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The $\bar{\partial}$-problem with support conditions on some weakly pseudoconvex domains

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Abstract. We consider a domain Ω with Lipschitz boundary, which is relatively compact in an n-dimensional Kähler manifold and satisfies some "$\log\delta$-pseudoconvexity" condition. We show that the $\bar{\partial}$-equation with exact support in Ω admits a solution in bidegrees (p,q), $1\le q\le n-1$. Moreover, the range of $\bar{\partial}$ acting on smooth $(p,n-1)$-forms with support in $\bar{\Omega}$ is closed. Applications are given to the solvability of the tangential Cauchy–Riemann equations for smooth forms and currents for all intermediate bidegrees on boundaries of weakly pseudoconvex domains in Stein manifolds and to the solvability of the tangential Cauchy–Riemann equations for currents on Levi flat CR manifolds of arbitrary codimension.

1. Introduction

Let us consider a complex manifold X and $\Omega\Subset X$ a relatively compact domain. In this article, we will study the following question:

Let f be a smooth (p,q)-form on X satisfying $\bar{\partial}f=0$ on X and $\operatorname{supp} f\subset\bar{\Omega}$ (in other words, f vanishes to infinite order at the boundary of Ω). Does the problem

$$(*)_{p,q}\begin{cases}\bar{\partial}u=f,\\ \operatorname{supp} u\subset\bar{\Omega}\end{cases}$$

admit a solution u, which is a smooth $(p,q-1)$-form on X?

The solvability of this $\bar{\partial}$-problem leads to extension results for $\bar{\partial}_b$-closed forms on the boundary of Ω, whenever $\partial\Omega$ is smooth, and can thus be used to understand the $\bar{\partial}_b$-cohomology of $\partial\Omega$.

More precisely, let (X,ω) be an n-dimensional Kähler manifold. We assume that Ω has Lipschitz boundary and is $\log\delta$-pseudoconvex, meaning roughly that the function $-\log(\text{boundary distance with respect to }\omega)$ admits a strictly plurisubharmonic extension to Ω. Then the $\bar{\partial}$-problem $(*)_{p,q}$ admits a solution for $1\le q\le n-1$, and the top degree $\bar{\partial}$-cohomology groups of smooth forms with support in $\bar{\Omega}$ are separated.

We will prove this by means of basic L^2 estimates on Ω with powers of the inverse of the boundary distance as weight functions. Sobolev estimates for elliptic operators whose symbol can be controlled by some power of the boundary distance will be deduced in order to prove regularity results for the minimal L^2 solutions of the $\bar{\partial}$-operator.

Examples of domains satisfying the above $\log\delta$-pseudoconvexity condition are weakly pseudoconvex domains in Stein manifolds and weakly pseudoconvex domains in Kähler manifolds with positive holomorphic bisectional curvature.

We would like to mention that the case of $\Omega\Subset\mathbf{C}^n$ with piecewise smooth boundary was already settled in [MS], using kernel methods. Moreover, if X is compact, then the solvability of the $\bar{\partial}$-problem $(*)_{p,q}$ is equivalent to the solvability of the $\bar{\partial}$-equation with smoothness up to the boundary in $X\setminus\bar{\Omega}$ in bidegree $(p,q-1)$, as long as the global $\bar{\partial}$-cohomology groups of X in bidegree (p,q) and $(p,q-1)$ vanishes. This has been studied in [HI]. For some related work, one should also consult [O].

By duality, we also solve the $\bar{\partial}$-equation for extensible currents on Ω. These currents were at first considered by Martineau [M]. The analogous results in the strictly pseudoconvex case can be found in [Sa] with a very different proof. We then deduce the solvability of the $\bar{\partial}$-equation in bidegree $(0,1)$ for smooth forms admitting a distribution boundary value and the vanishing of the Čech cohomology groups of the sheaf of germs of holomorphic functions admitting a distribution boundary value.

Moreover, we can apply the solvability of the $\bar{\partial}$-equation for extensible currents to deduce the vanishing of some tangential Cauchy–Riemann cohomology groups for currents on Levi flat CR manifolds of arbitrary codimension embedded in a Stein manifold.

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2. A regularity theorem for elliptic operators

In this section, we will study the regularity of the equation $Lu=f$, where L is an elliptic operator on a bounded open set in $\mathbf{R}^n$ whose principal symbol can be controlled by some power of the boundary distance.

More precisely, let Ω be an open set in $\mathbf{R}^n$, and let $L=\sum_{|\alpha|=m} a_\alpha(x)D^\alpha+\sum_{|\beta|<m} b_\beta(x)D^\beta$ be a differential operator of order m with smooth coefficients $a_\alpha, b_\beta\in\mathcal{C}^\infty(\Omega)$ on Ω. Let $\Delta:\Omega\to\mathbf{R}^+$ be a smooth function on Ω.

We say that L is an *elliptic operator of polynomial growth with respect to* Δ *on*

Ω if there exist $k, l \in \mathbf{N}$ such that

$$\left|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\right| \gtrsim \Delta^k(x)|\xi|^m \quad \text{for every } \xi \in \mathbf{R}^n \tag{2.1}$$

and

$$|D^\gamma a_\alpha(x)| \lesssim \Delta^{-l-|\gamma|}(x), \quad |D^\gamma b_\beta(x)| \lesssim \Delta^{-l-|\gamma|}(x) \tag{2.2}$$

for all multiindices α, β and γ.

Here we write $a \lesssim b$ (resp. $b \gtrsim a$), if there exists an *absolute* constant $C>0$ such that $a \le Cb$ (resp. $b \ge Ca$); $a \sim b$ signifies $a \lesssim b$ and $a \gtrsim b$.

Let us now recall some of the basic properties of Sobolev spaces.

Let $\mathcal{D}(\mathbf{R}^n)$ be the space of $\mathcal{C}^\infty$ functions on $\mathbf{R}^n$ with compact support, and $\mathcal{S}$ the Schwartz space of rapidly decreasing functions on $\mathbf{R}^n$. The Sobolev norms $\|\cdot\|_s$ of order s on $\mathbf{R}^n$, $s \in \mathbf{R}$, are defined by

$$\|u\|_s^2 = \int_{\mathbf{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi,$$

where $u \in \mathcal{S}$ and $\hat{u}$ is the Fourier transform of u. If k is a positive integer, we have

$$\|u\|_k^2 \sim \sum_{0 \le |\alpha| \le k} \|D^\alpha u\|_0^2 \quad \text{for all } u \in \mathcal{S}.$$

As usual the Sobolev space $H_s = H_s(\mathbf{R}^n)$ is the completion of $\mathcal{S}$ under the norm $\|\cdot\|_s$.

