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Time-global smoothing estimates for a class of dispersive equations with constant coefficients

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Abstract. We discuss smoothing effects of homogeneous dispersive equations with constant coefficients. In the case where the characteristic root is positively homogeneous, time-global smoothing estimates are known. It is also known that a dispersiveness condition is necessary for smoothing effects. We show time-global smoothing estimates where the characteristic root is not necessarily homogeneous. Our results give a sufficient condition so that lower order terms can be absorbed by the principal part, and also indicate that smoothing effects may be caused by lower order terms in the case where the dispersiveness condition fails to hold.

1. Introduction

We will consider the initial value problem for homogeneous pseudodifferential equations with constant coefficients

$$D_t u - a(D)u = 0 \quad \text{in } \mathbb{R}^{1+n}, \tag{1.1}$$

$$u(0,x) = \phi(x) \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $u(t,x)$ is a complex-valued unknown function of $(t,x)=(t,x_1,...,x_n)\in\mathbb{R}^{1+n}$, $n\geq 1$, and

$$D_t = -i\frac{\partial}{\partial t}, \quad D=(D_1,...,D_n), \quad D_j = -i\frac{\partial}{\partial x_j},$$

where i always denotes the imaginary unit. Here, $a(D)$ is a differential operator defined by

$$a(D)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}a(\xi)u(y)\,dy\,d\xi.$$

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Throughout this paper, we assume that the symbol $a(\xi)\in C^1(\mathbb{R}^n)$ is real-valued and has at most polynomial growth at infinity. The solution to the initial value problem (1.1)–(1.2) is given by

$$e^{ita(D)}\phi(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi+ita(\xi)}\phi(y)\,dy\,d\xi.$$

Smoothing effects of dispersive equations have been studied by many authors. First, Sjölin [11] showed a local estimate in the case where $a(\xi)=|\xi|^m$, $m>1$. Let us focus our attention on time-global L^2-estimates for homogeneous equations. Originally, the smoothing effects of dispersive equations such as (1.1)–(1.2) had been established for the usual Laplacian i.e., the case $a(D)=-\sum_{j=1}^n \partial^2/\partial x_j^2$, that is, $a(\xi)=|\xi|^2$. Then several extensions for the general symbol a were studied. See [2], [4], [5], [6], [7], [8], [9], [10], [12], [14], [15] and [16] and references therein.

To explain the details, we introduce the notation of function spaces. Let Ω be a subset of a Euclidean space. For $s\in\mathbb{N}\cup\{0\}$, let $C^s(\Omega)$ denote the set of all s times continuously differentiable real-valued functions on Ω. Let $L^2(\Omega)$ denote the set of all square integrable functions f on Ω. Set

$$\|f\|_{L^2(\Omega)}=\left(\int_\Omega |f(x)|^2\,dx\right)^{1/2}.$$

A local smoothing effect for positively homogeneous symbols is established as follows. Set $\langle x\rangle=(1+|x|^2)^{1/2}$.

Theorem 1.1. (Chihara [4, Theorem 1.1]) *Let $n\geq 1$. Suppose that $a\in C^1(\mathbb{R}^n)$ is positively homogeneous of degree $m>1$, and satisfies the dispersiveness condition*

$$\nabla a(\xi)\neq 0 \quad \text{for } \xi\in\mathbb{R}^n\setminus\{0\}.$$

Let $\delta>\frac{1}{2}$. Then there exists $C>0$ such that

$$\|\langle x\rangle^{-\delta}|D|^{(m-1)/2}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\leq C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for all $\phi\in L^2(\mathbb{R}^n)$.

Other types of local smoothing estimates are known. See [8] and the references therein. Roughly speaking, local smoothing effects are caused by the dispersiveness condition, which is equivalent to the nontrapping condition of classical orbits, that is, $X(t;x,\xi)=x+t\nabla a(\xi)$ tends to infinity, as $t\to\pm\infty$, for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$. Hoshiro recently proved that the dispersiveness condition is necessary for a local smoothing effect.

Theorem 1.2. (Hoshiro [6, Theorem 1.1]) *Let $a(\xi)=\sum_{|\alpha|\leq m}c_\alpha\xi^\alpha$ be a real polynomial of degree $m>1$. Let $a_m(\xi)$ be the principal part of the symbol: $a_m(\xi)=\sum_{|\alpha|=m}c_\alpha\xi^\alpha$. Let $\chi\in C^\infty(\mathbb{R}^n)$ be a compactly supported function satisfying $\chi(x)=1$*

for $x\in U$, where U is a nonempty bounded open set. Suppose that there exist positive constants C and T such that

$$\int_0^T \|\langle D\rangle^{(m-1)/2}\chi e^{ita(D)}\phi\|_{L^2(\mathbb{R}^n)}^2\,dt \le C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for all $\phi\in L^2(\mathbb{R}^n)$. Then the dispersiveness condition

$$\nabla a_m(\xi)\ne 0 \quad \text{for } \xi\in\mathbb{R}^n\setminus\{0\}$$

holds.

