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Equivariant Schubert calculus

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Abstract. We describe T-equivariant Schubert calculus on $G(k,n)$, T being an n-dimensional torus, through derivations on the exterior algebra of a free A-module of rank n, where A is the T-equivariant cohomology of a point. In particular, T-equivariant Pieri's formulas will be determined, answering a question raised by Lakshmibai, Raghavan and Sankaran (Equivariant Giambelli and determinantal restriction formulas for the Grassmannian, *Pure Appl. Math. Quart.* **2** (2006), 699–717).

1. Introduction

In this paper we deal with $G(k,n)$, the complex Grassmannian variety parameterizing k-dimensional vector subspaces of $\mathbb{C}^n$, equipped with a certain linear action of an n-dimensional torus T (either $(S^1)^n$ or $(\mathbb{C}^*)^n$) such that the fixed point locus is isolated. It turns out that $H_T^*(G(k,n))$, the integral T-equivariant cohomology ring of $G(k,n)$, is a finite free module over the ring $A:=H_T^*(\mathrm{pt})$, the T-equivariant cohomology of a point. Our main result (Theorem 2.2) shows that the multiplicative structure of the A-algebra $H_T^*(G(k,n))$, for all $1\le k\le n$, can be described through derivations on the Grassmann algebra of a free A-module of rank n, in the same spirit as [2], [5], [6] and [7]. We stress that the description of the product structure of $H_T^*(G(k,n))$ heavily depends on the particular choice of an A-basis for it. If $\sigma:=(\sigma_I)$ is any such A-basis, where I runs over a certain finite index set, the product structure of $H_T^*(G(k,n))$ concerns the structure constants $C_{IJ}^K(\sigma)$ in the product expansion $\sigma_I\sigma_J=\sum_K C_{IJ}^K(\sigma)\sigma_K$. The notation reflects the dependence of the structure constants on the basis σ. As the considered T-action on the Grassmannian has only isolated fixed points, it turns out that $G(k,n)$ can be decomposed into the disjoint union of T-invariant complex cells $\{B_I\}$. A natural A-basis

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$Y:=\{Y_I\}$ for $H^*_T(G(k,n))$ is then obtained by letting Y_I be the equivariant cohomology class of the closure of B_I. Such a basis Y is used in the important work [9], by Knutson and Tao, which sheds light on the deep relationship between the beautiful combinatorics of puzzles and the product structure of the ring $H^*_T(G(k,n))$. The main result of [9], as the authors put it, is that "puzzles compute equivariant Schubert calculus", i.e., more precisely, the constant structures $C^K_{IJ}(Y)\in A$. Alternatively we may phrase our main result by saying that *derivations on a Grassmann algebra compute (equivariant) Schubert calculus*. Still using the basis Y, Mihalcea achieves, in a couple of important papers ([16] and [17]), a detailed description of the equivariant small quantum deformation of $H^*_T(G(k,n))$. In particular (see [17]) he produces a quantum equivariant Giambelli's formula, exploiting the combinatorics of the factorial Schur functions. In [10], instead, the authors aim to a formulation of T-equivariant Schubert calculus closer to the classical one, based on "equivariant versions" of Giambelli's and Pieri's formulas. Indeed they prove a determinantal formula for the restriction to a torus fixed point of each element Y_I of the basis Y.

An elegant, new and illuminating approach is that pursued by Laksov ([12]; see also [11]): based on a previous joint work with Thorup [13], [14], he shows that the T-equivariant Schubert calculus on $G(k,n)$ is recovered by the unique symmetric structure on the kth exterior power of $H^*_T(\mathbb{P}^{n-1})$, the quotient of the ring $A[X]$ by a certain monic polynomial of degree n.

Our work is mostly related to Laksov's, but we put the emphasis on the exterior algebra of $A[X]$ rather than on a single exterior power. In our formulation, Schubert calculus takes place in a subring $\mathcal{A}^*(\bigwedge M)$ of the ring of endomorphisms of the exterior algebra of a free A-module M of rank n, as in [7], generated by derivations of $\bigwedge M$. The (equivariant) Schubert calculus on $G(k,n)$ is obtained by considering the $\mathcal{A}^*(\bigwedge M)$-module structure of $\bigwedge^k M$, the degree-k direct summand of $\bigwedge M$.

The paper is organized as follows. We put our main result at the end of Section 2, whose first part lists the basics we believe are necessary to keep the paper as self-contained as possible. In Section 3 we write down a full set of "equivariant" Pieri's formulas (proposing a solution, analogous to Laksov's one, for the question raised in [10]) recovering, in particular, the equivariant Pieri's rule as in [9], p. 237. As most expositions on (equivariant) Schubert calculus do, including [9] and [16], we give a quick look, at the end of the paper, to the example of $G(2,4)$, the first Grassmannian which is not a projective space. We revisit the example at p. 231 of [9], depicting the basis of the equivariant cohomology of $G(2,4)$, using solely Leibniz's rule and a bit of *integration by parts* (if one wants to avoid the general Giambelli's formula due to Laksov and Thorup [13], Theorem 0.1 (2)).

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2. Preliminaries and notation

2.1. *T*-equivariant cohomology

Let T be an n-dimensional torus, either $T=(S^1)^n$ or $T=(\mathbb{C}^*)^n$. The ring $A:=H_T^*(\mathrm{pt})$, the integral equivariant cohomology of a point, coincides with the polynomial ring $\mathbb{Z}[y_1,...,y_n]$. Let $A_i:=H_T^i(\mathrm{pt})$. Then, for each $i\geq 0$, $A_{2i+1}=0$ and A_{2i} is the $\mathbb{Z}$-submodule of A generated by the monomials of degree i. Thus $A=\bigoplus_{i\geq 0} A_i$. We apply the theory exposed in [7] to the T-equivariant cohomology of $G(k,n)$, in the case that the T-action is induced by the diagonal linear action

$$(t_1,...,t_n)(z_1,...,z_n)=(t_1z_1,...,t_nz_n)$$

of T on $\mathbb{C}^n$. If F is an equivariant vector bundle over $G(k,n)$, denote by $c^T(F)$ its equivariant Chern polynomial $\sum_{i\geq 0} c_i^T(F)t^i$, where, for each $i\geq 0$, $c_i^T(F)$ is the T-equivariant ith Chern class of F. All the bundles occurring in the tautological exact sequence over $G(k,n)$, $0\to S_k\to F_k\to Q_k\to 0$, are equivariant and then it makes sense to speak of their equivariant Chern classes. Set $c^T(Q_k-F_k)=c^T(Q_k)/c^T(F_k)$, the ratio taken into the ring of formal power series with coefficients in $H_T^*(G(k,n))$. The equality

$$c^T(Q_k-F_k)=\sum_{i\geq 0} c_i(Q_k-F_k)t^i$$

defines $c_i(Q_k-F_k)$ (see [4] for further foundational details).

