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The length of a 3–cocycle of the 5–dihedral quandle

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We determine the length of the Mochizuki 3–cocycle of the 5–dihedral quandle. This induces that the 2–twist-spun figure-eight knot and the 2–twist-spun $(2,5)$–torus knot have the triple point number eight.

57Q45; 57Q35

Dedicated to Professor Taizo Kanenobu on the occasion of his 60*th* *birthday*

1 Introduction

The triple point number is one of the elementary invariants of a surface-knot analogous to the crossing number of a classical knot. It is defined to be the minimal number of triple points for all possible diagrams of the surface-knot. An S^2–knot has the triple point number zero if and only if it is of ribbon type (see Yajima [21]), and the author showed [15; 17] that there is no S^2–knot whose triple point number is equal to one, two or three. The author also showed [16] that some nonorientable surface-links have positive triple point numbers determined by using the knot group and normal Euler number.

In 2004, Shima and the author [18] gave a lower bound of the triple point number in terms of the cocycle invariant with respect to the 3–dihedral quandle, and proved that the 2–twist-spun trefoil knot has the triple point number four. We introduced the notion of the length of a cocycle of a quandle, and proved that the 3–twist-spun trefoil knot has the triple point number six [19]. Oshiro [14] used a symmetric quandle to determine the triple point numbers of some nonorientable surface-links.

This paper is motivated by the study of Hatakenaka [8]. She proves that the length of the Mochizuki 3–cocycle of 5–dihedral quandle [13] is greater than or equal to six. The aim of this paper is to prove the following.

Theorem 1.1 *The length of the Mochizuki* 3*–cocycle of the* 5*–dihedral quandle is equal to eight. As a consequence, the* 2*–twist-spun figure-eight knot and the* 2*–twist-spun* $(2,5)$*–torus knot have the triple point number eight.*

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This paper is organized as follows. In Section 2, we define the length of a 3-(co)cycle, and prove Theorem 1.1 by assuming the theorem on the length of the Mochizuki 3–cocycle with an additional structure (Theorem 2.3). In Section 3, we introduce graphs which visualize 3–cycles. In Section 4, the reverse and reflection of a 3–chain are defined. By using these notions, we can reduce the number of cases to consider. In fact, we divide the 3–cycles with length at most seven into eight cases I–VIII in Section 5. Sections 6, 7 and 8 are devoted to studying the cases I–IV, V and VI, and VII and VIII, respectively. In Section 9, we give a complete list of 3–cycles with length at most seven up to sign, reverse and reflection, and prove Theorem 2.3. In Section 10, we give an example of surface-link whose triple point number is equal to eight.

2 Preliminaries

A nonempty set X with a binary operation $(a,b)\mapsto a^b$ is called a *quandle* [2; 5; 10; 12] if it satisfies the following:

- $a^a=a$ for any $a\in X$.
- For any $a,b\in X$, there is a unique element $x\in X$ such that $x^a=b$.
- $(a^b)^c=(a^c)^{b^c}$ for any $a,b,c\in X$.

We use the notations $(a^b)^c=a^{bc}$, $((a^b)^c)^d=a^{bcd}$, and so on.

The *associated group* $G(X)$ of a quandle X is a group generated by the elements of X with the relations $x^y=y^{-1}xy$ for any $x,y\in X$. A set S is called an *X–set* [11] if $G(X)$ acts on S from the right. We denote the action by $(s,g)\mapsto s^g$ for $s\in S$ and $g\in G(X)$. It holds that

$$s^{gg'}=(s^g)^{g'} \quad \text{and} \quad s^e=s$$

for any elements $g,g'\in G(X)$ and the identity element $e\in G(X)$. For any X–sets S and S', the product $S\times S'$ is also an X–set naturally.

Let $C_n(X)_S$ be the free abelian group generated by the $(n+1)$–tuples of the set

$$U_n=\{(s;x_1,\dots,x_n)\mid s\in S \text{ and } x_i\in X \text{ with } x_i\neq x_{i+1}\ (1\le i\le n-1)\}.$$

Any nonzero element $\gamma\in C_n(X)_S$ has a unique *reduced* presentation $\gamma=\sum_{i=1}^{\ell}\gamma_i$ such that $\gamma_i\in\pm U_n$ $(i=1,2,\dots,\ell)$ and $\gamma_i\neq-\gamma_j$ for any $i\neq j$, where $\pm U_n=U_n\cup(-U_n)$. The number ℓ of terms is called the *length* of γ and denoted by $\ell=\ell(\gamma)$. Throughout this paper, we may assume that a presentation of γ is reduced.

The homology group $H_n(X)_S$ of a pair (X, S) is defined from the chain group $C_n(X)_S$ and the boundary operation $\partial_n\colon C_n(X)_S \to C_{n-1}(X)_S$ defined by

$$\partial_n(s; x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i (s; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ + \sum_{i=1}^{n} (-1)^{i+1} (s^{x_i}; x_1^{x_i}, \dots, x_{i-1}^{x_i}, x_{i+1}, \dots, x_n).$$

Here, if an $(n-1)$–term $\pm(t; y_1, \dots, y_{n-1})$ in the right hand side satisfies $y_i = y_{i+1}$ for some i $(1 \le i \le n-2)$, then we remove it from the sum; see [6; 7; 11]. The cohomology theory $H^n(X; A)_S$ with an abelian group A is developed from the cochain group $C^n(X; A)_S = \mathrm{Hom}(C_n(X)_S, A)$ in a standard manner. If S consists of a single element s with the trivial action $s^g = s$ for any $g \in G(X)$, we omit s in $(s; x_1, x_2, \dots, x_n)$ and S in the subscripts of the groups. We denote by $Z_n(X; A)_S$ and $Z^n(X; A)_S$ the n–cycle and n–cocycle groups, respectively.

Let $\langle\ ,\ \rangle\colon C_n(X) \times C^n(X; A) \to A$ be the Kronecker product, and $\varphi\colon C_n(X)_S \to C_n(X)$ the chain homomorphism defined by

$$\varphi(s; x_1, \dots, x_n) = (x_1, \dots, x_n).$$

For an n–cocycle $\theta \in Z^n(X; A)$, we put

$$\ell(\theta, S) = \min\{\ell(\gamma) \mid \gamma \in Z_n(X)_S \text{ with } \langle \varphi(\gamma), \theta \rangle \ne 0\}.$$

If the set in the right hand side is empty, then we put $\ell(\theta, S) = 0$.

Definition 2.1 The *length* of an n–cocycle $\theta \in Z^n(X; A)$ is defined by

$$\ell(\theta) = \max\{\ell(\theta, S) \mid S \text{ an } X\text{–set}\}.$$

If the maximum does not exist, then we put $\ell(\theta) = \infty$.

Let F be an oriented surface-knot, and D a diagram of F. An (X, S)*–coloring* (see [11]) is a usual X–coloring for D together with a shadow S–coloring for the complementary regions in $\mathbb{R}^3$. See Figure 1. In particular, the X–set $S = \mathbb{Z}$ with the action $s^x = s + 1$ for any $s \in \mathbb{Z}$ and $x \in X$ corresponds to an Alexander numbering [4], $S = \mathbb{Z}_2$ with $s^x = s + 1 \pmod 2$ corresponds to a checkerboard coloring, and $S = X$ corresponds to the original shadow coloring; see [3]. Let $\mathrm{Col}_{X,S}(D)$ denote the set of (X, S)–colorings for D. Every (X, S)–coloring defines a 3–cycle $\gamma_C \in Z_3(X)_S$. The *cocycle invariant* of F associated with a 3–cocycle $\theta \in Z^3(X; A)$ is given by

$$\Phi_\theta(F) = \big\{ \langle \varphi(\gamma_C), \theta \rangle \in A \mid C \in \mathrm{Col}_{X,S}(D) \big\}$$

as a multiset [11].

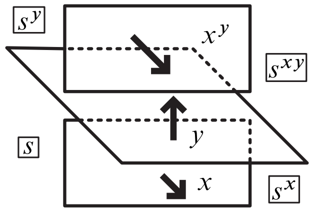


Figure 1

We denote by $t(D)$ the number of triple points of a diagram D, and $\mathrm{t}(F)$ the minimal number of $t(D)$ for all possible diagrams D of a surface-knot F, which is called the *triple point number* of F. The following is a generalization of the lower bound of $\mathrm{t}(F)$ for $S = \mathbb{Z}_2$ given in [19]. Since the proof is almost the same as the original one, we omit and leave it to the reader.

Theorem 2.2 *Let F be an oriented surface-knot, and θ a 3–cocycle in $Z^3(X;A)$. If the cocycle invariant $\Phi_\theta(F)$ of F associated with θ contains a nonzero element, then we have $\mathrm{t}(F) \geq \ell(\theta)$.* □

The 5*–dihedral quandle* $X = R_5$ is the set $\mathbb{Z}_5 = \{0, 1, \ldots, 4\}$ equipped with the binary operation $a^b \equiv 2b - a \pmod 5$. The map $\theta_{\mathrm{M}}\colon C_3(R_5) \to \mathbb{Z}_5$ defined by

$$\theta_{\mathrm{M}}(x, y, z) = (x - y)\frac{y^5 + (2z - y)^5 - 2z^5}{5}$$

is a 3–cocycle in $Z^3(R_5; \mathbb{Z}_5)$ and is called the *Mochizuki* 3*–cocycle* of R_5 [13]. Let $S = \mathbb{Z} \times R_5$ be the R_5–set whose action is given by $(n, w)^x = (n + 1, 2x - w)$ for $n \in \mathbb{Z}$ and $w, x \in R_5$. In Section 3 and after, we prove the following.

Theorem 2.3 $$\ell(\theta_{\mathrm{M}}, \mathbb{Z} \times R_5) \geq 8.$$

By using this theorem, we have Theorem 1.1 as follows.

Proof of Theorem 1.1 Let F be the 2–twist-spun trefoil knot or the 2–twist-spun $(2,5)$–torus knot. Since F is presented by a diagram with eight triple points [16; 22], we have $\mathrm{t}(F) \leq 8$. On the other hand, by the calculations in [1; 9], the cocycle invariant $\Phi_{\theta_{\mathrm{M}}}(F)$ is

$$\{\underbrace{0, \ldots, 0}_{5}, \underbrace{1, \ldots, 1}_{10}, \underbrace{4, \ldots, 4}_{10}\} \quad \text{or} \quad \{\underbrace{0, \ldots, 0}_{5}, \underbrace{2, \ldots, 2}_{10}, \underbrace{3, \ldots, 3}_{10}\},$$

respectively. Since the invariant contains a nonzero element, it follows by Theorems 2.2 and 2.3 that $\mathrm{t}(F) \geq \ell(\theta_{\mathrm{M}}) \geq \ell(\theta_{\mathrm{M}}, \mathbb{Z} \times R_5) \geq 8$. Therefore, we have $\mathrm{t}(F) = \ell(\theta_{\mathrm{M}}) = \ell(\theta_{\mathrm{M}}, \mathbb{Z} \times R_5) = 8$. □

3 Graphs of 3–cycles

To prove Theorem 2.3, we will construct the complete list of nonzero 3–cycles $\gamma \in Z_3(R_5)_{\mathbb{Z}\times R_5}$ with $\ell(\gamma) \le 7$ (Theorem 9.1). We put $C_k = C_k(R_5)_{\mathbb{Z}\times R_5}$ $(k=2,3)$ and $Z_3 = Z_3(R_5)_{\mathbb{Z}\times R_5}$.

Recall that the third chain group C_3 is generated by

$$U_3 = \{(n, w; x, y, z) \mid n \in \mathbb{Z} \text{ and } w, x, y, z \in R_5 \text{ with } x \neq y \neq z\},$$

and the second chain group C_2 is generated by

$$U_2 = \{(n, w; x, y) \mid n \in \mathbb{Z} \text{ and } w, x, y \in R_5 \text{ with } x \neq y\}.$$

An element in $\pm U_k$ is called a *k–term* $(k = 2, 3)$. For a 3–term $\gamma = \varepsilon(n, w; x, y, z)$ with $\varepsilon = \pm$, we call ε, n, w and (x, y, z) the *sign*, *degree*, *index* and *color* of γ, respectively. We use the same terminologies for a 2–term $\varepsilon(n, w; x, y)$, where the color is (x, y). The *type* of a 3–term $\varepsilon(n, w; x, y, z)$ is defined to be

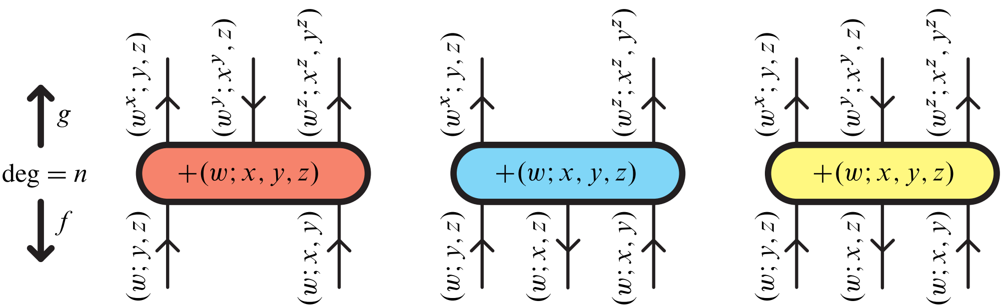
- type 1 if $x = z$,
- type 2 if $x^y = z$, and
- type 3 if $x \neq z$ and $x^y \neq z$.