Set $B_R=\{x\in\mathbb{R}^n;|x|<R\}$ and $\overline{B}_R=\{x\in\mathbb{R}^n;|x|\le R\}$ for $R>0$. A real polynomial a of order m is said to be of *real principal type* if there exists $R>0$ such that

$$|\nabla a(\xi)|\ge C(1+|\xi|)^{m-1} \quad \text{for } \xi\in\mathbb{R}^n\setminus B_R.$$

Note that in particular, for a real simply characteristic polynomial a, we have

$$\nabla a(\xi)\ne 0 \quad \text{for } \xi\in\mathbb{R}^n\setminus B_R \tag{1.3}$$

for some $R>0$. The following theorem gives a time-global smoothing estimate for real-principal-type operators.

Theorem 1.3. (Ben-Artzi and Devinatz [1, Theorem 3.1]) *Let a be a real simply characteristic polynomial and assume that a satisfies*

$$\lim_{\substack{|\xi|\to\infty\\ \xi\in M_\beta}} \frac{|\nabla^2 a(\xi)|}{|\nabla a(\xi)|}=0 \quad \text{for all } \beta>0,$$

where

$$M_\beta=\{\xi\in\mathbb{R}^n\,;|a(\xi)|\ge\beta\tilde a(\xi)\},\quad \tilde a(\xi)=\sum_{\alpha\in(\mathbb{N}\cup\{0\})^n}|D^\alpha a(\xi)|.$$

Choose $R>0$ such that (1.3) *holds. Let b be a real function of class $C^1(\mathbb{R}^n)$ satisfying*

$$|b(\xi)|(|b(\xi)|+|\nabla b(\xi)|)\le c|\nabla a(\xi)| \quad \text{for } \xi\in\mathbb{R}^n\setminus B_R.$$

For $\phi\in L^2(\mathbb{R}^n)$, decompose $\phi=\phi_1+\phi_2$, where their Fourier transforms $\hat\phi_1(\xi)=\chi_{B_R}(\xi)\hat\phi(\xi)$ and $\hat\phi_2(\xi)=(1-\chi_{B_R}(\xi))\hat\phi(\xi)$. Then there exists $C>0$ such that

$$\sup_{(t,x)\in\mathbb{R}^{1+n}}|D^\alpha e^{ita(D)}\phi_1(t,x)|\le C\|\phi\|_{L^2(\mathbb{R}^n)}$$

and

$$\|\langle x\rangle^{-1/2}b(D)e^{ita(D)}\phi_2\|_{L^2(\mathbb{R}^{1+n})}\le C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for all $\phi\in L^2(\mathbb{R}^n)$.

As is stated in [1, Corollary 3.2], if a is a real-principal-type polynomial of order m, then we can choose $b(\xi)=\langle\xi\rangle^{(m-1)/2}$ to obtain the estimate

$$\|\langle x\rangle^{-1/2}\langle D\rangle^{(m-1)/2}e^{ita(D)}\phi_2\|_{L^2(\mathbb{R}^{1+n})}\le C\|\phi\|_{L^2(\mathbb{R}^n)}.$$

This gives a better weight $\delta=-\frac{1}{2}$ than that of Theorem 1.1.

The aim of this paper is to show time-global smoothing estimates where the characteristic root is not necessarily homogeneous. The symbols which have so far been studied are mainly positively homogeneous functions and polynomials. We show a smoothing effect where the symbol $a(\xi)$ need not be a positively homogeneous function nor a polynomial. Let $a_m(\xi)$ be the principal part of the symbol $a(\xi)$ where a_m is positively homogeneous of order m. While $\nabla a_m(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$ is assumed in Theorem 1.1, we allow a to contain the lower part and we assume that $\nabla a(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$.

Our main result is the following. Set $S^{n-1}=\{x\in\mathbb{R}^n;|x|=1\}$.

Theorem 1.4. *Let $n\ge 1$. Suppose the following*:

(A1) $a\in C^1(\mathbb{R}^n)$;

(A2) $\nabla a(\xi)\neq 0$ *for all* $\xi\in\mathbb{R}^n\setminus\{0\}$;

(A3) *there exist* $m>1$ *and* $a_m\in C^1(\mathbb{R}^n)$ *such that*

(i) $\nabla a_m(\xi)\neq 0$ *for all* $\xi\in\mathbb{R}^n\setminus\{0\}$;

(ii) $a_m(\lambda\xi)=\lambda^m a_m(\xi)$ *for all* $\lambda>0$ *and* $\xi\in\mathbb{R}^n\setminus\{0\}$;

