widetilde{\mathrm{KB}}_g^{\mathrm{ab}} &\cong \mathbb{Z} \otimes_{\mathbb{Z}(q(\phi(\mathrm{KB}_g)))} (\mathbb{Z}(F_g^a) \otimes \mathbb{Z}\{b_1, b_2, \dots, b_g\}) \\ &\cong \mathbb{Z}(\hat{\pi}) \otimes \mathbb{Z}\{b_1, b_2, \dots, b_g\}\end{aligned}$$

because $\hat{\pi} = F_g^a / q(\phi(\mathrm{KB}_g))$. The second type relation gives the equation in the statement and hence the result. □

The following lemma is a well known fact; see Brown [5, Proposition II 5.4]. For readers' convenience, we include a proof here.

Lemma 3.6 *Let $N \hookrightarrow F \twoheadrightarrow G$ be a short exact sequence of groups such that F is a free group. Then there is an exact sequence of modules over $\mathbb{Z}(G)$*

$$0 \longrightarrow N^{\mathrm{ab}} \longrightarrow \mathbb{Z}(G) \otimes F^{\mathrm{ab}} \longrightarrow \mathbb{Z}(G) \overset{\epsilon}{\twoheadrightarrow} \mathbb{Z},$$

where ϵ is the augmentation and $\mathbb{Z}(G)$–action on N^{ab} is induced by the conjugation action of G on N^{ab}.

Proof Let $IG=\mathrm{Ker}(\epsilon\colon \mathbb{Z}(G)\to\mathbb{Z})$ be the augmentation ideal. Since F is a free group, its classifying space $BF\simeq\Sigma X$ for a pointed set X (as a discrete topological space), where each nonbasepoint $x_\alpha\in X$ determines a loop in ΣX and $F=\pi_1(\Sigma X)$ has a basis $\{x_\alpha\mid x_\alpha \text{ nonbasepoint in } X\}$. Let $V=\mathbb{Z}\{x_\alpha\mid x_\alpha \text{ nonbasepoint in } X\}$. From the short exact sequence of groups

$$N\hookrightarrow F\twoheadrightarrow G,$$

there is a principal G–bundle

$$G\hookrightarrow BN\overset{p}{\twoheadrightarrow}\Sigma X.$$

Let $C_+X=\mathrm{IM}([0,1/2]\times X\to\Sigma X=S^1\wedge X)$ and $C_-X=\mathrm{IM}([1/2,1]\times X\to\Sigma X=S^1\wedge X)$. Then the restricted bundles

$$G\hookrightarrow p^{-1}(C_+X)\overset{p|}{\twoheadrightarrow}C_+X\quad\text{and}\quad G\hookrightarrow p^{-1}(C_-X)\overset{p|}{\twoheadrightarrow}C_-X$$

are trivial bundles because the cones C_+X and C_-X are contractible. It follows that

$$p^{-1}(C_+X)\cong G\times C_+X\quad\text{and}\quad p^{-1}(C_-X)\cong G\times C_-X$$

with $BN=p^{-1}(C_+X)\cup p^{-1}(C_-X)$ and $p^{-1}(C_+X)\cap p^{-1}(C_-X)=G\times X$. Thus there is a cofibre sequence of G–spaces

$$G\simeq G\times C_+X\hookrightarrow BN\twoheadrightarrow BN/p^{-1}(C_+X)\cong(G\times C_-X)/(G\times X)\cong G\rtimes\Sigma X.$$

By applying the homology to the above cofibre sequence, there is a short exact sequence of $\mathbb{Z}(G)$–modules

$$H_1(BN)=N^{\mathrm{ab}}\hookrightarrow H_1(G\rtimes\Sigma X)=\mathbb{Z}(G)\otimes V\overset{\partial}{\twoheadrightarrow}\bar{H}_0(G)\cong IG.\tag{3-5}$$

For each nonbasepoint $x_\alpha\in X$, the corresponding loop in $\Sigma X=\bigvee_\alpha S^1_\alpha$ lifts to a path $\tilde{\lambda}\colon[0,1]\to BN$ such that $\tilde{\lambda}(0)=*$. Then $\tilde{\lambda}(1)$ defines an element in G. Regard x_α as an element in $F=\pi_1(\Sigma X)$. By applying the singular chain complexes to the above cofibre sequence, $\partial(x_\alpha)=\hat{x}_\alpha-1$, where $\hat{x}_\alpha$ is the image of x_α in G under the quotient homomorphism $F\to G$.

