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Spin structures on almost-flat manifolds

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We give a necessary and sufficient condition for almost-flat manifolds with cyclic holonomy to admit a Spin structure. Using this condition we find all 4–dimensional orientable almost-flat manifolds with cyclic holonomy that do not admit a Spin structure.

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1 Introduction

An *almost-flat* manifold is a closed manifold M with the property that for any $\epsilon > 0$ there exists a Riemannian metric g_ϵ on M such that $|K_\epsilon| \operatorname{diam}(M, g_\epsilon)^2 < \epsilon$, where K_ϵ is the sectional curvature and $\operatorname{diam}(M, g_\epsilon)$ is the diameter of M. In [10], Gromov gave a topological description of almost-flat manifolds, showing that every such manifold is finitely covered by a nilmanifold, ie it is a quotient of a connected, simply connected nilpotent Lie group by a uniform lattice. Ruh [17] later improved on Gromov's result by deducing that in fact every almost-flat manifold is infra-nil. Conversely, every infra-nilmanifold has an almost-flat structure, since it is finitely covered by a nilmanifold and every nilmanifold has an almost-flat structure (see Gromov [10] and Buser and Karcher [2]).

Given a connected and simply connected nilpotent Lie group N, the group of affine transformations of N is defined as $\operatorname{Aff}(N) = N \rtimes \operatorname{Aut}(N)$. This group acts on N by

$$(n, \phi) \cdot m = n\phi(m) \quad \text{for } m, n \in N \text{ and } \phi \in \operatorname{Aut}(N).$$

Let C be a maximal compact subgroup of $\operatorname{Aut}(N)$ and consider the subgroup $N \rtimes C$ of $\operatorname{Aff}(N)$. A discrete subgroup $\Gamma \subset N \rtimes C$ that acts cocompactly on N is called an *almost-crystallographic group*. In addition, if Γ is torsion-free then it is said to be *almost-Bieberbach*. In this case, the quotient N/Γ is a closed manifold called an *infra-nilmanifold* (modeled on N). If in addition $\Gamma \subset N$, then N/Γ is called a *nilmanifold*.

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Almost-flat manifolds occur naturally in the study of Riemannian manifolds with negative sectional curvature. It is well-known that every complete noncompact finite-volume manifold with pinched negative sectional curvature has finitely many cusps, all of which are diffeomorphic to manifolds of the form $M \times [0,\infty)$, where M is an almost-flat manifold (see Buser and Karcher [2, Section 1]). They also play a crucial role in the study of collapsing manifolds with uniformly bounded sectional curvature. By a deep theorem of Cheeger, Fukaya and Gromov [3], if a manifold is sufficiently collapsed relative to the size of its diameter, then it admits a local fibration structure whose fibers are almost-flat manifolds.

In this paper we study the problem of determining the existence of Spin structures on almost-flat manifolds. The existence of Spin structures on flat manifolds and related invariants have been investigated by the third author and others for the special case of flat manifolds (see for example Dekimpe, Sadowski and Szczepański [6], Gąsior and Szczepański [8], Hiss and Szczepański [11], Miatello and Podestá [13; 14], Putrycz and Szczepański [16], and Szczepański [19]). Our results represent the first modest step towards understanding this problem in the more general setting of almost-flat manifolds.

We will always assume that an almost-flat manifold comes equipped with the structure of an infra-nilmanifold when discussing its topological properties. Before stating our main result, let us recall the definition of a Spin structure on a smooth orientable manifold.

We denote by $\mathrm{SO}(n)$ the real special orthogonal group of rank n and by $\mathrm{Spin}(n)$ its universal covering group. We also write $\lambda_n\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ for the (double) covering homomorphism. A *Spin structure* on a smooth orientable manifold M is an equivariant lift of its orthonormal frame bundle via the covering λ_n. The existence of such a lift is equivalent to the existence of a lift $\widetilde{\tau}\colon M \to B\,\mathrm{Spin}(n)$ of the classifying map of the tangent bundle $\tau\colon M \to B\,\mathrm{SO}(n)$ such that $B_{\lambda_n} \circ \widetilde{\tau} = \tau$. Equivalently, M has a Spin structure if and only if the second Stiefel–Whitney class $w_2(TM)$ vanishes (see Kirby [12, pages 33–34]).

It is well-known that infra-nilmanifolds are classified by their fundamental group which is almost-crystallographic. A classical result of Auslander [1] asserts that every almost-crystallographic subgroup $\Gamma \subset \mathrm{Aff}(N)$ fits into an extension

$$1 \to \Lambda \to \Gamma \xrightarrow{q} F \to 1,$$

where $\Lambda = \Gamma \cap N$ is a uniform lattice in N and F is a finite subgroup of C called the *holonomy group* of the corresponding infra-nilmanifold N/Γ. The conjugation action of Γ on Λ induces an action of the holonomy group F on the factor groups of the adapted lower central series (see (1)) of the nilpotent lattice Λ. This gives us a representation $\theta\colon F \hookrightarrow \mathrm{GL}(n,\mathbb{Z})$, where n is the dimension of N.

