Convergence and multiplicities
for the Lempert function

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Abstract. Given a domain $\Omega \subset \mathbb{C}^n$, the Lempert function is a functional on the space $\mathrm{Hol}(\mathbb{D}, \Omega)$ of analytic disks with values in Ω, depending on a set of poles in Ω. We generalize its definition to the case where poles have multiplicities given by local indicators (in the sense of Rashkovskii) to obtain a function which still dominates the corresponding Green function, behaves relatively well under limits, and is monotonic with respect to the local indicators. In particular, this is an improvement over the previous generalization used by the same authors to find an example of a set of poles in the bidisk so that the (usual) Green and Lempert functions differ.

1. Introduction

We assume throughout that Ω is a bounded domain in $\mathbb{C}^n$. Let $\mathbb{D}$ stand for the unit disk in $\mathbb{C}$. The classical Lempert function with pole at $a \in \Omega$, [7], is defined by

$$\ell_a(z) := \inf\{\log|\zeta| : \text{there exists } \varphi \in \mathrm{Hol}(\mathbb{D}, \Omega) \text{ such that } \varphi(0) = z \text{ and } \varphi(\zeta) = a\}.$$

Given a finite number of points $a_j \in \Omega$, $j = 1, ..., N$, Coman [3] extended this to

$$(1) \quad \ell(z) := \ell_{\{a_1, ..., a_N\}}(z) := \inf\left\{ \sum_{j=1}^{N} \log|\zeta_j| : \text{there exists } \varphi \in \mathrm{Hol}(\mathbb{D}, \Omega) \right.$$
$$\left. \text{such that } \varphi(0) = z \text{ and } \varphi(\zeta_j) = a_j, j = 1, ..., N \right\}.$$

The *Green function* for the same poles is

$$g := \sup\{u \in \mathrm{PSH}(\Omega, \mathbb{R}_-) : u(z) \leq \log|z - a_j| + C_j$$
$$\text{for } z \text{ in a neighborhood of } a_j, j = 1, ..., N\},$$

where $\mathrm{PSH}(\Omega, \mathbb{R}_-)$ stands for the set of all negative plurisubharmonic functions in Ω. The inequality $g(z) \leq \ell(z)$ always holds, and it is known that it can be strict, see [2], [9] and [14]. If ℓ ever turns out to be plurisubharmonic itself, then ℓ must be equal to g, [3].

There are natural extensions of the definition of the Green function. In one dimension, considering a finite number of poles in the same location a, say m poles, it has a natural interpretation in terms of multiplicities: the point mass in the Riesz measure of the Green function is multiplied by m. Locally, the Green function behaves like $\log |f|$, where f is a holomorphic function vanishing at a with multiplicity m.

Lelong and Rashkovskii [6] and [10] defined a generalized Green function. The function $\log |z|$ was replaced by "local indicators", i.e. circled plurisubharmonic functions Ψ whose Monge–Ampère measure $(dd^c \Psi)^n$ is concentrated at the origin, such that whenever $\log |w_j| = c \log |z_j|$ for all $j \in \{1, ..., n\}$, then $\Psi(w) = c\Psi(z)$. This has the advantage of allowing for the consideration of non-isotropic singularities such as $\max(2 \log |z_1|, \log |z_2|)$, but the "circled" condition privileges certain coordinate axes, so that the class is not invariant under linear changes of variables. We will have to remove this restriction to obtain a class large enough to describe some natural limits.

In several complex variables, we would like to know which notion of multiplicity can arise when we take limits of ordinary Green (or Lempert) functions with several poles tending to the same point. This idea was put to use in [14] to exhibit an example where a Lempert function with four poles is different from the corresponding Green function. The definition of a generalized Lempert function chosen in [14] had some drawbacks – essentially, it was not monotonic with respect to its system of poles (in an appropriate sense) [14, Proposition 4.3] and did not pass to the limit in some very simple situations [13, Theorem 6.3]. We recall that monotonicity holds when no multiplicities are present, or more generally when a subset of the original set of poles is considered with the same indicators, see [17] and [14, Proposition 3.1] for the convex case, and the more recent [8] for arbitrary domains and weighted Lempert functions.

In Section 2, we successively define a class of indicators, a subclass which is useful to produce "monomial" examples, a notion of multiplicity for values attained by an analytic disk, and a generalization of Coman's Lempert function to systems of poles with indicators, different from [14]. In Section 3, we state our two main results: monotonicity, and convergence under certain restrictive (but, we hope, natural) conditions. Further sections are devoted to the proofs of those results.

Finally, in Section 7 we summarize the differences between our new definition and that given in [14].

2. Definitions

Definition 2.1. ([6]) Let $\Psi \in \mathrm{PSH}(\mathbb{D}^n)$. We call Ψ a *local indicator* and write $\Psi \in \mathcal{I}_0$ if

    1. Ψ is bounded from above on $\mathbb{D}^n$;

    2. Ψ is circled, i.e. $\Psi(z_1, ..., z_n)$ depends only on $(|z_1|, ..., |z_n|)$;

    3. for any $c > 0$, $\Psi(|z_1|^c, ..., |z_n|^c) = c\Psi(|z_1|, ..., |z_n|)$.

    As a consequence, $(dd^c\Psi)^n = \tau_\Psi \delta_0$ for some $\tau_\Psi \geq 0$.

Notice that if $\Psi_1 \in \mathrm{PSH}(\mathbb{D}^n)$, $\Psi_2 \in \mathrm{PSH}(\mathbb{D}^m)$, and they are both local indicators, then

$$\Psi(z, z') := \max(\Psi_1(z), \Psi_2(z'))$$

defines a local indicator on $\mathbb{D}^{n+m}$.

We need to remove the restriction to a single coordinate system in Definition 2.1.

Definition 2.2. We call Ψ a *generalized local indicator*, and we write $\Psi \in \mathcal{I}$ if there exists a neighborhood U of 0, $\Psi_0 \in \mathcal{I}_0$ and a one-to-one linear map L of $\mathbb{C}^n$ to itself such that $L(U) \subset \mathbb{D}^n$ and $\Psi = \Psi_0 \circ L$.

We will concentrate on a class of simple examples. Given two vectors $z, w \in \mathbb{C}^n$, their standard Hermitian product is denoted by $z \cdot \overline{w} := \sum_{j=1}^n z_j \overline{w}_j$. We also write $\|z\| := |z \cdot \bar{z}|^{1/2}$.

Definition 2.3. We say that Ψ is an *elementary local indicator* if there exists a basis $\{v_1, ..., v_n\}$ of vectors of $\mathbb{C}^n$ and scalars $m_j \in \mathbb{R}_+$, $1 \leq j \leq n$, such that for $z \in \mathbb{D}^n$,

(2) $$\Psi(z) = \max_{1 \leq j \leq n} m_j \log |z \cdot \bar{v}_j|.$$

One easily checks that any elementary local indicator is a generalized local indicator. The most interesting case is the one for which the basis is orthonormal. In fact, it is essentially the only case.

Lemma 2.4. *Given an elementary local indicator Ψ as in Definition 2.3 there exists an orthonormal basis $\{\tilde{v}_1, ..., \tilde{v}_n\}$ of $\mathbb{C}^n$ such that the associated indicator $\widetilde{\Psi}(z) := \max_{1 \leq j \leq n} m_j \log |z \cdot \bar{\tilde{v}}_j|$ satisfies $\widetilde{\Psi} - \Psi \in L^\infty(\mathbb{D}^n)$.*

As a consequence, we could have restricted the map L in Definition 2.2 to be unitary, and it would not have changed things in any essential way.

The proof of Lemma 2.4 is given in Section 4 below.

Lemma 2.5. ([6, example in Section 3] and [10]) *If Ψ is an elementary local indicator, then $\tau_\Psi = m_1 ... m_n$.*

We take the same definition of the generalized Green function as in [6].

Definition 2.6. Let Ω be a bounded domain in $\mathbb{C}^n$. Given

$$S := \{(a_j, \Psi_j) : 1 \leq j \leq N\}, \quad \text{where } a_j \in \Omega, a_j \neq a_k \text{ for } j \neq k, \text{ and } \Psi_j \in \mathcal{I},$$

its *Green function* is

$$G_S := \sup\{u \in \mathrm{PSH}(\Omega, \mathbb{R}_-) : u(z) \leq \Psi_j(z) + C_j,$$
$$\text{for } z \text{ in a neighborhood of } a_j, j = 1, ..., N\}.$$

To generalize the Lempert function, the first step is to quantify the way in which an analytic disk, i.e. an element of $\mathrm{Hol}(\mathbb{D}, \Omega)$, meets a pole provided with a generalized local indicator.