As is well known, $H_T^*(\mathbb{P}^{n-1}):=H_T^*(G(1,n))$ is a free A-module of rank n, spanned by $1,\xi,...,\xi^{n-1}$, where $\xi:=c_1^T(O_{\mathbb{P}^{n-1}}(1))$, the first T-equivariant Chern class of the hyperplane bundle. Let $\mathtt{p}$ be the minimal polynomial of the "multiplying-by-ξ" endomorphism of $H_T^*(\mathbb{P}^{n-1})$, i.e. $\mathtt{p}\in A[X]$ is the monic polynomial of minimal degree such that $\mathtt{p}(\xi)=0$ in $H_T^*(\mathbb{P}^{n-1})$. Then, obviously, the map $A[X]\to H_T^*(\mathbb{P}^{n-1})$, sending $X\mapsto\xi$, is an epimorphism with kernel $\mathtt{p}$, inducing an isomorphism

$$\iota_1\colon A[X]/\mathtt{p}\longmapsto H_T^*(\mathbb{P}^{n-1}).$$

2.2. Schubert calculus on a Grassmann algebra

Let $M{:=}XA[X]$ and $M(\mathtt{p}){=}M/\mathtt{p}M$. The quotient $M(\mathtt{p})$ is a free A-module generated by $\varepsilon^i{:=}X^i{+}\mathtt{p}M$, $i{\geq}1$: in particular an A-basis of $M(\mathtt{p})$ is $\boldsymbol{\varepsilon}{:=}(\varepsilon^1,...,\varepsilon^n)$. Furthermore, $M(\mathtt{p})$ is a free $A[X]/\mathtt{p}$ module of rank 1 generated by ε^1. For each $h{\geq}0$ define an A-endomorphism s_h of $M(\mathtt{p})$ through the equality

$$s_h\varepsilon^j = \left(X^h{+}\mathtt{p}\right)\left(X^j{+}\mathtt{p}\right) = X^{h+j}{+}\mathtt{p} = \varepsilon^{i+j},$$

for each $j{\geq}1$ (in the equalities above $X^h{+}\mathtt{p}$ is seen as an element of $A[X]/\mathtt{p}$). Let k be a positive integer and let $\mathcal{I}^k_n$ be the set of all strictly increasing sequences of positive integers not bigger than n. Then $\bigwedge^k M(\mathtt{p})$, the kth exterior power of $M(\mathtt{p})$, is freely generated over A by $\bigwedge^k \boldsymbol{\varepsilon}{:=}\{\bigwedge^I \boldsymbol{\varepsilon} \mid I{\in}\mathcal{I}^k_n\}$ where, if $I{:=}(i_1,...,i_k){\in}\mathcal{I}^k_n$, the notation $\bigwedge^I \boldsymbol{\varepsilon}$ stands for $\varepsilon^{i_1}{\wedge}...{\wedge}\varepsilon^{i_k}$.

Let $\bigwedge M(\mathtt{p}){:=}\bigoplus_k \bigwedge^k M(\mathtt{p})$ be the exterior algebra of $M(\mathtt{p})$. Identify $M(\mathtt{p})$ with the submodule $\bigwedge^1 M(\mathtt{p})$ of $\bigwedge M(\mathtt{p})$. By [7], Proposition 2.7, there exists a unique sequence $(D_0,D_1,...)$ of A-endomorphisms of $\bigwedge M(\mathtt{p})$ such that, for each $\alpha,\beta{\in}\bigwedge M(\mathtt{p})$, each $h{\geq}0$ and $i{\geq}1$, (the hth order) Leibniz's rule

$$D_h(\alpha{\wedge}\beta) = \sum_{\substack{h_1+h_2=h\\ h_1,h_2\geq 0}} D_{h_1}\alpha{\wedge}D_{h_2}\beta \tag{1}$$

holds and the initial conditions $D_{h|_{M(\mathtt{p})}}{=}s_h$ are satisfied. If one defines

$$D_t := \sum_{i\geq 0} D_i t^i : \ \bigwedge M(\mathtt{p}) \longrightarrow \bigwedge M(\mathtt{p})[[t]],$$

equation (1) can be more compactly phrased by saying that D_t is an A-algebra homomorphism ([5], [6], [7], [18]), i.e., more explicitly, that

$$D_t(\alpha{\wedge}\beta) = D_t\alpha{\wedge}D_t\beta.$$

2.3. The Poincaré isomorphism

If $A[\mathbf{T}]{:=}A[T_1,T_2,...]$ is the polynomial ring in infinitely many indeterminates, let $\mathcal{A}^*(\bigwedge M(\mathtt{p}))$ be its image in $\mathrm{End}_A(\bigwedge M(\mathtt{p}))$ via the natural evaluation map $T_i{\mapsto}D_i$. It is a commutative A-subalgebra of $\mathrm{End}_A(\bigwedge M(\mathtt{p}))$. Let

$$\mathrm{ev}_{\varepsilon^1\wedge...\wedge\varepsilon^k} : \mathcal{A}^*\left(\bigwedge M(\mathtt{p})\right) \longrightarrow \bigwedge^k M(\mathtt{p})$$

be the A-module homomorphism given by $P(D)\mapsto P(D)\varepsilon^1\wedge\ldots\wedge\varepsilon^k$, $P\in A[\mathbf{T}]$. By [7], the map $\mathrm{ev}_{\varepsilon^1\wedge\ldots\wedge\varepsilon^k}$ is surjective. This can be seen either by *integration by parts* (as in [7]) or by noticing, as in [13], that Giambelli's formula

$$\bigwedge^I \boldsymbol{\varepsilon}=\Delta_I(D)\cdot\varepsilon^1\wedge\ldots\wedge\varepsilon^k \tag{2}$$