To see that the $\mathbb{Z}(G)$–module structure in Equation (3-5) coincides with the conjugation action of $\mathbb{Z}(G)$ on N^{ab}, let K be the kernel of $IF\to IG$. Then K is the (left) ideal of $\mathbb{Z}(F)$ generated by IN. Then there is a commutative diagram

$$\begin{tikzcd}
K \otimes V \arrow[r, equal] \arrow[d, "\text{multi}"] & K \otimes V \arrow[d] & \\
K \arrow[r, hook] \arrow[d, two heads] & \mathbb{Z}(F) \otimes V \cong IF \arrow[r, two heads] \arrow[d, two heads] & IG \arrow[d, equal] \\
K/(K \cdot V) \arrow[r, hook] & \mathbb{Z}(G) \otimes V \arrow[r, two heads, "\text{multi}"] & IG,
\end{tikzcd}$$

where the rows are exact. Now the composite

$$IN \hookrightarrow K \twoheadrightarrow K/(K \cdot V) = K/(K \cdot IF)$$

is an epimorphism as K is the left ideal generated by IN. Moreover the above composite factors through IN/I^2N because $I^2N = IN \cdot IN \subseteq K \cdot IF$. Thus there is a commutative diagram

$$\begin{tikzcd}
IN \arrow[r, hook] \arrow[d, two heads] & K \arrow[d, two heads] \\
N^{\mathrm{ab}} \cong IN/I^2N \arrow[r, two heads] & K/(K \cdot V).
\end{tikzcd}$$

From Equation (3-5), the resulting homomorphism $N^{\mathrm{ab}} \to K/(K\cdot V)$ is an isomorphism. Now let $x \in N$ and $y \in F$. In the group algebra $\mathbb{Z}(F)$, write $x = 1 + \bar{x}$ and $y = 1 + \bar{y}$ with $\bar{x} \in IN$ and $\bar{y} \in IF$. Then, in IF,

$$\begin{aligned}
yxy^{-1} - 1 &= (y(1+\bar{x})y^{-1} - 1 \\
&= y\bar{x}(1 + \overline{y^{-1}}) \\
&= y \cdot \bar{x} + y\bar{x}\overline{y^{-1}}.
\end{aligned}$$

Since $$y\bar{x}\overline{y^{-1}} \in K \cdot IF,$$

we have $$\overline{yxy^{-1}} \equiv y \cdot \bar{x}$$

in $K/(K \cdot V) = K/(K \cdot IF) \cong N^{\mathrm{ab}}$. It follows that the conjugation action of $\mathbb{Z}(G)$ on N^{ab} coincides with $\mathbb{Z}(G)$–module structure on $N^{\mathrm{ab}} \cong K/(K \cdot V)$ and hence the result. □

Proof of Theorem 1.1 (1) Let $i\colon S_g \to V_g$ be the canonical inclusion. By Seifert–van Kampen Theorem, there is a push-out diagram of groups

$$\begin{tikzcd}
\pi_1(V_g) \arrow[r] & \pi_1(M) \\
\pi_1(S_g) \arrow[u, "i_*"] \arrow[r, two heads, "(i\circ\phi)_*"] & \pi_1(V_g) \arrow[u]
\end{tikzcd} \tag{3-6}$$

and hence assertion (1).

(2) Let $\pi = \pi_1(S_g)$. Observe that the automorphism $\phi_*\colon \pi_1(S_g) \to \pi_1(S_g)$ sends $\phi_*^{-1}(\mathrm{KB}_g)$ and KB_g to KB_g and $\phi_*(\mathrm{KB}_g)$, respectively. There is an isomorphism

$$(\mathrm{KB}_g \cap \phi_*^{-1}(\mathrm{KB}_g))[\mathrm{KB}_g, \phi_*^{-1}\,\mathrm{KB}_g] \xrightarrow[\cong]{\phi_*} (\mathrm{KB}_g \cap \phi_*(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi_*(\mathrm{KB}_g)].$$

Consider Diagram (3-1). By applying Lemma 3.6 to the short exact sequence

$$q(\phi_*(\mathrm{KB}_g)) \hookrightarrow F_g^a \twoheadrightarrow \hat{\pi} = \pi_1(M)$$

in the right column of Diagram (3-1), there is an exact sequence of $\mathbb{Z}(\pi_1(M))$–modules

$$0 \longrightarrow q(\phi_*(\mathrm{KB}_g))^{\mathrm{ab}} \longrightarrow \mathbb{Z}(\pi_1(M)) \otimes (F_g^a)^{\mathrm{ab}} \xrightarrow{\theta} \mathbb{Z}(\pi_1(M)) \xrightarrow{\epsilon} \mathbb{Z}.$$