We consider two kinds of homomorphisms $f, g\colon C_3 \to C_2$ such that a generator $\gamma = +(n, w; x, y, z)$ is mapped to

$$\begin{cases} f(\gamma) = -(n, w; y, z) \underline{+ (n, w; x, z)} - (n, w; x, y), \text{ and} \\ g(\gamma) = +(n+1, w^x; y, z) \underline{\underline{- (n+1, w^y; x^y, z)}} + (n+1, w^z; x^z, y^z), \end{cases}$$

where the underlined or doubly underlined 2–term is removed if γ is of type 1 or type 2, respectively. It follows by definition that the boundary map $\partial_3\colon C_3 \to C_2$ coincides with $f + g$. We remark that f does not change the degree, and g increases the degree by one. We describe the maps f and g schematically as shown in Figure 2, where the degrees are omitted in each term, and the orientations of edges are defined by the signs of 2–terms. In the figure, we color a 3–term of type 1, 2 or 3 red, blue or yellow, respectively.

As mentioned in Section 2, every 3–chain $\gamma \in C_3$ is presented by a reduced form $\gamma = \sum_{i=1}^{\ell} \gamma_i$. Let n_i be the degree of γ_i. The minimal and maximal numbers among $n_1, \dots, n_\ell$ are called the *minimal* and *maximal degree* of γ, and denoted by $\operatorname{mindeg}(\gamma)$ and $\operatorname{maxdeg}(\gamma)$, respectively. For an integer k, we denote by $T_k = T_k(\gamma)$ the set of 3–terms among $\gamma_1, \dots, \gamma_\ell$ of γ whose degrees are equal to k.



type 1 ($x = z$) type 2 ($x^y = z$) type 3 (otherwise)

Figure 2

Lemma 3.1 *For a* 3*–chain* $\gamma = \sum_{i=1}^{\ell} \gamma_i \in C_3$, *the following are equivalent:*

(i) γ *is a* 3*–cycle in* Z_3, *that is,* $\partial_3(\gamma) = 0$.

(ii) $\sum_{\gamma_i \in T_{k-1}} g(\gamma_i) + \sum_{\gamma_i \in T_k} f(\gamma_i) = 0$ *for any* $k \in \mathbb{Z}$.

In particular, if γ *is a* 3*–cycle in* Z_3, *then we have the following:*

(iii) $\sum_{\gamma_i \in T_k} f(\gamma_i) = 0$ *for* $k = \mathrm{mindeg}(\gamma)$.

(iv) $\sum_{\gamma_i \in T_k} g(\gamma_i) = 0$ *for* $k = \mathrm{maxdeg}(\gamma)$.

Proof This follows by definition immediately. □

To describe elements of R_5 in general, we use the following notation: for any different elements a_0 and a_1 of R_5, we put

$$a_2 = a_0 + 2s, \quad a_3 = a_0 + 3s \quad \text{and} \quad a_4 = a_0 + 4s,$$

where $s = a_0 - a_1 (\neq 0)$. Then it is easy to see that

- $R_5 = \{a_0, a_1, a_2, a_3, a_4\}$, and
- $a_i^{a_j} = a_{2j-i}$, where the subscripts are taken in $\mathbb{Z}_5$.

For example, it holds that $a_1^{a_0 a_4} = a_3^{a_2 a_0}$, since

$$a_1^{a_0 a_4} = (a_1^{a_0})^{a_4} = a_4^{a_4} = a_4 \quad \text{and} \quad a_3^{a_2 a_0} = (a_3^{a_2})^{a_0} = a_1^{a_0} = a_4,$$

and that $w^{a_0 a_1} = w^{a_4 a_0}$ for any $w \in R_5$, since

$$w^{a_0 a_1} = 2a_1 - (2a_0 - w) = w + 2s \quad \text{and} \quad w^{a_4 a_0} = 2a_0 - (2a_4 - w) = w + 2s.$$

Example 3.2 Let $\gamma=\sum_{i=1}^{6}\gamma_i\in C_3$ be a 3–chain with

$$\begin{aligned}
\gamma_1&=+(n,w;a_0,a_1,a_0), & \gamma_4&=+(n+1,w^{a_0};a_1,a_0,a_4),\\
\gamma_2&=-(n,w;a_4,a_0,a_1), & \gamma_5&=-(n+1,w^{a_1};a_3,a_2,a_0),\\
\gamma_3&=-(n,w;a_4,a_1,a_0), & \gamma_6&=-(n+1,w^{a_4};a_0,a_1,a_0).
\end{aligned}$$

Then it holds that $T_n(\gamma)=\{\gamma_1,\gamma_2,\gamma_3\}$ and $T_{n+1}=\{\gamma_4,\gamma_5,\gamma_6\}$. The 3–terms γ_1 and γ_6 are of type 1, γ_2 and γ_4 are of type 2, and γ_3 and γ_5 are of type 3. We see that γ is a 3–cycle in Z_3. The equation $\partial_3(\gamma)=0$ can be visualized by the graph as shown in Figure 3.

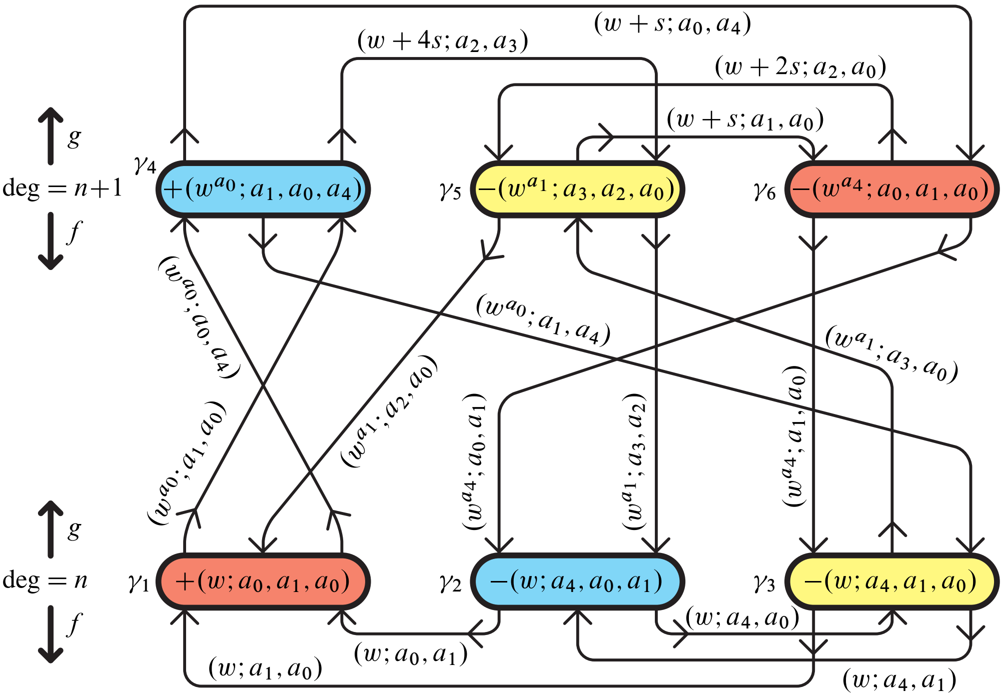


Figure 3

For example, since

$$\begin{cases} f(\gamma_1)=-(n,w;a_1,a_0)-(n,w;a_0,a_1), \text{ and}\\ g(\gamma_1)=+(n+1,w^{a_0};a_1,a_0)-(n+1,w^{a_1};a_2,a_0)+(n+1,w^{a_0};a_0,a_4),\end{cases}$$

the 3–term γ_1 is incident to two incoming edges of degree n and three (two outgoing and one incoming) edges of degree $n+1$.

The 3–terms γ_1, γ_2 and γ_3 are connected by four edges of degree n, or equivalently,

$$f(\gamma_1)+f(\gamma_2)+f(\gamma_3)=0,$$

which ensures Lemma 3.1(iii). Similarly, by observing the edges of degree $n+1$ and $n+2$, it holds that

$$g(\gamma_1)+g(\gamma_2)+g(\gamma_3)+f(\gamma_4)+f(\gamma_5)+f(\gamma_6)=0 \quad \text{and} \quad g(\gamma_4)+g(\gamma_5)+g(\gamma_6)=0.$$

Here, we use the equations $w^{a_0a_1}=w^{a_4a_0}=w+s$, $w^{a_0a_4}=w^{a_1a_0}=w+4s$, $w^{a_1a_3}=w^{a_4a_1}=w+2s$ and $w^{a_1a_2}=w^{a_4a_0}=w+s$ for the indices of edges of degree $n+2$.

4 Reverse and reflection

For a 3–term $\gamma=\varepsilon(n,w;x,y,z)$, we define the *reverse* of γ by

$$\overline{\gamma}=\varepsilon(-n,w^{xyz};x^{yz},y^z,z).$$

We extend it to the reverse of a 3–chain naturally. Similarly, the reverse of a 2–term $\delta=\varepsilon(n,w;x,y)$ is defined by

$$\overline{\delta}=\varepsilon(-n,w^{xy};x^y,y)$$

and extended to that of a 2–chain.

Let $\sigma\colon C_2\to C_2$ be an automorphism of C_2 defined by

$$\sigma(n,w;x,y)=(n+1,w;x,y),$$

which increases the degree of a 2–term by one. Then we have the following:

Lemma 4.1 *Let $\gamma\in C_3$ be a 3–chain.*

(i) $\overline{\overline{\gamma}}=\gamma$.

(ii) γ *is a 3–term of type* 1, 2 *or* 3 *if and only if* $\overline{\gamma}$ *is of type* 2, 1 *or* 3, *respectively.*

(iii) $f(\overline{\gamma})=\sigma(\overline{g(\gamma)})$, $g(\overline{\gamma})=\sigma(\overline{f(\gamma)})$, *and* $\partial_3(\overline{\gamma})=\sigma(\overline{\partial_3(\gamma)})$.

(iv) $\gamma\in Z_3$ *if and only if* $\overline{\gamma}\in Z_3$.

Proof The proof is straightforward. We remark that, since

$$a^{bb}=a,\quad a^{cbc}=a^{b^c}\quad\text{and}\quad a^{dbcd}=a^{b^dc^d}$$

hold for any $a,b,c,d\in R_5$, we have

$$\begin{aligned}(w^{xyz})^{x^{yz}y^zz}&=(w^{xy})^{zx^{yz}y^zz}=(w^{xy})^{(x^{yz})^z(y^z)^z}=(w^{xy})^{x^yy}\\&=(w^x)^{yx^yy}=(w^x)^{(x^y)^y}=(w^x)^x=w.\end{aligned}$$

□

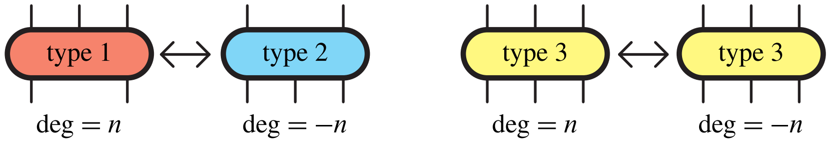


Figure 4

Roughly speaking, the reverse-operation changes the graph of a 3–cycle upside down with respect to the degree. See Figure 4.

For a 3–term $\gamma = \varepsilon(n, w; x, y, z)$, we define the *reflection* of γ by

$$\gamma^* = \varepsilon\big(n, (-1)^{n+1} w; (-1)^n(z-w), (-1)^n(y-w), (-1)^n(x-w)\big).$$

We extend it to the reflection of a 3–chain naturally. Similarly, the reflection of a 2–term $\delta = \varepsilon(n, w; x, y)$ is defined by

$$\delta^* = \varepsilon\big(n, (-1)^{n+1} w; (-1)^n(y-w), (-1)^n(x-w)\big)$$

and extended to that of a 2–chain.

Lemma 4.2 *Let $\gamma \in C_3$ be a 3–chain.*

(i) $\gamma^{**} = \gamma$.

(ii) γ *and* γ^* *are of the same type.*

(iii) $f(\gamma^*) = f(\gamma)^*$, $g(\gamma^*) = g(\gamma)^*$, *and* $\partial_3(\gamma^*) = \partial_3(\gamma)^*$.

(iv) $\gamma \in Z_3$ *if and only if* $\gamma^* \in Z_3$.

Proof The proof is straightforward. We remark that the equations

$$((-1)^{n+1} w)^{(-1)^n(x-w)} = 2(-1)^n(x-w) - (-1)^{n+1} w = (-1)^n w^x$$

and

$$(-1)^{n+1}(x^z - w^z) = (-1)^{n+1}((2z-x) - (2z-w)) = (-1)^n(x-w)$$

hold in R_5. □

Roughly speaking, the reflection-operation changes the color (x, y, z) of a 3–term into (z, y, x) by a slight modification.