(iii)

$$\lim_{\lambda\to\infty}\max_{\omega\in S^{n-1}}|\lambda^{-m+1}\nabla a(\lambda\omega)-\nabla a_m(\omega)|=0;$$

(A4) *there exists a continuous function* $a_0\colon S^{n-1}\to\mathbb{R}^n$ *such that*

$$\lim_{\lambda\searrow 0}\max_{\omega\in S^{n-1}}\left|\frac{\nabla a(\lambda\omega)}{|\nabla a(\lambda\omega)|}-a_0(\omega)\right|=0.$$

Let $\delta>\frac{1}{2}$. *Then there exists* $C>0$ *such that*

$$\|\langle x\rangle^{-\delta}|(\nabla a)(D)|^{1/2}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\le C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for all $\phi\in L^2(\mathbb{R}^n)$.

Roughly speaking, Theorem 1.4 corresponds to Theorem 1.3 in the case where $R=0$ formally and $b(\xi)=|(\nabla a)(D)|^{1/2}$. We also remark that in Theorem 1.4 we do not assume that a is simply characteristic. Theorem 1.4 shows that even if the lower part exists, we can gain a smoothing effect whenever it satisfies appropriate conditions. Namely, we give a sufficient condition that the lower order term is absorbed by the principal part. For instance, consider

$$a(\xi)=\sum_{j=1}^n\xi_j^4+|\xi|^2.$$

It satisfies the assumption in Theorem 1.4. Namely, $a(D)$ is a real-principal-type operator.

On the other hand, even if the dispersiveness condition fails to hold, our method will work well in the particular case where the symbol is the sum of one-dimensional functions.

Theorem 1.5. *Let $n\geq 2$. Let*

$$a(\xi)=g(h(\xi)),\quad h(\xi)=\sum_{j=1}^{n} a_j(\xi_j),$$

where $g\in C^1(\mathbb{R})$ has at most polynomial growth at infinity, $g'(h(\xi))\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$, and for all $j=1,...,n$,

(i) *$a_j\in C^1(\mathbb{R})$ has at most polynomial growth at infinity;*

(ii) *$|a_j'|$ is nonincreasing on $(-\infty,0)$ and nondecreasing on $(0,\infty)$;*

(iii) *$a_j'(\rho)=0$ if and only if $\rho=0$.*

Let $\delta>\frac{1}{2}$. Then there exists $C>0$ such that

$$\|\langle x\rangle^{-\delta}|(\nabla a)(D)|^{1/2}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\leq C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for all $\phi\in L^2(\mathbb{R}^n)$.

Theorem 1.5 shows that even if the condition for the principal symbol $\nabla a_m(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$ fails, if the lower part $a-a_m$ helps to hold $\nabla a(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$, then we can also gain a smoothing effect whenever it satisfies appropriate conditions. In other words, we can gain a smoothing effect for some operators which are not of real principal type. For instance, let $n\geq 2$ and consider

$$a(\xi)=\sum_{j=1}^{n-1}\xi_j^4+|\xi|^2.$$

Then all of the assumptions in Theorem 1.5 are satisfied. However, it does not satisfy the assumption (A2) in Theorem 1.4. It seems reasonable to conclude that the lower order term ξ_n^2 besides the principal part $\sum_{j=1}^{n-1}\xi_j^4$ causes a smoothing effect.

Other types of smoothing effect are studied by using the limiting absorption principle. See [3] and [13] for example. Ben-Artzi and Nemirovsky [3, Theorem 3A] investigated the continuity of the resolvent and proved the large-time decay

$$\|\langle x\rangle^{-1}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\leq C\|\phi\|_{L^2(\mathbb{R}^n)}$$

for some suitable symbols a, including the relativistic Schrödinger operator $a(D)=\langle D\rangle$.

We give an outline of our method. We here define the Fourier transform in $(t,x)\in\mathbb{R}^{1+n}$ by setting

$$\tilde{f}(\tau,\xi)=(2\pi)^{-(1+n)/2}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} f(t,x)e^{-it\tau-ix\cdot\xi}\,dt\,dx.$$

Generally speaking, the time-global smoothing estimate

$$\|\langle x\rangle^{-\delta}|(\nabla a)(D)|^{1/2}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\leq C\|\phi\|_{L^2(\mathbb{R}^n)}$$

is equivalent to a Fourier restriction inequality

$$\left(\int_{\mathbb{R}^n}|\nabla a(\xi)|\,|\tilde{f}(a(\xi),\xi)|^2\,d\xi\right)^{1/2}\leq C\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})} \tag{1.4}$$

by duality. For homogeneous symbols, Chihara [4] decomposed the Fourier phase space $\mathbb{R}^n$ into finite connected cones according to nonvanishing entries of $\nabla a(\xi)$, and obtained (1.4) by some change of variables in each cone. Since our symbols are not necessarily homogeneous, we need to decompose the Fourier phase space $\mathbb{R}^n$ into finite connected curved regions to show (1.4) in our proof of Theorem 1.5. In Theorem 1.4, when $|\xi|$ is large enough, we can decompose it in the same way as in [4] since $a(\xi)$ can be approximated by a homogeneous function.

