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Hybrid mean value on the difference between an integer and its inverse modulo q

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Abstract. The main purpose of this paper is to study the asymptotic property of one class of number-theoretic functions and give a sharp hybrid mean-value formula by using the generalized Bernoulli numbers, Gauss sums and the mean-value theorems of Dirichlet L-functions.

1. Introduction

Let $q>2$ and c be two integers with $(c,q)=1$. For each integer $1\le a\le q$ with $(a,q)=1$, we know that there exists one and only one integer $1\le b\le q$ with $(b,q)=1$ such that $ab\equiv c \bmod q$. Let

$$M(q,k,c)=\mathop{{\sum}'}_{a=1}^{q}\mathop{{\sum}'}_{\substack{b=1\\ ab\equiv c \bmod q}}^{q}(a-b)^{2k},$$

where $\sum_{a=1}^{\prime q}$ denotes the summation over all a such that $(a,q)=1$. In Zhang [5], the second author used the estimates for Kloosterman sums and trigonometric sums to obtain a sharp asymptotic formula for $M(q,k,c)$, and to prove the following result.

Proposition 1. *Let $q>2$ and c be two integers with $(c,q)=1$. Then for any positive integer k, we have the asymptotic formula*

$$M(q,k,c)=\frac{1}{(2k+1)(k+1)}\phi(q)q^{2k}+O(4^k q^{(4k+1)/2}d^2(q)\log^2 q),$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the divisor function.

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Tianping Zhang [4] got the following general result.

Proposition 2. *Let $q>2$ and c be two integers with $(c,q)=1$. For any positive integers k and $n>1$, let*

$$M(q,k,c,n)=\sideset{}{'}\sum_{a_1=1}^{q}\cdots\sideset{}{'}\sum_{a_n=1}^{q}\sideset{}{'}\sum_{\substack{b=1\\ a_1\dots a_n b\equiv c \bmod q}}^{q}(a_1\dots a_n-b)^{2k}.$$

Then we have

$$M(q,k,c,n)=\frac{1}{(2k+1)^n}\phi^n(q)q^{2kn}+O(4^k q^{(2k+1)n-1/2}d^2(q)\log q).$$

The error term in Proposition 1 is the best possible. In fact for $k=1$, let

$$M(q,1,c)=\frac{1}{6}\phi(q)q^2+\frac{1}{3}q\prod_{p|q}(1-p)+F(q,1,c),$$

where $\prod_{p|q}$ denotes the product over all distinct prime divisors of q. The second author [7] used the properties of Dedekind sums and Cochrane sums to give a sharp mean-value formula for $F(q,1,c)$.

Proposition 3. *For any integer $q>2$, we have the asymptotic formula*

$$\sideset{}{'}\sum_{c=1}^{q}F^2(q,1,c)=\frac{5}{36}q^3\phi^3(q)\prod_{p^\alpha\|q}\frac{(p+1)^3/p(p^2+1)-1/p^{3\alpha-1}}{1+1/p+1/p^2}$$
$$+O\left(q^5\exp\left(\frac{4\log q}{\log\log q}\right)\right),$$

where $\prod_{p^\alpha\|q}$ denotes the product over all prime divisors of q with $p^\alpha|q$ and $p^{\alpha+1}\nmid q$.

For a general integer $k\geq 2$, let $F(q,k,c)$ denote the error term in $M(q,k,c)$, i.e.

$$M(q,k,c)=\frac{1}{(2k+1)(k+1)}\phi(q)q^{2k}+F(q,k,c).$$

In [7], the second author asked us to find whether there exists an asymptotic formula for the mean value $\sum_{c=1}^{\prime q}F^2(q,k,c)$.

It is interesting that there exists some relation between the error term in $M(q,k,c)$ and Kloosterman sums

$$K(m,n;q)=\sum_{b=1}^{q}{}' e\left(\frac{mb+n\bar{b}}{q}\right),$$

where $e(y)=e^{2\pi iy}$ and $\bar{b}$ is defined by the equation $b\bar{b}\equiv 1 \bmod q$. Let

$$M(q,k,c)=\frac{1}{\phi(q)}\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'(a-b)^{2k}+E(q,k,c).$$

It is obvious that $E(q,1,c)=F(q,1,c)$ and $E(q,k,c)=F(q,k,c)+O(q^{2k})$, $k\geq 2$.