Let $\gamma_1, \ldots, \gamma_k$ be 3–terms of the same degree such that $\sum_{i=1}^k f(\gamma_i) = 0$. We say that they are *f–splittable* if there is a nonempty proper subset I of $\{1, 2, \ldots, k\}$ such that

$$\sum_{i \in I} f(\gamma_i) = 0 \quad \text{and} \quad \sum_{i \notin I} f(\gamma_i) = 0.$$

If $\gamma_1,\dots,\gamma_k$ are not f–splittable, then they are called *f–connected*. The notions of *g–splittability* and *g–connectivity* are defined similarly. In Example 3.2, the 3–terms γ_1, γ_2 and γ_3 are f–connected, and γ_4, γ_5 and γ_6 are g–connected.

Lemma 4.3 *Let $\gamma_i=\varepsilon_i(n,w_i;x_i,y_i,z_i)$ $(i=1,\dots,k)$ be 3–terms.*

(i) *The following are equivalent:*
- $\gamma_1,\dots,\gamma_k$ *are f–connected.*
- $\overline{\gamma}_1,\dots,\overline{\gamma}_k$ *are g–connected.*
- $\gamma_1^*,\dots,\gamma_k^*$ *are f–connected.*

(ii) *If $\gamma_1,\dots,\gamma_k$ are f–connected, then $w_1=\cdots=w_k$.*

(iii) *If $\gamma_1,\dots,\gamma_k$ are g–connected, then $w_1^{x_1y_1z_1}=\cdots=w_k^{x_ky_kz_k}$.*

Proof The lemma follows by Lemmas 4.1 and 4.2 immediately. □

5 Degrees of 3–terms

The aim of this section is to study the degrees of 3–terms in a 3–cycle whose length is at most seven.

Lemma 5.1 *For any 3–term $\gamma=\varepsilon(n,w;x,y,z)$, it holds that $f(\gamma)\neq 0$.*

Proof The lemma follows from the definition of f. □

Lemma 5.2 *Let γ_1 and γ_2 be 3–terms of degree n with $\gamma_1\neq-\gamma_2$. If $\sum_{i=1}^2 f(\gamma_i)=0$, then their indices are the same, say w, and $\sum_{i=1}^2\gamma_i$ is equal to*

$$+(n,w;a,b,a)-(n,w;b,a,b)$$

for some $a\neq b\in R_5$.

Proof Since γ_1 and γ_2 are f–connected by Lemma 5.1, we have $w_1=w_2\,(=w)$ by Lemma 4.3(ii).

The sum of the signs of 2–terms in $f(\gamma_i)$ is equal to $-2\varepsilon_i$ if γ_i is of type 1 and $-\varepsilon_i$ if γ_i is of type 2 or 3. Therefore, we have

- $\varepsilon_1=-\varepsilon_2$ (we may assume that $\varepsilon_1=+1$ and $\varepsilon_2=-1$), and
- γ_1 and γ_2 are both of type 1, or both of type 2 or 3.

Case 1 Assume that γ_1 and γ_2 are both of type 1. We may take

$$\sum_{i=1}^{2} \gamma_i = +(n, w; a, b, a) - (n, w; x_2, y_2, x_2),$$

where $a \neq b \in R_5$. Then it holds that

$$\sum_{i=1}^{2} f(\gamma_i) = -(n, w; b, a) - (n, w; a, b) + (n, w; y_2, x_2) + (n, w; x_2, y_2) = 0.$$

Since $(x_2, y_2) \neq (a, b)$, we have $(x_2, y_2) = (b, a)$. See Figure 5.

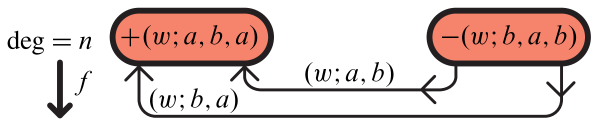


Figure 5

Case 2 Assume that γ_1 and γ_2 are both of type 2 or 3. We may take

$$\sum_{i=1}^{2} \gamma_i = +(n, w; a, b, c) - (n, w; x_2, y_2, z_2),$$

where $a, b, c \in R_5$ are mutually different. Then it holds that

$$\sum_{i=1}^{2} f(\gamma_i) = -(n, w; b, c) + (n, w; a, c) - (n, w; a, b) \\ + (n, w; y_2, z_2) - (n, w; x_2, z_2) + (n, w; x_2, y_2) = 0.$$

The 2–term $+(n, w; a, c)$ must be canceled with $-(n, w; x_2, z_2)$; that is, $(x_2, z_2) = (a, c)$. See Figure 6. Then $y_2 = b$, which contradicts the condition $\gamma_1 \neq -\gamma_2$. $\square$

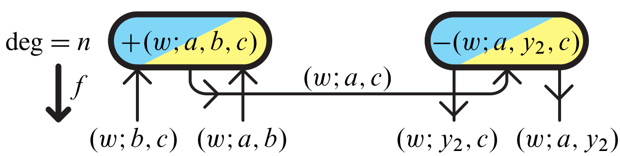


Figure 6

Lemma 5.3 *For the 3–chain $\sum_{i=1}^{2} \gamma_i$ in Lemma 5.2, we have $\sum_{i=1}^{2} g(\gamma_i) \neq 0$. Moreover, if k 3–terms $\gamma_3, \dots, \gamma_{k+2}$ of degree $n+1$ satisfy*

$$\sum_{i=1}^{2} g(\gamma_i) + \sum_{i=3}^{k+2} f(\gamma_i) = 0,$$

then we have $k \geq 4$.

Proof For $\sum_{i=1}^{2} \gamma_i = +(n, w; a, b, a) - (n, w; b, a, b)$, it holds that

$$\sum_{i=1}^{2} g(\gamma_i) = +(n+1, w^a; b, a) - (n+1, w^b; a^b, a) + (n+1, w^a; a, b^a) \\ -(n+1, w^b; a, b) + (n+1, w^a; b^a, b) - (n+1, w^b; b, a^b) \neq 0.$$

We remark that there is no canceling pair among the above six 2–terms. Since the three 2–terms of index w^a have all the positive sign, there are at least two 3–terms of index w^a among $\gamma_3, \dots, \gamma_{k+2}$. See Figure 7. Similarly, we see that there are at least two 3–terms of index w^b. Therefore, we have $k \geq 4$. □

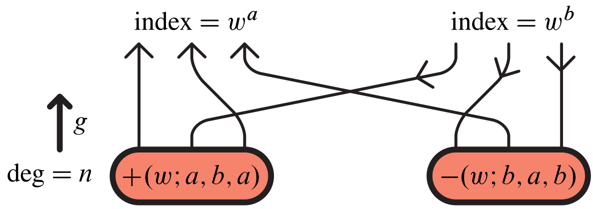


Figure 7

Lemma 5.4 *Let γ_1, γ_2 and γ_3 be 3–terms of degree n. If $\sum_{i=1}^{3} f(\gamma_i) = 0$, then their indices are the same, say w, and $\sum_{i=1}^{3} \gamma_i$ is equal to*

(i) $+(n, w; a, b, a) - (n, w; c, a, b) - (n, w; c, b, a)$, *or*

(ii) $+(n, w; a, b, a) - (n, w; b, a, c) - (n, w; a, b, c)$,

up to sign, where $a, b, c \in R_5$ are mutually different. Moreover, the reflection of case (i) *is coincident with* (ii) *under a suitable transformation of variables.*

Proof Since γ_1, γ_2 and γ_3 are f–connected by Lemma 5.1, we have $w_1 = w_2 = w_3\ (= w)$ by Lemma 4.3(ii). Let N_k^ε $(k = 1, 2, 3, \varepsilon = \pm)$ be the number of 3–terms among γ_1, γ_2 and γ_3 whose types are k and signs are ε. Put $N_{23}^\varepsilon = N_2^\varepsilon + N_3^\varepsilon$. Since the sum of the signs of 2–terms in $\sum_{i=1}^{3} f(\gamma_i)$ is equal to zero, it holds that

$$2(N_1^+ - N_1^-) + (N_{23}^+ - N_{23}^-) = 0 \quad \text{and} \quad \sum_{k,\varepsilon} N_k^\varepsilon = 3.$$

Therefore, we have $(N_1^+, N_1^-, N_{23}^+, N_{23}^-) = (1, 0, 0, 2)$ or $(0, 1, 2, 0)$.

We may assume that

$$\sum_{i=1}^{3} \gamma_i = +(n, w; a, b, a) - (n, w; x_2, y_2, z_2) - (n, w; x_3, y_3, z_3)$$

up to sign, where $a \neq b \in R_5$. Then we have

$$\begin{aligned}\sum_{i=1}^{3} f(\gamma_i) = &-(n,w;b,a)-(n,w;a,b)\\ &+(n,w;y_2,z_2)-(n,w;x_2,z_2)+(n,w;x_2,y_2)\\ &+(n,w;y_3,z_3)-(n,w;x_3,z_3)+(n,w;x_3,y_3)=0.\end{aligned}$$

By taking the first factors of the colors of the above eight 2–terms, it holds that

$$\{y_2,x_2,y_3,x_3\}=\{b,a,x_2,x_3\}, \quad \text{that is,} \quad \{y_2,y_3\}=\{a,b\}.$$

We may assume that $y_2=a$ and $y_3=b$. Then the above equation is

$$(b,a)+(a,b)+(x_2,z_2)+(x_3,z_3)=(a,z_2)+(x_2,a)+(b,z_3)+(x_3,b),$$

where we omit the degree n and index w for simplicity. It is not difficult to see that

(i) $z_2=b$, $z_3=a$ and $x_2=x_3$, or

(ii) $x_2=b$, $x_3=a$ and $z_2=z_3$.

See Figure 8.

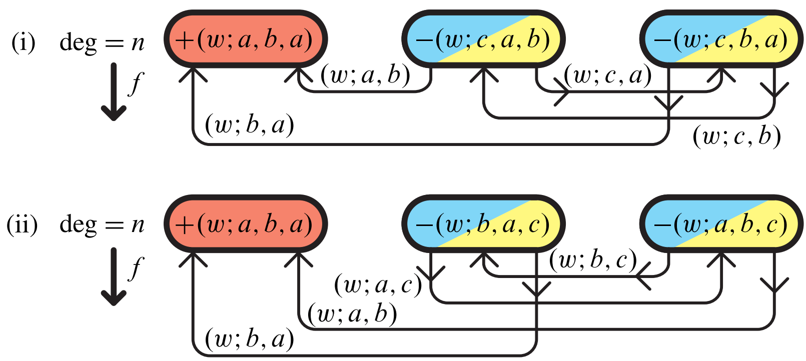


Figure 8

Moreover, the reflection of $\sum_{i=1}^{3}\gamma_i$ in (i) is

$$\begin{aligned}&+\big(n,(-1)^{n+1}w;(-1)^n(a-w),(-1)^n(b-w),(-1)^n(a-w)\big)\\ &-\big(n,(-1)^{n+1}w;(-1)^n(b-w),(-1)^n(a-w),(-1)^n(c-w)\big)\\ &-\big(n,(-1)^{n+1}w;(-1)^n(a-w),(-1)^n(b-w),(-1)^n(c-w)\big).\end{aligned}$$

By putting

$$(-1)^{n+1}w=w', \quad (-1)^n(a-w)=a', \quad (-1)^n(b-w)=b', \quad \text{and} \quad (-1)^n(c-w)=c',$$

we have case (ii): $+(n,w';a',b',a')-(n,w';b',a',c')-(n,w';a',b',c')$. □

We remark that at least one of the second and third 3–terms of γ in Lemma 5.4 is of type 3: in fact, it holds that $c^a \neq b$ or $c^b \neq a$ for any different $a, b, c \in R_5$.

Lemma 5.5 *For the 3–chain $\sum_{i=1}^{3} \gamma_i$ in Lemma 5.4, we have $\sum_{i=1}^{3} g(\gamma_i) \neq 0$. Moreover, if k 3–terms $\gamma_4, \dots, \gamma_{k+3}$ with degree $n+1$ satisfy*

$$\sum_{i=1}^{3} g(\gamma_i) + \sum_{i=4}^{k+3} f(\gamma_i) = 0,$$

then we have $k \geq 3$.