Main theorem *Let M be an almost-flat manifold with holonomy group F. Then M is orientable if and only if $\det\theta = 1$. Suppose M is orientable and a 2–Sylow subgroup of F is cyclic, ie $C_{2^m} = \langle t \mid t^{2^m} = 1\rangle$ for some $m \geq 0$. Let Γ_{ab} denote the abelianization of the fundamental group Γ of M.*

(a) *If $\frac{1}{2}(n - \mathrm{Trace}(\theta(t)^{2^{m-1}})) \not\equiv 2 \pmod 4$, then M has a Spin structure.*

(b) *If $\frac{1}{2}(n - \mathrm{Trace}(\theta(t)^{2^{m-1}})) \equiv 2 \pmod 4$, then M has a Spin structure if and only if the epimorphism $q_*\colon \Gamma_{ab} \to C_{2^m}$ induced by the projection $q\colon \Gamma \to C_{2^m}$ factors through a cyclic group of order 2^{m+1}.*

The conditions arising in the theorem are quite practical to check given a finite presentation of the fundamental group of the almost-flat manifold, ie the associated almost-Bieberbach group. We illustrate this by finding all 4–dimensional almost-flat manifolds whose holonomy group has a cyclic 2–Sylow subgroup that do not admit a Spin structure.

Corollary *There are exactly four families of 4–dimensional almost-flat manifolds with cyclic holonomy group that do not admit a Spin structure. In each family, any two distinct almost-Bieberbach groups Γ_1 and Γ_2 are modeled on the same nilpotent Lie group N but have nonisomorphic nilpotent sublattices, $\Gamma_1 \cap N \ncong \Gamma_2 \cap N$. The holonomy group is always isomorphic to C_2.*

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2 Results

We first show that the classifying map of the tangent bundle of an almost-flat manifold M factors through the classifying space of the holonomy group F and is induced by a representation $\rho\colon F \to \mathrm{O}(n)$. Let us describe this representation.

Define $\mathfrak{n}$ to be the Lie algebra corresponding to the nilpotent Lie group N modeling M. Since N is a connected and simply connected nilpotent Lie group, the differential defines an isomorphism $d\colon \mathrm{Aut}(N) \to \mathrm{Aut}(\mathfrak{n})$. Choose an inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$.

Since $d(C)$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{n})$, we can define a new inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathfrak{n}$ that is also invariant under the action of $d(C)$ by letting

$$\langle\!\langle v, w\rangle\!\rangle = \int_{d(C)} \langle xv, xw\rangle \mu(x) \quad \text{for } v, w \in \mathfrak{n},$$

where μ is a left-invariant Haar measure on $d(C)$.

Now we select basis on $\mathfrak{n}$ orthonormal with respect to the new inner product. Identifying this basis with the standard basis in $\mathbb{R}^n$ defines a vector space isomorphism $\eta\colon \mathfrak{n} \to \mathbb{R}^n$ and a monomorphism $\delta\colon \mathrm{Aut}(\mathfrak{n}) \to \mathrm{GL}(n)$ such that $\delta \circ d(C) \subseteq \mathrm{O}(n)$. We define $\rho\colon F \hookrightarrow \mathrm{O}(n)$ by restricting the domain and the codomain of the composite homomorphism

$$C \hookrightarrow \mathrm{Aut}(N) \xrightarrow{d} \mathrm{Aut}(\mathfrak{n}) \xrightarrow{\delta} \mathrm{GL}(n)$$

to F and $\mathrm{O}(n)$, respectively. It is crucial to note that ρ is well-defined up to isomorphism of representations. That is, for a different choice of the inner product and the orthonormal basis of $\mathfrak{n}$, one obtains a representation that is isomorphic to $\rho\colon F \hookrightarrow \mathrm{O}(n)$.