In this paper, we use the generalized Bernoulli numbers, Gauss sums and the mean-value theorems of Dirichlet L-functions to study the hybrid mean value $\sum_{c=1}^{\prime q} E(q,k,c)K(c,1;q)$, and give a sharp asymptotic formula.

Theorem. *Let $q>2$ and $k>0$ be integers, then we have*

$$\sum_{c=1}^{q}{}' E(q,k,c)K(c,1;q)=2q^{2k+1}\phi(q)\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)$$
$$\times\left[\sum_{n=2}^{2k-2}C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(-1)^{n+m+l}(2\pi)^{2m+2l}}\right.$$
$$\left.-\sum_{n=1}^{2k-1}C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(-1)^{n+m+l}(2\pi)^{2m+2l+2}}\right]$$
$$+O(q^{2k+1+\varepsilon}),$$

where $C_n^m=n!/m!(n-m)!$ is the binomial coefficient, ε is any fixed positive number, and $[y]$ denotes the greatest integer not larger than y.

2. Some lemmas

To complete the proof of the theorem, we need the following lemmas.

Lemma 1. *Let $q>2$ and $n>0$ be two integers, then for any non-principal character χ modulo q we have that*

$$\sum_{a=1}^{q} a^n\chi(a)=\begin{cases} \dfrac{1}{n+1}\displaystyle\sum_{m=1}^{[n/2]} C_{n+1}^{2m} B_{2m,\chi} q^{n+1-2m}, & \textit{if } \chi(-1)=1; \\ \dfrac{1}{n+1}\displaystyle\sum_{m=0}^{[(n-1)/2]} C_{n+1}^{2m+1} B_{2m+1,\chi} q^{n-2m}, & \textit{if } \chi(-1)=-1, \end{cases}$$

where $B_{n,\chi}$ is the generalized Bernoulli number.

Proof. From [3] we can easily get this identity. □

Lemma 2. *Let $q>2$ and c be two integers with $(c,q)=1$, then for any positive integer k we have*

$$E(q,k,c)=\frac{4q^{2k}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}\overline{\chi}(c)\sum_{n=2}^{2k-2}C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(2\pi)^{2m+2l}(-1)^{n+m+l}}\times\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{G(r,\chi)G(s,\chi)}{r^{2m}s^{2l}}$$
$$-\frac{4q^{2k}}{\phi(q)}\sum_{\chi(-1)=-1}\overline{\chi}(c)\sum_{n=1}^{2k-1}C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(2\pi)^{2m+2l+2}(-1)^{n+m+l}}\times\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{G(r,\chi)G(s,\chi)}{r^{2m+1}s^{2l+1}},$$

where $G(n,\chi)=\sum_{a=1}^{q}\chi(a)e(an/q)$ is the Gauss sum, $\sum_{\chi\neq\chi_0,\chi(-1)=1}$ denotes the summation over all non-principal even characters modulo q, and $\sum_{\chi(-1)=-1}$ denotes the summation over all odd characters modulo q.

Proof. The generalized Bernoulli numbers can be expressed in terms of Bernoulli polynomials as

$$B_{n,\chi}=q^{n-1}\sum_{a=1}^{q}\chi(a)B_n\left(\frac{a}{q}\right).$$

From Theorem 12.19 of [1] we also have that

$$B_n(x)=-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{e(rx)}{r^n},\quad \text{if } 0<x\leq 1.$$

Therefore

$$B_{n,\chi}=q^{n-1}\sum_{a=1}^{q}\chi(a)\left[-\frac{n!}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{e(ar/q)}{r^n}\right]=-\frac{n!q^{n-1}}{(2\pi i)^n}\sum_{\substack{r=-\infty\\ r\neq 0}}^{+\infty}\frac{G(r,\chi)}{r^n}. \tag{1}$$