Proof By Lemma 4.2(iii), we may assume that $\sum_{i=1}^{3} \gamma_i$ satisfies (i) in Lemma 5.4. Then it holds that

$$\begin{aligned}\sum_{i=1}^{3} g(\gamma_i) = &+(n+1, w^a; b, a) - (n+1, w^b; a^b, a) + (n+1, w^a; a, b^a) \\ &- (n+1, w^c; a, b) \underline{+ (n+1, w^a; c^a, b)} - (n+1, w^b; c^b, a^b) \\ &- (n+1, w^c; b, a) \underline{+ (n+1, w^b; c^b, a)} - (n+1, w^a; c^a, b^a) \neq 0.\end{aligned}$$

Here, if γ_2 or γ_3 is of type 2, then the underlined 2–term is removed from the equation above. See Figure 9. Therefore, there is at least one 3-term of each index w^a, w^b and w^c among $\gamma_4, \dots, \gamma_{k+3}$. □

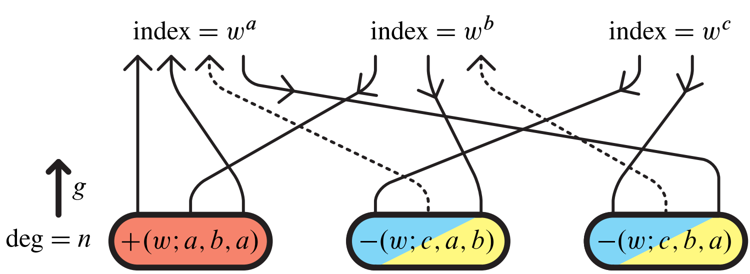


Figure 9

Proposition 5.6 *If $\gamma = \sum_{i=1}^{\ell} \gamma_i \in Z_3$ is a 3–cycle with $1 \leq \ell \leq 7$, then we have the following eight cases up to reverse, where $n = \mathrm{mindeg}(\gamma)$.*

(I) $\ell = 4$ *with* $|T_n| = 4$.

(II) $\ell = 5$ *with* $|T_n| = 5$.

(III) $\ell = 6$ *with* $|T_n| = 6$.

(IV) $\ell = 7$ *with* $|T_n| = 7$.

(V) $\ell = 6$ *with* $|T_n| = 2$ *and* $|T_{n+1}| = 4$.

(VI) $\ell = 7$ *with* $|T_n| = 2$ *and* $|T_{n+1}| = 5$.

(VII) $\ell = 6$ *with* $|T_n| = 3$ *and* $|T_{n+1}| = 3$.

(VIII) $\ell = 7$ *with* $|T_n| = 3$ *and* $|T_{n+1}| = 4$.

Proof Put $N = \operatorname{maxdeg}(\gamma)$. If $n = N$, then we have $|T_n| \geq 4$ by Lemmas 5.1, 5.3 and 5.5 to obtain the cases I, II, III and IV. If $n < N$, then we have

$$|T_n| \geq 2, \quad |T_N| \geq 2 \quad \text{and} \quad |T_n| + |T_N| \leq 7$$

by Lemmas 3.1(iii), (iv) and 5.1. By taking the reverse of γ if necessary, we may assume that $|T_n| \leq |T_N|$ by Lemma 4.1(iv). If $|T_n| = 2$, then we have $|T_{n+1}| \geq 4$ by Lemma 5.3 to obtain the cases V and VI with $N = n+1$. If $|T_n| = 3$, then we have $|T_{n+1}| \geq 3$ by Lemma 5.5 to obtain the cases VII and VIII with $N = n+1$. □

6 Cases I, II, III and IV

Throughout this section, we assume that $\gamma = \sum_{i=1}^{\ell} \gamma_i \in Z_3$ is a 3–cycle whose degrees are the same. We omit the degree n in presentation of 2– and 3–terms.

Lemma 6.1 *If $\gamma_1, \dots, \gamma_\ell$ are f–connected and g–connected, then we have $\ell \geq 8$.*

Proof Since $\gamma_1, \dots, \gamma_\ell$ are f–connected, their indices are the same by Lemma 4.3(ii), say w. Since $\gamma_1, \dots, \gamma_\ell$ are also g–connected, we have $w^{x_1 y_1 z_1} = \cdots = w^{x_\ell y_\ell z_\ell}$ by Lemma 4.3(iii); that is,

$$x_1 - y_1 + z_1 = \cdots = x_\ell - y_\ell + z_\ell.$$

Put $a_0 = x_i - y_i + z_i$. Then each term γ_i has a form $\pm(w; a_p, a_q, a_r)$ such that $a_p - a_q + a_r = a_0$. Since $p - q + r = 0$, each γ_i is one of the following 32 terms, where we omit the index w:

$$\begin{array}{llll}
\pm(a_1, a_0, a_4), & \pm(a_2, a_0, a_3), & \pm(a_3, a_0, a_2), & \pm(a_4, a_0, a_1), \\
\pm(a_1, a_2, a_1), & \pm(a_2, a_1, a_4), & \pm(a_3, a_1, a_3), & \pm(a_4, a_1, a_2), \\
\pm(a_1, a_3, a_2), & \pm(a_2, a_3, a_1), & \pm(a_3, a_2, a_4), & \pm(a_4, a_2, a_3), \\
\pm(a_1, a_4, a_3), & \pm(a_2, a_4, a_2), & \pm(a_3, a_4, a_1), & \pm(a_4, a_3, a_4).
\end{array}$$

We rewrite $\gamma = \sum \alpha_{pqr}(a_p, a_q, a_r)$ for $\alpha_{pqr} \in \mathbb{Z}$, where the sum is taken for $p \neq q \neq r$ and $p - q + r = 0$. The coefficient of the 2–term (a_1, a_0) in $f(\gamma)$ is equal to α_{104}: in fact, there is no 3–term other than (a_1, a_0, a_4) which satisfies $(a_1, a_0, *)$, $(a_1, *, a_0)$ or $(*, a_1, a_0)$ in the above table. Therefore, we have $\alpha_{104} = 0$ by $f(\gamma) = 0$. Similarly, it holds that $\alpha_{203} = \alpha_{302} = \alpha_{401} = 0$. In other words, γ has no 3–term of type 2.

Since the reverse $\overline{\gamma}$ has the same property as γ, $\overline{\gamma}$ has no 3–term of type 2 by the above argument. Therefore, γ has no 3–term of type 1.

Finally, we obtain a presentation

$$\begin{aligned}\gamma = {} & \alpha_{214}(a_2, a_1, a_4) + \alpha_{412}(a_4, a_1, a_2) + \alpha_{132}(a_1, a_3, a_2) + \alpha_{231}(a_2, a_3, a_1) \\ & + \alpha_{324}(a_3, a_2, a_4) + \alpha_{423}(a_4, a_2, a_3) + \alpha_{143}(a_1, a_4, a_3) + \alpha_{341}(a_3, a_4, a_1).\end{aligned}$$

It follows by $f(\gamma) = g(\gamma) = 0$ that

$$\alpha_{214} = -\alpha_{412} = -\alpha_{132} = \alpha_{231} = \alpha_{324} = -\alpha_{423} = -\alpha_{143} = \alpha_{341},$$

that is,

$$\begin{aligned}\gamma = k\big[& (a_2, a_1, a_4) - (a_4, a_1, a_2) - (a_1, a_3, a_2) + (a_2, a_3, a_1) \\ & + (a_3, a_2, a_4) - (a_4, a_2, a_3) - (a_1, a_4, a_3) + (a_3, a_4, a_1)\big]\end{aligned}$$

for some $k \neq 0 \in \mathbb{Z}$. Therefore, we have $\ell(\gamma) = 8|k| \geq 8$. □

Proposition 6.2 *There is no 3–cycle $\gamma = \sum_{i=1}^{4} \gamma_i \in Z_3$ in case* I.

Proof By Lemma 6.1, $\gamma_1, \dots, \gamma_4$ are f–splittable or g–splittable. By taking the reverse if necessary, we may assume that they are f–splittable. Then we can take

$$\gamma = +(w; a, b, a) - (w; b, a, b) + (v; p, q, p) - (v, q, p, q)$$

for $a \neq b$ and $p \neq q$ by Lemmas 5.1 and 5.2. It follows by $g(\gamma) = 0$ that $w^a = v^q$ and $w^b = v^p$. Furthermore, we have

$$\big\{(b, a), (a, b^a), (b^a, b)\big\} = \big\{(p^q, p), (p, q), (q, p^q)\big\}$$

and

$$\big\{(a^b, a), (a, b), (b, a^b)\big\} = \big\{(q, p), (p, q^p), (q^p, q)\big\}.$$

See Figure 10. In particular, we have

$$\{a, b, b^a\} = \{p, q, p^q\} \quad \text{and} \quad \{a, b, a^b\} = \{p, q, q^p\}$$

by observing the first factors. Since γ has no canceling pair of 3–terms, we have $(a, b) \neq (q, p)$. Then it is easy to see that

$$a = p, \quad b = q, \quad b^a = p^q \quad \text{and} \quad a^b = q^p.$$

Since $b^a \neq a^b$ holds in R_5, we have a contradiction. □

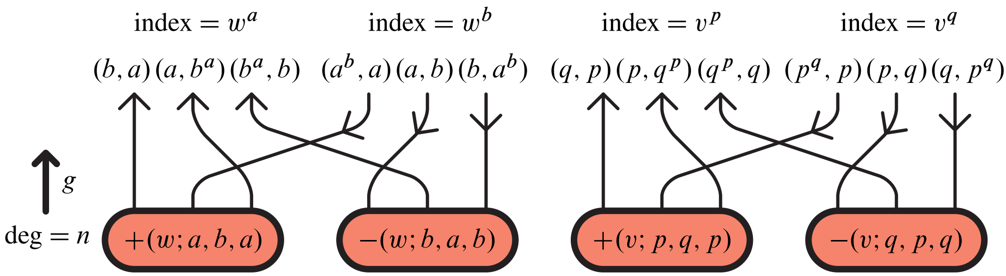


Figure 10

Proposition 6.3 *There is no 3–cycle $\gamma = \sum_{i=1}^{5} \gamma_i \in Z_3$ in case* II.

Proof By Lemma 6.1, it is sufficient to consider the case that $\gamma_1, \ldots, \gamma_5$ are f–splittable. We may assume that

$$\gamma = +(w;a,b,a) - (w;b,a,b) + (v;p,q,p) - (v;r,p,q) - (v;r,q,p)$$

up to sign and reflection for some $a \neq b$ and mutually different p,q,r by Lemmas 5.1, 5.2 and 5.4. See Figure 11. Therefore, the number of positive 2–terms in $g(\gamma)$ is at most seven, and that of negative 2–terms is equal to eight. This contradicts the assumption $g(\gamma) = 0$. □

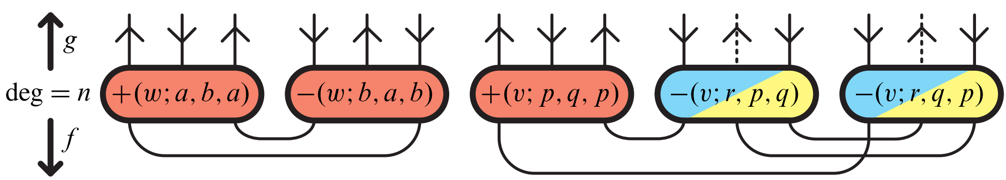


Figure 11

Proposition 6.4 *There is no 3–cycle $\gamma = \sum_{i=1}^{6} \gamma_i \in Z_3$ in case* III.

Proof It is sufficient to consider the case that $\gamma_1, \ldots, \gamma_6$ are f–splittable. By Lemma 5.1, we have the following three cases.

(a) The six 3–terms are divided into three sets, each of which consists of two f–connected 3–terms ($6 = 2 + 2 + 2$).

(b) The six 3–terms are divided into two sets, each of which consists of three f–connected 3–terms ($6 = 3 + 3$).

(c) The six 3–terms are divided into two sets which consist of two and four f–connected 3–terms, respectively ($6 = 2 + 4$).

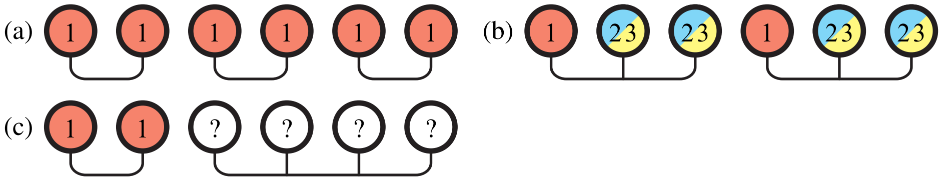


Figure 12

See Figure 12. We see that the γ_i are g–splittable: In fact, for cases (a) and (c), γ contains

$$+(w;a,b,a) \quad \text{and} \quad -(w;b,a,b)$$

for some $a \neq b$ by Lemma 5.2. Since $w^{aba} \neq w^{bab}$, the γ_i are g–splittable by Lemma 4.3(iii). Similarly, for case (b), γ contains

$$+(w;a,b,a), \quad -(w;c,a,b) \quad \text{and} \quad -(w;c,b,a)$$

for some mutually different a,b,c by Lemma 5.4, up to sign and reflection, which satisfies $w^{aba} \neq w^{cba}$.

Let N_k be the number of 3–terms among the γ_i with type $k(=1,2,3)$. Since the γ_i and $\overline{\gamma}_i$ are both f–splittable, we have $N_1 \geq 2$ and $N_2 \geq 2$.