holds for each $I\in\mathcal{I}_n^k$, where if $I=(i_1,\ldots,i_k)$ then $\Delta_I(D):=\det(D_{i_j-i})\in\mathcal{A}^*(\bigwedge M(\mathtt{p}))$. There is an A-module isomorphism (*Poincaré isomorphism*)

$$\Pi_k\colon \mathcal{A}^*\left(\bigwedge^k M(\mathtt{p})\right)\longrightarrow\bigwedge^k M(\mathtt{p}),$$

where $\mathcal{A}^*(\bigwedge^k M(\mathtt{p})):=\mathcal{A}^*(\bigwedge M(\mathtt{p}))/\ker\mathrm{ev}_{\varepsilon^1\wedge\ldots\wedge\varepsilon^k}$. For each A-basis $\boldsymbol{\nu}=(\nu^1,\ldots,\nu^n)$ of $M(\mathtt{p})$ and each $I\in\mathcal{I}_n^k$, let $G_I^{\boldsymbol{\nu}}(D)=\Pi_k^{-1}(\bigwedge^I\boldsymbol{\nu})$. Then $G^{\boldsymbol{\nu}}(D):=\{G_I^{\boldsymbol{\nu}}(D)\,|\,I\in\mathcal{I}_n^k\}$ is an A-basis of the ring $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))$. A representative in $\mathcal{A}^*(\bigwedge M(\mathtt{p}))$ of $G_I^{\boldsymbol{\nu}}(D)$ can be explicitly written via the general determinantal formula by Laksov and Thorup ([13], Theorem 0.1 (2)). In particular, if $\boldsymbol{\nu}=\boldsymbol{\varepsilon}$, one may take $G_I^{\boldsymbol{\varepsilon}}(D)$ as the class of $\Delta_I(D)$ modulo $\ker\mathrm{ev}_{\varepsilon^1\wedge\ldots\wedge\varepsilon^k}$.

2.4. Schubert calculus on an exterior power

Let $C_{IJ}^K(\boldsymbol{\nu})\in A$ be defined by the equality $G_I^{\boldsymbol{\nu}}(D)G_J^{\boldsymbol{\nu}}(D)=\sum_{K\in\mathcal{I}_n^k}C_{IJ}^K(\boldsymbol{\nu})G_K^{\boldsymbol{\nu}}(D)$ for each triple $I,J,K\in\mathcal{I}_n^k$. Since each $G_I^{\boldsymbol{\nu}}(D)$ is a polynomial expression in $(D_1, D_2,\ldots)$, it turns out that $\{C_{IJ}^K(\boldsymbol{\nu})\}$ can be computed once one knows a ***ν**-Pieri's formula*, i.e. a rule to expand the product $D_hG_I^{\boldsymbol{\nu}}(D)$ as a linear combination of elements of $G^{\boldsymbol{\nu}}(D)$. Notice that if $D_hG_I^{\boldsymbol{\nu}}(D)=\sum_J C_{h,I}^J(\boldsymbol{\nu})G_J^{\boldsymbol{\nu}}(D)$, then

$$D_h\left(\bigwedge^I\boldsymbol{\nu}\right)=D_hG_I^{\boldsymbol{\nu}}(D)\nu^1\wedge\ldots\wedge\nu^k=\sum_J C_{hI}^J(\boldsymbol{\nu})G_J^{\boldsymbol{\nu}}(D)\nu^1\wedge\ldots\wedge\nu^k$$
$$=\sum_J C_{hI}^J(\boldsymbol{\nu})\cdot\bigwedge^J\boldsymbol{\nu}$$

i.e. a $\boldsymbol{\nu}$-Pieri's formula in the ring $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))$ amounts to knowing the expansion of $D_h(\bigwedge^I\boldsymbol{\nu})$ as a linear combination of the elements of the basis $\bigwedge^k\boldsymbol{\nu}$. If $I,H\in\mathbb{N}^k$, denote by $I+H$ their componentwise sum. When $\boldsymbol{\nu}=\boldsymbol{\varepsilon}$, Pieri's formula looks the same as that holding in the classical (i.e. non-equivariant) case (see e.g. [5]).

Theorem 2.1. ($\boldsymbol{\varepsilon}$-Pieri's formula ([5] and [13])) *For each $h\geq 0$ and each $I:=(i_1,\ldots,i_k)\in\mathcal{I}_n^k$,*

$$D_h\bigwedge^I\boldsymbol{\varepsilon}=\sum_{H\in\mathcal{P}(I,h)}\bigwedge^{I+H}\boldsymbol{\varepsilon}, \tag{3}$$

where $\mathcal{P}(I,h)$ denotes the set of all $H{:=}(h_1,\dots,h_k){\in}\mathbb{N}^k$ such that $h_1{+}\dots{+}h_k{=}h$ and

$$1\le i_1\le i_1+h_1<i_2\le i_2+h_2<\dots\le i_{k-1}+h_{k-1}<i_k.$$

Abusing notation, denote by the same symbol both $\Delta_I(D)\in\mathcal{A}^*(\bigwedge M(\mathtt{p}))$ and its class modulo $\ker(\varepsilon^1\wedge\dots\wedge\varepsilon^k)$ (cf. Section 2.3). Then, by (2), $\{\Delta_I(D)\,|\,I\in\mathcal{I}_n^k\}$ is an A-basis of $\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))$ and the A-linear extension of the map sending $\Delta_I(D)\mapsto\Delta_I(c^T(Q_k-F_k))$ defines an A-module homomorphism

$$\iota_k\colon \mathcal{A}^*\left(\bigwedge^k M(\mathtt{p})\right)\longmapsto H_T^*(G(k,n)). \tag{4}$$

Our main result, below, relates derivations on a Grassmann algebra to the equivariant cohomology of $G(k,n)$.