Now let $\Omega \subset \mathbf{R}^n$ be a bounded open set in $\mathbf{R}^n$ and m a nonnegative integer. The Sobolev space $H_m(\Omega)$ is the completion of the space of all those $\mathcal{C}^\infty$ functions $f\colon \Omega \to \mathbf{C}$ such that

$$\|f\|_{m,\Omega}^2 := \sum_{|\alpha| \le m} \int_\Omega |D^\alpha f|^2 \, dx < +\infty$$

relative to the norm $\|\cdot\|_{m,\Omega}$. The completion of the space $\mathcal{D}(\Omega)$ of $\mathcal{C}^\infty$ functions with compact support in Ω relative to $\|\cdot\|_{m,\Omega}$ is denoted by $\mathring{H}_m(\Omega)$. If Ω has Lipschitz boundary, then if f is of class $\mathcal{C}^k$ on $\mathbf{R}^n$ and supported in Ω, then $f \in \mathring{H}_k(\Omega)$ (see [G]). On the other hand, if $f \in \mathring{H}_s(\Omega)$ and $s > k + \frac{1}{2}n$, then it follows from the Sobolev lemma that f is of class $\mathcal{C}^k$ on $\mathbf{R}^n$ and supported in $\overline{\Omega}$.

We define $\mathcal{C}^r(\mathbf{R}^n, \overline{\Omega}) := \{f \in \mathcal{C}^r(\mathbf{R}^n) \mid \operatorname{supp} f \subset \overline{\Omega}\}$.

Theorem 2.1. *Let L be a differential operator of order m with smooth coefficients on an open set $\Omega \Subset \mathbf{R}^n$, which is elliptic of polynomial growth with respect to a smooth function $\Delta \in \mathcal{C}^\infty(\Omega, \mathbf{R}^+)$. Assume that Δ has essentially the same features as the regularized boundary distance function of Ω (as cited before Proposition 3.1), i.e. $\Delta \sim d$ and $|D^\alpha \Delta| \lesssim d^{1-|\alpha|}$ for every multiindex α.*

Then we have the following a priori estimate

$$\|u\|_{s,\Omega}^2 \lesssim \|\Delta^{-ts} Lu\|_{s-m,\Omega}^2 + \|\Delta^{-Ts^2} u\|_{0,\Omega}^2 \tag{2.3}$$

for some $t, T \in \mathbf{N}$, all $s \gg 1$ and $u \in \mathcal{C}^\infty(\Omega)$.

Moreover, let Ω have Lipschitz boundary and let $u \in \mathcal{C}^\infty(\Omega)$ satisfy

$$\int_\Omega |u(x)|^2 \Delta^{-N}(x)\, d\lambda(x) < +\infty$$

and $Lu \in \mathcal{C}^N(\mathbf{R}^n, \bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$. Then $u \in \mathcal{C}^{s(N)}(\mathbf{R}^n, \bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, where $s(N) \sim \sqrt{N}$ for all $N \gg 1$.

Proof. We will first show that it suffices to prove the a priori estimate (2.3). Let $u \in \mathcal{C}^\infty(\Omega)$ satisfy $\int_\Omega |u(x)|^2 \Delta^{-N}(x)\, d\lambda(x) < +\infty$ and $Lu \in \mathcal{C}^N(\mathbf{R}^n, \bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$. We want to show that $u \in \mathcal{C}^{s(N)}(\mathbf{R}^n, \bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$ with $s(N) \sim \sqrt{N}$ for all $N \gg 1$. As noted above, it suffices by the Sobolev lemma to show that $u \in \mathring{H}_{s(N)}(\Omega)$.

Since Ω has Lipschitz boundary, it follows from a general result of Grisvard that

$$\mathcal{C}^k(\mathbf{R}^n, \bar{\Omega}) \subset \left\{ f \in \mathcal{C}^k(\Omega) \,\middle|\, \int_U |f|^2\, d^{-2k} d\lambda < +\infty \right\}$$

(see [G, Theorem 1.4.4.4]). Hence the a priori estimate (2.3) together with the assumptions on u yields $u \in H_{s(N)}(\Omega)$ with $s(N) \sim \sqrt{N}$.

Next, we define the open sets $\Omega_j \subset \Omega$ by

$$\Omega_j = \left\{ z \in \Omega \,\middle|\, d(z) > \frac{1}{j-1} \right\} \Subset \Omega_{j+1}.$$

For every $j \in \mathbf{N}$, it is then possible to construct $\chi_j \in \mathcal{C}^\infty(\mathbf{R}^n)$ with compact support in Ω_{j+1} such that $\chi_j \equiv 1$ in a neighborhood of $\bar{\Omega}_j$, and moreover, for every multiindex α,

$$\sup_{x \in \mathbf{R}^n} |D^\alpha \chi_j(x)| \le N_{|\alpha|} j^{2|\alpha|}. \tag{2.4}$$

We can also find functions $\eta_j \in \mathcal{C}^\infty(\mathbf{R}^n)$ satisfying $0 \le \eta_j \le 1$, $\eta_j \equiv 1$ in a neighborhood of $\bar{\Omega}_{j+1} \setminus \Omega_j$, $\operatorname{supp} \eta_j \subset \Omega_{j+2} \setminus \bar{\Omega}_{j-1}$ and

$$\sup_{x \in \mathbf{R}^n} |D^\alpha \eta_j(x)| \le M_{|\alpha|} j^{2|\alpha|}$$

for every multiindex α.

Let us now estimate $\|u-\chi_j u\|^2_{s,\Omega}$. Using the a priori estimate (2.3) and (2.4), we obtain

$$\begin{aligned}\|u-\chi_j u\|^2_{s,\Omega} &\lesssim \|\Delta^{-ts}L(u-\chi_j u)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}\\ &\lesssim \|\Delta^{-ts}(Lu-\chi_j Lu)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}+j^{cs}\|\eta_j u\|^2_{s-1,\Omega}\end{aligned}$$

for some large $c\in\mathbf{N}$.

We also have

$$\begin{aligned}\|\eta_j u\|^2_{s-1,\Omega} &\sim \int_{\mathbf{R}^n}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u}(\xi)|^2\,d\xi\\ &= \int_{\{\xi|1+|\xi|^2\geq j^{(4+c)s+1}\}}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u}(\xi)|^2\,d\xi\\ &\quad+\int_{\{\xi|1+|\xi|^2\leq j^{(4+c)s+1}\}}(1+|\xi|^2)^{s-1}|\widehat{\eta_j u}(\xi)|^2\,d\xi\\ &\lesssim j^{-(4+c)s-1}\|\eta_j u\|^2_{s,\Omega}+j^{c's^2}\|\eta_j u\|^2_{0,\Omega}\\ &\lesssim j^{-(4+c)s-1}\|\eta_j u\|^2_{s,\Omega}+j^{-cs-1}\|\Delta^{-c''s^2}u\|^2_{0,\Omega}\\ &\lesssim j^{-cs-1}(\|u\|^2_{s,\Omega}+\|\Delta^{-c''s^2}u\|^2_{0,\Omega})\end{aligned}$$

for some large $c',c''\in\mathbf{N}$; note that $j\lesssim\Delta^{-1}$ on $\Omega\setminus\Omega_j$.