Case 1 Consider the case that γ satisfies (a). Then $N_1 = 6$ and $N_2 = N_3 = 0$. This contradicts $N_2 \geq 2$, and so case 1 does not happen.

Case 2 Consider the case that γ satisfies (b). We may assume that $\overline{\gamma}$ satisfies (b) or (c). By Lemma 5.4, we can take

$$\gamma_1 = +(w;a,b,a), \quad \gamma_2 = -(w;c,a,b) \quad \text{and} \quad \gamma_3 = -(w;c,b,a)$$

up to sign and reflection. We remark that at least one of γ_2 and γ_3 is of type 3. Since $N_2 \geq 2$, one of γ_2 and γ_3 is of type 2, and the other is of type 3. Put $a = a_0$ and $b = a_1$.

If γ_2 is of type 2, then $c^a = b$, that is, $c = b^a = a_4$. Therefore, we have

$$\gamma_1 = +(w;a_0,a_1,a_0), \quad \gamma_2 = -(w;a_4,a_0,a_1) \quad \text{and} \quad \gamma_3 = -(w;a_4,a_1,a_0).$$

The indices of $\overline{\gamma}_1$, $\overline{\gamma}_2$ and $\overline{\gamma}_3$ are

$$w^{a_0a_1a_0} = w^{a_4}, \quad w^{a_4a_0a_1} = w^{a_0} \quad \text{and} \quad w^{a_4a_1a_0} = w^{a_3},$$

respectively. Since they are mutually different, $\overline{\gamma}$ must satisfy case (a). This contradicts the assumption.

If γ_3 is of type 2, then $c^b = a$, that is, $c = a^b = a_2$. Therefore, we have

$$\gamma_1 = +(w; a_0, a_1, a_0), \quad \gamma_2 = -(w; a_2, a_0, a_1) \quad \text{and} \quad \gamma_3 = -(w; a_2, a_1, a_0).$$

The indices of $\overline{\gamma}_1$, $\overline{\gamma}_2$ and $\overline{\gamma}_3$ are

$$w^{a_0 a_1 a_0} = w^{a_4}, \quad w^{a_2 a_0 a_1} = w^{a_3} \quad \text{and} \quad w^{a_2 a_1 a_0} = w^{a_1},$$

respectively, and hence, we have a contradiction by a similar argument as above. Therefore, case 2 does not happen.

Case 3 Consider the case that γ satisfies (c). We may assume that $\overline{\gamma}$ also satisfies (c). We can take

$$\gamma_1 = +(w; a, b, a) \quad \text{and} \quad \gamma_2 = -(w; b, a, b)$$

for some $a \neq b$. Since $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are of type 2, we may assume that

- γ_5 and γ_6 are of type 2,
- $\gamma_1, \dots, \gamma_4$ are g–connected, and
- γ_5 and γ_6 are g–connected.

See Figure 13. Since $w^{aba} \neq w^{bab}$, it must be that $\gamma_1, \dots, \gamma_4$ are g–splittable. This is a contradiction. Therefore, case 3 does not happen. This completes the proof. $\square$

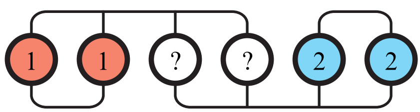


Figure 13

Proposition 6.5 *There is no 3–cycle $\gamma = \sum_{i=1}^{7} \gamma_i \in Z_3$ in case* IV.

Proof It is sufficient to consider the case that $\gamma_1, \dots, \gamma_7$ are f–splittable. By Lemma 5.1, we have the following three cases.

(a) The seven 3–terms are divided into three sets consisting of two, two and three f–connected 3–terms, respectively $(7 = 2 + 2 + 3)$.

(b) The seven 3–terms are divided into two sets consisting of two and five f–connected 3–terms, respectively $(7 = 2 + 5)$.

(c) The seven 3–terms are divided into two sets consisting of three and four f–connected 3–terms, respectively $(7 = 3 + 4)$.

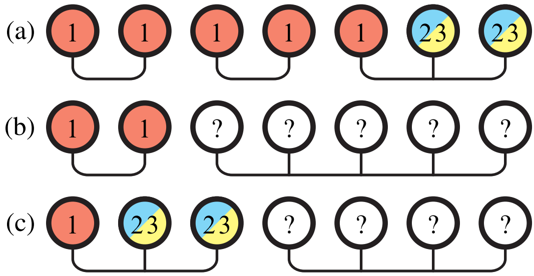


Figure 14

See Figure 14. Similarly to the proof of Proposition 6.4, we see that the γ_i are g–splittable. Let N_k be the number of 3–terms among the γ_i with type $k(=1,2,3)$. Since the γ_i and $\overline{\gamma}_i$ are both f–splittable, we have $N_1 \geq 1$ and $N_2 \geq 1$.

Case 1 Consider the case that γ satisfies (a). We may assume that

- γ_1 and γ_2 are f–connected,
- γ_3 and γ_4 are f–connected, and
- $\gamma_5, \gamma_6, \gamma_7$ are f–connected.

By Lemmas 5.2 and 5.4, it holds that $N_1 = 5$. Similarly to case 2 in the proof of Proposition 6.4, the indices of the reverses $\overline{\gamma}_5, \overline{\gamma}_6, \overline{\gamma}_7$ are mutually different. Therefore, $\overline{\gamma}$ must satisfy (a), which implies that $N_2 = 5$. This contradicts $N_1 + N_2 + N_3 = 7$, and so case 1 does not happen.

Case 2 Consider the case that both γ and $\overline{\gamma}$ satisfy (b). We may assume that

- γ_1 and γ_2 are f–connected, and
- $\gamma_3, \dots, \gamma_7$ are f–connected.

Since γ_1 and γ_2 are of type 1, we may also assume that

- γ_6 and γ_7 are g–connected such that γ_6 and γ_7 are of type 2, and
- $\gamma_1, \dots, \gamma_5$ are g–connected.

See Figure 15. On the other hand, as the indices of $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are different, $\gamma_1, \dots, \gamma_5$ must be g–splittable. This is a contradiction, and so case 2 does not happen.

Case 3 Consider the case that γ and $\overline{\gamma}$ satisfy (c) and (b), respectively. We may assume that

- $\gamma_1, \gamma_2, \gamma_3$ are f–connected such that γ_1 is of type 1, and
- $\gamma_4, \dots, \gamma_7$ are f–connected.

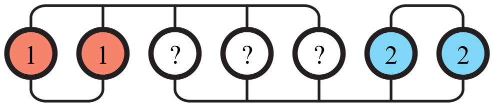


Figure 15

We see that both γ_2 and γ_3 are of type 3: in fact, if one of them is of type 2, then the indices of $\overline{\gamma}_1$, $\overline{\gamma}_2$ and $\overline{\gamma}_3$ are mutually different, which contradicts that $\overline{\gamma}$ satisfies (b). Therefore, we may also assume that

- γ_6 and γ_7 are g–connected such that γ_6 and γ_7 are of type 2, and
- $\gamma_1, \dots, \gamma_5$ are g–connected.

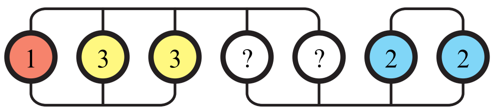


Figure 16

See Figure 16. We can take

$$\gamma_1 = +(w; a_0, a_1, a_0), \quad \gamma_2 = -(w; c, a_0, a_1) \quad \text{and} \quad \gamma_3 = -(w; c, a_1, a_0)$$

up to sign and reflection. Since γ_2 and γ_3 are of type 3, we have $c = a_3$. Then the indices of $\overline{\gamma}_1$, $\overline{\gamma}_2$, and $\overline{\gamma}_3$ are

$$w^{a_0 a_1 a_0} = w^{a_4}, \quad w^{a_3 a_0 a_1} = w^{a_4} \quad \text{and} \quad w^{a_3 a_1 a_0} = w^{a_2},$$

respectively. This contradicts that $\gamma_1, \dots, \gamma_5$ are g–connected, and so case 3 does not happen.

Case 4 Consider the case that both γ and $\overline{\gamma}$ are of type 3. We may assume that

- $\gamma_1, \gamma_2, \gamma_3$ are f–connected such that γ_1 is of type 1, and
- $\gamma_4, \dots, \gamma_7$ are f–connected.

Similarly to case 3, we see that both γ_2 and γ_3 are of type 3. Then we can take

$$\gamma_1 = +(w; a_0, a_1, a_0), \quad \gamma_2 = -(w; a_3, a_0, a_1) \quad \text{and} \quad \gamma_3 = -(w; a_3, a_1, a_0)$$

up to sign and reflection. The indices of $\overline{\gamma}_1$, $\overline{\gamma}_2$ and $\overline{\gamma}_3$ are w^{a_4}, w^{a_4} and w^{a_2}, respectively. Therefore, since $\overline{\gamma}$ satisfies (c), we may also assume that

- $\gamma_3, \gamma_4, \gamma_5$ are g–connected such that γ_5 is of type 2, and
- $\gamma_1, \gamma_2, \gamma_6, \gamma_7$ are g–connected.

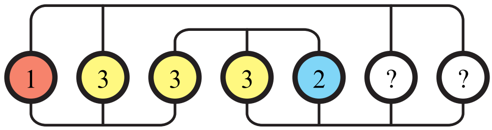


Figure 17

See Figure 17. Since $\overline{\gamma}_3 = -(w^{a_2}; a_1, a_4, a_0)$ and $\overline{\gamma}_3$, $\overline{\gamma}_4$ and $\overline{\gamma}_5$ are f–connected, we have

$$\overline{\gamma}_4 = -(w^{a_2}; a_4, a_1, a_0) \quad \text{or} \quad \overline{\gamma}_4 = -(w^{a_2}; a_1, a_0, a_4).$$

by Lemma 5.4. Since $\overline{\gamma}_4$ is of type 3, we have $\overline{\gamma}_4 = -(w^{a_2}; a_4, a_1, a_0)$. Therefore, it holds that

$$\overline{\gamma}_5 = +(w^{a_2}; a_1, a_4, a_1) \quad \text{or} \quad \overline{\gamma}_5 = +(w^{a_2}; a_4, a_1, a_4).$$

We remark that the index of γ_4 is equal to $w^{a_2 a_4 a_1 a_0} = w + s$, and that of γ_5 is equal to $w^{a_2 a_1 a_4 a_1} = w + s$ or $w^{a_2 a_4 a_1 a_4} = w$. Since $\gamma_4, \dots, \gamma_7$ are f–connected, we have $\overline{\gamma}_5 = +(w^{a_2}; a_1, a_4, a_1)$. This implies that we have

$$\gamma_4 = -(w + s; a_2, a_4, a_0) \quad \text{and} \quad \gamma_5 = +(w + s; a_0, a_3, a_1).$$

It follows by $f(\gamma_4 + \cdots + \gamma_7) = 0$ that

$$\begin{aligned} f(\gamma_6 + \gamma_7) &= -f(\gamma_4 + \gamma_5) \\ &= -(w + s; a_4, a_0) + (w + s; a_2, a_0) - (w + s; a_2, a_4) \\ &\qquad + (w + s; a_3, a_1) - (w + s; a_0, a_1) + (w + s; a_0, a_3). \end{aligned}$$

It is not difficult to see that

$$\gamma_6 + \gamma_7 = +(w + s; a_2, a_4, a_0) - (w + s; a_0, a_3, a_1) = -(\gamma_4 + \gamma_5).$$

This is a contradiction, and so case 4 does not happen. □

7 Cases V and VI

Throughout this section, we assume that $\gamma = \sum_{i=1}^{\ell} \gamma_i \in Z_3$ $(\ell = 6, 7)$ is a 3–cycle such that

- $\gamma_1 = +(n, w; a_0, a_1, a_0)$, $\gamma_2 = -(n, w; a_1, a_0, a_1)$, and
- the degrees of $\gamma_3, \dots, \gamma_\ell$ are equal to $n + 1$.

It holds that

$$\begin{aligned} f(\gamma_3+\cdots+\gamma_\ell) &= -g(\gamma_1+\gamma_2) \\ &= -(n{+}1, w^{a_0}; a_1, a_0) - (n{+}1, w^{a_0}; a_0, a_4) - (n{+}1, w^{a_0}; a_4, a_1) \\ &\quad + (n{+}1, w^{a_1}; a_2, a_0) + (n{+}1, w^{a_1}; a_0, a_1) + (n{+}1, w^{a_1}; a_1, a_2). \end{aligned}$$

Proposition 7.1 *If $\gamma = \sum_{i=1}^{6} \gamma_i \in Z_3$ is a 3–cycle in case* V, *then*

$$\begin{aligned} \gamma = &+(n, w; a_0, a_1, a_0) - (n, w; a_1, a_0, a_1) \\ &+(n+1, w^{a_0}; a_1, a_0, a_1) + (n+1, w^{a_0}; a_0, a_4, a_1) \\ &-(n+1, w^{a_1}; a_2, a_1, a_2) - (n+1, w^{a_1}; a_2, a_0, a_1) \end{aligned}$$

up to reverse and reflection.