Theorem 2.2. *The A-module homomorphism (4) is an A-algebra isomorphism.*

Proof. Let $\delta_I=\Delta_I(c^T(Q_k-F_k))$. Using well-known approximations of the equivariant cohomology of $G(k,n)$ by the ordinary cohomology of finite-dimensional spaces [1], [4], one easily shows that δ_I is indeed an A-basis of $H_T^*(G(k,n))$ and that the same kind of Pieri's rule as displayed in [3], p. 266 (by replacing partitions of length $\le k$ with elements of $\mathcal{I}_n^k$) holds in the equivariant case:

$$c_h^T(Q_k-F_k)\cup\delta_I=\sum_{H\in\mathcal{P}(I,h)}\delta_{I+H}.$$

We are hence left to prove that $\iota_k(\Delta_I(D)\cdot\Delta_J(D))=\iota_k(\Delta_I(D))\cup_T\iota_k(\Delta_J(D))$ ($\cup_T$ denotes the T-equivariant cup product). It suffices to check the equality when the first factor is of the form D_h (corresponding to $I{:=}(1,\dots,k-1,k+h)$). One has

$$\begin{aligned}\iota_k(D_h\Delta_I(D))&=\iota_k\left(\sum_{H\in\mathcal{P}(I,h)}\Delta_{I+H}(D)\right)\\&=\sum_{H\in\mathcal{P}(I,h)}\delta_{I+H}=c_h^T(Q_k-F_k)\cup_T\delta_I=\iota_k(D_h)\cup_T\iota_k(\Delta_I(D)),\end{aligned}$$

and the proof is complete. $\square$

3. Equivariant Pieri's formulas

3.1. Equivariant cohomology of $\mathbb{P}^{n-1}$

Let $A:=H_T^*(\mathrm{pt})=\mathbb{Z}[y_1,\dots,y_n]$ be the T-equivariant cohomology ring of a point. If P is a T-fixed point of $\mathbb{P}^{n-1}$, let $\{P\}\hookrightarrow\mathbb{P}^{n-1}$ be the canonical inclusion. It induces an injective map ([9], p. 232)

$$H_T^*(\mathbb{P}^{n-1})\longmapsto \bigoplus_{T\text{-fixed points of }\mathbb{P}^{n-1}} H_T^*(P). \tag{5}$$

Let $A^{\oplus n}$ be the direct sum of n copies of A, a ring with respect to the natural componentwise product structure. By [9], p. 230, $H_T^*(\mathbb{P}^{n-1})$ can be identified with the subring of $A^{\oplus n}$ satisfying certain Goresky–Kottwitz–MacPherson (GKM) conditions [8]. For each $1\le i\le n$ let $Y_i:=y_i-y_1$ (in particular $Y_1=0$), and define monic polynomials $\mathtt{p}_i\in A[X]$ by setting $p_0=1$ and $\mathtt{p}_i=\prod_{j=1}^{i}(X-Y_j)$, if $1\le i\le n$. The polynomial $\mathtt{p}_n$ will simply be denoted by $\mathtt{p}$. Let:

$$\mathfrak{S}_1=\begin{pmatrix} Y_1\\ \vdots\\ Y_n\end{pmatrix}\in A^{\oplus n}$$

and $\mathfrak{S}_i=\mathtt{p}_i(\mathfrak{S}_1)$. An easy check shows that $\mathfrak{S}_0=\mathbf{1}_{A^{\oplus n}},\mathfrak{S}_1,\dots,\mathfrak{S}_{n-1}$ are A-linearly independent n-tuples of polynomials. All of them satisfy the GKM conditions (as in [9], p. 230). Since $H_T^*(G(1,n))$ is free of rank n over A, they span $H_T^*(G(1,n))$, identified with the image of the map (5). Furthermore $H_T^*(G(1,n))$ is generated by $\mathfrak{S}_1$ as an A-algebra, and because $\mathtt{p}$ is the A-minimal polynomial of $\mathfrak{S}_1$, one has indeed the isomorphism $H_T^*(G(1,n))\cong A[\mathfrak{S}_1]/\mathtt{p}$. Let $M(\mathtt{p})$ be as at the beginning of Section 2.2. Then $\iota_1:\mathcal{A}^*(M(\mathtt{p}))\longrightarrow H_T^*(G(1,n))$, defined by $D_1\mapsto\mathfrak{S}_1$, is a ring isomorphism. It follows that $\boldsymbol{\mu}:=(\mu^1,\dots,\mu^n)$ defined by (notation as in Sections 2.3 and 2.4)

$$\mu^i:=\Pi_1\circ\iota^{-1}(\mathfrak{S}_{i-1})=X\cdot\mathtt{p}_{i-1}+\mathtt{p}M\in M(\mathtt{p}) \tag{6}$$

is an A-basis of $M(\mathtt{p})$. Our next task is to explicitly write down the (equivariant) Pieri's formulas with respect to the basis $\boldsymbol{\mu}$ (cf. Section 2.4). To this purpose, first notice that

$$D_1\mu^j=X(X\mathtt{p}_{j-1})+\mathtt{p}M=(X-Y_j)Xp_{j-1}+Y_jXp_{j-1}+\mathtt{p}M=\mu^{j+1}+Y_j\mu^j. \tag{7}$$

We also need the following result.

Lemma 3.1. *For all* $i>0$,

$$D_i(\mu^j)=D_1^i(\mu^j)=\sum_{l=0}^{i} h_{i-l}(Y_j,\dots,Y_{j+l})\mu^{j+l}, \quad 1\le j\le n, \tag{8}$$

where $h_0=1$ *and, for each* $m\ge 1$, $h_m(X_1,\dots,X_k)$ *is the complete homogeneous symmetric polynomial in* $X_1,\dots,X_k$ *of degree* m [15].

Proof. The proof is by induction over the integer i. If $i=1$, one has, by (7)

$$D_1\mu^j=\mu^{j+1}+Y_j\mu^j=\mu^{j+1}+h_1(Y_j)\mu^j. \tag{9}$$

Suppose that the formula holds for $i\ge 1$. Since $D_i\mu^j=D_1D_1^{i-1}\mu^j$, by the inductive hypothesis

$$D_i(\mu^j)=D_1\left(\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})\mu^{j+l}\right)=\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})D_1\mu^{j+l}. \tag{10}$$