Definition 2.7. Let $\alpha \in \mathbb{D}$, $a \in \Omega$ and $\Psi \in \mathcal{I}$. Then the *multiplicity* of $\varphi \in \mathrm{Hol}(\mathbb{D}, \Omega)$ at α, with respect to a, is given by

$$m_{\varphi, a, \Psi}(\alpha) := \begin{cases} \min\left(\tau_\Psi, \liminf_{\zeta \to 0} \dfrac{\Psi(\varphi(\alpha + \zeta) - a)}{\log |\zeta|}\right), & \text{if } \varphi(\alpha) = a, \\ 0, & \text{if } \varphi(\alpha) \neq a. \end{cases}$$

Notice that if $\Psi_1 - \Psi_2$ is locally bounded near the origin, then $m_{\varphi, a, \Psi_1}(\alpha) = m_{\varphi, a, \Psi_2}(\alpha)$.

The quantity $\liminf_{\zeta \to 0} \Psi(\varphi(\alpha + \zeta) - a) / \log |\zeta|$ is exactly the Lelong number at 0 of the subharmonic function $\Psi \circ \varphi$, compare with [11, pp. 334–335]. Truncating at the level of the local Monge–Ampère mass τ_Ψ will turn out to be convenient in Definition 2.10, and the proofs that use it.

It is useful to see what this means in the case of elementary local indicators.

Elementary example. Suppose that $\alpha=0$, $a=0$, and that

$$\Psi(z) = \max_{1 \leq j \leq n} m_j \log |z_j|.$$

We write

$$\varphi(\zeta) = (\varphi_1(\zeta), ..., \varphi_n(\zeta)),$$

and define the valuations

$$\nu_j := \nu_j(0, \varphi) := \min\left\{ k : \left(\frac{d}{d\zeta}\right)^k \varphi_j(0) \neq 0 \right\}.$$

Then we have

(3) $$m_{\varphi, 0, \Psi}(0) = \min\left(\min_{1 \leq j \leq n} m_j \nu_j, \prod_{j=1}^n m_j \right).$$

Example 2.8. If $m_j=1$ for all j, then

$$m_{\varphi, 0, \Psi}(0) = \begin{cases} 1, & \text{if } \varphi(0)=0, \\ 0, & \text{otherwise.} \end{cases}$$

This is the basic case where one just records whether a point has been hit by the analytic disk or not.

Example 2.9. In more general cases, the use of an indicator will impose higher-order differential conditions on the map φ. For instance, if $m_1=2$ and $m_j=1$, $2 \leq j \leq n$, then

$$m_{\varphi, 0, \Psi}(0) = \begin{cases} 0, & \text{if } \varphi(0) \neq 0, \\ 1, & \text{if } \varphi(0)=0 \text{ and } \varphi'_j(0) \neq 0 \text{ for some } j \in \{2, ..., n\}, \\ 2, & \text{if } \varphi(0)=0 \text{ and } \varphi'_j(0)=0 \text{ for any } j \in \{2, ..., n\}. \end{cases}$$

Definition 2.10. Given a system S as in Definition 2.6, we write $\tau_j := \tau_{\Psi_j}$.

Let $\varphi \in \mathrm{Hol}(\mathbb{D}, \Omega)$ and $A_j \subset \mathbb{D}$, $1 \leq j \leq N$. We say that $(\varphi, \{A_j\}_{1 \leq j \leq N})$ is *admissible* (*for S and z*) if

$$\varphi(0) = z; \ A_j \subset \varphi^{-1}(a_j) \text{ and } \sum_{\alpha \in A_j} m_{\varphi, a_j, \Psi_j}(\alpha) \leq \tau_j, \quad 1 \leq j \leq N.$$

In this case, we write (with the convention that $0 \cdot \infty = 0$)

$$\mathcal{S}(\varphi, \{A_j\}_{1 \leq j \leq N}) := \sum_{j=1}^N \sum_{\alpha \in A_j} m_{\varphi, a_j, \Psi_j}(\alpha) \log |\alpha|.$$

Then the generalized Lempert function is defined by

$$\mathcal{L}_S^\Omega(z) := \mathcal{L}_S(z) := \inf\{\mathcal{S}(\varphi, \{A_j\}_{1 \leq j \leq N}) : (\varphi, \{A_j\}_{1 \leq j \leq N}) \text{ is admissible for } S, z\}.$$

Notice that we allow any of the A_j to be the empty set (in which case the jth term drops from the sum).

Consider the *single poles case* where

(4) $$\Psi_j(z) = \max_{1 \leq l \leq n} \log |z_l| \text{ or } \Psi_j(z) = \log \|z\| \quad \text{for each } j$$

– it is the same, since both functions differ by a bounded term near 0; in fact, one could use any norm that is homogeneous under complex scalar multiplication.

In this case, $\tau_j = 1$ for every j. With a slight abuse of notation, we write $S = \{a_1, ..., a_N\}$. Then $\mathcal{L}_S(z) = \min_{S' \subset S} \ell_{S'}(z)$, where ℓ_S was defined in (1). And in fact $\min_{S' \subset S} \ell_{S'}(z) = \ell_S(z)$ [8] (see also [16] and [17] for the case when the domain Ω is convex).

The Lempert function is different from the functionals considered by Poletsky and others in that it is restricted to one pre-image per pole a_j (thus the Lempert function can fail to be equal to the corresponding Green function). In our definition, the number of pre-images per pole is bounded above by the Monge–Ampère mass at that pole of its generalized local indicator. In [14], each pole could only have one pre-image, but (essentially) φ had to hit the pole with maximum multiplicity at that pre-image.

Although Definition 2.10 may seem contrived, it is required to obtain the reasonable convergence Theorem 3.3. See the discussion in Section 7.

We remark right away that the usual relationship holds between this generalized Lempert function and the corresponding Green function.

Lemma 2.11. *For a bounded domain Ω, for any system S as in Definition 2.6, for any $z \in \Omega$, $G_S(z) \leq \mathcal{L}_S(z)$.*

Proof. If $\varphi \in \text{Hol}(\mathbb{D}, \Omega)$, and $u \in \text{PSH}_-(\Omega)$ is a member of the defining family for the Green function of S, then $u \circ \varphi$ is subharmonic and negative on $\mathbb{D}$. Furthermore, if $(\varphi, \{A_j\}_{1 \leq j \leq N})$ is admissible (for S and z) and $\alpha \in A_j$, then given any $\varepsilon > 0$, for $|\zeta|$ small enough,

$$u \circ \varphi(\alpha + \zeta) \leq C_j + \Psi_j(\varphi(\alpha + \zeta) - a_j) \leq C_j + (m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon) \log |\zeta|.$$

So $u \circ \varphi$ is a member of the defining family for the Green function on $\mathbb{D}$ with poles α and weights $m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon$ at α. This implies that

$$u \circ \varphi(\zeta) \leq \sum_{j=1}^{N} \sum_{\alpha \in A_j} (m_{\varphi, a_j, \Psi_j}(\alpha) - \varepsilon) \log \left| \frac{\alpha - \zeta}{1 - \zeta \bar{\alpha}} \right|.$$

Letting ε tend to 0 and setting $\zeta = 0$, we get $u(z) \leq \mathcal{S}(\varphi, \{A_j\}_{1 \leq j \leq N})$.

Passing to the supremum over u, and then to the infimum over $(\varphi, \{A_j\}_{1 \leq j \leq N})$, we get the lemma. $\square$

3. Main results

We start with a remark.

Lemma 3.1. *If S is as in Definition 2.6, $1 \leq N' \leq N$, and*

$$S' := \{(a_j, \Psi_j) : 1 \leq j \leq N'\},$$

then for any $z \in \Omega$, $\mathcal{L}_{S'}(z) \geq \mathcal{L}_S(z)$.