Proof By Lemma 5.3, we may assume that $w_3 = w_4 = w^{a_0}$ and $w_5 = w_6 = w^{a_1}$.

First, we consider the equation

$$f(\gamma_3+\gamma_4) = -(n+1, w^{a_0}; a_1, a_0) - (n+1, w^{a_0}; a_0, a_4) - (n+1, w^{a_0}; a_4, a_1).$$

It is not difficult to see that there are six possibilities for $\gamma_3+\gamma_4$; that is,

$$\gamma_3+\gamma_4 = +(n+1, w^{a_0}; z, x, z) + (n+1, w^{a_0}; x, y, z)$$

or

$$\gamma_3+\gamma_4 = +(n+1, w^{a_0}; x, z, x) + (n+1, w^{a_0}; x, y, z)$$

for $(x, y, z) = (a_1, a_0, a_4)$, (a_0, a_4, a_1), or (a_4, a_1, a_0).

We see that $\gamma_3, \dots, \gamma_6$ are g–connected: In fact, if they are g–splittable, then they are of type 2 by Lemmas 5.1 and 5.2. This contradicts that γ_3 or γ_4 is of type 1.

If $\gamma_3+\gamma_4 = +(n+1, w^{a_0}; z, x, z) + (n+1, w^{a_0}; x, y, z)$, then $w^{a_0 zxz} = w^{a_0 xyz}$, or equivalently, $2x = y+z$. Therefore, it holds that $(x, y, z) = (a_0, a_4, a_1)$ and

$$\gamma_3+\gamma_4 = +(n+1, w^{a_0}; a_1, a_0, a_1) + (n+1, w^{a_0}; a_0, a_4, a_1).$$

Similarly, if $\gamma_3+\gamma_4 = +(n+1, w^{a_0}; x, z, x) + (n+1, w^{a_0}; x, y, z)$, then we have $w^{a_0 xzx} = w^{a_0 xyz}$, or equivalently, $2z = x+y$. Therefore, it holds that $(x, y, z) = (a_4, a_1, a_0)$ and

$$\gamma_3+\gamma_4 = +(n+1, w^{a_0}; a_4, a_0, a_4) + (n+1, w^{a_0}; a_4, a_1, a_0).$$

We remark that the second solution is coincident with the reflection of the first one under a suitable transformation of variables.

Next, we consider the equation

$$f(\gamma_5+\gamma_6) = +(n+1, w^{a_1}; a_2, a_0) + (n+1, w^{a_1}; a_0, a_1) + (n+1, w^{a_1}; a_1, a_2).$$

The solutions of $\gamma_5+\gamma_6$ are obtained from those of $\gamma_3+\gamma_4$ with opposite sign by replacing a_i with a_{1-i}. Therefore, we have

$$\gamma_5+\gamma_6 = -(n+1, w^{a_1}; a_0, a_1, a_0) - (n+1, w^{a_1}; a_1, a_2, a_0)$$

or

$$\gamma_5+\gamma_6 = -(n+1, w^{a_1}; a_2, a_1, a_2) - (n+1, w^{a_1}; a_2, a_0, a_1).$$

Finally, by taking the reflection if necessary, we may assume that $\gamma_3+\gamma_4$ satisfies the first solution. Since $\gamma_3,\dots,\gamma_6$ are g–connected, we see that $\gamma_5+\gamma_6$ must satisfy the second solution. Then it is easy to see that $g(\gamma_3+\dots+\gamma_6)=0$. See Figure 18. □

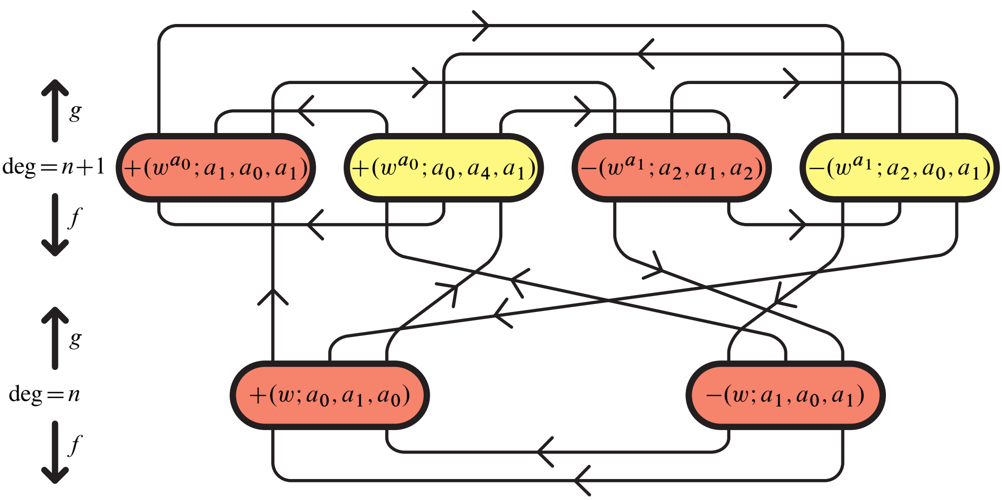


Figure 18

Proposition 7.2 *There is no 3–cycle $\gamma=\sum_{i=1}^{7}\gamma_i \in Z_3$ in case* VI.

Proof Let N_k^{ε} be the number of 3–terms among $\gamma_3,\dots,\gamma_7$ with type $k(=1,2,3)$ and sign $\varepsilon(=\pm)$. Since it holds that

- $N_1^+,\dots,N_3^- \geq 0$,
- $N_1^+ + \dots + N_3^- = 5$,
- $2(N_1^+ - N_1^-) + (N_2^+ - N_2^-) + (N_3^+ - N_3^-) = 0$, and
- $(N_1^+ - N_1^-) + 2(N_2^+ - N_2^-) + (N_3^+ - N_3^-) = 0$,

we have that $(N_1^+, N_1^-, N_2^+, N_2^-, N_3^+, N_3^-)$ is equal to either

$$(1,0,1,0,0,3) \quad \text{or} \quad (0,1,0,1,3,0).$$

By a similar argument to the proof of Proposition 7.1, we may assume that

$$\gamma_3 + \gamma_4 = +(n+1, w^{a_0}; z, x, z) + (n+1, w^{a_0}; x, y, z)$$

for $(x,y,z) = (a_1, a_0, a_4)$, (a_0, a_4, a_1) or (a_4, a_1, a_0), and $w_5 = w_6 = w_7 = w^{a_1}$.

We see that $\gamma_3, \dots, \gamma_7$ are g–connected: In fact, if they are g–splittable, then we have that $\overline{\gamma}_3, \dots, \overline{\gamma}_7$ are f–splittable and divided into two sets consisting of two and three f–connected 3–terms, respectively $(5 = 2+3)$. By Lemmas 5.2 and 5.4, we see that

$$N_2^+ + N_2^- = 3.$$

This is a contradiction. See Figure 19.

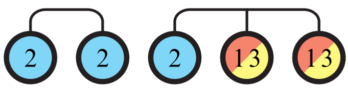


Figure 19

Since the indices of $\overline{\gamma}_3$ and $\overline{\gamma}_4$ are the same, that is, $w^{a_0 zxz} = w^{a_0 xyz}$, we have $(x,y,z) = (a_0, a_4, a_1)$ and

$$\gamma_3 + \gamma_4 = +(n+1, w^{a_0}; a_1, a_0, a_1) + (n+1, w^{a_0}; a_0, a_4, a_1).$$

This implies that

$$N_1^+ \geq 1 \quad \text{and} \quad N_3^+ \geq 1,$$

and we have a contradiction. □

8 Cases VII and VIII

Throughout this section, we assume that $\gamma = \sum_{i=1}^{\ell} \gamma_i \in Z_3$ $(\ell = 6, 7)$ is a 3–cycle such that

- $\gamma_1 = +(n, w; a_0, a_1, a_0)$, $\gamma_2 = -(n, w; x, a_0, a_1)$, and $\gamma_3 = -(n, w; x, a_1, a_0)$ for $x \neq a_0, a_1$, up to sign and reflection, and
- $n_4 = \cdots = n_\ell = n+1$.

It holds that

$$\begin{aligned}
f(\gamma_4+\cdots+\gamma_\ell)&\\
&=-g(\gamma_1+\gamma_2+\gamma_3)\\
&=-(n+1,w^{a_0};a_1,a_0)-(n+1,w^{a_0};a_0,a_4)\underline{-(n+1,w^{a_0};x^{a_0},a_1)}\\
&\quad+(n+1,w^{a_0};x^{a_0},a_4)+(n+1,w^{a_1};a_2,a_0)+(n+1,w^{a_1};x^{a_1},a_2)\\
&\quad\underline{\underline{+(n+1,w^{a_1};x^{a_1},a_0)}}+(n+1,w^{x};a_0,a_1)+(n+1,w^{x};a_1,a_0).
\end{aligned}$$

Here, the underlined (or doubly underlined) 2–term is removed if $x=a_4$ (or $x=a_2$).

Proposition 8.1 *If $\gamma=\sum_{i=1}^{6}\gamma_i\in Z_3$ is a 3–cycle in case* VII, *then*

$$\begin{aligned}
\gamma&=+(n,w;a_0,a_1,a_0)-(n,w;a_4,a_0,a_1)-(n,w;a_4,a_1,a_0)\\
&\quad+(n+1,w^{a_0};a_1,a_0,a_4)-(n+1,w^{a_1};a_3,a_2,a_0)-(n+1,w^{a_4};a_0,a_1,a_0)
\end{aligned}$$

up to sign, reverse and reflection.

Proof By Lemma 5.5, we may assume that $w_4=w^{a_0}$, $w_5=w^{a_1}$ and $w_6=w^{x}$. The 3–term γ_4 satisfies

$$\begin{aligned}
f(\gamma_4)=-(n+1,w^{a_0};a_1,a_0)-(n+1,w^{a_0};a_0,a_4)&\\
\underline{-(n+1,w^{a_0};x^{a_0},a_1)}+(n+1,w^{a_0};x^{a_0},a_4)&.
\end{aligned}$$

Therefore, the underlined 2–term is removed with $x=a_4$, and therefore, we have $\gamma_4=+(n+1,w^{a_0};a_1,a_0,a_4)$. Then γ_5 satisfies

$$f(\gamma_5)=+(n+1,w^{a_1};a_2,a_0)+(n+1,w^{a_1};a_3,a_2)-(n+1,w^{a_1};a_3,a_0),$$

so that $\gamma_5=-(n+1,w^{a_1};a_3,a_2,a_0)$. We remark that the indices of $\overline{\gamma}_4$ and $\overline{\gamma}_5$ are

$$w^{a_0a_1a_0a_4}=w^{a_1a_3a_2a_0}=w.$$

On the other hand, since the 3–term γ_6 satisfies

$$f(\gamma_6)=+(n+1,w^{a_4};a_0,a_1)+(n+1,w^{a_4};a_1,a_0),$$

we have $\gamma_6=-(n+1,w^{a_4};a_0,a_1,a_0)$ or $-(n+1,w^{a_4};a_1,a_0,a_1)$. The index of $\overline{\gamma}_6$ is either

$$w^{a_4a_0a_1a_0}=w\quad\text{or}\quad w^{a_4a_1a_0a_1}=w+3s.$$

Since γ_4, γ_5 and γ_6 are g–connected, we have $\gamma_6=-(n+1,w^{a_4};a_0,a_1,a_0)$.

It is easy to see that $g(\gamma_4+\gamma_5+\gamma_6)=0$. See Figure 3. □

Proposition 8.2 *If* $\gamma = \sum_{i=1}^{7} \gamma_i \in Z_3$ *is a* 3*–cycle in case* VIII*, then*

$$\begin{aligned}\gamma = &+(n,w;a_0,a_1,a_0)-(n,w;a_2,a_0,a_1)-(n,w;a_2,a_1,a_0)\\ &+(n+1,w^{a_0};a_3,a_1,a_0)+(n+1,w^{a_0};a_3,a_0,a_4)\\ &-(n+1,w^{a_1};a_0,a_2,a_0)-(n+1,w^{a_2};a_0,a_1,a_0)\end{aligned}$$

or

$$\begin{aligned}\gamma = &+(n,w;a_0,a_1,a_0)-(n,w;a_3,a_0,a_1)-(n,w;a_3,a_1,a_0)\\ &+(n+1,w^{a_0};a_2,a_1,a_0)+(n+1,w^{a_0};a_2,a_0,a_4)\\ &-(n+1,w^{a_1};a_4,a_2,a_0)-(n+1,w^{a_3};a_0,a_1,a_0)\end{aligned}$$

up to sign, reverse and reflection.

Proof We divide the proof into three cases with respect to $x = a_2, a_3, a_4$.