Note that the group F_g^a is the free group with a basis $a_1, a_2, \dots, a_g$. We have

$$\mathbb{Z}(\pi_1(M)) \otimes (F_g^a)^{\mathrm{ab}} = \mathbb{Z}(\pi_1(M))\{y_1, \dots, y_g\},$$

where y_i is the image of a_i in $(F_g^a)^{\mathrm{ab}}$. By the proof of Lemma 3.6, $\theta(y_i) = a_i - 1$ as an element in $\mathbb{Z}(\pi_1(M))$ for $1 \le i \le g$.

By Proposition 3.1, there is a short exact sequence

$$(\mathrm{KB}_g \cap \phi_*(\mathrm{KB}_g))/[\mathrm{KB}_g, \phi_*(\mathrm{KB}_g)] \hookrightarrow \bigl(\phi_*(\mathrm{KB}_g)/[\mathrm{KB}_g, \phi_*(\mathrm{KB}_g)]\bigr)^{\mathrm{ab}} \xrightarrow{f'} q(\phi_*(\mathrm{KB}_g))^{\mathrm{ab}}.$$

According to Proposition 3.5,

$$\bigl(\phi_*(\mathrm{KB}_g)/[\mathrm{KB}_g, \phi(\mathrm{KB}_g)]\bigr)^{\mathrm{ab}} = \mathbb{Z}(\pi_1(M))\{x_1, \dots, x_g\}/J,$$

where x_i is the letter $\phi_*(b_i)$. By identifying the image $q_\phi(\mathrm{KB}_g)^{\mathrm{ab}}$ as a subgroup of $\mathbb{Z}(\pi_1(M)) \otimes (F_g^a)^{\mathrm{ab}} = \mathbb{Z}(\pi_1(M))\{y_1, \dots, y_g\}$, $f'(x_i)$ is the image of $q(\phi_*(b_i)) - 1 \in IF_g^a$ in its quotient group $\mathbb{Z}(\pi_1(M)) \otimes (F_g^a)^{\mathrm{ab}}$. From the fundamental theorem of free calculus,

$$q(\phi_*(b_i)) - 1 = \sum_{j=1}^{g} \frac{\partial q(\phi_*(b_i))}{\partial a_j}(a_j - 1) \in IF_g^a.$$

Thus f' is the same as the $\mathbb{Z}(\pi_1(M))$–morphism

$$T^{\phi}\colon \mathbb{Z}(\pi_1(M))\{x_1,\dots,x_g\}/J \longrightarrow \mathbb{Z}(\pi_1(M))\{y_1,\dots,y_g\}$$

with $T^{\phi}(x_i) = \sum_{j=1}^{g} \partial_j(\phi_*(b_i))y_j$. The proof is finished. □

4 Intersecting kernel of the connected sum of Heegaard splittings and the proof of Theorem 1.3

A Heegaard splitting $\mathcal{M} = (M; V, W; S)$ is *reducible* if there exist essential disks $D \subset V$ and $E \subset W$ with $\partial D = \partial E$.

It is a well-known result by Haken (see Jaco [11]) that any Heegaard splitting of a reducible 3–manifold is reducible.

Proposition 4.1 *A Heegaard splitting $\mathcal{M} = (M; V, W; S)$ is reducible if and only if there exists an essential simple closed curve C in S such that $[C] \in K(\mathcal{M})$.*

Proof One direction follows from the definition, the other direction follows from Dehn's Lemma (refer to Hempel [9] or Jaco [11]). □