Proposition 2.1 *Let M be an n–dimensional almost-flat manifold modeled on a connected and simply connected nilpotent Lie group N. Denote by Γ the fundamental group of M and let*

$$1 \to \Lambda \to \Gamma \xrightarrow{q} F \to 1$$

be the standard extension of Γ. Then the classifying map $\tau\colon M \to B\mathrm{O}(n)$ of the tangent bundle of M factors through BF and is induced by a composite homomorphism

$$\rho \circ q\colon \Gamma \to F \xrightarrow{\rho} \mathrm{O}(n).$$

Proof Let $\rho\colon F \hookrightarrow \mathrm{O}(n)$ be the representation constructed above. This yields a map of the classifying spaces $B_\rho\colon BF \to B\mathrm{O}(n)$ that is well-defined up to homotopy. Denote by σ the pullback of the universal n–dimensional vector bundle on $B\mathrm{O}(n)$ under the map B_ρ. Its total space is the Borel construction $EF \times_F \mathbb{R}^n$, ie the quotient of $EF \times \mathbb{R}^n$ by the action of F given by $f \cdot (x, v) = (fx, \rho(f)v)$ for all $f \in F$ and $(x, v) \in EF \times \mathbb{R}^n$.

We claim that the pullback bundle $B_q^*(\sigma)$ of σ under the map $B_q\colon B\Gamma \to BF$ is isomorphic to tangent bundle $TM \to M$. To see this, let $L_g\colon N \to N$, $h \mapsto gh$ be the left multiplication by an element g in N. It is a standard fact from Lie group theory that the map

$$\phi\colon TN \to N \times \mathfrak{n}, \quad (g, v) \mapsto (g, dL_{g^{-1}}(v)) \quad \text{for } g \in N,\ v \in \mathfrak{n}$$

gives a trivialization of the tangent bundle of N. A quick computation shows that this map is equivariant with respect to the action of Γ on $N \times \mathfrak{n}$ given by

$$\gamma \cdot (g, v) = (\gamma g, d \circ q(\gamma)(v)) \quad \text{for } \gamma \in \Gamma \text{ and } (g, v) \in N \times \mathfrak{n}.$$

Hence, we obtain a commutative diagram

$$\begin{tikzcd} TN \arrow[r, "\phi"] \arrow[d, "/\Gamma"] & N \times \mathfrak{n} \arrow[d, "/\Gamma"] \\ TM \arrow[r, "\overline{\phi}"] & N \times_\Gamma \mathfrak{n} \end{tikzcd}$$

where the resulting map $\overline{\phi}\colon TM \to N \times_\Gamma \mathfrak{n}$ gives an isomorphism between the tangent bundle of M and $\overline{\mathrm{pr}}_1\colon N \times_\Gamma \mathfrak{n} \to N/\Gamma = M$. But since N is a model for $E\Gamma$, we also have a commutative diagram

$$\begin{tikzcd} N \times_\Gamma \mathfrak{n} \arrow[r, "\psi"] \arrow[d, "\overline{\mathrm{pr}}_1"] & EF \times_F \mathbb{R}^n \arrow[d, "\sigma"] \\ M \arrow[r, "B_q"] & BF \end{tikzcd}$$

for $\psi\colon N \times_\Gamma \mathfrak{n} \to EF \times_F \mathbb{R}^n$, $\{g, v\} \mapsto \{E_q(g), \eta(v)\}$, where $E_q\colon N \to EF$ is an equivariant map covering B_q. This finishes the claim and the proposition follows. □

Remark 2.2 If the manifold M is orientable, then in the statement of the proposition the structure group $\mathrm{O}(n)$ can be replaced by $\mathrm{SO}(n)$.

With the previous notation, we define the *classifying representation* of an oriented almost-flat manifold M to be the composite homomorphism $\rho \circ q\colon \Gamma \to \mathrm{SO}(n)$. Recall that it is well-defined up to isomorphism of representations.

Corollary 2.3 *Let M be an orientable almost-flat manifold of dimension n with fundamental group Γ. Then M has a Spin structure if and only if there exists a homomorphism $\epsilon\colon \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \circ \epsilon = \rho \circ q$.*

Proof The manifold M has a Spin structure if and only if the classifying map $\tau = B_{\rho \circ q}\colon M \to B\mathrm{O}(n)$ has a lift $\widetilde{\tau}\colon M \to B\,\mathrm{Spin}(n)$ such that $B_{\lambda_n} \circ \widetilde{\tau} = B_{\rho \circ q}$. Since $M = B\Gamma$, a homomorphism $\epsilon\colon \Gamma \to \mathrm{Spin}(n)$ satisfying $\lambda_n \circ \epsilon = \rho \circ q$ yields a map $B_\epsilon\colon M \to B\,\mathrm{Spin}(n)$ such that $B_{\lambda_n} \circ B_\epsilon = B_{\rho \circ q}$. Hence, M has a Spin structure.