In this paper, Sections 2 and 3 describe the proofs of Theorems 1.4 and 1.5, respectively.

2. An estimate for real-principal-type operators

In this section, we give a proof of Theorem 1.4.

Remark 2.1. By substituting $e^{ita(0)}u(t,x)$ for $u(t,x)$, we may assume that $a(0)=0$ without loss of generality.

Lemma 2.2. *Let $n\geq 2$. Suppose* (A1)–(A3). *Set*

$$\Gamma_j=\left\{\xi\in\mathbb{R}^n\setminus\{0\}\,;\left|\frac{\partial a_m}{\partial\xi_j}(\xi)\right|>(2n)^{-1/2}|\nabla a_m(\xi)|\right\}$$

for $j=1,\dots,n$. Then there exists $R>1$ such that

$$\left|\frac{\partial a}{\partial\xi_j}(\xi)\right|\geq C|\nabla a(\xi)|$$

for all $\xi\in\Gamma_j\setminus B_{R-1}$.

We note that (A3) (i) implies that

$$\bigcup_{j=1}^{n} \Gamma_j = \mathbb{R}^n \setminus \{0\} \tag{2.1}$$

when $n \geq 2$.

Proof. It follows from (A3) (iii) that there exists $R>1$ such that

$$\frac{|\nabla a(\xi) - \nabla a_m(\xi)|}{|\xi|^{m-1}} \leq \frac{(2n)^{-1/2}}{10} \min_{\omega \in S^{n-1}} |\nabla a_m(\omega)|$$

for $\xi \in \mathbb{R}^n \setminus B_{R-1}$. Then,

$$\begin{aligned} |\nabla a(\xi)| &\leq |\nabla a_m(\xi)| + \frac{(2n)^{-1/2}|\xi|^{m-1}}{10} \min_{\omega \in S^{n-1}} |\nabla a_m(\omega)| \\ &= |\nabla a_m(\xi)| + \frac{(2n)^{-1/2}}{10} \min_{\xi \in \mathbb{R}^n \setminus \{0\}} |\nabla a_m(\xi)| \\ &\leq \tfrac{11}{10} |\nabla a_m(\xi)| \end{aligned}$$

for $\xi \in \mathbb{R}^n \setminus B_{R-1}$. Using the above inequality, we have that

$$\begin{aligned} \left| \frac{\partial a}{\partial \xi_j}(\xi) \right| &\geq \left| \frac{\partial a_m}{\partial \xi_j}(\xi) \right| - \frac{(2n)^{-1/2}|\xi|^{m-1}}{10} \min_{\omega \in S^{n-1}} |\nabla a_m(\omega)| \\ &\geq (2n)^{-1/2} |\nabla a_m(\xi)| - \frac{(2n)^{-1/2}}{10} \min_{\xi \in \mathbb{R}^n \setminus \{0\}} |\nabla a_m(\xi)| \\ &\geq \frac{9(2n)^{-1/2}}{10} |\nabla a_m(\xi)| \\ &\geq \frac{9(2n)^{-1/2}}{11} |\nabla a(\xi)| \end{aligned}$$

for $\xi \in \Gamma_j \setminus B_{R-1}$. □

The next lemma provides a decomposition of the Fourier phase space $\mathbb{R}^n$ into finite convex regions on which specific directional derivatives of a do not vanish.

Lemma 2.3. *Let* $n \geq 1$. *Suppose* (A1)–(A4). *Then there exist* $l \in \mathbb{N}$, *convex sets* $\Lambda_1, ..., \Lambda_l \subset \mathbb{R}^n \setminus \{0\}$ *and* $\omega_1, ..., \omega_l \in S^{n-1}$ *such that*

(i) $\bigcup_{k=1}^{l} \Lambda_k = \mathbb{R}^n \setminus \{0\}$;

(ii) $\nabla a(\xi) \cdot \omega_k \geq C |\nabla a(\xi)|$ *for all* $\xi \in \Lambda_k$.