Let $\mathcal{M}_1 = (M_1; V_1, W_1; S_1)$, $\mathcal{M}_2 = (M_2; V_2, W_2; S_2)$ be two Heegaard splittings with $M_1 = V_1 \cup_{S_1} W_1$, $M_2 = V_2 \cup_{S_2} W_2$. Define the connected sum $\mathcal{M}_1 \# \mathcal{M}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ in a natural way as follows: take a small 3–ball B_i in M_i such that $B_i \cap S_i$ is a properly embedded disk in B_i, $i = 1, 2$. Let $h\colon \partial B_1 \to \partial B_2$ be a homeomorphism which takes the disk $D_1 = (\partial B_1) \cap V_1$ to the disk $D_2 = (\partial B_2) \cap V_2$ and $E_1 = (\partial B_1) \cap W_1$ to $E_2 = (\partial B_2) \cap W_2$. Thus when we set $V_1' = \overline{V_1 - B_1}$, $W_1' = \overline{W_1 - B_1}$, $S_1' = V_1' \cap W_1'$, and $V_2' = \overline{V_2 - B_2}$, $W_2' = \overline{W_2 - B_2}$, $S_2' = V_2' \cap W_2'$, we have $V = V_1' \cup_{D_1 = D_2} V_2'$, $W = W_1' \cup_{E_1 = E_2} W_2'$, and $S = \overline{(S_1 - B_1 \cap S_1)} \cup \overline{(S_2 - B_2 \cap S_2)} = S_1 \# S_2$. Then we have a Heegaard splitting $(M_1 \# M_2; V, W; S)$, which is called the connected sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ and is denoted by $\mathcal{M}_1 \# \mathcal{M}_2$.

Set $C = \partial S_1' = \partial S_2'$. Then it is clear that $[C] \in K(\mathcal{M}_1 \# \mathcal{M}_2)$.

Remark The following construction shows that once there exists an essential separating simple closed curve $C \subset S$ with $[C] \in K(M; V, W; S)$, then there exist infinitely many such curves in S.

Construction [12] Use the notation as above. Choose a pair of parallel essential disks Δ_1, Δ_2 in V_1', such that $\Delta_i \cap S_1' = \partial \Delta_i \cap S_1' = d_i$ and $\Delta_i \cap D = \partial \Delta_i \cap D = d_i'$ are two arcs in $\partial \Delta_i$, and $\partial \Delta_i = d_i \cup d_i'$, $i = 1, 2$. Denote $\partial d_i = \partial d_i' = \{p_{i1}, p_{i2}\}$,

$i = 1, 2$. Similarly, choose a pair of parallel essential disks Σ_1, Σ_2 in W_2', such that $\Sigma_i \cap S_2' = \partial\Sigma_i \cap S_2' = e_i$ and $\Sigma_i \cap E = \partial\Sigma_i \cap E = e_i'$ are two arcs in $\partial\Sigma_i$, and $\partial\Sigma_i = e_i \cup e_i'$, $i = 1, 2$. Furthermore, assume $\partial e_i = \partial e_i' = \{p_{1i}, p_{2i}\}$, $i = 1, 2$; see Figure 1 below.

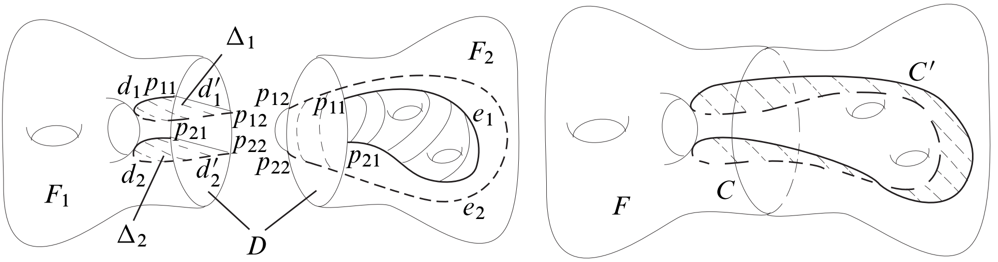

Figure 1: From C with $[C] \in K(K; V, W; S)$ to get C'

Let $C' = d_1 \cup e_1 \cup d_2 \cup e_2$, then C' is a simple closed curve on S. It is easy to see that C' is essential and separating on S. Since e_1 and e_2 are parallel on S_2', we can find a proper disk Σ' in V_2' such that $\partial\Sigma' = d_1' \cup e_1 \cup d_2' \cup e_2$. Thus C' bounds a disk $\Delta_1 \cup \Sigma' \cup \Delta_2$ in V. Similarly, C' also bounds a disk in W. So $[C'] \in K(\mathcal{M}_1 \# \mathcal{M}_2)$. Clearly, C' and C are not isotopic on F, and there are infinitely many such ways to construct such curves.

Next we consider how the intersecting kernel of the connected sum of two Heegaard splittings are related to those of its two factors. Suppose $\mathcal{M}_1 = (M_1; V_1, W_1; S_1)$, $\mathcal{M}_2 = (M_2; V_2, W_2; S_2)$, and $\mathcal{M}_1 \# \mathcal{M}_2 = (M; V, W; S)$, where $M = M_1 \# M_2$. Use the notation as before.