For the other direction, assume M has a Spin structure. Then $w_2(TM) = 0$ and it is the image of the generator of $\mathrm{H}^2(B\,\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2$ under the homomorphism

$B^*_{\rho\circ q}\colon \mathrm{H}^2(B\,\mathrm{SO}(n),\mathbb{Z}_2)\to\mathrm{H}^2(B\Gamma,\mathbb{Z}_2)$. Let $\mathrm{SO}(n)^\delta$ denote the group $\mathrm{SO}(n)$ but with the discrete topology. Note that the Friedlander–Milnor conjecture holds for $\mathrm{SO}(n)$ (see [18]), ie the forgetful map $f\colon \mathrm{SO}(n)^\delta\to\mathrm{SO}(n)$ induces an isomorphism of cohomology groups of $B\,\mathrm{SO}(n)^\delta$ and $B\,\mathrm{SO}(n)$ with mod 2 coefficients. This implies that the homomorphism $B^*_{\rho\circ q}$ can be identified with $(\rho\circ q)^*\colon \mathrm{H}^2(\mathrm{SO}(n),\mathbb{Z}_2)\to\mathrm{H}^2(\Gamma,\mathbb{Z}_2)$ and therefore the image of the generator of $\mathrm{H}^2(\mathrm{SO}(n),\mathbb{Z}_2)$ is zero. Reinterpreting the statement using group extensions, gives us a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[r] & \mathrm{Spin}(n) \arrow[r, "\lambda_n"] & \mathrm{SO}(n) \\
\mathbb{Z}_2 \arrow[u, "\mathrm{id}"] \arrow[r] & \widetilde{\Gamma} \arrow[u, "\omega"] \arrow[r, "\pi"] & \Gamma \arrow[u, "\rho\circ q"'] \arrow[l, bend left, "s"]
\end{tikzcd}$$

where $\pi\circ s=\mathrm{id}_\Gamma$. Setting $\epsilon=\omega\circ s$ we have $\lambda_n\circ\epsilon=\rho\circ q$ as desired. □

Next we will show that the representation $\rho\colon F\hookrightarrow \mathrm{O}(n)$ is isomorphic in $\mathrm{GL}(n)$ to a representation that arises from the action of the holonomy group on the factor groups of a certain adapted lower central series of the nilpotent lattice Λ. This representation will turn out to be more suitable for applications.

To this end, we denote by

$$\Lambda=\gamma_1(\Lambda)>\gamma_2(\Lambda)>\cdots>\gamma_{c+1}(\Lambda)=1,$$

the lower central series of Λ, ie $\gamma_{i+1}(\Lambda)=[\Lambda,\gamma_i(\Lambda)]$ for $1\le i\le c$. By [5, Lemma 1.2.6], we have that $\sqrt[\Lambda]{\gamma_i(\Lambda)}=\Lambda\cap\gamma_i(N)$. By [5, Lemmas 1.1.2–3], the resulting *adapted lower central series*

$$\Lambda=\sqrt[\Lambda]{\gamma_1(\Lambda)}>\sqrt[\Lambda]{\gamma_2(\Lambda)}>\cdots>\sqrt[\Lambda]{\gamma_{c+1}(\Lambda)}=1 \tag{1}$$

has torsion-free factor groups

$$Z_i=\frac{\sqrt[\Lambda]{\gamma_i(\Lambda)}}{\sqrt[\Lambda]{\gamma_{i+1}(\Lambda)}},\quad 1\le i\le c.$$

Thus, each $Z_i\cong\mathbb{Z}^{k_i}$ for some positive integer k_i. Just as in the case when Λ is abelian, conjugation in Γ induces an action of the holonomy group F on each factor group Z_i. This gives a faithful representation

$$\theta\colon F\hookrightarrow \mathrm{GL}(k_1,\mathbb{Z})\times\cdots\times\mathrm{GL}(k_c,\mathbb{Z})\hookrightarrow\mathrm{GL}(n,\mathbb{Z}),\quad k_1+\cdots+k_c=n.$$

The representation is indeed faithful since its kernel is a finite unipotent group and is therefore trivial.

Proposition 2.4 *The representations $\theta \otimes \mathbb{R}\colon F \hookrightarrow \mathrm{GL}(n)$ and $\rho\colon F \hookrightarrow \mathrm{O}(n) \subset \mathrm{GL}(n)$ are isomorphic.*

Proof Since F is finite, it suffices to show that the two representations have equal characters (see [4, Corollary 30.14]). Let $\mathcal{C}\colon \mathrm{Aff}(N) \to \mathrm{Aut}(N)$ denote the homomorphism defined by the conjugation action of the group of affine transformations on the normal subgroup N. Note that restricted to the standard subgroup $\mathrm{Aut}(N)$ of $\mathrm{Aff}(N)$, this is just the identity homomorphism. Let $\exp\colon \mathfrak{n} \to N$ be the exponential map. Recall that for any homomorphism $\phi\colon N \to N$ there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{d\phi} & \mathfrak{n} \\ \Big\downarrow \exp & & \Big\downarrow \exp \\ N & \xrightarrow{\phi} & N. \end{array}$$

Moreover each subgroup $\gamma_i(N)$ in the lower central series of N is characteristic in N and one has $\exp(\gamma_i(\mathfrak{n})) = \gamma_i(N)$ (see [5, Lemma 1.2.5]).