Proof. Step 1. For $\omega_0 \in S^{n-1}$ and $r>0$, set

$$\Lambda_{\omega_0, r} = \left\{ \lambda\omega \,; 0 < \lambda < r,\ \omega \in S^{n-1} \text{ and } \omega \cdot \omega_0 > \tfrac{9}{10} \right\},$$

which is convex and open. It follows from (A4) that $a_0(\lambda\omega)=\nabla a(\lambda\omega)/|\nabla a(\lambda\omega)|$ is a uniformly continuous function on $[0,1]\times S^{n-1}$. Therefore, there exists $r\in(0,1]$ such that

$$\left|a_0(\lambda\omega)-a_0\left(\frac{r\omega_0}{2}\right)\right|<\frac{1}{10}$$

for $\lambda\omega\in\Lambda_{\omega_0,r}$. Then

$$\begin{aligned}a_0(\lambda\omega)\cdot a_0\left(\frac{r\omega}{2}\right)&=\left|a_0\left(\frac{r\omega}{2}\right)\right|^2+\left(a_0(\lambda\omega)-a_0\left(\frac{r\omega}{2}\right)\right)\cdot a_0\left(\frac{r\omega}{2}\right)\\&\geq 1-\left|a_0(\lambda\omega)-a_0\left(\frac{r\omega}{2}\right)\right|>\frac{9}{10}\end{aligned}$$

for $\lambda\omega\in\Lambda_{\omega_0,r}$. Now, if we set $\eta_{\omega_0}=a_0\left(\frac{1}{2}r\omega_0\right)$, then

$$\nabla a(\lambda\omega)\cdot\eta_{\omega_0}>\tfrac{9}{10}|\nabla a(\lambda\omega)|$$

for $\lambda\omega\in\Lambda_{\omega_0,r}$. Since $\Lambda_{\omega_0,r}\ni\frac{1}{2}r\omega_0$, we have

$$\bigcup_{\omega_0\in S^{n-1}}\Lambda_{\omega_0,r}\supset\frac{1}{2}rS^{n-1}=\left\{\frac{1}{2}r\omega\,;\omega\in S^{n-1}\right\}.$$

As $\frac{1}{2}rS^{n-1}$ is compact, there exist $l\in\mathbb{N}$ and $\omega_{0,1},\dots,\omega_{0,l}\in S^{n-1}$ such that

$$\bigcup_{k=1}^{l}\Lambda_{\omega_{0,k},r}\supset\frac{1}{2}rS^{n-1}.$$

Since $\Lambda_{\omega_{0,k},r}$ is a circular cone, we have $\bigcup_{k=1}^{l}\Lambda_{\omega_{0,k},r}=B_r\setminus\{0\}$. Finally, set $\Lambda_k=\Lambda_{\omega_{0,k},r}$ and $\omega_k=\eta_{\omega_{0,k}}$. Then, $\bigcup_{k=1}^{l}\Lambda_k=B_r\setminus\{0\}$ and $\nabla a(\xi)\cdot\omega_k\geq C|\nabla a(\xi)|$ for $\xi\in\Lambda_k$.

Step 2. (a) The case $n\geq 2$. Let $R>1$ be as in Lemma 2.2. Let $\omega_0\in S^{n-1}$. Note that (2.1) implies that $\omega_0\in\Gamma_j$ for some j. Therefore, we can choose $U_{\omega_0}\subset S^{n-1}\cap\Gamma_j$ which is an open neighborhood of ω_0 in S^{n-1} such that the cone $W_{\omega_0}=\{\lambda\omega;\lambda>0 \text{ and } \omega\in U_{\omega_0}\}\subset\Gamma_j$ is convex. We have that $\bigcup_{\omega_0\in S^{n-1}}U_{\omega_0}=S^{n-1}$. As S^{n-1} is compact, there exist $l\in\mathbb{N}$ and $\omega_{0,1},\dots,\omega_{0,l}\in S^{n-1}$ such that $\bigcup_{k=1}^{l}U_{\omega_{0,k}}=S^{n-1}$. Then, $\bigcup_{k=1}^{l}W_{\omega_{0,k}}=\mathbb{R}^n\setminus\{0\}$. Covering $W_{\omega_{0,k}}$ with finite thin enough cones (we use the same notation $\{W_{\omega_{0,k}}\}_{k=1}^{l}$ again) if necessary, we can choose a convex set Λ_k such that $W_{\omega_{0,k}}\setminus\overline{B}_R\subset\Lambda_k\subset W_{\omega_{0,k}}\setminus\overline{B}_{R-1}$. By virtue of Lemma 2.2, the set $\{(\partial a/\partial\xi_j)(\xi);\xi\in\Lambda_k\}$ is an interval which does not contain 0. For each $k=1,\dots,l$, let $\omega_k=e_j=(0,\dots,0,1_{(j)},0,\dots,0)$ with j defined above if $(\partial a/\partial\xi_j)(\xi)$ is positive on Λ_k, and let $\omega_k=-e_j$ if $(\partial a/\partial\xi_j)(\xi)$ is negative on Λ_k. Then, $\bigcup_{k=1}^{l}\Lambda_k\supset\mathbb{R}^n\setminus\overline{B}_R$ and $\nabla a(\xi)\cdot\omega_k\geq C|\nabla a(\xi)|$ for $\xi\in\Lambda_k$.