Combining this with the above inequalities, we obtain

$$\begin{aligned}\|u-\chi_j u\|^2_{s,\Omega} &\lesssim \|\Delta^{-ts}(Lu-\chi_j Lu)\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}(u-\chi_j u)\|^2_{0,\Omega}\\ &\quad+\frac{1}{j}(\|u\|^2_{s,\Omega}+\|\Delta^{-c''s^2}u\|^2_{0,\Omega}).\end{aligned}$$

We have already shown that for some $s\sim\sqrt{N}$, $\|u\|^2_{s,\Omega}<+\infty$. By the hypothesis on u, we also have $\|\Delta^{-c''s^2}u\|^2_{0,\Omega}<+\infty$ for some $s\sim\sqrt{N}$, thus the last term in the above inequality tends to zero, as $j\to+\infty$. Moreover, the assumptions on u imply that also the first two terms tend to zero, as $j\to+\infty$, for some $s\sim\sqrt{N}$ (see [G, Theorem 1.4.4.4]). We have therefore proved the last assertion of the theorem.

Now, let us finally turn to the proof of the a priori estimate (2.3). We prove this estimate by simply making explicit the dependence on Δ of all the constants involved in the classical proof of the hypoellipticity of uniformly elliptic operators (see [F]).

Let us fix $x_0\in\Omega$ and let $B_\delta(x_0)$ be the ball of radius $\delta\ll 1$ centered at x_0. Let u be a smooth function with support in $B_\delta(x_0)$.

First, we assume that $b_\beta=0$ for every multiindex β. Then we have

$$\widehat{L_{x_0}u}(\xi)=i^m\sum_{|\alpha|=m}a_\alpha(x_0)\xi^\alpha\hat{u}(\xi),$$

where $L_{x_0}=L(x_0)$ is the differential operator with frozen coefficients at x_0.

This implies

$$\begin{aligned}(1+|\xi|^2)^s|\hat{u}(\xi)|^2&\le 2^m(1+|\xi|^2)^{s-m}(1+|\xi|^{2m})|\hat{u}(\xi)|^2\\&\lesssim(1+|\xi|^2)^{s-m}|\hat{u}(\xi)|^2+\Delta^{-2k}(x_0)(1+|\xi|^2)^{s-m}|\widehat{L_{x_0}u}(\xi)|^2\end{aligned}$$

by (2.1). Integrating both sides and using the inequality $\|u\|_{s-m,\Omega}\le\|u\|_{s-1,\Omega}$, one obtains

$$\|u\|^2_{s,\Omega}\lesssim\Delta^{-2k}(x_0)\|L_{x_0}u\|^2_{s-m,\Omega}+\|u\|^2_{s-1,\Omega}.$$

Hence there exists $C_0>0$ such that

$$\|u\|^2_{s,\Omega}\le C_0\Delta^{-2k}(x_0)(\|L_{x_0}u\|^2_{s-m,\Omega}+\|u\|^2_{s-1,\Omega}). \tag{2.5}$$

We now wish to estimate

$$\|L_xu-L_{x_0}u\|^2_{s-m,\Omega}=\left\|\sum_{|\alpha|=m}(a_\alpha(x)-a_\alpha(x_0))D^\alpha u\right\|^2_{s-m,\Omega}.$$

The estimates (2.2) yield

$$|a_\alpha(x)-a_\alpha(x_0)|\le C_1\Delta^{-l-1}(x_0)|x_0-x|$$

for some $C_1>0$, all α, x and x_0 with $\frac{1}{2}\Delta(x_0)\le\Delta(x)\le 2\Delta(x_0)$.

Set $\delta=(8C_0C_1^2n^m\Delta^{-2k-2l-2}(x_0))^{-1/2}$ and assume that x_0 is close enough to the boundary of Ω in order that $\delta\le\frac{1}{2}\Delta(x_0)$. Fix $\phi\in\mathcal{D}(B_{2\delta}(0))$ with $0\le\phi\le 1$ and $\phi\equiv 1$ on $B_\delta(0)$. Suppose u is a smooth function supported in $B_\delta(x_0)$. Then

$$(a_\alpha(x)-a_\alpha(x_0))D^\alpha u(x)=\phi(x_0-x)(a_\alpha(x)-a_\alpha(x_0))D^\alpha u(x)$$

and

$$\sup_{x\in B_\delta(x_0)}|\phi(x_0-x)(a_\alpha(x)-a_\alpha(x_0))|^2\le 4C_1^2\Delta^{-2l-2}(x_0)\delta^2=\frac{1}{2n^mC_0\Delta^{-2k}(x_0)}.$$

Hence by (2.2)

$$\|(a_\alpha(x)-a_\alpha(x_0))D^\alpha u\|^2_{s-m,\Omega}\le\frac{\|u\|^2_{s,\Omega}}{2n^mC_0\Delta^{-2k}(x_0)}+C_2\Delta^{-s_1s-s_0}(x_0)\|u\|^2_{s-1,\Omega}$$

for some $C_2>0$ and $s_0, s_1\in\mathbf{N}$.

Thus, since there are at most n^m multiindices α with $|\alpha|=m$, we have

$$\|L_x u - L_{x_0}u\|^2_{s-m,\Omega} \leq \frac{\|u\|^2_{s,\Omega}}{2C_0\Delta^{-2k}(x_0)} + n^m C_2 \Delta^{-s_1 s - s_0}(x_0)\|u\|^2_{s-1,\Omega}.$$

Combining this with (2.5), we then obtain

$$\|u\|^2_{s,\Omega} \leq C_0\Delta^{-2k}(x_0)(\|Lu\|^2_{s-m,\Omega} + n^m C_2\Delta^{-s_1 s-s_0}(x_0)\|u\|^2_{s-1,\Omega}) + \tfrac{1}{2}\|u\|^2_{s,\Omega},$$

hence

$$\|u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_0)\|Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s-k_0}(x_0)\|u\|^2_{s-1,\Omega}$$

for some $m_0, k_0\in\mathbf{N}$.

Next, we consider the case $b_\beta\not\equiv 0$. Replacing m_0 and k_0 by larger integers if necessary, we can absorb the additional terms of Lu in the term $\Delta^{-m_0 s-k_0}(x_0)\|u\|^2_{s-1,\Omega}$ and still have the estimate

$$\|u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_0)\|Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s-k_0}(x_0)\|u\|^2_{s-1,\Omega}.$$

We emphasize that all the constants involved are independent of $x_0\in\Omega$.