Now we choose a Mal'cev basis for $\mathfrak{n}$ so that the images of its elements under the exponential map generate the lattice Λ. By construction, the subspaces $V_i = \eta(\gamma_i(\mathfrak{n}))$, $1 \leq i \leq c$ give us a filtration

$$0 = V_{c+1} \subset V_c \subset \cdots \subset V_1 = \mathbb{R}^n$$

with $\dim V_i = k_i + \cdots + k_c$ and each V_i is left invariant under the action by the image of the homomorphism $\delta\colon \mathrm{Aut}(\mathfrak{n}) \to \mathrm{GL}(n)$. For each $1 \leq i \leq c$, this defines a representation

$$\delta_i\colon \mathrm{Aut}(\mathfrak{n}) \to \mathrm{GL}(V_i / V_{i+1}).$$

Let $\widetilde{\rho}_i\colon \Gamma \to \mathrm{GL}(k_i)$ denote the composition

$$\Gamma \hookrightarrow \mathrm{Aff}(N) \xrightarrow{\mathcal{C}} \mathrm{Aut}(N) \xrightarrow{d} \mathrm{Aut}(\mathfrak{n}) \xrightarrow{\delta_i} \mathrm{GL}(V_i / V_{i+1}).$$

Since Λ is in the kernel of $\widetilde{\rho}_i$, it gives rises to the representation $\rho_i\colon F \to \mathrm{GL}(k_i)$. Since δ and $\delta_1 \oplus \cdots \oplus \delta_c$ have equal characters, ρ and $\rho_1 \oplus \cdots \oplus \rho_c$ have equal characters.

On the other hand, the representation θ is isomorphic to $\theta_1 \oplus \cdots \oplus \theta_c$ where $\theta_i\colon F \to \mathrm{GL}(k_i, \mathbb{Z})$ is induced from $\widetilde{\theta}_i\colon \Gamma \to \mathrm{GL}(k_i, \mathbb{Z})$ and the latter is defined by

$$\Gamma \xrightarrow{\mathcal{C}|_\Gamma} \mathrm{Aut}(\Lambda) \to \mathrm{GL}(k_1, \mathbb{Z}) \times \cdots \times \mathrm{GL}(k_c, \mathbb{Z}) \xrightarrow{\mathrm{pr}_i} \mathrm{GL}(k_i, \mathbb{Z}),$$

for each $1 \le i \le c$. So, to finish the proof it suffices to show that $\widetilde{\rho}_i$ and $\widetilde{\theta}_i \otimes \mathbb{R}$ have equal characters for each $1 \le i \le c$.

By taking a closer look at $\widetilde{\rho}_i$, it is not difficult to see that it is isomorphic to the composition

$$\Gamma \xrightarrow{\mathcal{C}|_\Gamma} \mathrm{Aut}(N) \to \mathrm{Aut}(\gamma_i(N)/\gamma_{i+1}(N)) \xrightarrow{d} \mathrm{GL}(\gamma_i(\mathfrak{n})/\gamma_{i+1}(\mathfrak{n}))$$

where we identify $\gamma_i(\mathfrak{n})/\gamma_{i+1}(\mathfrak{n})$ and V_i/V_{i+1} via the isomorphism $\eta\colon \mathfrak{n} \to \mathbb{R}^n$ and where the second homomorphism is the natural map arising from the action of the automorphism group of N on the lower central series of N.

On the other hand, the representation $\widetilde{\theta}_i$ can be defined by the composition

$$\Gamma \xrightarrow{\mathcal{C}|_\Gamma} \mathrm{Aut}(\Lambda) \to \mathrm{Aut}\Big(\sqrt[\Lambda]{\gamma_i(\Lambda)}/\sqrt[\Lambda]{\gamma_{i+1}(\Lambda)}\Big),$$

where the second homomorphism is the natural map arising from the action of the automorphism group of Λ on the adapted lower central series of Λ.