Proof. If we take $A_j = \varnothing$ for $N'+1 \leq j \leq N$, any member of the defining family for $\mathcal{L}_{S'}(z)$ becomes a member of the defining family for $\mathcal{L}_S(z)$, and the sum remains the same. $\square$

The above lemma goes in the direction of monotonicity of the Lempert function with respect to its system of poles. For the Green function, it is immediate that the more poles there are, the more negative the function must be. More generally the more negative the generalized local indicators are (removing a pole corresponds to replacing a local indicator by 0), the more negative the function must be. This is not immediately apparent in Definition 2.10, but it does hold for elementary local indicators.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{C}^n$,*

$$S := \{(a_j, \Psi_j) : 1 \leq j \leq N\}, \quad S' := \{(a_j, \Psi_j') : 1 \leq j \leq N\}, \quad where \; a_j \in \Omega,$$

and Ψ_j and Ψ_j' are elementary local indicators such that $\Psi_j \leq \Psi_j' + C_j$ in a neighborhood of 0, $C_j \in \mathbb{R}$, $1 \leq j \leq N$. Then $\mathcal{L}_{S'}(z) \geq \mathcal{L}_S(z)$ for all $z \in \Omega$.

The proof is given in Section 5.

Now we turn to a result about the convergence of some families of (ordinary) Lempert functions with single poles, whose limits can be described naturally as generalized Lempert functions. Note that the proof of this next theorem does not require the relatively difficult Theorem 3.2, only the easy Lemma 3.1.

For $z \in \mathbb{C}^n \setminus \{0\}$, we denote by $[z]$ the equivalence class of z in the complex projective space $\mathbb{P}^{n-1}$.

Theorem 3.3. *Let Ω be a bounded and convex domain in $\mathbb{C}^n$. Let $0 \leq M \leq N$ be integers. For ε belonging to a neighborhood of 0 in $\mathbb{C}$, using the simplified notation of the single pole case (4), let*

$$S(\varepsilon) := \{a_j(\varepsilon) : 1 \leq j \leq M\} \cup \{a_j'(\varepsilon), a_j''(\varepsilon) : M+1 \leq j \leq N\} \subset \Omega.$$

Suppose that all the points of $S(\varepsilon)$ are distinct for any fixed ε, that

$$\lim_{\varepsilon\to 0} a_j(\varepsilon) = a_j \in \Omega, \quad 1 \le j \le M,$$

$$\lim_{\varepsilon\to 0} a_j'(\varepsilon) = \lim_{\varepsilon\to 0} a_j''(\varepsilon) = a_j \in \Omega, \quad M+1 \le j \le N,$$

and that

(5)
$$\lim_{\varepsilon\to 0}[a_j''(\varepsilon) - a_j'(\varepsilon)] = [v_j],$$

where the limit is with respect to the distance in $\mathbb{P}^{n-1}$ and the representative v_j is chosen of unit norm. Let $\Psi_j(z):=\log\|z\|$, $1\le j\le M$. Denote by π_j the orthogonal projection onto $\{v_j\}^\perp$, $M+1\le j\le N$, and by Ψ_j the generalized local indicator

$$\Psi_j(z) := \max(\log\|\pi_j(z)\|, 2\log|z\cdot\bar{v}_j|), \quad M+1 \le j \le N.$$

Set $S:=\{(a_j, \Psi_j):1\le j\le N\}$. Then

$$\lim_{\varepsilon\to 0}\ell_{S(\varepsilon)}(z) = \lim_{\varepsilon\to 0}\mathcal{L}_{S(\varepsilon)}(z) = \mathcal{L}_S(z) \quad \textit{for all } z\in\Omega.$$

Remarks. (a) As in the comments after (4), one could replace Ψ_j by an elementary local indicator; (b) The convexity requirement is imposed by Lemma 6.1, and we conjecture that it is not essential.

Note that in the case where $a_j'(\varepsilon)=a_j$ does not depend on ε, the hypothesis (5) means that the point $a_j''(\varepsilon)$ converges to a limit in the blow-up of $\mathbb{C}^n$ around the point a_j.

It seems to us that this is the only reasonable convergence result that can be obtained for a family of ordinary Lempert functions. If (5) is not satisfied, one can find two distinct limit points for our family of Lempert functions. Thus hypothesis (5) is required.

We are restricting ourselves to the case where no more than two points converge to the same point: examples where three points converge to the origin in the bidisk are explicitly studied in [12], and show that the situation leads to results that probably cannot be described in terms of our generalized local indicators.

The proof is given in Section 6.

4. Proof of Lemma 2.4

Multiplying one of the vectors v_j by a scalar only modifies the function Ψ by a bounded additive term, so it will be enough to exhibit an orthogonal basis of vectors complying with the conclusion of the lemma.

Renumber the vectors v_j so that we have $0 \leq m_1 \leq ... \leq m_n$. Using the Gram–Schmidt orthogonalization process, we produce an orthogonal system of vectors $\tilde{v}_k$ such that $\mathrm{Span}(\tilde{v}_1, ..., \tilde{v}_k) = \mathrm{Span}(v_1, ..., v_k)$ for any k, $1 \leq k \leq n$.

We proceed by induction on the dimension n. When $n=1$ the property is immediate. Assume that the result holds up to dimension $n-1$. Write

$$\Psi_1(z) := \max_{1 \leq j \leq n-1} m_j \log |z \cdot \bar{v}_j| \quad \text{and} \quad \widetilde{\Psi}_1(z) := \max_{1 \leq j \leq n-1} m_j \log |z \cdot \bar{\tilde{v}}_j|.$$

Set $z_n := z \cdot \bar{\tilde{v}}_n$.

It is enough to obtain the estimates on a neighborhood U of 0. We choose it so that $\Psi_1(z) \leq 0$ and $\widetilde{\Psi}_1(z) \leq 0$ for $z \in U$ with $|z_n| \leq 1$. Since $v_n = \tilde{v}_n - w$, where $w \in \mathrm{Span}(v_1, ..., v_{n-1})$, we have

(6) $\Psi(z) = \max(\Psi_1(z'), m_n \log |z_n - z' \cdot \overline{w}|), \quad \widetilde{\Psi}(z) = \max(\widetilde{\Psi}_1(z'), m_n \log |z_n|),$

where z' is the orthogonal projection of z onto $\mathrm{Span}(v_1, ..., v_{n-1}) = \mathrm{Span}(\tilde{v}_1, ..., \tilde{v}_{n-1})$. By the induction hypothesis, $\Psi_1 = \widetilde{\Psi}_1 + O(1)$, so it is enough to prove that

$$\Psi'(z) := \max(\Psi_1(z'), m_n \log |z_n|)$$

differs from $\Psi(z)$ by a bounded additive term.

There is a constant $C_0 > 0$ such that $\Psi_1(z') \geq m_{n-1} \log \|z'\| - \log C_0$, for $z' \in U$. Choose a constant $A > 1$ large enough so that $\|w\|(C_0/A)^{1/m_{n-1}} < \frac{1}{2}$.

Then, since $\Psi_1(z) \leq 0$ and $m_{n-1} \leq m_n$,

(7) $|z' \cdot \overline{w}| \leq \|w\| C_0^{1/m_{n-1}} \exp\left(\frac{\Psi_1(z')}{m_{n-1}}\right) \leq \|w\| C_0^{1/m_{n-1}} \exp\left(\frac{\Psi_1(z')}{m_n}\right).$

Case 1. $\Psi_1(z') \geq m_n \log |z_n| - \log A$.

By the inequality above, $\Psi'(z) \leq \Psi_1(z') + \log a \leq \Psi(z) + \log A$. On the other hand, using (7), we get

$$|z_n - z' \cdot \overline{w}|^{m_n} \leq (A^{1/m_n} + \|w\| C_0^{1/m_{n-1}})^{m_n} \exp(\Psi_1(z')),$$

so $\Psi(z) \leq \Psi_1(z') + O(1) \leq \Psi'(z) + O(1)$.

Case 2. $\Psi_1(z') \leq m_n \log |z_n| - \log A$.