(b) The case $n=1$. It follows from (A3) that there exists $R>1$ such that

$$\frac{|a'(\xi)-a'_m(\xi)|}{|\xi|^{m-1}}\leq\frac{1}{10}\min\{|a'_m(1)|,|a'_m(-1)|\}$$

for $\xi\in(-\infty,-R+1]\cup[R-1,\infty)$. Then,

$$\begin{aligned}|a'(\xi)| &\geq |a'_m(\xi)| - \frac{|\xi|^{m-1}}{10}\min\{|a'_m(1)|, |a'_m(-1)|\} \\ &= |a'_m(\xi)| - \tfrac{1}{10}\min_{\xi\in\mathbb{R}\setminus\{0\}} |a'_m(\xi)| \geq \tfrac{9}{10}|a'_m(\xi)| > 0\end{aligned}$$

for $\xi\in(-\infty,-R+1]\cup[R-1,\infty)$. Set now $\Lambda_1=(R-1,\infty)$, $\Lambda_2=(-\infty,-R+1)$, $\omega_1=\operatorname{sgn} a'(R)$ and $\omega_2=\operatorname{sgn} a'(-R)$. Then we obtain the same conclusion as in the case $n\geq 2$.

Step 3. Let $r\in(0,1]$ and $R>1$ be as in Steps 1 and 2, respectively. Fix an arbitrary $\xi\in\overline{B}_R\setminus B_r$. Set $\omega_\xi=\nabla a(\xi)/|\nabla a(\xi)|$. The function

$$\frac{\nabla a(\zeta+t\omega_\xi)\cdot\omega_\xi}{|\nabla a(\zeta+t\omega_\xi)|}$$

is continuous with respect to $(t,\zeta)\in\mathbb{R}\times\mathbb{R}^n$ except on a neighborhood of $\zeta+t\omega_\xi=0$, and it attains 1 at $(t,\zeta)=(0,\xi)$. Therefore, there exist $d_1, d_2>0$ such that

$$\nabla a(\zeta+t\omega_\xi)\cdot\omega_\xi \geq \tfrac{1}{2}|\nabla a(\zeta+t\omega_\xi)|$$

for $\zeta+t\omega_\xi\in\Lambda_\xi$, where we set

$$\Lambda_\xi = \{\zeta+t\omega_\xi\,;\zeta\in\mathbb{R}^n,\ |\zeta-\xi|<d_1 \text{ and } -d_2<t<d_2\},$$

which is convex and open. Since $\Lambda_\xi\ni\xi$, we have that $\bigcup_{\xi\in\overline{B}_R\setminus B_r}\Lambda_\xi\supset\overline{B}_R\setminus B_r$. As $\overline{B}_R\setminus B_r$ is compact, there exist $l\in\mathbb{N}$ and $\xi_1,...,\xi_l\in\overline{B}_R\setminus B_r$ such that $\bigcup_{k=1}^l\Lambda_{\xi_k}\supset\overline{B}_R\setminus B_r$. Set $\Lambda_k=\Lambda_{\xi_k}$ and $\omega_k=\omega_{\xi_k}$. Then, $\bigcup_{k=1}^l\Lambda_k\supset\overline{B}_R\setminus B_r$ and $\nabla a(\xi)\cdot\omega_k\geq\frac{1}{2}|\nabla a(\xi)|$ for $\xi\in\Lambda_k$.

Combining Steps 1–3, we obtain the desired conclusion. □

We prove Theorem 1.4 using Lemma 2.3.

Proof of Theorem 1.4. First we show (1.4). We split the integral in the left-hand side of (1.4) into integrals on Λ_k by Lemma 2.3(i). Namely, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|\nabla a(\xi)|\,|\tilde f(a(\xi),\xi)|^2\,d\xi\right)^{1/2} &= \left(\int_{\bigcup_{k=1}^l\Lambda_k}|\nabla a(\xi)|\,|\tilde f(a(\xi),\xi)|^2\,d\xi\right)^{1/2} \\ &\leq \sum_{k=1}^l\left(\int_{\Lambda_k}|\nabla a(\xi)|\ |\tilde f(a(\xi),\xi)|^2\,d\xi\right)^{1/2}.\end{aligned}\tag{2.2}$$