From the choice of the Mal'cev basis on $\mathfrak{n}$ and the fact that $\sqrt[\Lambda]{\gamma_i(\Lambda)}/\sqrt[\Lambda]{\gamma_{i+1}(\Lambda)}$ is a lattice in the Euclidean group $\gamma_i(N)/\gamma_{i+1}(N)$, it follows that $\widetilde{\theta}_i \otimes \mathbb{R}$ is isomorphic to the composition

$$\Gamma \xrightarrow{\mathcal{C}|_\Gamma} \mathrm{Aut}(N) \to \mathrm{Aut}(\gamma_i(N)/\gamma_{i+1}(N))$$

and hence to $\widetilde{\rho}_i$. This finishes the proof. □

Remark 2.5 It follows that the almost-flat manifold M is orientable if and only if the image of the representation $\theta\colon F \hookrightarrow \mathrm{GL}(n,\mathbb{Z})$ lies inside $\mathrm{SL}(n,\mathbb{Z})$.

Lemma 2.6 *Let M be an orientable almost-flat manifold with holonomy group F. Let S be a 2–Sylow subgroup of F and set $M(2) = N/q^{-1}(S)$. Then M has a* Spin *structure if and only if $M(2)$ has a* Spin *structure.*

Proof Recall that the second Stiefel–Whitney class $w_2(TM)$ is the obstruction for the existence of a Spin structure on M. The inclusion $i\colon q^{-1}(S) \hookrightarrow \Gamma$ induces a homomorphism $i^*\colon \mathrm{H}^2(M,\mathbb{Z}_2) \to \mathrm{H}^2(M(2),\mathbb{Z}_2)$. This is a monomorphism because $q^{-1}(S)$ is a subgroup of Γ of odd index. Since $w_2(TM(2)) = i^*(w_2(TM))$, we obtain that $w_2(TM) = 0$ if and only if $w_2(TM(2)) = 0$. □

We also need the following lemma that will help us determine whether almost-flat manifolds with cyclic holonomy group admit Spin structures.

Lemma 2.7 *Let* $A \in \mathrm{SO}(n)$ *be of order* 2^m, $m > 0$. *Then there is an element in* $\lambda_n^{-1}(\langle A \rangle)$ *of order* 2^{m+1} *if and only if*

$$\tfrac{1}{2}(n - \mathrm{Trace}(A^{2^{m-1}})) \equiv 2 \pmod 4.$$

Proof The case $m = 1$ is well-known (see [9; 7]). The general case follows easily from this case where we replace the matrix A with $A^{2^{m-1}}$. □

We are now ready to prove the main theorem.

Proof of Main theorem **(a)** Suppose $\frac{1}{2}(n - \mathrm{Trace}(\theta(t)^{2^{m-1}})) \not\equiv 2 \pmod 4$. Then, by Lemma 2.7 and Proposition 2.4, we have $\lambda_n^{-1}(\rho(C_{2^m})) \cong C_2 \times C_{2^m}$. So, the restriction $\lambda_n\colon \lambda_n^{-1}(\rho(C_{2^m})) \to \rho(C_{2^m})$ splits and hence the classifying homomorphism $\rho \circ q\colon \Gamma \to \mathrm{SO}(n)$ lifts to the universal covering group $\mathrm{Spin}(n)$ of $\mathrm{SO}(n)$. This, by Proposition 2.1, insures that M has a Spin structure.

(b) In view of Lemma 2.6, we can assume the C_{2^m} is the whole holonomy group of M. Thus, M has a Spin structure if and only if there is a lift $l\colon \Gamma \to \mathrm{Spin}(n)$ of the composite homomorphism

$$\rho \circ q\colon \Gamma \xrightarrow{q} C_{2^m} \xrightarrow{\rho} \mathrm{SO}(n).$$

But by our assumption and Lemma 2.6, the preimage $\lambda_n^{-1}(\rho(C_2))$ is isomorphic to $C_{2^{m+1}}$. This shows that there is lift $l\colon \Gamma \to \mathrm{Spin}(n)$ if and only if $q\colon \Gamma \to C_{2^m}$ factors through $C_{2^{m+1}}$ which happens if and only if $q_*\colon \Gamma_{ab} \to C_{2^m}$ factors through $C_{2^{m+1}}$. □

3 Applications

It is well-known that all closed orientable manifolds of dimension at most 3 have a Spin structure (see [12, page 35; 15, Exercise 12.B and VII, Theorem 2]). Next we give a list of 4–dimensional orientable almost-flat manifolds modeled on a connected, simply connected nilpotent Lie group N that cannot have a Spin structure. This list is complete in the sense that, up to dimension 4, it gives all possible examples of orientable almost-flat manifolds whose holonomy has a cyclic 2–Sylow subgroup not admitting a Spin structure (see [16]). In fact, we will see that in each of these examples the holonomy group is C_2. In contrast, all flat manifolds with holonomy C_2 have a Spin structure (see [11, Theorem 3.1(3)]).

For this purpose, we use the classification of the associated almost-Bieberbach groups given in [5, Sections 7.2 and 7.3].