Next, one can cover Ω by balls $B_{\delta_j}(x_j)$ of the above type, $j\in\mathbf{N}$, such that there exists a partition of unity $(\theta_j)_{j\in\mathbf{N}}$ with respect to this covering satisfying $\sum_{|\alpha|\leq s}|D^\alpha\theta_j|^2\leq\theta_j|P_s(\delta_j^{-2})|$, where P_s is a polynomial of degree s in one variable. One has

$$\|\theta_j u\|^2_{s,\Omega} \lesssim \Delta^{-2k}(x_j)\|L\theta_j u\|^2_{s-m,\Omega} + \Delta^{-m_0 s-k_0}(x_j)\|\theta_j u\|^2_{s-1,\Omega}$$

for every smooth function u on Ω.

Replacing m_0 and k_0 by larger integers if necessary, we get

$$\begin{aligned}\|\theta_j u\|^2_{s,\Omega} &\leq C\biggl(\Delta^{-k_0}(x_j)\|\theta_j Lu\|^2_{s-m,\Omega} + \Delta^{-m_0 s-k_0}(x_j)\|\theta_j u\|^2_{s-1,\Omega} \\ &\qquad + \Delta^{-m_0 s-k_0}(x_j)\int_\Omega \theta_j|u|^2\,d\lambda\biggr) \\ &\leq M\Delta^{-m_0 s-k_0}(x_j)\biggl(\sum_{|\alpha|\leq s-m}\int_\Omega \theta_j|D^\alpha(Lu)|^2\,d\lambda \\ &\qquad + \|\theta_j u\|^2_{s-1,\Omega} + \int_\Omega \theta_j|u|^2\,d\lambda\biggr)\end{aligned} \tag{2.6}$$

for some $C, M>0$; note that $\delta_j^{-2}\sim\Delta^{-2k-2l-2}(x_j)$.

Moreover,

$$\begin{aligned}
&M\Delta^{-m_0 s-k_0}(x_j)\|\theta_j u\|^2_{s-1,\Omega}\\
&\quad= M\Delta^{-m_0 s-k_0}(x_j)\int_{\mathbf{R}^n}(1+|\xi|^2)^{s-1}|\widehat{\theta_j u}(\xi)|^2\,d\lambda\\
&\quad= M\Delta^{-m_0 s-k_0}(x_j)\int_{\{\xi\,|\,1+|\xi|^2\geq 2M\Delta^{-m_0 s-k_0}(x_j)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_j u}(\xi)|^2\,d\lambda\\
&\qquad+M\Delta^{-m_0 s-k_0}(x_j)\int_{\{\xi\,|\,1+|\xi|^2\leq 2M\Delta^{-m_0 s-k_0}(x_j)\}}(1+|\xi|^2)^{s-1}|\widehat{\theta_j u}(\xi)|^2\,d\lambda\\
&\quad\leq \tfrac{1}{2}\|\theta_j u\|^2_{s,\Omega}+C'\Delta^{-m_0 s^2+m_0 s-k_0 s+k_0}(x_j)\|\theta_j u\|^2_{0,\Omega}
\end{aligned}$$

for some $C'>0$. Thus, by (2.6),

$$\|\theta_j u\|^2_{s,\Omega}\lesssim \sum_{|\alpha|\leq s-m}\int_\Omega \theta_j\Delta^{-2ts}|D^\alpha(Lu)|^2\,d\lambda+\int_\Omega\theta_j\Delta^{-2Ts^2}|u|^2\,d\lambda$$

for some $t,T\in\mathbf{N}$ and $s\gg 1$. So

$$\|u\|^2_{s,\Omega}=\Big\|\sum_{j\in\mathbf{N}}\theta_j u\Big\|^2_{s,\Omega}\leq\sum_{j\in\mathbf{N}}\|\theta_j u\|^2_{s,\Omega}\lesssim\|\Delta^{-ts}Lu\|^2_{s-m,\Omega}+\|\Delta^{-Ts^2}u\|^2_{0,\Omega},$$

which completes the proof. □

3. Some L^2 cohomology groups of the $\bar\partial$-operator on $\log\delta$-pseudoconvex domains

In order to prove a solvability result for the $\bar\partial$-problem with exact support in pseudoconvex domains, we have to make a global assumption on the ambient complex manifold as well as an additional assumption on the domain itself.

We let (X,ω) be an n-dimensional Kähler manifold. Let $\Omega\Subset X$ be an open set. Let $\delta(z)$ be the distance from $z\in\Omega$ to the boundary of Ω with respect to the metric ω.

Definition. We say that Ω is *$\log\delta$-pseudoconvex*, if there exists a smooth bounded function ψ on Ω such that

$$i\partial\bar\partial(-\log\delta+\psi)\geq\omega\quad\text{in }\Omega. \tag{3.1}$$

In particular every $\log\delta$-pseudoconvex domain Ω admits a strictly plurisubharmonic exhaustion function, therefore Ω is a Stein manifold.

Example 1. Let X be a Stein manifold and let $\Omega \Subset X$ be a domain which is locally Stein, i.e. for every $x \in \partial\Omega$, there exists a neighborhood U_x of x in X such that $\Omega \cap U_x$ is Stein. It was shown in [E] that there exists a Kähler metric ω on X such that Ω is $\log \delta$-pseudoconvex.

The same remains true if X is only assumed to admit a strictly plurisubharmonic function (see [E]).

In particular, every bounded weakly pseudoconvex domain with smooth boundary in $\mathbf{C}^n$ is $\log \delta$-pseudoconvex.

Example 2. Let (X, ω) be a Kähler manifold with *positive holomorphic bisectional curvature*, that is $T^{1,0}X$ is positive in the sense of Griffiths. Then every (weakly) pseudoconvex domain $\Omega \Subset X$ is $\log \delta$-pseudoconvex (see [T], for the case $X = \mathbf{P}^n$, [E] and [Su]).

In particular, the complex projective space $\mathbf{P}^n$ equipped with the Fubini–Study metric is a Kähler manifold with positive holomorphic bisectional curvature. By [SY] we moreover know that a compact Kähler manifold with positive holomorphic bisectional curvature is isomorphic to $\mathbf{P}^n$.

In general, δ is not a smooth function in Ω. However, in [St, p. 171], the existence of a regularized distance having essentially the same profile as δ is proved:

There exists a function $\Delta \in \mathcal{C}^\infty(\Omega, \mathbf{R})$ satisfying

$$c_1 \delta(x) \le \Delta(x) \le c_2 \delta(x) \quad \textit{and} \quad \left| \frac{\partial^\alpha}{\partial x^\alpha} \Delta(x) \right| \le B_\alpha (\delta(x))^{1-|\alpha|},$$

where $x = (x_1, \dots, x_{2n})$ are local coordinates on X. The constants B_α, c_1 and c_2 are independent of Ω.