Then (7) and the choice of A imply that

$$|z' \cdot \overline{w}| \leq \|w\| C_0^{1/m_{n-1}} \exp\left(\log |z_n| - \frac{\log A}{m_{n-1}}\right) \leq \frac{1}{2}|z_n|,$$

and thus (6) implies that

$$\Psi'(z) + m_n \log \tfrac{1}{2} \leq \Psi(z) \leq \Psi'(z) + m_n \log \tfrac{3}{2}.$$

5. Proof of Theorem 3.2

Without loss of generality, we may assume that $\tau_j' > 0$ for all j. We have $\Psi_j \leq \Psi_j' + C_j$ in a neighborhood of 0 and

$$\operatorname{supp}(dd^c \Psi_j)^n \subset \{0\} \quad \text{and} \quad \operatorname{supp}(dd^c \Psi_j')^n \subset \{0\}.$$

Thus it follows from Bedford and Taylor's comparison theorem [1], [5, p. 126, Theorem 3.7.1], that $\tau_j \geq \tau_j' > 0$. For any α and a_j,

$$(8) \qquad\qquad m_{\varphi, a_j, \Psi_j}(\alpha) \geq m_{\varphi, a_j, \Psi_j'}(\alpha).$$

Therefore

$$(9) \qquad\qquad \sum_{j=1}^N \sum_{\alpha \in A_j} m_{\varphi, a_j, \Psi_j}(\alpha) \log |\alpha| \leq \sum_{j=1}^N \sum_{\alpha \in A_j} m_{\varphi, a_j, \Psi_j'}(\alpha) \log |\alpha|.$$

To finish the proof, it suffices to show that the family over which we take the infimum is smaller for $\mathcal{L}_{S'}(z)$ than the one for $\mathcal{L}_S(z)$. This can be checked for each j separately, hence we drop the index j.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{C}^n$. If Ψ and Ψ' are elementary local indicators such that $\Psi \leq \Psi' + C$ and $\tau' := \tau_{\Psi'} > 0$, if $A \subset \mathbb{D}$, $a \in \Omega$ and $\varphi \in \operatorname{Hol}(\mathbb{D}, \Omega)$ satisfy*

$$\sum_{\alpha \in A} m_{\varphi, a, \Psi'}(\alpha) \leq \tau',$$

then

$$\sum_{\alpha \in A} m_{\varphi, a, \Psi}(\alpha) \leq \tau := \tau_{\Psi}.$$

Proof. Since the point a plays no role, we assume $a = 0$ and write $m_{\varphi, 0, \Psi}(\alpha) = m_{\varphi, \Psi}(\alpha)$. By (8), we may assume that $m_{\varphi, \Psi}(\alpha) > 0$ and the sums in (9) will not change.

Using Lemma 2.4, we reduce ourselves to the case where the elementary local indicators are given by orthonormal systems of vectors. We use the same "valuations" as in the elementary example:

$$\nu_j(\alpha) := \nu_j(\alpha, \varphi) := \min\left\{ k : \left(\frac{d}{d\zeta}\right)^k (\varphi(\zeta) \cdot \bar{v}_j)(\alpha) \neq 0 \right\},$$

and $\nu_j'(\alpha)$ is defined analogously using the vectors v_j'.

Case 1. There exists α_0 such that $m_{\varphi,\Psi'}(\alpha_0)=\tau'$.

Then the hypothesis of Lemma 5.1 implies that for all $\alpha\in A\setminus\{\alpha_0\}$, $m_{\varphi,\Psi'}(\alpha)=0$, so that $\min_{1\leq k\leq n} m'_k\nu'_k(\alpha)=0$. Since $\tau'>0$, we have $m'_k>0$ for all k, so there must exist k such that $\nu'_k(\alpha)=0$. Then $\varphi(\alpha)\neq 0$, which implies that $m_{\varphi,\Psi}(\alpha)=0$, and

$$\sum_{\alpha\in A} m_{\varphi,\Psi}(\alpha) = m_{\varphi,\Psi}(\alpha_0) \leq \tau,$$

by definition of the multiplicity.

Case 2. For all $\alpha\in A$, $m_{\varphi,\Psi'}(\alpha)<\tau'$.

Therefore $m_{\varphi,\Psi'}(\alpha)=\min_{1\leq k\leq n} m'_k\nu'_k(\alpha)$, and since we always have $m_{\varphi,\Psi}(\alpha)\leq \min_{1\leq k\leq n} m_k\nu_k(\alpha)$, it becomes enough to work with the quantities in (9). By dividing by τ and τ', respectively, it will be enough to prove the following lemma. $\square$

Lemma 5.2. *Under the hypotheses of Lemma* 5.1 *and Case* 2 *above, for each* $\alpha\in A$,

$$\frac{\min_{1\leq k\leq n} m'_k\nu'_k(\alpha)}{\prod_{k=1}^n m'_k} \geq \frac{\min_{1\leq k\leq n} m_k\nu_k(\alpha)}{\prod_{k=1}^n m_k}.$$

Proof. Since we are now dealing with a single α, we also drop it from the notation.

We introduce a binary relation on the index set $\{1,...,n\}$.

Definition 5.3. Given $k,l\in\{1,...,n\}$, we say that $k\mathcal{R}l$ if and only if $v_k\cdot\bar{v}'_l\neq 0$.

Lemma 5.4. *If* $\Psi'+C\geq\Psi$ *and* $k\mathcal{R}l$, *then* $m_k\geq m'_l$.

Proof. For any nonzero $\lambda\in\mathbb{C}$,

$$\Psi'(\lambda v'_l) = m'_l\log|\lambda|+m'_l\log\|v'_l\|^2,$$

while, for $|\lambda|$ small enough,

$$\Psi(\lambda v'_l) = \max_{1\leq j\leq n}\left(m_j(\log|\lambda|+\log|v_j\cdot\bar{v}'_l|)\right) = \left(\min_{k\mathcal{R}l} m_k\right)\log|\lambda|+O(1),$$

therefore by letting λ tend to 0 we see that $\min_{k\mathcal{R}l} m_k\geq m'_l$. $\square$

Lemma 5.5. *If* $\Psi'+C\geq\Psi$, *then*
1. $\nu'_l\geq\min\{\nu_k:k\mathcal{R}l\}$,
2. $\nu_k\geq\min\{\nu'_l:k\mathcal{R}l\}$.

Proof. We will use and prove Part 1 only. Part 2 has a similar proof.

Since v_l' is orthogonal to v_k unless $k\mathcal{R}l$, we must have complex scalars c_k such that $v_l' = \sum_{k\mathcal{R}l} c_k v_k$. Thus for φ as in Lemma 5.1,

$$\varphi(\zeta)\cdot\bar{v}_l' = \sum_{k\mathcal{R}l} \bar{c}_k \varphi(\zeta)\cdot\bar{v}_k.$$

Now take $m < \nu_k = \nu_k(\alpha,\varphi)$ for all k such that $k\mathcal{R}l$. Then

$$\left(\frac{d}{d\zeta}\right)^m (\varphi\cdot\bar{v}_l')(\alpha) = \sum_{k\mathcal{R}l} \bar{c}_k \left(\frac{d}{d\zeta}\right)^m (\varphi(\zeta)\cdot\bar{v}_k)(\alpha) = 0,$$

so we must have $\nu_l' > m$, which proves the result. $\square$

Now renumber the vectors v_l' so that $\min_{1\leq k\leq n}(m_k'\nu_k') = m_1'\nu_1'$. Pick an index k_0 such that $k_0\mathcal{R}1$ and $\nu_{k_0} = \min_{1\leq k\leq n}\{\nu_k : k\mathcal{R}1\}$. By renumbering the vectors v_k, we may assume that $k_0 = 1$. By Lemma 5.5, we may assume that $\nu_1' \geq \nu_1$.

The conclusion of Lemma 5.2 thus reduces to

(10)
$$\frac{\nu_1'}{\prod_{k=2}^n m_k'} \geq \frac{\nu_1}{\prod_{k=2}^n m_k}.$$

This is a consequence of the next result.