3.1 N is 2–step nilpotent

The only family of almost-flat manifolds without a Spin structure are classified by number 5, $Q = C2$ on page 171 of [5].

For each integer $k > 0$, the almost-Bieberbach group Γ_k has the presentation

$$\Gamma_k = \left\langle a,b,c,d,\alpha \;\middle|\; \begin{array}{ccc} [b,a]=1, & [c,a]=d^k, & [d,a]=1, \\ [c,b]=d^k, & [d,b]=1, & [d,c]=1, \\ \alpha^2=d, & \alpha a\alpha^{-1}=b^{-1}, & \alpha b\alpha^{-1}=a^{-1}, \\ & \alpha d\alpha^{-1}=d, & \alpha c\alpha^{-1}=c^{-1} \end{array} \right\rangle$$

where $\Lambda = \langle a,b,c,d\rangle$ and $\sqrt[\Lambda]{[\Lambda,\Lambda]} = \langle d\rangle$. Since the representation $\theta\colon C_2 \hookrightarrow \mathrm{GL}(4,\mathbb{Z})$ arises from the conjugation by the element α on Λ, it is generated by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

which lies in $\mathrm{SL}(4,\mathbb{Z})$. So, by Remark 2.5, M_k is orientable for all $k > 0$.

The abelianization of Γ_k has the presentation

$$(\Gamma_k)_{ab} = \langle \bar{a}, \bar{c}, \bar{\alpha} \mid \bar{c}^2 = \bar{\alpha}^{2k} = 1\rangle = C_\infty \times C_2 \times C_{2k}.$$

The map $q_*\colon (\Gamma_k)_{ab} \to C_2$ can then be seen as the epimorphism arising from the projection of the C_{2k}–factor onto C_2. Therefore, it does not factor through C_4 if and only if k is odd. So, by the Main theorem(b), M_k does not have a Spin structure if and only if k is odd.

3.2 N is 3–step nilpotent

In this case, we find 3 families of almost-flat manifolds without a Spin structure.

The first family is classified by number 3, $Q = \langle(2l,1)\rangle$ on page 220 of [5]. For each $k,l > 0$, the associated almost-Bieberbach group $\Gamma_{k,l}$ has the presentation

$$\Gamma_{k,l} = \left\langle a,b,c,d,\alpha \;\middle|\; \begin{array}{ccc} [b,a]=c^{2l}d^{(2l-1)k}, & [c,a]=1, & [d,a]=1, \\ [c,b]=d^{2k}, & [d,b]=1, & [d,c]=1, \\ \alpha^2=d, & \alpha a=a\alpha c, & \alpha b=b^{-1}\alpha, \\ \alpha d\alpha^{-1}=d, & \alpha c\alpha^{-1}=c^{-1} & \end{array} \right\rangle$$

where $\Lambda = \langle a,b,c,d\rangle$, $\sqrt[\wedge]{[\Lambda,\Lambda]} = \langle c,d\rangle$ and $\sqrt[\wedge]{\gamma_3(\Lambda)} = \langle d\rangle$. The representation $\theta\colon C_2 \hookrightarrow \mathrm{GL}(4,\mathbb{Z})$ is generated by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which lies in $\mathrm{SL}(4,\mathbb{Z})$. So, $M_{k,l}$ is orientable for all $k > 0$.

The abelianization of $\Gamma_{k,l}$ has the presentation

$$(\Gamma_{k,l})_{ab} = \langle \bar{a}, \bar{b}, \bar{\alpha} \mid \bar{b}^2 = \bar{\alpha}^{2k} = 1\rangle = C_\infty \times C_2 \times C_{2k}.$$

The map $q_*\colon (\Gamma_{k,l})_{ab} \to C_2$ is the epimorphism arising from the projection of the C_{2k}–factor onto C_2. Therefore, it does not factor through C_4 if and only if k is odd. So, by the Main theorem(b), $M_{k,l}$ does not have a Spin structure if and only if k is odd.

The second family is classified by number 5, $Q = \langle(2l,0)\rangle$, on page 222 of [5]. For each $k,l > 0$, the associated almost-Bieberbach group $\Gamma_{k,l}$ has the presentation

$$\Gamma_{k,l} = \left\langle a,b,c,d,\alpha \;\middle|\; \begin{array}{ccc} [b,a] = c^{2l}, & [c,a] = d^{k}, & [d,a] = 1, \\ [c,b] = d^{-k}, & [d,b] = 1, & [d,c] = 1, \\ \alpha^2 = d, & \alpha a = b\alpha, & \alpha b = a\alpha, \\ & \alpha d\alpha^{-1} = d, & \alpha c\alpha^{-1} = c^{-1} \end{array} \right\rangle$$

where $\Lambda = \langle a,b,c,d\rangle$, $\sqrt[\wedge]{[\Lambda,\Lambda]} = \langle c,d\rangle$ and $\sqrt[\wedge]{\gamma_3(\Lambda)} = \langle d\rangle$. The representation $\theta\colon C_2 \hookrightarrow \mathrm{GL}(4,\mathbb{Z})$ is generated by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which lies in $\mathrm{SL}(4,\mathbb{Z})$. So, $M_{k,l}$ is orientable for all $k > 0$.