Let $i_1\colon S_1 \hookrightarrow V_1$, $j_1\colon S_1 \hookrightarrow W_1$ and $i_2\colon S_2 \hookrightarrow V_2$, $j_2\colon S_2 \hookrightarrow W_2$ be inclusions. Let $K = K(\mathcal{M})$, $K_1 = K(\mathcal{M}_1)$ and $K_2 = K(\mathcal{M}_2)$. By contracting the 2–sphere $F = \partial B_1 = \partial B_2$ in M to a point, we get a continuous map $f\colon M \to M_1 \vee M_2$ and $g = f|_S\colon S \to S_1 \vee S_2$. Clearly, $g_*\colon \pi_1(S) \to \pi_1(S_1 \vee S_2) = \pi_1(S_1) * \pi_1(S_2)$ is surjective. Now consider $\rho = g_*|_K\colon K \to \pi_1(S_1) * \pi_1(S_2)$.

Lemma 4.2 *$\rho = g_*|_K\colon K \to K_1 * K_2$ is surjective.*

Proof By contracting $V_2' \cup W_2'$ in M to a point, we get a continuous onto map $f_1\colon M \to V_1 \cup_{S_1} W_1$ which induces an epimorphism $f_{1*}\colon \pi_1(M) \to \pi_1(M_1)$. Let $g_1 = f_1|_S\colon S \to S_1$, and $g_{1*}\colon \pi_1(S) \to \pi_1(S_1)$. Let $i\colon S \hookrightarrow V$, $j\colon S \hookrightarrow W$, and $i_1\colon S_1 \hookrightarrow V_1$, $j_1\colon S_1 \hookrightarrow W_1$ be inclusions. Then $K = \operatorname{Ker} i_* \cap \operatorname{Ker} j_*$ and

$K_1 = \operatorname{Ker} i_{1*} \cap \operatorname{Ker} j_{1*}$. Set $g_{1*}' = g_{1*}|_K\colon K \to \pi_1(S_1)$. We have the commutative graph as follows:

$$\begin{tikzcd}
\pi_1(W) \arrow[d, "f_1|_{W*}"] & \pi_1(S) \arrow[l, "j_*"'] \arrow[r, "i_*"] \arrow[d, "g_{1*}"'] & \pi_1(V) \arrow[d, "f_1|_{V*}"] \\
\pi_1(W_1) & \pi_1(S_1) \arrow[l, "j_{1*}"'] \arrow[r, "i_{1*}"] & \pi_1(V_1)
\end{tikzcd}$$

For any $\alpha \in \operatorname{Ker} i_*$, we have $i_*(\alpha) = 1$, so $i_{1*}g_{1*}(\alpha) = i_* f|_{V*}(\alpha) = 1$, $g_*(\alpha) \in \operatorname{Ker} i_{1*}$, thus $g_{1*}(\operatorname{Ker} i_*) \subset \operatorname{Ker} i_{1*}$. Similarly, $g_{1*}(\operatorname{Ker} j_*) \subset \operatorname{Ker} j_{1*}$. Therefore $g_{1*}K = g_{1*}(\operatorname{Ker} i_* \cap \operatorname{Ker} j_*) = g_{1*}(\operatorname{Ker} i_*) \cap g_{1*}(\operatorname{Ker} j_*) \subset \operatorname{Ker} i_{1*} \cap \operatorname{Ker} j_{1*} = K_1$. Hence $g_{1*}|_K = g'\colon K \to K_1$ is well-defined.

We show that $g'\colon K \to K_1$ is surjective.

Observe that the inclusion $S_1' \hookrightarrow S_1$ induces an epimorphism $\pi_1(S_1') \to \pi_1(S_1)$. Note that the inclusions $V_1' \hookrightarrow V_1$ and $W_1' \hookrightarrow W_1$ are homotopy equivalence. Thus there is a commutative diagram

$$\begin{tikzcd}
\pi_1(V_1') \arrow[d, "\cong"] & \pi_1(S_1') \arrow[l] \arrow[r] \arrow[d] & \pi_1(W_1') \arrow[d, "\cong"] \\
\pi_1(V_1) & \pi_1(S_1) \arrow[l] \arrow[r] & \pi_1(W_1).
\end{tikzcd}$$