Lemma 5.6. *There exists a bijection σ from $\{2,...,n\}$ onto itself such that for any $l\in\{2,...,n\}$, $\sigma(l)\mathcal{R}l$.*

This lemma will be proved below. It implies that

$$\prod_{k=2}^n m_k = \prod_{l=2}^n m_{\sigma(l)} \geq \prod_{l=2}^n m_l',$$

by Lemma 5.4, so (10) holds and this concludes the proof of Lemma 5.2. $\square$

Proof of Lemma 5.6.   Set $A := \{a_{kl}\}_{2\leq k,l\leq n} := \{v_k\cdot\bar{v}_l'\}_{2\leq k,l\leq n}$. First we prove that this matrix is non-singular. Let π be the orthogonal projection on $\{v_1'\}^\perp$. If $\mathrm{rank}\{\pi(v_k):2\leq k\leq n\} < n-1$, there exists $w\in\{v_1'\}^\perp$, $w\neq 0$, such that $w\perp\pi(v_k)$, $2\leq k\leq n$. This implies that $w\perp v_k$, $2\leq k\leq n$. Since we have orthogonal bases, $v_1 = \lambda w$ for some $\lambda\in\mathbb{C}$. So $v_1\cdot\bar{v}_1' = 0$, which contradicts the fact that $1\mathcal{R}1$.

We construct the bijection σ by induction on n. For $n=2$ it is obvious. Suppose that the property holds for $n-1$. Then

$$0 \neq \det A = \sum_{k=2}^n (-1)^k a_{k2} \det A_k,$$

where A_k stands for the minor matrix with the first column and the kth row removed. There must be some k for which $a_{k2} \det A_k \neq 0$. Let $\sigma(2)=k$; the induction hypothesis gives us a bijection σ' from $\{3,...,n\}$ to $\{2,...,n\} \setminus \{k\}$ such that $a_{\sigma'(l)l} \neq 0$, and this finishes the proof. $\square$

6. Proof of Theorem 3.3

First observe that we can relax the conditions used in Definition 2.10.

Lemma 6.1. *Let Ω be a convex bounded domain in $\mathbb{C}^n$ containing the origin, and let $z \in \Omega$.*

(i) Let $a_j \in \Omega$, $\Psi_j \in \mathcal{I}$ and, as in Definition 2.6,

$$S := \{(a_j, \Psi_j) : 1 \leq j \leq N\}.$$

Suppose that for any $\delta > 0$, there exists a map φ^δ holomorphic from $\mathbb{D}$ to $(1+\delta)\Omega$ and sets $\{A_j(\delta)\}_{1 \leq j \leq N}$ such that $(\varphi^\delta, \{A_j(\delta)\}_{1 \leq j \leq N})$ is admissible for S and z with respect to $(1+\delta)\Omega$ and

$$\mathcal{S}(\varphi^\delta, \{A_j(\delta)\}_{1 \leq j \leq N}) \leq \ell + h(\delta),$$

where $h(\delta) \geq 0$ and $\lim_{\delta \to 0} h(\delta)=0$. Then $\mathcal{L}_S^\Omega(z) \leq \ell$.

(ii) For ε in a neighborhood V of 0 in $\mathbb{C}$, let $a_j(\varepsilon) \in \Omega$, $1 \leq j \leq N$, and

$$S(\varepsilon) := \{(a_j(\varepsilon), \Psi_j) : 1 \leq j \leq N\}.$$

Let $g \colon V \to \mathbb{R}_+^*$ be such that $\lim_{\varepsilon \to 0} g(\varepsilon)=0$. Then

$$\limsup_{\varepsilon \to 0} \mathcal{L}_{S(\varepsilon)}^\Omega(z) \leq \limsup_{\varepsilon \to 0} \mathcal{L}_{S(\varepsilon)}^{(1+g(\varepsilon))\Omega}(z).$$

Proof. Without loss of generality, we may assume that $z=0$. Let

$$\Omega_r := \{\varphi(\zeta) : \varphi \in \mathrm{Hol}(\mathbb{D}, \Omega), \varphi(0)=0 \text{ and } |\zeta| < r\}.$$

A bounded convex domain is Kobayashi complete hyperbolic [4, Proposition 6.9(b), p. 88], so Ω_r is relatively compact in Ω. Let ρ_Ω stand for the Minkowski function of Ω:

$$\rho_\Omega(z) := \inf\left\{r > 0 : \frac{z}{r} \in \Omega\right\}.$$

We set $\gamma(r) := \sup_{\Omega_r} \rho_\Omega$. The function γ is increasing and continuous from $(0, 1)$ to itself.

For any $\mu \in (0,1)$ and $\phi \in \mathrm{Hol}(\mathbb{D}, \mathbb{C}^n)$, set $\phi_\mu(\zeta) := \phi(\mu\zeta)$. Note that for any points and indicators, $m_{\phi_\mu, a, \Psi}(\alpha/\mu) = m_{\phi, a, \Psi}(\alpha)$.

Take φ^δ as in Part (i) of the lemma, in particular $\varphi^\delta(0) = 0$, so by the construction of γ,

$$\frac{1}{(1+\delta)} \varphi_\mu^\delta(\mathbb{D}) \subset \gamma(\mu)\Omega.$$

Choose some $\mu(\delta)$ such that $\gamma(\mu(\delta)) = (1+\delta)^{-1}$, and set $\widetilde{\varphi}^\delta := \varphi_{\mu(\delta)}^\delta$, then $\widetilde{\varphi}^\delta \in \mathrm{Hol}(\mathbb{D}, \Omega)$. Note that $\lim_{\delta \to 0} \mu(\delta) = 1$, by the relative compactness of each Ω_r.

Let

$$\tilde{A}_j(\delta) := \left\{ \frac{\alpha}{\mu(\delta)} : \alpha \in A_j(\delta) \text{ and } |\alpha| < \mu(\delta) \right\}.$$

Then

(11)

$$|\mathcal{S}(\widetilde{\varphi}^\delta, \{\tilde{A}_j(\delta)\}_{1 \le j \le N}) - \mathcal{S}(\varphi^\delta, \{A_j(\delta)\}_{1 \le j \le N})|$$

$$= \left| \sum_{j=1}^{N} \sum_{\substack{\alpha \in A_j \\ |\alpha| < \mu(\delta)}} m_{\varphi^\delta, a_j, \Psi_j}(\alpha) |\log \mu(\delta)| - \sum_{j=1}^{N} \sum_{\substack{\alpha \in A_j \\ |\alpha| \ge \mu(\delta)}} m_{\varphi^\delta, a_j, \Psi_j}(\alpha) \log |\alpha| \right|$$

$$\le 2 \left(\sum_{j=1}^{N} \tau_{\Psi_j} \right) |\log \mu(\delta)|,$$

and this last quantity tends to 0, which concludes the proof of (i).

To prove (ii), take maps φ^ε and systems of points $\{A_j(\varepsilon)\}_{1 \le j \le N}$, admissible for $S(\varepsilon)$, such that

$$\lim_{\varepsilon \to 0} \mathcal{S}(\varphi^\varepsilon, \{A_j(\varepsilon)\}_{1 \le j \le N}) = \limsup_{\varepsilon \to 0} \mathcal{L}_{S(\varepsilon)}^{(1+g(\varepsilon))\Omega}(0).$$

Use the above proof with $\delta = g(\varepsilon)$ to construct maps $\widetilde{\varphi}^\varepsilon$ into Ω and systems of points $\{\tilde{A}_j(\varepsilon)\}_{1 \le j \le N}$, admissible for $S(\varepsilon)$, such that

$$|\mathcal{S}(\widetilde{\varphi}^\varepsilon, \{\tilde{A}_j(\varepsilon)\}_{1 \le j \le N}) - \mathcal{S}(\varphi^\varepsilon, \{A_j(\varepsilon)\}_{1 \le j \le N})| \le 2 \left(\sum_{j=1}^{N} \tau_{\Psi_j} \right) |\log \mu(g(\varepsilon))|,$$

and by definition $\mathcal{S}(\widetilde{\varphi}^\varepsilon, \{\tilde{A}_j(\varepsilon)\}_{1 \le j \le N}) \ge \mathcal{L}_{S(\varepsilon)}^\Omega(0)$. $\square$