On the other hand, from the definition of $M(q,k,c)$ we have that

$$\begin{aligned} E(q,k,c) &= \sideset{}{'}\sum_{\substack{a=1\\ ab\equiv c \bmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q}(a-b)^{2k}-\frac{1}{\phi(q)}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}(a-b)^{2k}\\ &=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(c)\sum_{a=1}^{q}\sum_{b=1}^{q}\chi(a)\chi(b)(a-b)^{2k}-\frac{1}{\phi(q)}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}(a-b)^{2k}\\ &=\frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}\overline{\chi}(c)\sum_{a=1}^{q}\sum_{b=1}^{q}\chi(a)\chi(b)(a-b)^{2k}. \end{aligned} \tag{2}$$

Note

$$\sum_{a=1}^{q}\chi(a)=0,\ \text{if } \chi\neq\chi_0, \quad \text{and} \quad \sum_{a=1}^{q}a\chi(a)=0,\ \text{if } \chi(-1)=1. \tag{3}$$

So from Lemma 1 and formulae (1), (2) and (3) we have that

$$\begin{aligned} E(q,k,c) &= \frac{1}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}\overline{\chi}(c)\sum_{n=2}^{2k-2}C_{2k}^{n}(-1)^n\left[\sum_{a=1}^{q}a^n\chi(a)\right]\left[\sum_{b=1}^{q}b^{2k-n}\chi(b)\right]\\ &\quad+\frac{1}{\phi(q)}\sum_{\chi(-1)=-1}\overline{\chi}(c)\sum_{n=1}^{2k-1}C_{2k}^{n}(-1)^n\left[\sum_{a=1}^{q}a^n\chi(a)\right]\left[\sum_{b=1}^{q}b^{2k-n}\chi(b)\right]\\ &=\frac{4q^{2k}}{\phi(q)}\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}}\overline{\chi}(c)\sum_{n=2}^{2k-2}C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(2\pi)^{2m+2l}(-1)^{n+m+l}}\\ &\qquad\qquad\times\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{G(r,\chi)G(s,\chi)}{r^{2m}s^{2l}}\\ &\quad-\frac{4q^{2k}}{\phi(q)}\sum_{\chi(-1)=-1}\overline{\chi}(c)\sum_{n=1}^{2k-1}C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(2\pi)^{2m+2l+2}(-1)^{n+m+l}}\\ &\qquad\qquad\times\sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty}\frac{G(r,\chi)G(s,\chi)}{r^{2m+1}s^{2l+1}}. \end{aligned}$$

This completes the proof. $\square$

Lemma 3. *Let q be a positive integer, then for any character χ modulo q, we have that*

$$\sum_{c=1}^{q} \overline{\chi}(c)K(c,1;q)=\tau^2(\overline{\chi}),$$

where $\tau(\chi)=G(1,\chi)$.

Proof. This is Lemma 2 of [8]. $\square$

Lemma 4. *For any integer $q\geq 3$, let χ be a non-principal character modulo q, and q^* denote the conductor of χ with $\chi\Leftrightarrow\chi^*$, where χ^* is the primitive character modulo q^*. If $(n,q)>1$, we have that*

$$G(n,\chi)=\begin{cases} \overline{\chi}^*\left(\frac{n}{(n,q)}\right)\chi^*\left(\frac{q}{q^*(n,q)}\right)\mu\left(\frac{q}{q^*(n,q)}\right)\phi(q)\phi^{-1}\left(\frac{q}{(n,q)}\right)\tau(\chi^*), & q^*=\dfrac{q_1}{(n,q_1)}; \\ 0, & q^*\neq\dfrac{q_1}{(n,q_1)}, \end{cases}$$

where $\mu(n)$ is the Möbius function, and q_1 is the largest divisor of q that has the same prime factors as q^*.

If $(n,q)=1$, then we have that

$$G(n,\chi)=\overline{\chi}^*(n)\chi^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\chi^*).$$

Proof. See [2]. $\square$

Lemma 5. *Let q and r be integers with $q\geq 2$ and $(r,q)=1$, and let χ be a Dirichlet character modulo q. Then we have the identities*

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(r)=\sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q)=\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^*_{\chi \bmod q}$ denotes the summation over all primitive characters mod q, and $J(q)$ denotes the number of primitive characters mod q.