Let (E, h) be a hermitian holomorphic vector bundle on X, and let $N \in \mathbf{Z}$. We denote by $L^2_{p,q}(\Omega, E, N)$ the Hilbert space of (p,q)-forms u with values in E which satisfy

$$\|u\|_N^2 := \int_\Omega |u|^2_{\omega,h} \Delta^N \mathrm{dV}_\omega < +\infty.$$

Here dV_ω is the canonical volume element associated with the metric ω, and $|\cdot|_{\omega,h}$ is the norm of (p,q)-forms induced by ω and h.

Proposition 3.1. *Let Ω be a relatively compact domain in a Kähler manifold (X, ω). We assume that Ω is $\log \delta$-pseudoconvex. Let (E, h) be a hermitian holomorphic vector bundle on X and let $N \gg 1$ and $1 \le q \le n$. Suppose $f \in L^2_{n,q}(\Omega, E, N) \cap \operatorname{Ker} \bar{\partial}$. Then there exists $u \in L^2_{n,q-1}(\Omega, E, N)$ such that $\bar{\partial} u = f$ and $\|u\|_N \lesssim \|f\|_N$.*

Proof. This follows immediately from standard L^2 estimates in the form of [D]. Indeed, since Δ has essentially the same features as $\delta\exp(-\psi)$ (cf. (3.1)), it suffices to prove the statement with Δ replaced by $\delta\exp(-\psi)$ in the definition of the spaces $L^2_{p,q}(\Omega,E,N)$. But for N sufficiently large, we clearly have

$$i\Theta(E)+Ni\partial\bar\partial(-\log\delta+\psi)\otimes\mathrm{Id}_E\geq\omega\otimes\mathrm{Id}_E$$

by (3.1), thus the desired vanishing result follows from [D]; note that $-\log\delta+\psi=-\log(\delta\exp(-\psi))$. $\square$

Proposition 3.2. *Let Ω be a relatively compact domain in an n-dimensional Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let (E,h) be a hermitian vector bundle on X and let $N\gg1$. Suppose $f\in L^2_{0,q}(\Omega,E,-N)\cap\operatorname{Ker}\bar\partial$, $1\leq q\leq n-1$. Then there exists $u\in L^2_{0,q-1}(\Omega,E,-N+2)$ such that $\bar\partial u=f$ and $\|u\|_{-N+2}\lesssim\|f\|_{-N}$.*

Proof. Suppose $1\leq q\leq n-1$ and let $f\in L^2_{0,q}(\Omega,E,-N)\cap\operatorname{Ker}\bar\partial$, $N\gg1$. We define the linear operator

$$\begin{aligned}L_f\colon\bar\partial L^2_{n,n-q}(\Omega,E^*,N-2)&\longrightarrow\mathbf{C}\\ \bar\partial\varphi&\longmapsto\int_\Omega f\wedge\varphi.\end{aligned}$$

Let us first show that L_f is well defined.

Indeed, let $\varphi_1,\varphi_2\in L^2_{n,n-q}(\Omega,E^*,N-2)$ so that $\bar\partial\varphi_1=\bar\partial\varphi_2$. Then $\bar\partial(\varphi_1-\varphi_2)=0$, and by Proposition 3.1, since $n-q\geq1$, there exists $\alpha\in L^2_{n,n-q-1}(\Omega,E^*,N-2)$ such that $\bar\partial\alpha=\varphi_1-\varphi_2$. But then

$$\begin{aligned}\int_\Omega f\wedge(\varphi_1-\varphi_2)&=\int_\Omega f\wedge\bar\partial\alpha=\lim_{\varepsilon\to0}(-1)^q\int_{\partial\Omega_\varepsilon}f\wedge\alpha\\ &=-\lim_{\varepsilon\to0}\int_{\Omega\setminus\Omega_\varepsilon}f\wedge\bar\partial\alpha=-\lim_{\varepsilon\to0}\int_{\Omega\setminus\Omega_\varepsilon}f\wedge(\varphi_1-\varphi_2)\end{aligned}$$

with $(\Omega_\varepsilon)_{\varepsilon>0}$ being an exhaustion of Ω by smoothly bounded domains such that $\Omega_\varepsilon\supset\{z\in\Omega\,|\,\Delta(z)>\varepsilon\}$. Here we have used Stoke's theorem several times. The third equality is obtained as follows: Fix $\varepsilon<0$ and choose for each large $j>2/\varepsilon$ a $\mathcal{C}^\infty$ function χ_j such that $\chi_j\equiv1$ on $\Omega_{2/j}$, $\chi_j\equiv0$ on $\Omega_{1/j}$, $0\leq\chi_j\leq1$ and $|D\chi_j|\leq Cj$. Set $\alpha_j=\chi_j\alpha\in\mathcal{D}^{n,n-q-1}(\Omega)$. Then we have

$$\int_{\Omega\setminus\Omega_\varepsilon}f\wedge\bar\partial\alpha_j=\int_{\Omega\setminus\Omega_\varepsilon}\chi_jf\wedge\bar\partial\alpha+\int_{\Omega\setminus\Omega_\varepsilon}f\wedge\bar\partial\chi_j\wedge\alpha$$

and

$$\left|\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\bar\partial\chi_j\wedge\alpha\right|^2 \le C\int_{\Omega\setminus\Omega_\varepsilon}|f|^2_\omega\Delta^{-N}\,dV_\omega\int_{\Omega\setminus\Omega_{2/j}} j^2|\alpha|^2_\omega\Delta^N\,dV_\omega$$
$$\le C\|f\|^2_{-N}\|\alpha\|^2_{N-2}.$$

Hence the dominated convergence theorem gives

$$\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\bar\partial\alpha = \lim_{j\to\infty}\int_{\Omega\setminus\Omega_\varepsilon} f\wedge\bar\partial\alpha_j = (-1)^q\lim_{j\to\infty}\int_{\Omega\setminus\Omega_\varepsilon}\bar\partial(f\wedge\alpha_j)$$
$$= -(-1)^q\lim_{j\to\infty}\int_{\partial\Omega_\varepsilon} f\wedge\alpha_j = -(-1)^q\int_{\partial\Omega_\varepsilon} f\wedge\alpha.$$

Moreover,

$$\left|\int_{\Omega\setminus\Omega_\varepsilon} f\wedge(\varphi_1-\varphi_2)\right| \le \left(\int_{\Omega\setminus\Omega_\varepsilon}|f|^2_\omega\Delta^{-N}\right)^{1/2}\left(\int_{\Omega\setminus\Omega_\varepsilon}|\varphi_1-\varphi_2|^2_\omega\Delta^N\right)^{1/2}\longrightarrow 0,$$

as $\varepsilon\to 0$. Thus $L_f(\varphi_1)=L_f(\varphi_2)$.