Consider as in Theorem 3.3 a bounded convex domain Ω, and distinct points $a_j \in \Omega$, $1 \le j \le N$. Let $z \in \Omega \setminus \{a_j : 1 \le j \le N\}$ (otherwise the property is trivially true). Again we may assume that $z = 0$. By Lemma 6.1 applied to $S(\delta) = S$ for any δ, to

show that

$$\mathcal{L}_S(z) \le \liminf_{\varepsilon \to 0} \mathcal{L}_{S(\varepsilon)}(z) =: \ell, \tag{12}$$

it will be enough to provide: some increasing function g such that $g(0)=0$ and, for any $\delta > 0$, maps $\varphi^\delta \in \mathrm{Hol}(\mathbb{D}, (1+g(\delta))\Omega)$ and subsets $\{A_j^\delta\}_{1 \le j \le N}$ of Ω such that $(\varphi^\delta, \{A_j^\delta\}_{1 \le j \le N})$ is admissible for S and z, and that

$$\mathcal{S}(\varphi^\delta, \{A_j^\delta\}_{1 \le j \le N}) = \ell.$$

The systems $S(\varepsilon)$ all have single poles, so the definition of ℓ means that there exist $\varphi_m \in \mathrm{Hol}(\mathbb{D}, \Omega)$, $\varepsilon_m \to 0$, and points $\alpha_{j,m}, \alpha'_{j,m}, \alpha''_{j,m} \in \mathbb{D}$ such that $\varphi_m(\alpha_{j,m}) = a_j(\varepsilon_m)$, $1 \le j \le M$, and $\varphi_m(\alpha'_{j,m}) = a'_j(\varepsilon_m)$, $\varphi_m(\alpha''_{j,m}) = a''_j(\varepsilon_m)$, $M+1 \le j \le N$; and they satisfy

$$\sum_{j=1}^{M} \log |\alpha_{j,m}| + \sum_{j=M+1}^{N} (\log |\alpha'_{j,m}| + \log |\alpha''_{j,m}|) = \ell + \delta(m),$$

with $\lim_{m \to \infty} \delta(m) = 0$.

Passing to a subsequence, for which we keep the same notation, we may assume that $\alpha_{j,m} \to \alpha_j \in \overline{\mathbb{D}}$, $\alpha'_{j,m} \to \alpha'_j \in \overline{\mathbb{D}}$, $\alpha''_{j,m} \to \alpha''_j \in \overline{\mathbb{D}}$ as $m \to \infty$, and that $\varphi_m \to \widetilde{\varphi} \in \mathrm{Hol}(\mathbb{D}, \overline{\Omega})$ uniformly on compact subsets of $\mathbb{D}$. Furthermore, by compactness of the unit circle, there exists $\tilde{v}_j \in [v_j] \cap S^{2n-1}$ such that, taking a further subsequence,

$$\lim_{m \to \infty} \frac{a''_j(\varepsilon_m) - a'_j(\varepsilon_m)}{\|a''_j(\varepsilon_m) - a'_j(\varepsilon_m)\|} = \tilde{v}_j.$$

By renumbering the points and exchanging a'_j and a''_j as needed, we may assume that there are integers $M' \le M \le N_1 \le N_2 \le N_3 \le N$ such that

$$
\begin{aligned}
\alpha_j &\in \mathbb{D} && \text{for } 1 \le j \le M', \\
\alpha_j &\in \partial\mathbb{D} && \text{for } M'+1 \le j \le M, \\
\alpha'_j &= \alpha''_j \in \mathbb{D} && \text{for } M+1 \le j \le N_1, \\
|\alpha'_j| &< |\alpha''_j| < 1 && \text{for } N_1+1 \le j \le N_2, \\
|\alpha'_j| &< 1, |\alpha''_j| = 1 && \text{for } N_2+1 \le j \le N_3, \\
|\alpha'_j| &= |\alpha''_j| = 1 && \text{for } N_3+1 \le j \le N.
\end{aligned}
$$

Then

$$
\begin{aligned}
\ell &= \lim_{m \to \infty} \left(\sum_{j=1}^{M} \log |\alpha_{j,m}| + \sum_{j=M+1}^{N} (\log |\alpha'_{j,m}| + \log |\alpha''_{j,m}|) \right) \\
&= \sum_{j=1}^{M'} \log |\alpha_j| + \sum_{j=M+1}^{N_1} 2 \log |\alpha'_j| + \sum_{j=N_1+1}^{N_2} (\log |\alpha'_j| + \log |\alpha''_j|) + \sum_{j=N_2+1}^{N_3} \log |\alpha'_j|.
\end{aligned}
$$

Now we choose

$$
\begin{aligned}
A_j &= \{\alpha_j\} && \text{for } 1 \le j \le M', \\
A_j &= \varnothing && \text{for } M'+1 \le j \le M, \\
A_j &= \{\alpha_j'\} && \text{for } M+1 \le j \le N_1, \\
A_j &= \{\alpha_j', \alpha_j''\} && \text{for } N_1+1 \le j \le N_2, \\
A_j &= \{\alpha_j'\} && \text{for } N_2+1 \le j \le N_3, \\
A_j &= \varnothing && \text{for } N_3+1 \le j \le N.
\end{aligned}
$$

Notice that $(\widetilde{\varphi}, \{A_j\}_{1 \le j \le N})$ hits the correct points but does not necessarily produce an admissible choice, because for some j, $N_1+1 \le j \le N_2$, we could have

$$
m_{\widetilde{\varphi}, a_j, \Psi_j}(\alpha_j') + m_{\widetilde{\varphi}, a_j, \Psi_j}(\alpha_j'') > 2 = \tau_j.
$$

So, in order to apply Lemma 6.1 with $\delta \to 0$, we set $A_j^\delta = A_j$ for any $\delta > 0$ and

$$
\widetilde{\varphi}^\delta(\zeta) := \widetilde{\varphi}(\zeta) + \delta\zeta \left[\prod_{j=1}^{M'} (\zeta - \alpha_j) \prod_{j=M+1}^{N_1} (\zeta - \alpha_j')^2 \prod_{j=N_1+1}^{N_2} (\zeta - \alpha_j')(\zeta - \alpha_j'') \prod_{j=N_2+1}^{N_3} (\zeta - \alpha_j') \right] v,
$$

where $v \in \mathbb{C}^n$ is a unit vector chosen such that $\pi_j(v) \ne 0$, $N_1+1 \le j \le N_3$. For any $\alpha \in \bigcup_{j=1}^{N} A_j$, $\widetilde{\varphi}^\delta(\alpha) = \widetilde{\varphi}(\alpha)$. There is a constant $C > 0$ such that $\widetilde{\varphi}^\delta(\mathbb{D}) \subset \Omega + C\delta B(0,1)$.

All the following considerations apply when δ is small enough.

For $1 \le j \le M'$, $m_{\widetilde{\varphi}^\delta, a_j, \Psi_j}(\alpha_j) = 1$, because $\widetilde{\varphi}^\delta$ takes the correct value, and the multiplicity cannot be more than $1 = \tau_j$ in these cases.

For $N_1+1 \le j \le N_3$, we have

$$
\pi_j((\widetilde{\varphi}^\delta)'(\alpha_j')) = \pi_j((\widetilde{\varphi})'(\alpha_j')) + \delta p_j \pi_j(v),
$$

where p_j is some complex scalar which does not depend on δ, so for $\delta > 0$ small enough, this projection does not vanish and we have $m_{\widetilde{\varphi}^\delta, a_j, \Psi_j}(\alpha_j') = 1$. An analogous reasoning shows that $m_{\widetilde{\varphi}^\delta, a_j, \Psi_j}(\alpha_j'') = 1$ for $N_1+1 \le j \le N_2$.

For $M+1 \le j \le N_1$, we have

$$
(\widetilde{\varphi}^\delta)'(\alpha_j') = (\widetilde{\varphi})'(\alpha_j'),
$$

and by the uniform convergence on compact sets,

$$
(\widetilde{\varphi})'(\alpha_j') = \lim_{m \to \infty} \frac{\varphi_m(\alpha_{j,m}') - \varphi_m(\alpha_{j,m}'')}{\alpha_{j,m}' - \alpha_{j,m}''} = \lim_{m \to \infty} \frac{a_j'(\varepsilon_m) - a_j''(\varepsilon_m)}{\alpha_{j,m}' - \alpha_{j,m}''},
$$

which must be colinear to v_j by definition. Therefore $m_{\widetilde{\varphi}^\delta, a_j, \Psi_j}(\alpha_j') = 2$ for $M+1 \le j \le N_1$.

Thus $(\widetilde{\varphi}^{\delta}, \{A_j\}_{1\leq j\leq N})$ is admissible for S and 0, and $\mathcal{S}(\widetilde{\varphi}^{\delta}, \{A_j\}_{1\leq j\leq N})=\ell$, which proves (12).