Proof. This is Lemma 3 of [6]. $\square$

Lemma 6. *Let $q=uv$, where $(u,v)=1$, u is a square, and v is a square-free number. Then for any real numbers $s,t\geq 1$, we have that*

$$\text{(I)}\quad \sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}\overline{\chi}(t_1t_2)L(s,\overline{\chi})L(t,\overline{\chi})$$
$$=\frac{q\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O(q^{2+\varepsilon});$$

$$\text{(II)}\quad \sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(t_1t_2)L(s,\overline{\chi})L(t,\overline{\chi})$$
$$=\frac{q\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)+O(q^{2+\varepsilon}).$$

Proof. We only prove (II), since we can deduce (I) similarly. We suppose that $t\geq s$ without loss of generality. Let $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$. For any non-principal character χ modulo ud, and parameter $N\geq q$, applying Abel's identity gives that

$$L(s,\overline{\chi})=\sum_{n=1}^{+\infty}\frac{\overline{\chi}(n)}{n^s}=\sum_{n=1}^{N}\frac{\overline{\chi}(n)}{n^s}+s\int_N^\infty\frac{\sum_{n=N+1}^{[y]}\overline{\chi}(n)}{y^{s+1}}\,dy$$
$$=\sum_{n=1}^{N}\frac{\overline{\chi}(n)}{n^s}+O\left(\frac{\sqrt{ud}\log ud}{N^s}\right).$$

For $(a,q)=1$, from Lemma 5 we get that

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(a)=\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod q}(1-\chi(-1))\chi(a)=\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod q}\chi(a)-\frac{1}{2}\sideset{}{^*}\sum_{\chi \bmod q}\chi(-a)$$
$$=\frac{1}{2}\sum_{\omega|(q,a-1)}\mu\left(\frac{q}{\omega}\right)\phi(\omega)-\frac{1}{2}\sum_{\omega|(q,a+1)}\mu\left(\frac{q}{\omega}\right)\phi(\omega).$$

Then we have that

$$\sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(t_1t_2)L(s,\overline{\chi})L(t,\overline{\chi})$$
$$=\sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t}$$
$$\times\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(t_1t_2)\sum_{n=1}^{N}\frac{\overline{\chi}(n)}{n^s}\sum_{m=1}^{N}\frac{\overline{\chi}(m)}{m^t}+O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right)$$