Now let

$$\varphi\in\operatorname{Dom}(\bar\partial\colon L^2_{n,n-q}(\Omega,E^*,N-2)\longrightarrow L^2_{n,n-q+1}(\Omega,E^*,N-2)).$$

By Proposition 3.1, there exists $\widetilde\varphi\in L^2_{n,n-q}(\Omega,E^*,N-2)$ satisfying $\bar\partial\widetilde\varphi=\bar\partial\varphi$ and $\|\widetilde\varphi\|_{N-2}\lesssim\|\bar\partial\varphi\|_{N-2}$. This yields

$$|L_f(\bar\partial\varphi)| = |L_f(\bar\partial\widetilde\varphi)| = \left|\int_\Omega f\wedge\widetilde\varphi\right| \le \|f\|_{-N}\|\widetilde\varphi\|_N$$
$$\le \|f\|_{-N}\|\widetilde\varphi\|_{N-2}\lesssim\|f\|_{-N}\|\bar\partial\varphi\|_{N-2}.$$

Thus L_f is a continuous linear operator of norm $\lesssim\|f\|_{-N}$ and therefore, using the Hahn–Banach theorem, L_f extends to a continuous linear operator with norm $\lesssim\|f\|_{-N}$ on the Hilbert space $L^2_{n,n-q+1}(\Omega,E^*,N-2)$. By the theorem of Riesz, there exists $u\in L^2_{0,q-1}(\Omega,E,-N+2)$ with $\|u\|_{-N+2}\lesssim\|f\|_{-N}$ such that for every $\varphi\in L^2_{n,n-q}(\Omega,E^*,N-2)$ we have

$$(-1)^q\int_\Omega u\wedge\bar\partial\varphi = L_f(\varphi) = \int_\Omega f\wedge\varphi,$$

i.e. $\bar\partial u=f$. $\square$

4. The $\bar\partial$-problem with exact support

In this section, we will show some vanishing and separation theorems for the $\bar\partial$-cohomology groups with values in a vector bundle E supported in $\bar\Omega$:

$$H^{p,q}(X,\bar\Omega,E)=\frac{\{f\in\mathcal{C}^\infty_{p,q}(X,E)\mid \operatorname{supp} f\subset\bar\Omega\}\cap\operatorname{Ker}\bar\partial}{\bar\partial\{f\in\mathcal{C}^\infty_{p,q-1}(X,E)\mid \operatorname{supp} f\subset\bar\Omega\}}.$$

This is done by solving the $\bar\partial$-equation in the L^2-sense as in the last section and then applying the results of Section 2 to the operator $\Box_{-N}=\bar\partial\bar\partial^*_{-N}+\bar\partial^*_{-N}\bar\partial$ for $N>0$. Here $\bar\partial^*_{-N}$ is the von Neumann adjoint of $\bar\partial$: $L^2_{p,q}(\Omega,E,-N+2)\to L^2_{p,q+1}(\Omega,E,-N)$. An easy computation shows that $\bar\partial^*_{-N}u=\Delta^{N-2}\bar\partial^*_\omega(\Delta^{-N}u)$, where $\bar\partial^*_\omega$ is the von Neumann adjoint of $\bar\partial$ with respect to the metric ω on X.

Theorem 4.1. *Assume that Ω has Lipschitz boundary. Let $u\in L^2_{p,q}(\Omega,E,-N)$ satisfy $\bar\partial u=f$ and $\bar\partial^*_{-N}u=0$ with $f\in\mathcal{C}^N_{p,q}(X,\bar\Omega,E)\cap\mathcal{C}^\infty_{p,q}(\Omega,E)$.*

Then $u\in\mathcal{C}^{s(N)}_{p,q}(X,\bar\Omega,E)\cap\mathcal{C}^\infty_{p,q}(\Omega,E)$, where $s(N)$ is a function proportional to $\sqrt{N}$, $N\gg1$.

Proof. The above theorem is a consequence of the results of Section 2. Indeed, since $\bar\partial^*_{-N}u=\Delta^{N-2}\bar\partial^*_\omega(\Delta^{-N}u)$, where $\bar\partial^*_\omega$ is the adjoint of $\bar\partial$ with respect to the metric ω on X, it is clear that $\Box_{-N}$ is an elliptic operator of polynomial growth with respect to Δ on Ω. Since $\bar\partial^*_{-N}u=0$, and $\bar\partial u=f$, we have $\Box_{-N}u=\bar\partial^*_{-N}f$. From general results on domains with Lipschitz boundaries (see [G, Section 1]), we deduce that $\bar\partial^*_{-N}f\in\mathcal{C}^{N-k_0}_{p,q-1}(X,\bar\Omega,E)\cap\mathcal{C}^\infty_{p,q-1}(\Omega,E)$ for some k_0 not depending on N. The result then follows from Theorem 2.1, using a finite partition of unity. $\Box$

We are now ready to prove the main theorem of this section.

Theorem 4.2. *Let Ω be a relatively compact domain with Lipschitz boundary in an n-dimensional Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let E be a holomorphic vector bundle on X. Then we have*

$$H^{p,q}(X,\bar\Omega,E)=0\quad \text{for } 1\le q\le n-1$$

and $H^{p,n}(X,\bar\Omega,E)$ is separated for the usual $\mathcal{C}^\infty$-topology. Moreover,

$$\bar\partial\mathcal{C}^\infty_{p,n-1}(X,\bar\Omega,E)=\bigcap_{N\in\mathbf{N}}\left\{f\in\mathcal{C}^\infty_{p,n}(X,\bar\Omega,E)\;\middle|\;\int_\Omega f\wedge h=0\right.$$
$$\left.\text{for all } h\in L^2_{n-p,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar\partial\right\}.$$

Proof. Replacing the vector bundle E by $\Lambda^p(T^{1,0}X)^*\otimes E$, it is no loss of generality to assume $p=0$.

We will begin by proving the following claim:

Let $f\in\mathcal{C}^k_{0,q}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q}(\Omega,E)\cap\operatorname{Ker}\bar{\partial}$, $1\le q\le n-1$, $k\gg 1$. *Then there exists* $u\in\mathcal{C}^{j(k)}_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$ *such that* $\bar{\partial}u=f$ *with* $j(k)\sim\sqrt{k}$.

Indeed, general results on Lipschitz domains (see e.g. [G, Theorem 1.4.4.4]) show that $f\in L^2_{0,q}(\Omega,E,-2k)$. Proposition 3.2 implies that there exists $u\in L^2_{0,q-1}(\Omega,E,-2k+2)$ such that $\bar{\partial}u=f$ in Ω. Moreover, choosing the minimal solution, we may assume $\bar{\partial}^*_{-2k}u=0$. Applying Theorem 4.1, we then have

$$u\in\mathcal{C}^{j(k)}_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$$

with $j(k)\sim\sqrt{k}$.