Now we need to show that

(13)
$$\mathcal{L}_S(z) \geq \limsup_{\varepsilon\to 0} \mathcal{L}_{S(\varepsilon)}(z).$$

We use Lemma 6.1(ii). For any $\delta>0$, we need to construct a positive function g such that $\lim_{\varepsilon\to 0} g(\varepsilon)=0$, and, for ε small enough, $\varphi^{\varepsilon}\in\mathrm{Hol}(\mathbb{D}(1+g(\varepsilon))\Omega)$ and sets $\{A_j(\varepsilon)\}_{1\leq j\leq N}$ such that $(\varphi^{\varepsilon}, \{A_j(\varepsilon)\}_{1\leq j\leq N})$ is admissible for $S(\varepsilon)$ and 0 and

$$\mathcal{S}(\varphi^{\varepsilon}, \{A_j(\varepsilon)\}_{1\leq j\leq N}) \leq \mathcal{L}_S(z)+\delta.$$

We start with an admissible choice $(\varphi, \{A_j\}_{1\leq j\leq N})$ for S, such that

$$\mathcal{S}(\varphi, \{A_j\}_{1\leq j\leq N}) \leq \mathcal{L}_S(z)+\frac{\delta}{2}.$$

To fix notation, suppose that, after renumbering and exchanging the points as needed, there exist integers $M'\leq M$ and $N_1, N_2, N_3\in\{M, ..., N\}$ such that

$$\begin{aligned}
A_j &= \{\alpha_j\} &&\text{for } 1\leq j\leq M', \\
A_j &= \varnothing &&\text{for } M'+1\leq j\leq M, \\
A_j &= \{\alpha_j'\}, m_{\varphi,a_j,\Psi_j}(\alpha_j')=2 &&\text{for } M+1\leq j\leq N_1, \\
A_j &= \{\alpha_j', \alpha_j''\}, \alpha_j'\neq\alpha_j'' &&\text{for } N_1+1\leq j\leq N_2, \\
A_j &= \{\alpha_j'\}, m_{\varphi,a_j,\Psi_j}(\alpha_j')=1 &&\text{for } N_2+1\leq j\leq N_3, \\
A_j &= \varnothing &&\text{for } N_3+1\leq j\leq N.
\end{aligned}$$

The definition of Ψ_j (see the computations performed in the elementary example) implies that, for $M+1\leq j\leq N_1$, $\varphi'(\alpha_j')\cdot\overline{w}=0$ for any $w\in v_j^{\perp}$. We perturb φ to make sure that, on the other hand, $\varphi'(\alpha_j')\cdot\overline{v}_j\neq 0$ in the same index range. For $\eta(\varepsilon)\in\mathbb{C}$ to be chosen later, set

$$\begin{aligned}
\widetilde{\varphi}(\zeta) := \varphi(\zeta)+\eta(\varepsilon)\Bigg[&\zeta\prod_{j=1}^{M'}(\zeta-\alpha_j) \prod_{j=N_1+1}^{N_2}(\zeta-\alpha_j')(\zeta-\alpha_j'') \prod_{j=N_2+1}^{N_3}(\zeta-\alpha_j')\Bigg] \\
&\times\Bigg(\sum_{j=M+1}^{N_1}\Bigg[(\zeta-\alpha_j')\prod_{\substack{M+1\leq k\leq N_1\\k\neq j}}(\zeta-\alpha_k')^2\Bigg]v_j\Bigg).
\end{aligned}$$

The map $\widetilde{\varphi}$ depends on ε and is again admissible.

We have positive constants C_1, C_2 and C_3 such that
1. $\widetilde{\varphi}'(\alpha_j')=\lambda_j v_j$, with $C_1^{-1}|\eta(\varepsilon)|\leq|\lambda_j|\leq C_1|\eta(\varepsilon)|$,
2. $\|\widetilde{\varphi}-\varphi\|_{\infty}\leq C_2|\eta(\varepsilon)|$,
3. $\widetilde{\varphi}(\mathbb{D})\subset(1+C_3|\eta(\varepsilon)|)\Omega$;

in particular $\widetilde{\varphi}$ will be bounded by constants independent of ε, along with all its derivatives on any given compact subset of $\mathbb{D}$.

For $M+1 \leq j \leq N$ and ε in a neighborhood of 0, $a''_j(\varepsilon) - a'_j(\varepsilon) = n_j(\varepsilon) v_j(\varepsilon)$, where $\|v_j(\varepsilon)\| = 1$, $\lim_{\varepsilon \to 0} v_j(\varepsilon) = v_j$ and $n_j(\varepsilon) \in \mathbb{C}$.

For $|\varepsilon|$ small enough, we now may define

$$A_j(\varepsilon) := A_j \qquad\qquad \text{for } 1 \leq j \leq M \text{ and } N_1 + 1 \leq j \leq N,$$

$$A_j(\varepsilon) := \left\{ \alpha'_j, \alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j} \right\} \quad \text{for } M+1 \leq j \leq N_1.$$

We shall need to add to $\widetilde{\varphi}$ a vector-valued correcting term obtained by Lagrange interpolation. To this end, we write $B(\varepsilon) := \bigcup_{j=1}^{N} A_j(\varepsilon)$, and values to be interpolated, $w(\alpha)$, for $\alpha \in B(\varepsilon)$. Let

$$w(\alpha_j) := a_j(\varepsilon) - a_j = a_j(\varepsilon) - \widetilde{\varphi}(\alpha_j) \quad \text{for } 1 \leq j \leq M',$$

$$w(\alpha'_j) := a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \widetilde{\varphi}(\alpha'_j) \quad \text{for } M+1 \leq j \leq N_1,$$

$$w\left(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j} \right) := a''_j(\varepsilon) - \widetilde{\varphi}\left(\alpha'_j + \frac{n_j(\varepsilon)}{\lambda_j} \right) \quad \text{for } M+1 \leq j \leq N_1,$$

$$w(\alpha'_j) := a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \widetilde{\varphi}(\alpha'_j) \quad \text{for } N_1 + 1 \leq j \leq N_2,$$

$$w(\alpha''_j) := a''_j(\varepsilon) - a_j = a''_j(\varepsilon) - \widetilde{\varphi}(\alpha''_j) \quad \text{for } N_1 + 1 \leq j \leq N_2,$$

$$w(\alpha'_j) := a'_j(\varepsilon) - a_j = a'_j(\varepsilon) - \widetilde{\varphi}(\alpha'_j) \quad \text{for } N_2 + 1 \leq j \leq N_3.$$

We denote by P_ε the solution to the interpolation problem

$$\{ P(\alpha) = w(\alpha) : \alpha \in B(\varepsilon) \}.$$

Let $\varphi^\varepsilon := \widetilde{\varphi} + P_\varepsilon \in \mathrm{Hol}(\mathbb{D}, \Omega^\varepsilon)$. The domain Ω^ε will be specified below. By construction $(\varphi^\varepsilon, \{A_j(\varepsilon)\}_{1 \leq j \leq N})$ is admissible for $S(\varepsilon)$, and for $|\varepsilon|$ small enough,

$$\mathcal{S}(\varphi^\varepsilon, \{A_j(\varepsilon)\}_{1 \leq j \leq N}) \leq \mathcal{L}_S(z) + \delta,$$

provided that, for $M+1 \leq j \leq N_1$,

$$(14) \qquad\qquad\qquad\qquad \lim_{\varepsilon \to 0} \frac{n_j(\varepsilon)}{\lambda_j} = 0.$$

Now we need to show that the correction is small, more precisely that we can choose $\eta(\varepsilon)$ so that the above condition is satisfied and $\lim_{\varepsilon \to 0} \|P_\varepsilon\|_\infty = 0$. Then we can choose a function g tending to 0 such that

$$\Omega^\varepsilon = (1 + g(\varepsilon)) \Omega \supset (1 + C_3 |\eta(\varepsilon)|) \Omega + B(0, \|P_\varepsilon\|_\infty).$$

Write Π_α for the unique (scalar) polynomial of degree less or equal to $d:=\#B(\varepsilon)-1$ (d does not depend on ε) such that

$$\Pi_\alpha(\alpha)=1 \quad \text{and} \quad \Pi_\alpha(\beta)=0 \text{ for any } \beta\in B(\varepsilon)\setminus\{\alpha\}.$$

Then

$$P_\varepsilon = \sum_{\alpha\in B(\varepsilon)} \Pi_\alpha w(\alpha).$$

For $\alpha\in\bigcup_{1\leq j\leq M,N_1+1\leq j\leq N} A_j$, the supremum norm $\|\Pi_\alpha\|_\infty$ is uniformly bounded, because $\mathrm{dist}(\alpha, B(\varepsilon)\setminus\{\alpha\})\geq\gamma>0$ with γ independent of ε. It also follows from the hypotheses of the theorem and the choice of w that

$$\lim_{\varepsilon\to0}\max\left\{\|w(\alpha)\|:\alpha\in\bigcup_{\substack{1\leq j\leq M\\N_1+1\leq j\leq N}} A_j\right\}=0.$$

For $M+1\leq j\leq N_1$, we need an elementary lemma about Lagrange interpolation.