$$
\begin{aligned}
&= \sum_{d|v} (ud)^2 \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t} \\
&\quad \times \sum_{n=1}^{N^2} \frac{\sigma_{s-t}(n)}{n^s} \sideset{}{^*}\sum_{\substack{\chi \bmod ud \\ \chi(-1)=-1}} \overline{\chi}(t_1t_2)\overline{\chi}(n) + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right) \\
&= \frac{1}{2} \sum_{d|v} (ud)^2 \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t} \\
&\quad \times \sum_{\substack{n=1 \\ (n,ud)=1}}^{N^2} \frac{\sigma_{s-t}(n)}{n^s} \sum_{\omega|(ud,t_1t_2n-1)} \mu\left(\frac{ud}{\omega}\right)\phi(\omega) \\
&\quad - \frac{1}{2} \sum_{d|v} (ud)^2 \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t} \\
&\quad \times \sum_{\substack{n=1 \\ (n,ud)=1}}^{N^2} \frac{\sigma_{s-t}(n)}{n^s} \sum_{\omega|(ud,t_1t_2n+1)} \mu\left(\frac{ud}{\omega}\right)\phi(\omega) + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right) \\
&= \frac{1}{2} \sum_{d|v} (ud)^2 \sum_{\omega|ud} \mu\left(\frac{ud}{\omega}\right)\phi(\omega) \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_2^{t-s}} \\
&\quad \times \sum_{\substack{n=1 \\ (n,ud)=1 \\ t_1t_2n\equiv 1 \bmod \omega}}^{N^2} \frac{\sigma_{s-t}(n)}{(t_1t_2n)^s} - \frac{1}{2} \sum_{d|v} (ud)^2 \sum_{\omega|ud} \mu\left(\frac{ud}{\omega}\right)\phi(\omega) \\
&\quad \times \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_2^{t-s}} \sum_{\substack{n=1 \\ (n,ud)=1 \\ t_1t_2n\equiv -1 \bmod \omega}}^{N^2} \frac{\sigma_{s-t}(n)}{(t_1t_2n)^s} + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right) \\
&= \frac{1}{2} \sum_{d|v} (ud)^2 J(ud) \\
&\quad + O\left(\sum_{d|v} (ud)^2 \sum_{\omega|ud} \phi(\omega) \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)}{t_2^{t-s}} \sum_{l=1}^{[((vN/d)^2-1)/\omega]} \frac{N^\varepsilon}{(l\omega+1)^s} \right) \\
&\quad + O\left(\sum_{d|v} (ud)^2 \sum_{\omega|ud} \phi(\omega) \sum_{t_1|v/d} \sum_{t_2|v/d} \frac{\phi(t_1)\phi(t_2)}{t_2^{t-s}} \sum_{l=[2/\omega]}^{[((vN/d)^2+1)/\omega]} \frac{N^\varepsilon}{(l\omega-1)^s} \right) \\
&\quad + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{u\phi^2(u)}{2}\sum_{d\mid v} d^2 J(d) + O(q^2 N^{\varepsilon}) + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right)\\
&= \frac{q\phi^2(q)}{2}\prod_{p\| q}\left(1-\frac{1}{p(p-1)}\right) + O(q^2 N^{\varepsilon}) + O\left(\frac{q^{7/2+\varepsilon}}{N^s}\right).
\end{aligned}
$$

Now taking $N=[q^{3/2}]$, we immediately get that

$$
\begin{aligned}
\sum_{d\mid v}(ud)^2 \sum_{t_1\mid v/d}\sum_{t_2\mid v/d} \frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^s t_2^t} \mathop{{\sum}^{*}}_{\substack{\chi \bmod ud\\ \chi(-1)=-1}} \overline{\chi}(t_1t_2)L(s,\overline{\chi})L(t,\overline{\chi})&\\
= \frac{q\phi^2(q)}{2}\prod_{p\| q}\left(1-\frac{1}{p(p-1)}\right) + O(q^{2+\varepsilon}). \quad \Box&
\end{aligned}
$$

3. Proof of the theorem

In this section, we complete the proof of the theorem. For integers $q>2$ and $k\geq 1$, by Lemmas 2 and 3 we have that

$$
\begin{aligned}
&\mathop{{\sum}'}_{c=1}^{q} E(q,k,c)K(c,1;q)\\
&= \frac{4q^{2k}}{\phi(q)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \tau^2(\overline{\chi}) \sum_{n=2}^{2k-2} C_{2k}^n \sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]} \frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(2\pi)^{2m+2l}(-1)^{n+m+l}}\\
&\quad\times \sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty} \frac{G(r,\chi)G(s,\chi)}{r^{2m}s^{2l}}\\
&\quad - \frac{4q^{2k}}{\phi(q)} \sum_{\chi(-1)=-1} \tau^2(\overline{\chi}) \sum_{n=1}^{2k-1} C_{2k}^n \sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]} \frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(2\pi)^{2m+2l+2}(-1)^{n+m+l}}\\
&\quad\times \sum_{r=1}^{+\infty}\sum_{s=1}^{+\infty} \frac{G(r,\chi)G(s,\chi)}{r^{2m+1}s^{2l+1}}.
\end{aligned}
$$