Using (9), the last member of (10) is thence equal to

$$\begin{aligned}
&\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})(\mu^{j+1}+h_1(Y_j)\mu^j)\\
&\qquad=\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})\mu^{j+l+1}+\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})h_1(Y_{j+l})\mu^{j+l}\\
&\qquad=\sum_{l'=1}^{i} h_{i-l'}(Y_j,\dots,Y_{j+l'-1})\mu^{j+l'}+\sum_{l=0}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})h_1(Y_{j+l})\mu^{j+l}\\
&\qquad=h_0(Y_j,\dots,Y_{j+i-1})\mu^{j+i}+\sum_{l'=1}^{i-1} h_{i-l'}(Y_j,\dots,Y_{j+l'-1})\mu^{j+l'}\\
&\qquad\quad+h_{i-1}(Y_j)h_1(Y_j)\mu^j+\sum_{l=1}^{i-1} h_{i-l-1}(Y_j,\dots,Y_{j+l})h_1(Y_{j+l})\mu^{j+l}\\
&\qquad=h_0(Y_j,\dots,Y_{j+i-1})\mu^{j+i}+h_i(Y_j)\mu^j\\
&\qquad\quad+\sum_{l'=1}^{i-1}(h_{i-l'}(Y_j,\dots,Y_{j+l'-1})+h_{i-l'-1}(Y_j,\dots,Y_{j+l'})h_1(Y_{j+l'}))\mu^{j+l'}\\
&\qquad=h_0(Y_j,\dots,Y_{j+i-1})\mu^{j+i}+h_i(Y_j)\mu^j+\sum_{l'=1}^{i-1} h_{i-l'}(Y_j,\dots,Y_{j+l'})\mu^{j+l'},
\end{aligned}$$

i.e. $D_i(\mu^j) = \sum_{l=0}^{i} h_{i-l}(Y_j, \dots, Y_{j+l})\mu^{j+l}$, $1 \le j \le n$, as desired. $\square$

Let $I \in \mathcal{I}_n^k$, $\bigwedge^I \boldsymbol{\mu} \in \bigwedge^k M(\mathtt{p})$ and $l \ge 0$. Our next task is to get an expression for $D_l(\bigwedge^I \boldsymbol{\mu})$. It will be an A-linear combination of elements of the basis $\bigwedge^k \boldsymbol{\mu}$ of $\bigwedge^k M(\mathtt{p})$. Predicting which terms cancel because of the skew-symmetry of the wedge product, will give us a $\boldsymbol{\mu}$-Pieri's formula (see Theorem 3.2 below). Leibniz's rule for D_l (cf. (1)) and an easy induction on $k \ge 1$ gives

$$D_l\Big(\bigwedge^I \boldsymbol{\mu}\Big) = \sum_{\substack{l_1+\dots+l_k=l \\ l_i \ge 0}} D_{l_1}\mu^{l_1} \wedge \dots \wedge D_{l_k}\mu^{l_k}.$$

Using (8),

$$\begin{aligned}
&\sum_{\substack{l_1+\dots+l_k=l \\ l_i \ge 0}} D_{l_1}\mu^{l_1} \wedge \dots \wedge D_{l_k}\mu^{l_k} \\
&\qquad = \sum_{\substack{l_1+\dots+l_k=l \\ l_i \ge 0}} \Bigg[\Bigg(\sum_{m_1=0}^{l_1} h_{l_1-m_1}(Y_{i_1}, Y_{i_1+1}, \dots, Y_{i_1+m_1})\mu^{i_1+m_1}\Bigg) \\
&\qquad\qquad \wedge \dots \wedge \Bigg(\sum_{m_k=0}^{l_k} h_{l_k-m_k}(Y_{i_k}, Y_{i_k+1}, \dots, Y_{i_k+m_k})\mu^{i_k+m_k}\Bigg)\Bigg].
\end{aligned}$$

Expanding the wedge products, the last member can be written as

$$\begin{aligned}
&= \sum_{\substack{l_1+\dots+l_k=l \\ l_i \ge 0}} \sum_{\substack{m_1+\dots+m_k=0 \\ m_i \ge 0}}^{l_1+\dots+l_k} \Bigg(\prod_{j=1}^{k} h_{l_j-m_j}(Y_{i_j}, Y_{i_j+1}, \dots, Y_{i_j+m_j})\mu^{i_1+m_1} \wedge \dots \wedge \mu^{i_k+m_k}\Bigg) \\
&= \sum_{\substack{m_1+\dots+m_k=0 \\ m_i \ge 0}}^{l} \Bigg(\sum_{\substack{l_1+\dots+l_k=l \\ l_i \ge 0}} \prod_{j=1}^{k} h_{l_j-m_j}(Y_{i_j}, Y_{i_j+1}, \dots, Y_{i_j+m_j})\Bigg)\mu^{i_1+m_1} \wedge \dots \wedge \mu^{i_k+m_k} \\
&= \sum_{\substack{m_1+\dots+m_k=0 \\ m_i \ge 0}}^{l} h_{l-\sum_{j=1}^{k} m_j}(Y_{i_1}, \dots, Y_{i_1+m_1}, \dots, Y_{i_k}, \dots, Y_{i_k+m_k})\mu^{i_1+m_1} \wedge \dots \wedge \mu^{i_k+m_k}.
\end{aligned}$$

The last equality uses basic properties of the complete symmetric polynomials (see e.g. [15]). Putting $u = l - \sum_{j=1}^{k} m_j$, one finally has

$$\begin{aligned}
D_l(\mu^{i_1} \wedge \dots \wedge \mu^{i_k}) = \sum_{u=0}^{l} \sum_{\substack{m_1+\dots+m_k+u=l \\ m_i \ge 0}} & h_u(Y_{i_1}, \dots, Y_{i_1+m_1}, \dots, Y_{i_k}, \dots, Y_{i_k+m_k}) \\
& \times \mu^{i_1+m_1} \wedge \dots \wedge \mu^{i_k+m_k}.
\end{aligned} \tag{11}$$

Our aim is now to rewrite the right-hand side of (11) keeping into account the cancellations due to the alternating feature of the $\wedge$-product.

Theorem 3.2. (See also [12].) *The following* T-equivariant Pieri's formula *holds*:

$$D_l\Big(\bigwedge^I \boldsymbol{\mu}\Big)=\sum_{u=0}^{l}\sum_{M\in\mathcal{P}(I,l-u)} h_u(Y_{i_1},\dots,Y_{i_1+h_1},\dots,Y_{i_k},\dots,Y_{i_k+h_k})\cdot\bigwedge^{I+M}\boldsymbol{\mu}, \tag{12}$$

where $M:=(m_1,\dots,m_k)\in\mathcal{P}(I,l-u)$ *is as in Theorem* 2.1.