Let us now prove the theorem.

That $H^{0,1}(X,\bar{\Omega},E)=0$ follows immediately from the above claim and the hypoellipticity of $\bar{\partial}$ in bidegree $(0,1)$.

Now assume that $1<q\le n-1$ and let $f\in\mathcal{C}^\infty_{0,q}(X,\bar{\Omega},E)\cap\operatorname{Ker}\bar{\partial}$. By induction, we will construct $u_k\in\mathcal{C}^k_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$ such that $\bar{\partial}u_k=f$ and so that $|u_{k+1}-u_k|_{j(k)-1}<2^{-k}$. It is then clear that $(u_k)_{k\in\mathbf{N}}$ converges to $u\in\mathcal{C}^\infty_{0,q-1}(X,\bar{\Omega},E)$ such that $\bar{\partial}u=f$.

Suppose that we have constructed $u_1,\dots,u_k$. By the above claim, since $f\in\mathcal{C}^\infty_{0,q}(X,\bar{\Omega},E)$, there exists $\alpha_{k+1}\in\mathcal{C}^{k+1}_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$ such that $f=\bar{\partial}\alpha_{k+1}$. We have $\alpha_{k+1}-u_k\in\mathcal{C}^k_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)\cap\operatorname{Ker}\bar{\partial}$. Once again by the above claim, there exists $g\in\mathcal{C}^{j(k)}_{0,q-2}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-2}(\Omega,E)$ satisfying $\alpha_{k+1}-u_k=\bar{\partial}g$. Since $\mathcal{C}^\infty_{0,q-2}(X,\bar{\Omega},E)$ is dense in $\mathcal{C}^{j(k)}_{0,q-2}(X,\bar{\Omega},E)$, there exists $g_{k+1}\in\mathcal{C}^\infty_{0,q-2}(X,\bar{\Omega},E)$ such that $|g-g_{k+1}|_{j(k)}<2^{-k}$. Define $u_{k+1}=\alpha_{k+1}-\bar{\partial}g_{k+1}\in\mathcal{C}^{k+1}_{0,q-1}(X,\bar{\Omega},E)\cap\mathcal{C}^\infty_{0,q-1}(\Omega,E)$. Then $\bar{\partial}u_{k+1}=f$ and $|u_{k+1}-u_k|_{j(k)-1}=|\bar{\partial}g-\bar{\partial}g_{k+1}|_{j(k)-1}\le|g-g_{k+1}|_{j(k)}<2^{-k}$.

Thus u_{k+1} has the desired properties.

It remains to show that

$$\bar{\partial}\mathcal{C}^\infty_{0,n-1}(X,\bar{\Omega},E)=\bigcap_{N\in\mathbf{N}}\left\{f\in\mathcal{C}^\infty_{0,n}(X,\bar{\Omega},E)\;\middle|\;\int_\Omega f\wedge h=0\right.$$
$$\left.\text{for all } h\in L^2_{n,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar{\partial}\right\}.$$

This clearly implies that $H^{0,n}(X,\bar{\Omega},E)$ is separated.

First of all, suppose $f=\bar{\partial}\alpha$ with $\alpha\in\mathcal{C}^\infty_{0,n-1}(X,\bar{\Omega},E)$ and let $h\in L^2_{n,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar{\partial}$. Then we have

$$\int_\Omega f\wedge h=\int_\Omega\bar{\partial}\alpha\wedge h=\lim_{\varepsilon\to0}\int_{\partial\Omega_\varepsilon}\alpha\wedge h=-\lim_{\varepsilon\to0}\int_{\Omega\setminus\Omega_\varepsilon}\bar{\partial}\alpha\wedge h=-\lim_{\varepsilon\to0}\int_{\Omega\setminus\Omega_\varepsilon}f\wedge h$$

with $\Omega_\varepsilon \supset \{z\in\Omega \,|\, \Delta(z)>\varepsilon\}$ and

$$\left|\int_{\Omega\setminus\Omega_\varepsilon} f\wedge h\right| \leq \left(\int_{\Omega\setminus\Omega_\varepsilon} |f|^2_\omega \Delta^{-N-2}\right)^{1/2} \left(\int_{\Omega\setminus\Omega_\varepsilon} |h|^2_\omega \Delta^{N+2}\right)^{1/2} \leq \varepsilon \|f\|_{-N-2}\|h\|_N \to 0,$$

as $\varepsilon\to 0$, which shows the inclusion $\subset$ (notice that $f\in\mathcal{C}^\infty_{0,n}(X,\overline{\Omega},E)$ implies that $f\in L^2_{0,n}(\Omega,E,-N-2)$ for all $N\in\mathbf{N}$, and see the proof of Theorem 3.2 for the justification of some of the equalities).

Now, let us take

$$f\in \bigcap_{N\in\mathbf{N}} \left\{ f\in\mathcal{C}^\infty_{0,n}(X,\overline{\Omega},E) \,\middle|\, \int_\Omega f\wedge h=0 \text{ for all } h\in L^2_{n,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar\partial \right\}.$$

We first show that for each $N\in\mathbf{N}$, $N\gg 1$, there exists $\beta_N\in L^2_{0,n-1}(\Omega,E,-N)$ satisfying $\bar\partial\beta_N=f$.

To see this, we define the linear operator

$$\begin{aligned} L_f\colon \operatorname{Im}(\bar\partial : L^2_{n,0}(\Omega,E^*,N) \longrightarrow L^2_{n,1}(\Omega,E^*,N)) &\longrightarrow \mathbf{C}, \\ \bar\partial\varphi &\longmapsto \int_\Omega f\wedge\varphi. \end{aligned}$$

First of all, notice that L_f is well-defined because of the moment conditions imposed on f.

By Proposition 3.1, L_f is a continuous linear operator and therefore extends to a continuous linear operator on the Hilbert space $L^2_{n,1}(\Omega,E^*,N)$ by the Hahn–Banach theorem. By the theorem of Riesz, there exists $\beta_N\in L^2_{0,n-1}(\Omega,E,-N)$ such that for every $\varphi\in L^2_{n,0}(\Omega,E^*,N)$ we have

$$(-1)^r\int_\Omega \beta_N\wedge\bar\partial\varphi = L_f(\varphi) = \int_\Omega f\wedge\varphi,$$

i.e. $\bar\partial\beta_N=f$.