Case 1 Consider the case $x = a_2$. It holds that

$$\begin{aligned}f(\gamma_4+\cdots+\gamma_7) = &-(n+1,w^{a_0};a_1,a_0)-(n+1,w^{a_0};a_0,a_4)\\ &-(n+1,w^{a_0};a_3,a_1)+(n+1,w^{a_0};a_3,a_4)\\ &+(n+1,w^{a_1};a_2,a_0)+(n+1,w^{a_1};a_0,a_2)\\ &+(n+1,w^{a_2};a_0,a_1)+(n+1,w^{a_2};a_1,a_0).\end{aligned}$$

Therefore, we may assume that $w_4 = w_5 = w^{a_0}$, $w_6 = w^{a_1}$, and $w_7 = w^{a_2}$. It is easy to see that

- $\gamma_6 = -(n+1, w^{a_1}; a_0, a_2, a_0)$ or $-(n+1, w^{a_1}; a_2, a_0, a_2)$, and the index of $\overline{\gamma}_6$ is $w+4s$ or $w+s$, respectively, and
- $\gamma_7 = -(n+1, w^{a_2}; a_0, a_1, a_0)$ or $-(n+1, w^{a_2}; a_1, a_0, a_1)$, and the index of $\overline{\gamma}_7$ is $w+4s$ or w, respectively.

Let N_k^{ε} be the number of 3–terms among $\gamma_4, \dots, \gamma_7$ with type $k(=1,2,3)$ and sign $\varepsilon(=\pm)$. Since

- $N_1^+, \dots, N_3^- \geq 0$,
- $N_1^+ + \cdots + N_3^- = 4$,
- $2(N_1^+ - N_1^-) + (N_2^+ - N_2^-) + (N_3^+ - N_3^-) = -2$, and
- $(N_1^+ - N_1^-) + 2(N_2^+ - N_2^-) + (N_3^+ - N_3^-) = 0$,

we have

$$(N_1^+, N_1^-, N_2^+, N_2^-, N_3^+, N_3^-) = (0,2,0,0,2,0).$$

We see that $\gamma_4,\dots,\gamma_7$ are g–connected: In fact, if they are g–splittable, then they are of type 2 by Lemmas 5.1 and 5.2. This contradicts $N_2^+ = N_2^- = 0$.

Since the indices of $\overline{\gamma}_6$ and $\overline{\gamma}_7$ are the same, we have

$$\gamma_6 = -(n+1, w^{a_1}; a_0, a_2, a_0) \quad \text{and} \quad \gamma_7 = -(n+1, w^{a_2}; a_0, a_1, a_0).$$

For γ_4 and γ_5, it holds that

$$f(\gamma_4+\gamma_5) = -(n+1, w^{a_0}; a_1, a_0) - (n+1, w^{a_0}; a_0, a_4) \\ -(n+1, w^{a_0}; a_3, a_1) + (n+1, w^{a_0}; a_3, a_4).$$

It is not difficult to see that

$$\gamma_4+\gamma_5 = +(n+1, w^{a_0}; a_3, a_1, a_0) + (n+1, w^{a_0}; a_3, a_0, a_4)$$

or

$$\gamma_4+\gamma_5 = +(n+1, w^{a_0}; a_1, a_0, a_4) + (n+1, w^{a_0}; a_3, a_1, a_4).$$

Since the indices of $\overline{\gamma}_4$ and $\overline{\gamma}_5$ are $w+4s$, we have the first solution. Then it holds that $g(\gamma_4+\cdots+\gamma_7) = 0$. See Figure 20.

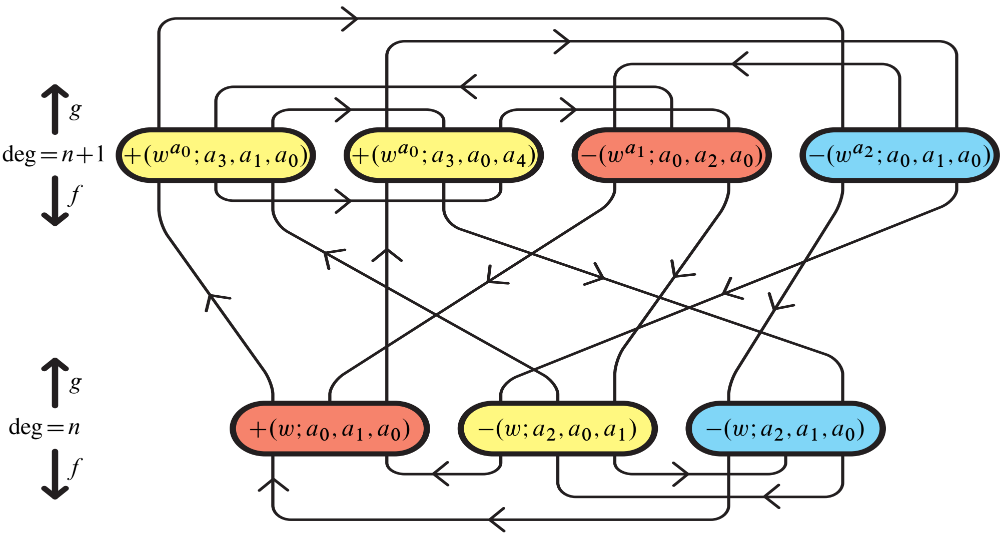


Figure 20

Case 2 Consider the case $x = a_4$. It holds that

$$f(\gamma_4+\cdots+\gamma_7) = -(n{+}1, w^{a_0}; a_1, a_0) - (n{+}1, w^{a_0}; a_0, a_4) + (n{+}1, w^{a_0}; a_1, a_4) \\ + (n{+}1, w^{a_1}; a_2, a_0) + (n{+}1, w^{a_1}; a_3, a_2) - (n{+}1, w^{a_1}; a_3, a_0) \\ + (n{+}1, w^{a_4}; a_0, a_1) + (n{+}1, w^{a_4}; a_1, a_0).$$

By a similar argument to case 1, we have

$$(N_1^+, N_1^-, N_2^+, N_2^-, N_3^+, N_3^-) = (0, 2, 0, 0, 2, 0).$$

There are at least two 3–terms of index w^{a_0} among $\gamma_4, \dots, \gamma_7$: In fact, if the number is just one, then the 3–term is $+(n+1, w^{a_0}; a_1, a_0, a_4)$. This implies that $N_1^+ \geq 1$, which is a contradiction.

Similarly, there are at least two 3–terms of index w^{a_1} among $\gamma_4, \dots, \gamma_7$: In fact, if the number is just one, then the 3–term is $-(n+1, w^{a_1}; a_3, a_2, a_0)$. This implies that $N_3^- \geq 1$, which is a contradiction.

On the other hand, there is at least one 3–term of index w^{a_4}. Therefore, we have $N_1^+ + \cdots + N_3^- \geq 5$, which is a contradiction.

Case 3 Consider the case $x = a_3$. It holds that

$$\begin{aligned} f(\gamma_4 + \cdots + \gamma_7) = &- (n{+}1, w^{a_0}; a_1, a_0) - (n{+}1, w^{a_0}; a_0, a_4) \\ &- (n{+}1, w^{a_0}; a_2, a_1) + (n{+}1, w^{a_0}; a_2, a_4) \\ &+ (n{+}1, w^{a_1}; a_2, a_0) + (n{+}1, w^{a_1}; a_4, a_2) - (n{+}1, w^{a_1}; a_4, a_0) \\ &+ (n{+}1, w^{a_3}; a_0, a_1) + (n{+}1, w^{a_3}; a_1, a_0). \end{aligned}$$

We may assume that $w_4 = w_5 = w^{a_0}$, $w_6 = w^{a_1}$, and $w_7 = w^{a_2}$. It is easy to see that

- $\gamma_6 = -(n+1, w^{a_1}; a_4, a_2, a_0)$, and the index of $\overline{\gamma}_6$ is $w + 2s$, and
- $\gamma_7 = -(n+1, w^{a_3}; a_0, a_1, a_0)$ or $-(n+1, w^{a_3}; a_1, a_0, a_1)$, and the index of $\overline{\gamma}_7$ is $w + 2s$ or $w + 3s$, respectively.

We see that $\gamma_4, \dots, \gamma_7$ are g–connected: In fact, if they are g–splittable, then they are all of type 2 by Lemmas 5.1 and 5.2. Since γ_7 is of type 1, we have a contradiction.

Since the indices of $\overline{\gamma}_6$ and $\overline{\gamma}_7$ are the same, we have

$$\gamma_6 = -(n+1, w^{a_1}; a_4, a_2, a_0) \quad \text{and} \quad \gamma_7 = -(n+1, w^{a_3}; a_0, a_1, a_0).$$

For γ_4 and γ_5, it holds that

$$\begin{aligned} f(\gamma_4 + \gamma_5) = -(n+1, w^{a_0}; a_1, a_0) - (n+1, w^{a_0}; a_0, a_4) \\ - (n+1, w^{a_0}; a_2, a_1) + (n+1, w^{a_0}; a_2, a_4). \end{aligned}$$

It is not difficult to see that

$$\gamma_4 + \gamma_5 = +(n+1, w^{a_0}; a_2, a_1, a_0) + (n+1, w^{a_0}; a_2, a_0, a_4)$$

or

$$\gamma_4 + \gamma_5 = +(n+1, w^{a_0}; a_1, a_0, a_4) + (n+1, w^{a_0}; a_2, a_1, a_4).$$

Since the indices of $\overline{\gamma}_4$ and $\overline{\gamma}_5$ are $w + 2s$, we have the first solution. Then it holds that $g(\gamma_4 + \cdots + \gamma_7) = 0$. See Figure 21. □

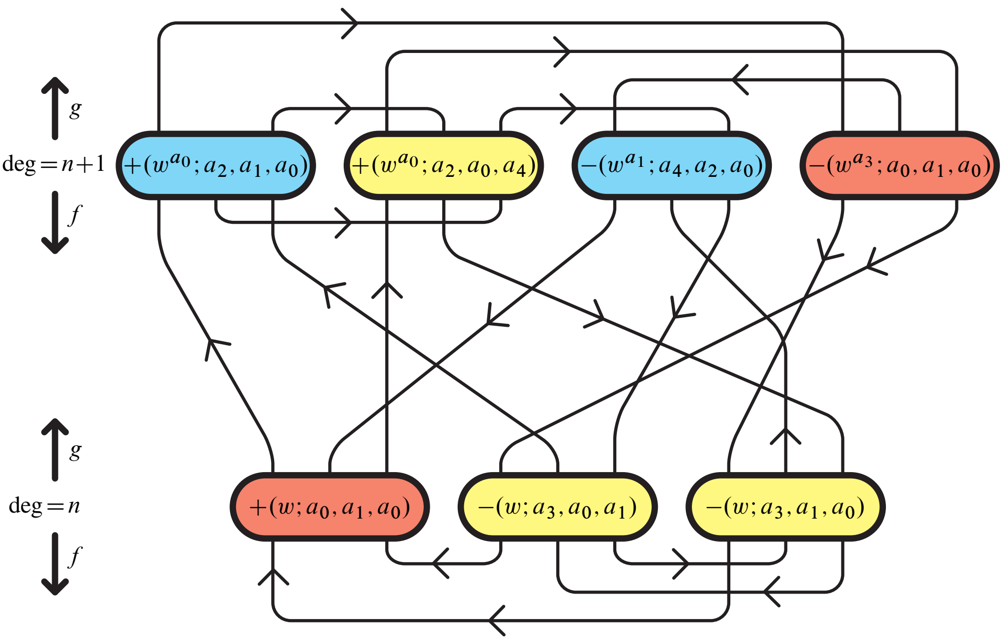


Figure 21

9 Mochizuki 3–cocycle

Theorem 9.1 *If a nonzero 3–cycle $\gamma = \sum_{i=1}^{\ell} \gamma_i \in Z_3$ satisfies $\ell \leq 7$, then we have the following up to sign, reverse and reflection.*

(i) $$\begin{aligned}\gamma = &+ (n, w; a_0, a_1, a_0) - (n, w; a_1, a_0, a_1)\\ &+ (n+1, w^{a_0}; a_1, a_0, a_1) + (n+1, w^{a_0}; a_0, a_4, a_1)\\ &- (n+1, w^{a_1}; a_2, a_1, a_2) - (n+1, w^{a_1}; a_2, a_0, a_1).\end{aligned}$$

(ii) $$\begin{aligned}\gamma = &+(n, w; a_0, a_1, a_0) - (n, w; a_4, a_0, a_1) - (n, w; a_4, a_1, a_0)\\ &+(n+1, w^{a_0}; a_1, a_0, a_4) - (n+1, w^{a_1}; a_3, a_2, a_0) - (n+1, w^{a_4}; a_0, a_1, a_0).\end{aligned}$$

(iii) $$\begin{aligned}\gamma = &+ (n, w; a_0, a_1, a_0) - (n, w; a_2, a_0, a_1) - (n, w; a_2, a_1, a_0)\\ &+ (n+1, w^{a_0}; a_3, a_1, a_0) + (n+1, w^{a_0}; a_3, a_0, a_4)\\ &- (n+1, w^{a_1}; a_0, a_2, a_0) - (n+1, w^{a_2}; a_0, a_1, a_0).\end{aligned}$$