Proof. (Cf. [5], Theorem 2.4.) The proof is by induction over the integer k. For $k=1$, formula (12) is trivially true. The first case where the alternating behavior of the wedge product comes into play, causing cancellations in the expansion (11), is $k=2$. This case we analyze directly. For each $l\geq 0$, let us split the sum (12) as

$$\begin{aligned} D_l(\mu^{i_1}\wedge\mu^{i_2}) &= \sum_{u=0}^{l}\sum_{\substack{m_1+m_2=l-u\\ m_i\geq 0}} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_1},\dots,Y_{i_2+m_2})\mu^{i_1+m_1}\wedge\mu^{i_2+m_2}\\ &= U+\overline{U}, \end{aligned}$$

where

$$U=\sum_{u=0}^{l}\sum_{\substack{m_1+m_2=l-u\\ m_i\geq 0\\ i_1+m_1<i_2}} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_2},\dots,Y_{i_2+m_2})\mu^{i_1+m_1}\wedge\mu^{i_2+m_2}$$

and

$$\overline{U}=\sum_{u=0}^{l}\sum_{\substack{m_1+m_2=l-u\\ m_i\geq 0\\ i_1+m_1<i_2}} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_2},\dots,Y_{i_2+m_2})\mu^{i_1+m_1}\wedge\mu^{i_2+m_2}.$$

We contend that $\overline{U}$ vanishes. In fact, on the finite set of all integers $i_2-i_1\leq a\leq l-u$, define the bijection $\gamma(a)=i_2-i_1+l-u-a$. Then

$$\begin{aligned} 2\overline{U} &= \sum_{u=0}^{l}\sum_{m_1=i_2-i_1}^{l-u} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_2},\dots,Y_{i_2+l-u-m_1})\mu^{i_1+m_1}\wedge\mu^{i_2+l-u-m_1}\\ &\quad+\sum_{u=0}^{l}\sum_{m_1=i_2-i_1}^{l-u} h_u(Y_{i_1},\dots,Y_{i_1+\gamma(m_1)},Y_{i_2},\dots,Y_{i_2+l-u-\gamma(m_1)})\\ &\quad\times\mu^{i_1+\gamma(m_1)}\wedge\mu^{i_2+l-u-\gamma(m_1)} \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{u=0}^{l}\sum_{m_1=i_2-i_1}^{l-u} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_2},\dots,Y_{i_2+l-u-m_1})\mu^{i_2+l-u-m_1}\wedge\mu^{i_2+m_1}\\
&\quad+\sum_{u=0}^{l}\sum_{m_1=i_2-i_1}^{l-u} h_u(Y_{i_1},\dots,Y_{i_1+m_1},Y_{i_2},\dots,Y_{i_2+l-u-m_1})\mu^{i_2+l-u-m_1}\wedge\mu^{i_1+m_1}\\
&= 0.
\end{aligned}
$$

Hence $\overline{U}=0$ and (12) holds for $k=2$. Suppose now that (12) holds for all $1\le k'\le k-1$. Then, for each $l\ge 0$,

$$
D_l(\mu^{i_1}\wedge\mu^{i_2}\wedge\dots\wedge\mu^{i_k}) = \sum_{l'_k+l_k=l} D_{l'_k}(\mu^{i_1}\wedge\dots\wedge\mu^{i_{k-1}})\wedge D_{l_k}\mu^{i_k}. \tag{13}
$$

By the inductive hypothesis, the right-hand side of (13) is equal to

$$
\sum_{u=0}^{l'_k}\mathcal{H}_{k-1}(u)\wedge\sum_{m_k=0}^{l_k} h_{l_k-m_k}(Y_{i_k},Y_{i_k+1},\dots,Y_{i_k+m_k})\mu^{i_k+m_k}, \tag{14}
$$

where for notational brevity we set $I':=(i_1,\dots,i_{k-1})$, $M'=(m_1,\dots,m_{k-1})$ and

$$
\mathcal{H}_{k-1}(u)=\sum_{M'\in\mathcal{P}(I',l'_k-u)} h_u(Y_{i_1},\dots,Y_{i_1+m_1},\dots,Y_{i_{k-1}},\dots,Y_{i_{k-1}+m_{k-1}})\bigwedge^{I'+M'}\boldsymbol{\mu}.
$$

But now (14) can be equivalently written as

$$
\sum_{u=0}^{l-l''}\mathcal{H}_{k-2}(u)\wedge D_{l''}(\mu^{i_{k-1}}\wedge\mu^{i_k}), \tag{15}
$$

where

$$
\begin{aligned}
\mathcal{H}_{k-2}(u) = \sum_{M''\in\mathcal{P}(I'',l-l''-u)} & h_u(Y_{i_1},\dots,Y_{i_1+m_1},\dots,Y_{i_{k-2}},\dots,Y_{i_{k-2}+m_{k-2}})\\
&\times\mu^{i_1+m_1}\wedge\dots\wedge\mu^{i_{k-2}+m_{k-2}},
\end{aligned}
$$

with $M''=(m_1,\dots,m_{k-2})$. The inductive hypothesis implies that

$$
\begin{aligned}
&D_{l''}(\mu^{i_{k-1}}\wedge\mu^{i_k})\\
&=\sum_{u=0}^{l''}\sum_{\substack{m_{k-1}+m_k=l''-u\\ i_{k-1}+m_{k-1}<i_k}} h_u(Y_{i_{k-1}},\dots,Y_{i_{k-1}+m_{k-1}},Y_{i_k},\dots,Y_{i_k+m_k})\mu^{i_{k-1}+m_{k-1}}\wedge\mu^{i_k+m_k},
\end{aligned}
$$

which, substituted into (15), yields precisely (12). $\square$

Example 3.3. (i) The coefficient of $\mu^2\wedge\mu^3\wedge\mu^7$, in the expansion of

$$D_3(\mu^2\wedge\mu^3\wedge\mu^5),$$

is

$$h_1(Y_2,Y_3,Y_5,Y_6,Y_7)=Y_2+Y_3+Y_5+Y_6+Y_7=y_2+y_3+y_5+y_6+y_7-5y_1.$$

(ii) For $l=1$, Pieri's formula (12) reads as

$$D_1(\mu^{i_1}\wedge\ldots\wedge\mu^{i_k})=\sum_{j=1}^{k}\mu^{i_1}\wedge\ldots\wedge\mu^{i_{j-1}}\wedge\mu^{i_j+1}\wedge\mu^{i_{j+1}}\wedge\ldots\wedge\mu^{i_k} + (Y_{i_1}+\ldots+Y_{i_k})\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}, \tag{16}$$

that is the coefficient of $\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}$ in the expansion of $D_1(\mu^{i_1}\wedge\ldots\wedge\mu^{i_k})$ is $Y_{i_1}+\ldots+Y_{i_k}$.