We deal with the right-hand side term by term. We may assume that $\omega_k=e_1=(1,0,...,0)$ without loss of generality by substituting $a(P\,\cdot\,)$ for a (and $P\Lambda_k$ for Λ_k), where P is the orthogonal matrix satisfying $P\omega_k = e_1$. We write $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Set $Z_k = \{(a(\xi), \xi'); \xi \in \Lambda_k\}$. The set $\{t \in \mathbb{R}; te_1 + \xi' \in \Lambda_k\}$ is an interval for all fixed $\xi \in \Lambda_k$ since Λ_k is convex. By virtue of Lemma 2.3,

$$\frac{d}{dt} a(te_1 + \xi') = \frac{\partial a}{\partial \xi_1}(te_1 + \xi') \geq C|\nabla(te_1 + \xi')| > 0$$

on the interval, which implies that $a(te_1 + \xi')$ is strictly increasing with respect to t. Then the mapping

$$\Lambda_k \ni \xi \longmapsto (\tau, \xi') = (a(\xi), \xi') \in Z_k \tag{2.3}$$

is bijective. We here denote its inverse by

$$Z_k \ni (\tau, \xi') \longmapsto (\Xi_k(\tau, \xi'), \xi') \in \Lambda_k.$$

We have

$$\left|\det \frac{\partial \xi}{\partial(\tau, \xi')}\right| = \left|\det \frac{\partial(\tau, \xi')}{\partial \xi}\right|^{-1} = \left|\frac{\partial a}{\partial \xi_1}(\xi)\right|^{-1} \leq C|\nabla a(\xi)| \tag{2.4}$$

for $\xi \in \Lambda_k$. Changing the variables by (2.3), using (2.4), the Minkowski inequality, the Plancherel–Parseval formula and the Schwarz inequality yield

$$\begin{aligned}
&\int_{\Lambda_k} |\nabla a(\xi)|\, |\tilde{f}(a(\xi), \xi)|^2 \, d\xi \\
&= \iint_{Z_k} |\nabla a(\Xi_k(\tau, \xi'), \xi')|\, |\tilde{f}(\tau, \Xi_k(\tau, \xi'), \xi')|^2 \left|\frac{\partial a}{\partial \xi_1}(\Xi_k(\tau, \xi'), \xi')\right|^{-1} d\tau \, d\xi' \\
&\leq C \iint_{Z_k} |\tilde{f}(\tau, \Xi_k(\tau, \xi'), \xi')|^2 \, d\tau \, d\xi' \\
&= C \iint_{Z_k} \left| \int_{-\infty}^{\infty} \exp(-ix_1 \Xi_k(\tau, \xi')) \mathcal{F}_{t,x'}[f](\tau, x_1, \xi') \, dx_1 \right|^2 d\tau \, d\xi' \\
&\leq C \iint_{Z_k} \left(\int_{-\infty}^{\infty} |\mathcal{F}_{t,x'}[f](\tau, x_1, \xi')| \, dx_1 \right)^2 d\tau \, d\xi' \\
&\leq C \left(\int_{-\infty}^{\infty} \left(\iint_{Z_k} |\mathcal{F}_{t,x'}[f](\tau, x_1, \xi')|^2 \, d\tau \, d\xi' \right)^{1/2} dx_1 \right)^2 \\
&\leq C \left(\int_{-\infty}^{\infty} \left(\iint_{\mathbb{R}^n} |\mathcal{F}_{t,x'}[f](\tau, x_1, \xi')|^2 \, d\tau \, d\xi' \right)^{1/2} dx_1 \right)^2 \\
&= C \left(\int_{-\infty}^{\infty} \left(\iint_{\mathbb{R}^n} |f(t, x)|^2 \, dt \, dx' \right)^{1/2} dx_1 \right)^2 \\
&\leq C \left(\int_{-\infty}^{\infty} (1 + x_1^2)^{-\delta/2} \left(\iint_{\mathbb{R}^n} |\langle x \rangle^{\delta} f(t, x)|^2 \, dt \, dx' \right)^{1/2} dx_1 \right)^2
\end{aligned} \tag{2.5}$$

$$\le C\|\langle x\rangle^{\delta} f\|^2_{L^2(\mathbb{R}^{1+n})}\int_{-\infty}^{\infty}(1+y^2)^{-\delta}\,dy$$
$$= C\|\langle x\rangle^{\delta} f\|^2_{L^2(\mathbb{R}^{1+n})},$$

where $\mathcal{F}_{t,\hat{x}_j}[f]$ denotes the Fourier transform of f in $(t,\hat{x}_j)\in\mathbb{R}^n$. Substituting (2.5) for (2.2), we obtain

$$\left(\int_{\mathbb{R}^n}|\nabla a(\xi)|\,|\tilde{f}(a(\xi),\xi)|^2\,d\xi\right)^{1/2} \le C\|\langle x\rangle^{\delta} f\|_{L^2(\mathbb{R}^{1+n})}. \tag{2.6}$$