(iv) $$\begin{aligned}\gamma = &+ (n, w; a_0, a_1, a_0) - (n, w; a_3, a_0, a_1) - (n, w; a_3, a_1, a_0)\\ &+ (n+1, w^{a_0}; a_2, a_1, a_0) + (n+1, w^{a_0}; a_2, a_0, a_4)\\ &- (n+1, w^{a_1}; a_4, a_2, a_0) - (n+1, w^{a_3}; a_0, a_1, a_0).\end{aligned}$$

Proof This follows from Propositions 5.6, 6.2–6.5, 7.1, 7.2, 8.1, and 8.2 directly. □

Proposition 9.2 *The reflection γ^* of γ in Theorem 9.1 is given as follows, where $b_i = (-1)^n(a_i - w)$ and $v = (-1)^{n+1}w$.*

(i) $$\begin{aligned}\gamma^* = &+(n,v;b_0,b_1,b_0)-(n,v;b_1,b_0,b_1)\\ &+(n+1,v^{b_0};b_4,b_0,b_4)+(n+1,v^{b_0};b_4,b_1,b_0)\\ &-(n+1,v^{b_1};b_0,b_1,b_0)-(n+1,v^{b_1};b_1,b_2,b_0).\end{aligned}$$

(ii) $$\begin{aligned}\gamma^* = &+(n,v;b_0,b_1,b_0)-(n,v;b_1,b_0,b_4)-(n,v;b_0,b_1,b_4)\\ &+(n+1,v^{b_0};b_1,b_0,b_4)-(n+1,v^{b_1};b_2,b_0,b_4)-(n+1,v^{b_4};b_3,b_2,b_3).\end{aligned}$$

(iii) $$\begin{aligned}\gamma^* = &+(n,v;b_0,b_1,b_0)-(n,v;b_1,b_0,b_2)-(n,v;b_0,b_1,b_2)\\ &+(n+1,v^{b_0};b_0,b_4,b_2)+(n+1,v^{b_0};b_1,b_0,b_2)\\ &-(n+1,v^{b_1};b_2,b_0,b_2)-(n+1,v^{b_2};b_4,b_3,b_4).\end{aligned}$$

(iv) $$\begin{aligned}\gamma^* = &+(n,v;b_0,b_1,b_0)-(n,v;b_1,b_0,b_3)-(n,v;b_0,b_1,b_3)\\ &+(n+1,v^{b_0};b_0,b_4,b_3)+(n+1,v^{b_0};b_1,b_0,b_3)\\ &-(n+1,v^{b_1};b_2,b_0,b_3)-(n+1,v^{b_3};b_1,b_0,b_1).\end{aligned}$$

Proof It holds that
$$b_i^{b_j} = 2b_j - b_i = (-1)^n(2a_j - a_i - w) = (-1)^n(a_{2j-i} - w) = b_{2j-i}$$
with $\{b_0, \dots, b_4\} = R_5$. Since
$$(-1)^{n+2}w^{a_i} = (-1)^{n+2}(2a_i - w) = 2b_i - v = v^{b_i}$$
and
$$(-1)^{n+1}(a_i - w^{a_j}) = (-1)^{n+1}(a_i - 2a_j + w) = 2b_j - b_i = b_i^{b_j},$$
the reflection of $\varepsilon(n,w;a_i,a_j,a_k)$ is
$$\varepsilon\big(n,(-1)^{n+1}w;(-1)^n(a_k-w),(-1)^n(a_j-w),(-1)^n(a_i-w)\big) = \varepsilon(n,v;b_k,b_j,b_i),$$
and that of $\varepsilon(n+1,w^{a_p};a_i,a_j,a_k)$ is
$$\begin{aligned}\varepsilon\big(n+1,(-1)^{n+2}w^{a_p};(-1)^{n+1}(a_k-w^{a_p}),(-1)^{n+1}(a_j-w^{a_p}),(-1)^{n+1}(a_i-w^{a_p})\big)\\ = \varepsilon(n+1,v^{b_p};b_k^{b_p},b_j^{b_p},b_i^{b_p}).\end{aligned}$$
Therefore, we have the conclusion. □

Recall that $\varphi\colon C_3 = C_3(R_5)_{\mathbb{Z}\times R_5} \to C_3(R_5)$ is the chain homomorphism defined by $\varphi(n,w;x,y,z) = (x,y,z)$, and $\langle\ ,\theta_{\mathrm{M}}\rangle\colon C_3(R_5) \to \mathbb{Z}_5$ is the evaluation by the Mochizuki 3–cocycle θ_{M}.

Lemma 9.3 *For any 3–chain $\gamma \in C_3$, it holds that $\langle \varphi(\overline{\gamma}), \theta_{\mathrm{M}} \rangle = \langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle$.*

Proof It is sufficient to prove the equality for a 3–term $\gamma = +(n, w; x, y, z)$. Recall that the reverse of γ is given by $\overline{\gamma} = +(-n, w^{xyz}; x^{yz}, y^z, z)$. We see that

$$x^{yz} - y^z = (2z - 2y + x) - (2z - y) = x - y$$

and $y^{zz} = y$. Therefore, we have

$$\begin{aligned}\langle \varphi(\overline{\gamma}), \theta_{\mathrm{M}} \rangle &= (x^{yz} - y^z)\frac{(y^z)^5 + (y^{zz})^5 - 2z^5}{5}\\ &= (x-y)\frac{(y^z)^5 + y^5 - 2z^5}{5} = \langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle.\end{aligned}$$ □

Proof of Theorem 2.3 We will prove that $\langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle = 0$ for any 3–cycle in Z_3 with $\ell(\gamma) \leq 7$. By Lemma 9.3, it is sufficient to consider the 3–cycles in Theorem 9.1 and Proposition 9.2. We put $a_i = a_0 + is$ and $b_i = b_0 + it$ for $0 \leq i \leq 4$ as integers.

For the 3–cycle γ in Theorem 9.1(i), it holds that

$$\begin{aligned}\langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle = (a_0 - a_1)&\frac{(a_1)^5 + (a_4)^5 - 2(a_0)^5}{5} - (a_1 - a_0)\frac{(a_0)^5 + (a_2)^5 - 2(a_1)^5}{5}\\ &+ (a_1 - a_0)\frac{(a_0)^5 + (a_2)^5 - 2(a_1)^5}{5} + (a_0 - a_4)\frac{(a_4)^5 + (a_3)^5 - 2(a_1)^5}{5}\\ &- (a_2 - a_1)\frac{(a_1)^5 + (a_3)^5 - 2(a_2)^5}{5} - (a_2 - a_0)\frac{(a_0)^5 + (a_2)^5 - 2(a_1)^5}{5}\\ = s(2(a_1)^5 &- (a_3)^5 - (a_4)^5) \in \mathbb{Z}.\end{aligned}$$

Since $k^5 \equiv k \pmod 5$, we have $\langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle \equiv s(2a_1 - a_3 - a_4) \equiv 0 \pmod 5$. For the reverse γ^* in Proposition 9.2(i), it holds that

$$\begin{aligned}\langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle = (b_0 - b_1)&\frac{(b_1)^5 + (b_4)^5 - 2(b_0)^5}{5} - (b_1 - b_0)\frac{(b_0)^5 + (b_2)^5 - 2(b_1)^5}{5}\\ &+ (b_4 - b_0)\frac{(b_0)^5 + (b_3)^5 - 2(b_4)^5}{5} + (b_4 - b_1)\frac{(b_1)^5 + (b_4)^5 - 2(b_0)^5}{5}\\ &- (b_0 - b_1)\frac{(b_1)^5 + (b_4)^5 - 2(b_0)^5}{5} - (b_1 - b_2)\frac{(b_2)^5 + (b_3)^5 - 2(b_0)^5}{5}\\ = t(-(b_0)^5 &+ (b_1)^5 + (b_3)^5 - (b_4)^5) \in \mathbb{Z}.\end{aligned}$$

Then we have $\langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle \equiv t(-b_0 + b_1 + b_3 - b_4) \equiv 0 \pmod 5$.

Similarly, for the 3–cycles in (ii)–(iv), we have

$$\text{(ii)}\quad \begin{cases} \langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle = s((a_0)^5 + (a_1)^5 - (a_2)^5 - (a_4)^5), \\ \langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle = 0, \end{cases}$$

$$\text{(iii)}\quad \begin{cases} \langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle = s(-(a_0)^5 + (a_1)^5 + (a_3)^5 - (a_4)^5), \\ \langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle = t(-(b_0)^5 + 2(b_2)^5 - (b_4)^5), \end{cases}$$

$$\text{(iv)}\quad \begin{cases} \langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle = s((a_0)^5 + (a_1)^5 - (a_2)^5 - (a_4)^5), \text{ and} \\ \langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle = t(-(b_2)^5 + 2(b_3)^5 - (b_4)^5). \end{cases}$$

Therefore, it holds that $\langle \varphi(\gamma), \theta_{\mathrm{M}} \rangle \equiv \langle \varphi(\gamma^*), \theta_{\mathrm{M}} \rangle \equiv 0 \pmod 5$. □

10 Example

Let $F = S \cup T$ be the 2–component surface-link presented by the diagram as shown in Figure 22, where S and T are components of F linking once. The component T is constructed by taking the product of the diagram of the figure-eight knot with periodicity two and a circle equipped with a half twist. We remark that each of S and T is unknotted; see [20].

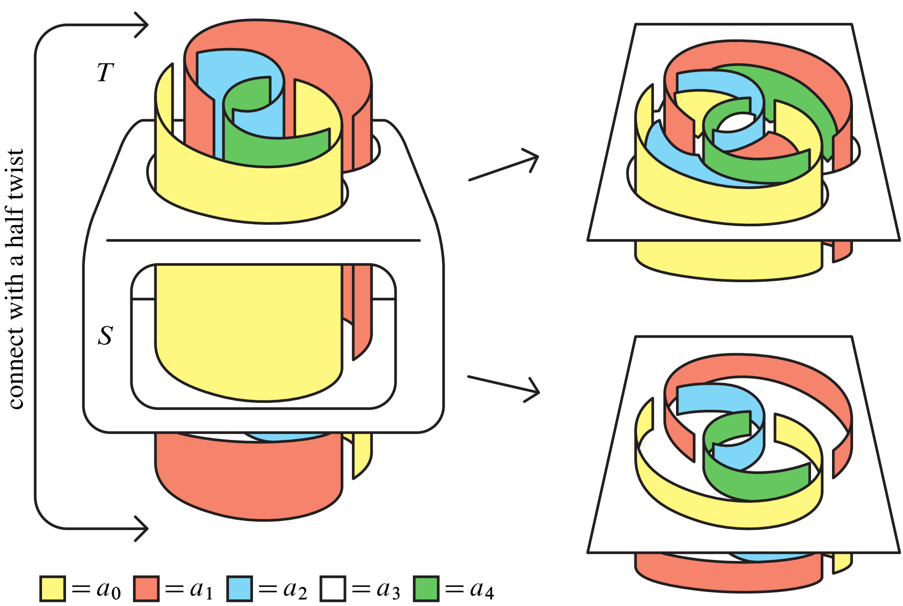


Figure 22

The diagram of F in the figure has eight triple points. For a suitable orientation of F, there is an $(R_5)_{\mathbb{Z}\times R_5}$–coloring such that the 3–cycle γ associated with the coloring is given as follows:

$$\begin{aligned}\gamma = &+(n,w;a_0,a_1,a_3)+(n,w;a_1,a_0,a_3)\\ &-(n,w;a_3,a_0,a_1)-(n,w;a_3,a_1,a_0)\\ &+(n+1,w^{a_0};a_2,a_1,a_4)-(n+1;w^{a_0};a_1,a_4,a_3)\\ &+(n+1,w^{a_1};a_4,a_0,a_2)-(n+1,w^{a_1};a_0,a_2,a_3).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\langle\varphi(\gamma),\theta_{\mathrm{M}}\rangle = &+(a_0-a_1)\frac{(a_1)^5+(a_0)^5-2(a_3)^5}{5}+(a_1-a_0)\frac{(a_0)^5+(a_1)^5-2(a_3)^5}{5}\\ &-(a_3-a_0)\frac{(a_0)^5+(a_2)^5-2(a_1)^5}{5}-(a_3-a_1)\frac{(a_1)^5+(a_4)^5-2(a_0)^5}{5}\\ &+(a_2-a_1)\frac{(a_1)^5+(a_2)^5-2(a_4)^5}{5}-(a_1-a_4)\frac{(a_4)^5+(a_2)^5-2(a_3)^5}{5}\\ &+(a_4-a_0)\frac{(a_0)^5+(a_4)^5-2(a_2)^5}{5}-(a_0-a_2)\frac{(a_2)^5+(a_4)^5-2(a_3)^5}{5}\\ =\ & s((a_0)^5+(a_1)^5-(a_2)^5-2(a_3)^5+(a_4)^5)\\ \equiv\ & s(a_0+a_1-a_2-2a_3+a_4)\equiv 2s^2\not\equiv 0 \pmod 5.\end{aligned}$$

Therefore, we have $\mathrm{t}(F)=8$.

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