3.2. The "equivariant Pieri's rule"

The *equivariant Pieri's rule* stated in [9], p. 237, can be inferred as follows. Let $1\le k\le n$ be a fixed integer and $\left\{ {n \atop k} \right\}$ the set of all functions $\mathbf{a}_{i_1\ldots i_k}\colon\{1,\ldots,n\}\to\{0,1\}$, $1\le i_1<\ldots<i_k\le n$, such that $\mathbf{a}_{i_1\ldots i_k}(j)=0$ if $j\in\{i_1,\ldots,i_k\}$ and $\mathbf{a}_{i_1\ldots i_k}(j)=1$ otherwise. Each $\lambda\in\left\{ {n \atop k} \right\}$ can also be represented by a string $\lambda(1)\lambda(2)\ldots\lambda(n)$ of zeros and ones only (as in [9]). If $\lambda=\mathbf{a}_{i_1\ldots i_k}$, write $\widetilde{S}_\lambda:=G^{\boldsymbol{\mu}}_{(i_1,\ldots,i_k)}(D)\in\mathcal{A}^*(\bigwedge^k M(\mathtt{p}))$ and $\widetilde{S}_{\mathrm{div}}:=\widetilde{S}_{\mathbf{a}_{12\ldots k-1,k+1}}=\widetilde{S}_{0\ldots010}$. It is easily seen that $\widetilde{S}_{\mathrm{div}}=D_1-\sum_{r=1}^{k}Y_r$ and then, using (7) and (16),

$$\begin{aligned}
&\widetilde{S}_{\mathrm{div}}\widetilde{S}_\lambda\mu^1\wedge\ldots\wedge\mu^k\\
&\quad=\widetilde{S}_{\mathrm{div}}\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}\\
&\quad=\left(D_1-\sum_{r=1}^{k}Y_r\right)\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}\\
&\quad=D_1\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}-\sum_{r=1}^{k}Y_r\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}\\
&\quad=\sum_{j=1}^{k}\mu^{i_1}\wedge\ldots\wedge\mu^{i_{j-1}}\wedge\mu^{i_j+1}\wedge\mu^{i_{j+1}}\wedge\ldots\wedge\mu^{i_k}+\left(\sum_{r=1}^{k}(Y_{i_r}-Y_r)\right)\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}\\
&\quad=\sum_{j=1}^{k}\mu^{i_1}\wedge\ldots\wedge\mu^{i_{j-1}}\wedge\mu^{i_j+1}\wedge\mu^{i_{j+1}}\wedge\ldots\wedge\mu^{i_k}+\left(\sum_{r=1}^{k}(y_{i_r}-y_r)\right)\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}.
\end{aligned}$$

We have hence proven that, as in [9],

$$\widetilde{S}_{\mathrm{div}}\widetilde{S}_\lambda=\sum_{\lambda':\lambda'\to\lambda}\widetilde{S}_{\lambda'}+\big(\widetilde{S}_{\mathrm{div}}|\lambda\big)\widetilde{S}_\lambda.$$

The notation $\lambda':\lambda'\to\lambda$ means that λ' differs from λ in only two spots i and $i+1$, where λ has 01 and λ' has 10 ([9], p. 236, Lemma 3), while the coefficient $(\widetilde{S}_{\mathrm{div}}|\lambda)$ is given by $(\widetilde{S}_{\mathrm{div}}|\lambda)=\sum_{j=1}^n(1-\lambda(j))y_j-\sum_{i=1}^k y_i$. Thus, if $\lambda=\mathbf{a}_{i_1,\ldots,i_k}$, one has, in fact

$$\big(\widetilde{S}_{\mathrm{div}}|\lambda\big)=(y_{i_1}+y_{i_2}+\ldots+y_{i_k})-(y_1+y_2+\ldots+y_k)=\sum_{r=1}^k(y_{i_r}-y_r).$$

Example 3.4. (Cf. [9], p. 231.) Let $A:=\mathbb{Z}[y_1,y_2,y_3,y_4]$, $Y_i=y_i-y_1$ $(1\le i\le 4)$,

$$\mathtt{p}=X(X-Y_2)(X-Y_3)(X-Y_4),$$

$M=XA[X]$ and $M(\mathtt{p})=M/\mathtt{p}M$. Let $\boldsymbol{\mu}=(\mu^1,\mu^2,\mu^3,\mu^4)$ as in (6). Using the isomorphism $\iota_2\colon \bigwedge^2 M(\mathtt{p})\to H_T^*(G(2,4))$ we shall explicitly compute a basis of the latter, recovering the same equivariant corrections listed in [9], p. 231. We do not need, however, to impose a priori the GKM conditions (cf. Section 3.1)—see below. This example will also show the efficiency of our methods for computations.

Let $G_{ij}(D):=G^{\boldsymbol{\mu}}_{ij}(D)=\Pi_2^{-1}(\mu^i\wedge\mu^j)\in\mathcal{A}^*(\bigwedge^2M(\mathtt{p}))$. “Integration by parts” (as in [7], Remark 3.12) or the general determinantal formula due to Laksov and Thorup [13], yields

$$\begin{aligned}
&G_{12}(D)=1, && G_{13}(D)=D_1-Y_2D_0,\\
&G_{14}(D)=D_2-(Y_2+Y_3)D_1+Y_2Y_3D_0, &&\\
&G_{23}(D)=D_1^2-D_2, && G_{24}(D)=(D_1-Y_4)G_{14}(D),\\
&G_{34}(D)=\big(D_2-Y_4^2\big)G^{\boldsymbol{\mu}}_{14}(D)-(Y_2+Y_4)G_{24}(D). &&
\end{aligned}$$