Finally we show that the Fourier restriction inequality (2.6) implies the time-global smoothing estimate. Using the Plancherel–Parseval formula and (2.6) yield

$$\begin{aligned}
&\|\langle x\rangle^{-\delta}|(\nabla a)(D)|^{1/2}e^{ita(D)}\phi\|_{L^2(\mathbb{R}^{1+n})}\\
&= \sup_{\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}=1}\left|\int_{\mathbb{R}^n}\int_{-\infty}^{\infty}(|(\nabla a)(D)|^{1/2}e^{ita(D)}\phi(x))\overline{f(t,x)}\,dt\,dx\right|\\
&= \sup_{\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}=1}\left|\int_{\mathbb{R}^n}\hat{\phi}(\xi)|(\nabla a)(\xi)|^{1/2}\int_{-\infty}^{\infty}\overline{e^{-ita(\xi)}\hat{f}(t,\xi)}\,dt\,d\xi\right|\\
&= (2\pi)^{1/2}\sup_{\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}=1}\left|\int_{\mathbb{R}^n}\hat{\phi}(\xi)|(\nabla a)(\xi)|^{1/2}\,\overline{\tilde{f}(a(\xi),\xi)}\,d\xi\right|\\
&\le (2\pi)^{1/2}\|\phi\|_{L^2(\mathbb{R}^{1+n})}\sup_{\|\langle x\rangle^{\delta}f\|_{L^2(\mathbb{R}^{1+n})}=1}\left(\int_{\mathbb{R}^n}|(\nabla a)(\xi)|\,|\tilde{f}(a(\xi),\xi)|^2\,d\xi\right)^{1/2}\\
&\le C\|\phi\|_{L^2(\mathbb{R}^{1+n})},
\end{aligned}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ in $x\in\mathbb{R}^n$. □

3. An estimate for operators which need not be of real principal type

In this section, we give the proof of Theorem 1.5.

Proof of Theorem 1.5. It follows from the definition of $a(\xi)$ that

$$\nabla a(\xi) = g'(h(\xi))(a_1'(\xi),...,a_n'(\xi)) \neq 0 \quad \text{for } \xi\in\mathbb{R}^n\setminus\{0\}. \tag{3.1}$$

Set

$$\begin{aligned}
\Gamma_j &= \left\{\xi\in\mathbb{R}^n\setminus\{0\}\,;\left|\frac{\partial a}{\partial \xi_j}(\xi)\right| > (n+1)^{-1/2}|\nabla a(\xi)|\right\}\\
&= \left\{\xi\in\mathbb{R}^n\setminus\{0\}\,;|a_j'(\xi_j)| > n^{-1/2}\left(\sum_{l\neq j}a_l'(\xi_l)^2\right)^{1/2}\right\}
\end{aligned}$$

and

$$\Gamma_{j,1}=\left\{\xi\in\mathbb{R}^n\setminus\{0\}\,;\,|a_j'(\xi_j)|>n^{-1/2}\left(\sum_{l\neq j}a_l'(\xi_l)^2\right)^{1/2}\text{ and }\xi_j>0\right\},$$

$$\Gamma_{j,2}=\left\{\xi\in\mathbb{R}^n\setminus\{0\}\,;\,|a_j'(\xi_j)|>n^{-1/2}\left(\sum_{l\neq j}a_l'(\xi_l)^2\right)^{1/2}\text{ and }\xi_j<0\right\}$$

for $j=1,...,n$. Our assumption implies that $\Gamma_j=\Gamma_{j,1}\cup\Gamma_{j,2}$. Then (3.1) implies that $\mathbb{R}^n\setminus\{0\}=\bigcup_{j=1}^n\bigcup_{k=1}^2\Gamma_{j,k}$.

We claim that each $\Gamma_{j,k}$ is either connected or empty. We only demonstrate that $\Gamma_{1,1}$ is connected if it is not empty; we can argue for the others similarly. We write $\xi=(\xi_1,\xi')\in\mathbb{R}\times\mathbb{R}^{n-1}$. Let ξ and η be arbitrary points in $\Gamma_{1,1}$. Define a mapping $\Psi\colon[0,1]\to\mathbb{R}^n\setminus\{0\}$ by

$$\Psi(\theta)=\begin{cases}(\xi_1,(1-3\theta)\xi'), & \text{if } 0\le\theta\le\frac{1}{3},\\ ((2-3\theta)\xi_1+(3\theta-1)\eta_1,0), & \text{if } \frac{1}{3}\le\theta\le\frac{2}{3},\\ (\eta_1,(3\theta-2)\eta'), & \text{if } \frac{2}{3}\le\theta\le 1.\end{cases}$$

It is easy to see that Ψ is continuous and maps $[0,1]$ to $\Gamma_{1,1}$. Therefore, $\Gamma_{1,1}$ is arcwise connected in $\mathbb{R}^n$ and hence connected. Moreover, the set

$$\{t\in\mathbb{R}\,;\,te_1+\xi\in\Gamma_{1,1}\}=\left\{t>-\xi_1\,;\,|a_1'(t+\xi_1)|>n^{-1/2}\left(\sum_{l\neq 1}a_l'(\xi_l)^2\right)^{1/2}\right\}$$

is an interval for all fixed $\xi\in\Gamma_{1,1}$.

After this, we can argue as in the proof of Theorem 1.4 to prove Theorem 1.5. We omit the details. □

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