Lemma 6.2. *Let $x_0,...,x_d\in\mathbb{D}$ and $w_0,w_1\in\mathbb{C}^n$. Suppose that there exists $\gamma>0$ such that $|x_0-x_1|\leq\gamma$ and $\mathrm{dist}([x_0,x_1],\{x_2,...,x_d\})\geq2\gamma$, where $[x_0,x_1]$ is the real line segment from x_0 to x_1.*

Let P be the unique ($\mathbb{C}^n$-valued) polynomial of degree less or equal to d such that

$$P(x_0)=w_0, \quad P(x_1)=w_1 \quad \text{and} \quad P(x_j)=0, \ 2\leq j\leq d.$$

Then there exist constants L_1 and L_0 depending only on γ and d such that

$$\sup_{\zeta\in\mathbb{D}}\|P(\zeta)\| \leq L_1\left\|\frac{w_1-w_0}{x_1-x_0}\right\|+L_0\|w_0\|.$$

We will prove this lemma a little later. It yields, for $M+1\leq j\leq N_1$,

$$\sup_{\zeta\in\mathbb{D}}\left\|\Pi_{\alpha_j'}(\zeta)w(\alpha_j')+\Pi_{\alpha_j'+n_j(\varepsilon)/\lambda_j}(\zeta)w\left(\alpha_j'+\frac{n_j(\varepsilon)}{\lambda_j}\right)\right\|$$

$$\leq L_1\left|\frac{\lambda_j}{n_j(\varepsilon)}\right|\left\|a_j''(\varepsilon)-\widetilde\varphi\left(\alpha_j'+\frac{n_j(\varepsilon)}{\lambda_j}\right)-(a_j'(\varepsilon)-\widetilde\varphi(\alpha_j'))\right\|+L_0\|a_j'(\varepsilon)-a_j\|.$$

We now estimate the first term in the last sum above. By the Taylor formula,

$$a_j''(\varepsilon)-\widetilde\varphi\left(\alpha_j'+\frac{n_j(\varepsilon)}{\lambda_j}\right)-(a_j'(\varepsilon)-\widetilde\varphi(\alpha_j'))=a_j''(\varepsilon)-a_j'(\varepsilon)-n_j(\varepsilon)v_j+R_2(\varepsilon)$$

$$=n_j(\varepsilon)(v_j(\varepsilon)-v_j)+R_2(\varepsilon),$$

where $\|R_2(\varepsilon)\|\leq C|n_j(\varepsilon)|^2|\lambda_j|^{-2}$ with a constant C independent of ε by the boundedness of the derivatives of $\widetilde{\varphi}$. Finally

$$\sup_{\zeta\in\mathbb{D}}\left\|\Pi_{\alpha'_j}(\zeta)w(\alpha'_j)+\Pi_{\alpha'_j+n_j(\varepsilon)/\lambda_j}(\zeta)w\left(\alpha'_j+\frac{n_j(\varepsilon)}{\lambda_j}\right)\right\|$$
$$\leq C(\|v_j(\varepsilon)-v_j\|\,|\eta(\varepsilon)|+|n_j(\varepsilon)|\,|\eta(\varepsilon)|^{-1}+\|a'_j(\varepsilon)-a_j\|).$$

To satisfy (14), we need to have $\lim_{\varepsilon\to0}n_j(\varepsilon)/\eta(\varepsilon)=0$; to make sure, in addition, that the whole sum above tends to 0 as ε tends to 0, it will be enough to choose $\eta(\varepsilon)$ going to zero, but more slowly than $|n_j(\varepsilon)|=\|a''_j(\varepsilon)-a'_j\|$ for $M+1\leq j\leq N_1$.

Proof of Lemma 6.2. Let

$$Q(X,Y):=\prod_{k=2}^{d}\frac{X-x_k}{Y-x_k}.$$

Then Q and all of its derivatives are bounded for $X\in\overline{\mathbb{D}}$ and $Y\in[x_0,x_1]$, and

$$P(X)=\frac{X-x_0}{x_1-x_0}Q(X,x_1)w_1+\frac{X-x_1}{x_0-x_1}Q(X,x_0)w_0$$
$$=\frac{w_1-w_0}{x_1-x_0}(X-x_0)Q(X,x_1)+\left(-Q(X,x_1)+(X-x_1)\frac{Q(X,x_1)-Q(X,x_0)}{x_0-x_1}\right)w_0.$$

Then the conclusion follows from the boundedness of Q and Q' and the mean-value theorem. $\square$

7. Comparison with previous results

Denote by L_S the old-style generalized Lempert function defined in [14]. Since it was not monotonic, we also needed the following definition.

Definition 7.1. Let

$$S:=\{(a_j,\Psi_j):1\leq j\leq N\}\quad\text{and}\quad S_1:=\{(a_j,\Psi_j^1):1\leq j\leq N\},$$

where $a_j\in\Omega$ and Ψ_j and Ψ_j^1 are local indicators. We define

$$\tilde{L}_S(z):=\inf\{L_{S^1}(z):\Psi_j^1\geq\Psi_j+C_j,1\leq j\leq N\}.$$

Lemma 7.2. *If* $S=\{(a_j,\Psi_j):1\leq j\leq N\}$, *where the* Ψ_j *are elementary local indicators, then for any* $z\in\Omega$, $\mathcal{L}_S(z)\leq\tilde{L}_S(z)$.

We skip the easy proof, which makes use of Theorem 3.2.

We now return to the study of the example presented in [14]. Let us recall the notation. For $z \in \mathbb{D}^2$,

$$\Psi_0(z) := \max(\log|z_1|, \log|z_2|) \quad \text{and} \quad \Psi_V(z) := \max(\log|z_1|, 2\log|z_2|).$$

For $a, b \in \mathbb{D}$ and $\varepsilon \in \mathbb{C}$, let

$$S_\varepsilon := \{((a,0), \Psi_0), ((b,0), \Psi_0), ((b,\varepsilon), \Psi_0), ((a,\varepsilon), \Psi_0)\},$$
$$S := \{((a,0), \Psi_V), ((b,0), \Psi_V)\}.$$

Here the Green functions are explicitly known and the following is proved in [14, p. 397].

Proposition 7.3. *If $b=-a$ and $|a|^2 < |\gamma| < |a|$, then $G_S(0,\gamma) < \tilde{L}_S(0,\gamma)$.*

It follows from our Theorem 3.3 that for any $z \in \mathbb{D}^2$, $\lim_{\varepsilon \to 0} L_{S_\varepsilon}(z) = \mathcal{L}_S(z)$. It is a consequence of [14, Theorem 5.1] that for $b=-a$ and $|a|^{3/2} < |\gamma| < |a|$, then $\mathcal{L}_S(0,\gamma) > G_S(0,\gamma)$; therefore $L_{S_\varepsilon}(0,\gamma) > G_{S_\varepsilon}(0,\gamma)$ for $|\varepsilon|$ small enough.

On the other hand, when $|\gamma| < |a|^{3/2}$, the old generalized Lempert function does not provide the correct limit of the single pole Lempert functions.

Proposition 7.4. *For $b=-a$ and $|a|^2 < |\gamma| < |a|^{3/2}$, $\mathcal{L}_S(0,\gamma) < \tilde{L}_S(0,\gamma)$.*

We omit the proof. We note that the mappings φ exhibiting the strict inequality are obtained with $\#A_1=2$ and $\#A_2=1$, but φ has multiplicity 2 at the single point of A_2.

A complete computation can be found in [15].

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