We now look for the matrices associated to $G_{ij}(D)$ in the basis $\bigwedge^2\boldsymbol{\mu}$. The matrix $(G_{12}(D))$ is just the 6×6 identity matrix. We show, for instance, how the computation of the matrix of $(G_{13}(D))$ is carried out by exhibiting the product $G^{\boldsymbol{\mu}}_{13}(D)\bigwedge^I\boldsymbol{\mu}$ in the first few cases. One has, e.g.

$$(D_1-Y_2D_0)\mu^1\wedge\mu^2=\mu^2\wedge\mu^2+\mu^1\wedge(\mu^3+Y_2\mu^2)-Y_2\mu^1\wedge\mu^2=\mu^1\wedge\mu^3$$

and

$$\begin{aligned}G_{13}(D)\mu^1\wedge\mu^3&=(D_1-Y_2D_0)\mu^1\wedge\mu^3=\mu^2\wedge\mu^3+\mu^1\wedge(\mu^4+Y_3\mu^2)-Y_2\mu^1\wedge\mu^3\\&=\mu^2\wedge\mu^3+\mu^1\wedge\mu^4+(Y_3-Y_2)\mu^1\wedge\mu^3.\end{aligned}$$

Continuing in the same way, the matrix of $G_{13}(D)$ in the basis $\bigwedge^2 \boldsymbol{\mu}$ is

$$(G_{13}(D)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & Y_3-Y_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & Y_4-Y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & Y_4 & 0 \\ 0 & 0 & 0 & 0 & 1 & Y_4+Y_3-Y_2 \end{pmatrix}.$$

Similarly, one can compute (by hand) the matrices of the remaining $G_{ij}(D)$, getting

$$(G_{14}(D)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & Y_4-Y_2 & (Y_4-Y_2)(Y_4-Y_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & Y_4-Y_3 & Y_4 & Y_4(Y_4-Y_3) & 0 \\ 0 & 0 & 1 & 0 & Y_4 & Y_4(Y_4-Y_2) \end{pmatrix},$$

$$(G_{23}(D)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & Y_3 & 0 & Y_2Y_3 & 0 & 0 \\ 0 & 1 & Y_4 & Y_2 & Y_2Y_4 & 0 \\ 0 & 0 & 0 & 1 & Y_4 & Y_3Y_4 \end{pmatrix},$$

$$(G_{24}(D)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & Y_4 & Y_4(Y_4-Y_3) & Y_2Y_4 & Y_2Y_4(Y_4-Y_3) & 0 \\ 0 & 1 & Y_4 & Y_4 & Y_4^2 & Y_4Y_3(Y_4-Y_2) \end{pmatrix},$$

$$(G_{34}(D)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & Y_4+Y_3-Y_2 & Y_4(Y_4-Y_2) & Y_4Y_3 & Y_4Y_3(Y_4-Y_2) & Y_3(Y_4-Y_2)Y_4(Y_3-Y_2) \end{pmatrix}.$$

Let $\{\widetilde{S}_{\mathbf{a}_{ij}} \mid (i,j)\in\mathcal{I}_4^2\}$ be the basis of $H_T^*(G(2,4))$ depicted in [9], p. 231, Figure 7. Then the ring isomorphism $\iota_2 : \mathcal{A}^*(\bigwedge^2 M(\mathtt{p})) \to H_T^*(G(2,4))$ (as in Theorem 2.2, for $k=2$) is explicitly given by $G_{ij}(D) \mapsto \widetilde{S}_{\mathbf{a}_{ij}}$. Notice that the diagonal entries of the matrices $(G_{ij}(D))$ satisfy the GKM conditions [9], which thus arise in a purely algebraic way.

References

1. Brion, M., Equivariant cohomology and equivariant intersection theory, in *Representation Theories and Algebraic Geometry* (*Montreal*, *QC*, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **514**, pp. 1–37, Kluwer, Dordrecht, 1997.
2. Cordovez, J., Gatto, L. and Santiago, L., Newton binomial formulas in Schubert calculus, *Rev. Mat. Complut.* **22** (2008), 129–152.
3. Fulton, W., *Intersection Theory*, Springer, Berlin, 1998.
4. Fulton, W., Equivariant Intersection Theory, Eilenberg Lectures at Columbia University, New York, 2007, notes by D. Anderson available at http://www.math.lsa.umich.edu/dandersen/eilenberg/.
5. Gatto, L., Schubert calculus via Hasse–Schmidt derivations, *Asian J. Math.* **9** (2005), 315–322.
6. Gatto, L., *Schubert Calculus*: *An Algebraic Introduction*, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2005.
7. Gatto, L. and Santiago, T., Schubert calculus on a Grassmann algebra, to appear in *Canad. Math. Bull.*
8. Goresky, M., Kottwitz, R. and MacPherson, R., Equivariant cohomology, Koszul duality, and the localization theorem, *Invent. Math.* **131** (1998), 25–83.
9. Knutson, A. and Tao, T., Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. J.* **119** (2003), 221–260.
10. Lakshmibai, V., Raghavan, K. N. and Sankaran, P., Equivariant Giambelli and determinantal restriction formulas for the Grassmannian, *Pure Appl. Math. Quart.* **2** (2006), 699–717.
11. Laksov, D., The formalism of equivariant Schubert calculus, *Preprint*, 2006.
12. Laksov, D., Schubert calculus and equivariant cohomology of Grassmannians, *Adv. Math.* **217** (2008), 1869–1888.
13. Laksov, D. and Thorup, A., A determinantal formula for the exterior powers of the polynomial ring, *Indiana Univ. Math. J.* **56** (2007), 825–845.
14. Laksov, D. and Thorup, A., Schubert calculus on Grassmannians and exterior products, *Indiana Univ. Math. J.* **58** (2009), 283–300.
15. Macdonald, I. G., *Symmetric Functions and Hall Polynomials*, 2nd ed., Clarendon, Oxford, 1979.
16. Mihalcea, L. C., Equivariant quantum Schubert calculus, *Adv. Math.* **203** (2006), 1–33.
17. Mihalcea, L. C., Giambelli formulae for the equivariant quantum cohomology of the Grassmannian, *Trans. Amer. Math. Soc.* **360** (2008), 2285–2301.
18. Santiago, T., *Schubert Calculus on a Grassmann Algebra.* Ph.D. Thesis, Politecnico di Torino, Torino, 2006.

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