Let $q=uv$, where $(u,v)=1$, u is a square, and v is a square-free number. Let q^* denote the conductor of χ with $\chi\Leftrightarrow\chi^*$, then

$$
\tau(\overline{\chi}) = \overline{\chi}^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\overline{\chi}^*) \neq 0
$$

if and only if $q^*=ud$, where $d|v$. So from Lemmas 4 and 6 and the above we have that

$$\begin{aligned}
&\sum_{c=1}^{q}{}' E(q,k,c)K(c,1;q)\\
&=\frac{4q^{2k}}{\phi(q)}\sum_{d|v}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}\overline{\chi}^2\Big(\frac{v}{d}\Big)\mu^2\Big(\frac{v}{d}\Big)\tau^2(\overline{\chi})\\
&\quad\times\sum_{n=2}^{2k-2}C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(2\pi)^{2m+2l}(-1)^{n+m+l}}\\
&\quad\times\Big[\sum_{t_1|v/d}\frac{\chi(v/dt_1)\mu(v/dt_1)\phi(q)\tau(\chi)L(2m,\overline{\chi})}{t_1^{2m}\phi(q/t_1)}\Big]\\
&\quad\times\Big[\sum_{t_2|v/d}\frac{\chi(v/dt_2)\mu(v/dt_2)\phi(q)\tau(\chi)L(2l,\overline{\chi})}{t_2^{2l}\phi(q/t_2)}\Big]\\
&\quad-\frac{4q^{2k}}{\phi(q)}\sum_{d|v}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}^2\Big(\frac{v}{d}\Big)\mu^2\Big(\frac{v}{d}\Big)\tau^2(\overline{\chi})\\
&\quad\times\sum_{n=1}^{2k-1}C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(2\pi)^{2m+2l+2}(-1)^{n+m+l}}\\
&\quad\times\Big[\sum_{t_1|v/d}\frac{\chi(v/dt_1)\mu(v/dt_1)\phi(q)\tau(\chi)L(2m+1,\overline{\chi})}{t_1^{2m+1}\phi(q/t_1)}\Big]\\
&\quad\times\Big[\sum_{t_2|v/d}\frac{\chi(v/dt_2)\mu(v/dt_2)\phi(q)\tau(\chi)L(2l+1,\overline{\chi})}{t_2^{2l+1}\phi(q/t_2)}\Big]\\
&=\frac{4q^{2k}}{\phi(q)}\sum_{n=2}^{2k-2}C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(2\pi)^{2m+2l}(-1)^{n+m+l}}\\
&\quad\times\sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^{2m}t_2^{2l}}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=1}}\overline{\chi}(t_1t_2)L(2m,\overline{\chi})L(2l,\overline{\chi})\\
&\quad-\frac{4q^{2k}}{\phi(q)}\sum_{n=1}^{2k-1}C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(2\pi)^{2m+2l+2}(-1)^{n+m+l}}\\
&\quad\times\sum_{d|v}(ud)^2\sum_{t_1|v/d}\sum_{t_2|v/d}\frac{\phi(t_1)\phi(t_2)\mu(t_1)\mu(t_2)}{t_1^{2m+1}t_2^{2l+1}}\sideset{}{^*}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}\overline{\chi}(t_1t_2)L(2m+1,\overline{\chi})L(2l+1,\overline{\chi})
\end{aligned}$$

$$
\begin{aligned}
&= 2q^{2k+1}\phi(q)\prod_{p\|q}\left(1-\frac{1}{p(p-1)}\right)\\
&\times\left[\sum_{n=2}^{2k-2} C_{2k}^{n}\sum_{m=1}^{[n/2]}\sum_{l=1}^{[(2k-n)/2]}\frac{C_n^{2m-1}C_{2k-n}^{2l-1}(2m-1)!(2l-1)!}{(-1)^{n+m+l}(2\pi)^{2m+2l}}\right.\\
&\left.\quad-\sum_{n=1}^{2k-1} C_{2k}^{n}\sum_{m=0}^{[(n-1)/2]}\sum_{l=0}^{[(2k-n-1)/2]}\frac{C_n^{2m}C_{2k-n}^{2l}(2m)!(2l)!}{(-1)^{n+m+l}(2\pi)^{2m+2l+2}}\right]+O(q^{2k+1+\varepsilon}).
\end{aligned}
$$

This completes the proof of the theorem.

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