The abelianization of $\Gamma_{k,l}$ has the presentation

$$(\Gamma_{k,l})_{ab} = \langle \bar{a}, \bar{c}, \bar{\alpha} \mid \bar{c}^2 = \bar{\alpha}^{2k} = 1\rangle = C_\infty \times C_2 \times C_{2k}.$$

The map $q_*\colon (\Gamma_{k,l})_{ab} \to C_2$ is the epimorphism resulting from projection of the C_{2k}–factor onto C_2. Therefore, it does not factor through C_4 if and only if k is odd. So, by the Main theorem(b) $M_{k,l}$ does not have a Spin structure if and only if k is odd.

The third family is classified by number 5, $Q = \langle (2l+1, 0) \rangle$, on page 222 of [5]. For each $k, l > 0$, the associated almost-Bieberbach group $\Gamma_{k,l}$ has the presentation

$$\Gamma_{k,l} = \left\langle a, b, c, d, \alpha \;\middle|\; \begin{array}{ccc} [b,a] = c^{2l+1}, & [c,a] = d^k, & [d,a] = 1, \\ [c,b] = d^{-k}, & [d,b] = 1, & [d,c] = 1, \\ \alpha^2 = d, & \alpha a = b\alpha, & \alpha b = a\alpha, \\ & \alpha d \alpha^{-1} = d, & \alpha c \alpha^{-1} = c^{-1} \end{array} \right\rangle$$

where $\Lambda = \langle a, b, c, d \rangle$, $\sqrt[\wedge]{[\Lambda, \Lambda]} = \langle c, d \rangle$ and $\sqrt[\wedge]{\gamma_3(\Lambda)} = \langle d \rangle$. The representation $\theta\colon C_2 \hookrightarrow \mathrm{GL}(4, \mathbb{Z})$ is generated by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which lies in $\mathrm{SL}(4, \mathbb{Z})$. So, $M_{k,l}$ is orientable for all $k > 0$.

The abelianization of $\Gamma_{k,l}$ has the presentation

$$(\Gamma_{k,l})_{ab} = \langle \bar{a}, \bar{\alpha} \mid \bar{\alpha}^{2k} = 1 \rangle = C_\infty \times C_{2k}.$$

The map $q_*\colon (\Gamma_{k,l})_{ab} \to C_2$ can then be seen as the epimorphism arising from projection of the C_{2k}–factor onto C_2. Therefore, it does not factor through C_4 if and only if k is odd. So, by the Main theorem(b), $M_{k,l}$ does not have a Spin structure if and only if k is odd.

[5, §7.2–3]	Q	class	Γ_{ab}	holonomy	parameters
5, page 171	$C2$	2	$C_\infty^2 \times C_{2k}$	C_2	k odd
3, page 220	$\langle (2l, 1) \rangle$	3	$C_\infty \times C_2 \times C_{2k}$	C_2	$l > 0$, k odd
5, page 222	$\langle (2l, 0) \rangle$	3	$C_\infty \times C_2 \times C_{2k}$	C_2	$l > 0$, k odd
5, page 222	$\langle (2l+1, 0) \rangle$	3	$C_\infty \times C_{2k}$	C_2	$l > 0$, k odd

Table 1: Almost-flat manifolds without Spin structures

We now summarize our investigations (see Table 1). Every 4–dimensional almost-Bieberbach group Γ fits into an extension

$$0 \to \mathbb{Z} \to \Gamma \to Q \to 1,$$

where Q is a 3–dimensional almost-crystallographic group (see [5, Section 6.3]). If N is 2–step nilpotent, then Q is in fact a crystallographic group. The first column of the table indicates the number of the associated almost-crystallographic group Q as shown in [5, Section 7.2–3] and the page number in [5] where the presentation of

Γ is given. The second column gives the classification of Q as in the International Tables for Crystallography (I T) or as in [5, Section 7.1]. The third column indicates the nilpotency class of the group N on which the almost-flat manifold is modeled. Columns four and five show the abelianization and the holonomy group, respectively. The last column indicates the exact parameters for which the associated almost-flat manifold cannot admit a Spin structure.

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