It follows that

$$\operatorname{Ker}(\pi_1(S_1') \to \pi_1(S_1)) \subseteq \operatorname{Ker}(\pi_1(S_1') \to \pi_1(V_1')) \cap \operatorname{Ker}(\pi_1(S_1') \to \pi_1(W_1')).$$

Let $K_1' = \operatorname{Ker}(\pi_1(S_1') \to \pi_1(V_1')) \cap \operatorname{Ker}(\pi_1(S_1') \to \pi_1(W_1'))$. Then, from the above commutative diagram, one can easily check that

$$K_1' \longrightarrow K_1$$

is an epimorphism. From the commutative diagram

$$\begin{tikzcd}
V_1' \arrow[d, hook] & S_1' \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & W_1' \arrow[d, hook] \\
V \arrow[d, hook, "f_1|_V"] & S \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook, "f_1|_S"] & W \arrow[d, hook, "f_1|_W"] \\
V_1 & S_1 \arrow[l, hook'] \arrow[r, hook] & W_1,
\end{tikzcd}$$

where the composites in the columns are inclusions, the epimorphism $K_1' \twoheadrightarrow K_1$ admits a decomposition $K_1' \to K \to K_1$. It follows that $g'\colon K \to K_1$ is an epimorphism.

Similarly, $g'' = g_{2*}|_K \colon K \to K_2$ is surjective, where $g_2 = f_2|_S \colon S \to S_2$, and $f_2 \colon M \to V_2 \cup_{S_2} W_2$ is a continuous onto map obtained by contracting $V_1' \cup W_1'$ in M to a point. The assertion follows. □

Proof of Theorem 1.3 By Lemma 4.2, $\rho\colon K \to K_1 * K_2$ is surjective. To show

$$\{1\} \longrightarrow \langle [C] \rangle^N \longrightarrow K(\mathcal{M}) \xrightarrow{\rho} K_1 * K_2 \longrightarrow \{1\}$$

is a short exact sequence, it suffices to show the kernel of ρ is the normal closure of $[C]$ in $\pi_1(S)$. Denote the normal closure of $[C]$ in $\pi_1(S)$ by $\langle [C] \rangle^N$. Then by the definition of $g\colon S \to S_1 \vee S_2$, $\operatorname{Ker} g_* = \langle [C] \rangle^N$, and $\pi_1(S)/\langle [C] \rangle^N \cong \pi_1(S_1) * \pi_1(S_2)$.

For all $\alpha \in \operatorname{Ker} \rho$, $\rho(\alpha) = 1 \in K_1 * K_2 \subset \pi_1(S_1) * \pi_1(S_2) \cong \pi_1(S)/\langle [C] \rangle^N$, $\alpha \in \langle [C] \rangle^N$, so $\operatorname{Ker} \rho \subset \langle [C] \rangle^N$.

On the other hand, for all $\beta \in \langle [C] \rangle^N$, β can be expressed as

$$\beta = y_1 [C]^{n_1} y_1^{-1} y_2 [C]^{n_2} y_2^{-1} \cdots y_m [C]^{n_m} y_m^{-1},$$

where $y_p \in \pi_1(S)$, $n_p \in \mathbb{Z}$, $1 \le p \le m$. Note that $[C] \in \operatorname{Ker} i_* \cap \operatorname{Ker} j_* = K$, so $i_* \beta = 1$ and $j_* \beta = 1$. Thus $\beta \in K$. Clearly, $\rho(\beta) = 1$, so $\beta \in \operatorname{Ker} \rho$, and $\langle [C] \rangle^N \subset \operatorname{Ker} \rho$. Hence $\operatorname{Ker} \rho = \langle [C] \rangle^N$.

This completes the proof of Theorem 1.3. □

Example 4.3 Let $\mathcal{M} = (M; V, W; S)$ be a Heegaard splitting of genus 2 for $M = S^3$, $L(p,q)$, or $L(p,q) \# L(r,s)$, and let C be a simple closed curve on S so that C cuts S into two once-punctured tori and C bounds disks in both V and W. As a direct consequence of Theorem 1.3, we have $K(\mathcal{M}) = \langle [C] \rangle^N$. If we choose another simple closed curve C' on S with the same property, we also have $K(\mathcal{M}) = \langle [C'] \rangle^N$.

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School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, China

Department of Mathematics, National University of Singapore
S17-06-02, 10 Lower Kent Ridge Road, Singapore 119076, Singapore

`fclei@dlut.edu.cn`, `matwuj@nus.edu.sg`

`http://www.math.nus.edu.sg/~matwujie`

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