Now the proof follows the same lines as above, and we construct $(u_k)_{k\in\mathbf{N}}\in \mathcal{C}^k_{0,n-1}(X,\overline{\Omega},E)$ converging to $u\in\mathcal{C}^\infty_{0,n-1}(X,\overline{\Omega},E)$ such that $\bar\partial u=f$, which concludes the proof. □

Corollary 4.3. *Let $\Omega\subsetneq X$ be a $\mathcal{C}^\infty$-smooth domain in a compact Kähler manifold (X,ω) of complex dimension n. We assume that Ω is* $\log\delta$*-pseudoconvex. Let E be a holomorphic vector bundle on X. Let $f\in\mathcal{C}^\infty_{p,q}(\partial\Omega,E)\cap\operatorname{Ker}\bar\partial_b$ satisfy the tangential Cauchy–Riemann equations on $\partial\Omega$, $q\leq n-2$.*

Then there exists $F\in\mathcal{C}^\infty_{p,q}(\overline{\Omega},E)$ such that $F|_{\partial\Omega}=f$ and $\bar\partial F=0$.

For $q=n-1$, the same holds true if there exists $\tilde f\in\mathcal{C}^\infty_{p,n-1}(\overline{\Omega},E)$ such that $\tilde f|_{\partial\Omega}=f$, $\bar\partial\tilde f$ vanishes to infinite order on $\partial\Omega$ and $\int_\Omega\bar\partial\tilde f\wedge h=0$ for all $h\in L^2_{n-p,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar\partial$, for all $N\in\mathbf{N}$.

Proof. Choose $\tilde f\in\mathcal{C}^\infty_{p,q}(X,E)$ such that $\tilde f|_{\partial\Omega}=f$ and such that $\bar\partial\tilde f$ vanishes to infinite order on $\partial\Omega$ (for the existence of such an extension see e.g. [Ra, Lemma 2.4]). Applying Theorem 4.2, there exists $h\in\mathcal{C}^\infty_{p,q}(X,\overline\Omega,E)$ such that $\bar\partial h=\bar\partial\tilde f$. The function $F:=\tilde f-h$ is then a $\bar\partial$-closed $\mathcal{C}^\infty$ extension of f to $\overline\Omega$. □

Corollary 4.4. (See [HI]) *Let $\Omega\subsetneq X$ be a $\mathcal{C}^\infty$-smooth domain in a compact Kähler manifold (X,ω) of complex dimension n. We assume that Ω is $\log\delta$-pseudoconvex. Let E be a holomorphic vector bundle on X. Assume that $H^{p,q}(X,E)=0$ and put $D=X\setminus\overline\Omega$.*

Then for every $\bar\partial$-closed form $f\in\mathcal{C}^\infty_{p,q}(\overline D,E)$, which is smooth up to the boundary, there exists $u\in\mathcal{C}^\infty_{p,q-1}(\overline D,E)$ such that $\bar\partial u=f$, $1\le q\le n-2$.

For $q=n-1$, the same holds true if there exists $\tilde f\in\mathcal{C}^\infty_{p,n-1}(X,E)$ such that $\tilde f|_{\overline D}=f$, $\bar\partial\tilde f$ vanishes to infinite order on $\partial\Omega$ and $\int_\Omega\bar\partial\tilde f\wedge h=0$ for all $h\in L^2_{n-p,0}(\Omega,E^*,N)\cap\operatorname{Ker}\bar\partial$, for all $N\in\mathbf{N}$.

Proof. Choose $\tilde f\in\mathcal{C}^\infty_{p,q}(X,E)$ such that $\tilde f|_{\overline D}=f$. Then $\bar\partial\tilde f$ vanishes to infinite order on $\partial\Omega$. By Theorem 4.2, there exists $h\in\mathcal{C}^\infty_{p,q}(X,\overline\Omega,E)$ such that $\bar\partial h=\bar\partial\tilde f$. The function $F:=\tilde f-h$ is then a $\bar\partial$-closed $\mathcal{C}^\infty$ extension of f to X. As $H^{p,q}(X,E)=0$, we have $F=\bar\partial u$ for some $u\in\mathcal{C}^\infty_{p,q-1}(X,E)$. Then $u|_{\overline D}$ has the desired properties. □

Remark. Corollary 4.3 is related to the nonexistence of smooth Levi flat hypersurfaces in $\mathbf{P}^n$, $n\ge3$, which was proved by Siu [S]. Indeed, as observed in [S], to establish this nonexistence result, it suffices to solve the equation $\bar\partial_b u=f$ on a Levi flat hypersurface M in P^n, where f is a smooth $\bar\partial_b$-closed $(0,1)$-form on M and u is a smooth function on M. Since $H^{0,1}(\mathbf{P}^n)=0$, it then follows from Corollary 4.3 that this $\bar\partial_b$-equation can be solved if $n\ge3$. We thus get a different proof of Siu's theorem.

5. The $\bar\partial$-equation for extensible currents

The results of the previous section will allow us to solve the $\bar\partial$-equation for extensible currents by duality.

Let $\Omega\subset X$ be an open set in an n-dimensional complex manifold X. A current T defined on Ω is said to be *extensible*, if T is the restriction to Ω of a current defined on X.

It was shown in [M] that if Ω satisfies $\mathring{\overline\Omega}=\Omega$ (which is always satisfied in our case), the vector space $\check{\mathcal{D}}'^{p,q}_\Omega(X)$ of extensible currents on Ω of bidegree (p,q) is the topological dual of $\mathcal{C}^\infty_{n-p,n-q}(X,\overline\Omega)\cap\mathcal{D}^{n-p,n-q}(X)$.

Theorem 5.1. *Let Ω be a relatively compact domain with Lipschitz boundary in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex.*

Let $T\in\check{\mathcal{D}}'^{p,q}_{\Omega}(X)$ be an extensible current on Ω of bidegree (p,q), $q\geq 1$, such that $\bar\partial T=0$ in Ω. Then there exists $S\in\check{\mathcal{D}}'^{p,q-1}_{\Omega}(X)$ satisfying $\bar\partial S=T$ in Ω.

Proof. Since Ω is relatively compact in X, we have

$$\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega)\cap\mathcal{D}^{n-p,n-q}(X)=\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega).$$

Let $T\in\check{\mathcal{D}}'^{p,q}_{\Omega}(X)$ be an extensible current on Ω of bidegree (p,q), $q\geq 1$, such that $\bar\partial T=0$ in Ω.

Consider the operator

$$\begin{aligned} L_T\colon \bar\partial\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega) &\longrightarrow \mathbf{C},\\ \bar\partial\varphi &\longmapsto \langle T,\varphi\rangle. \end{aligned}$$

We first notice that L_T is well defined. Indeed, let $\varphi\in\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega)$ be such that $\bar\partial\varphi=0$.

If $q=n$, the analytic continuation principle for holomorphic functions yields $\varphi=0$, so $\langle T,\varphi\rangle=0$.

If $1\leq q\leq n-1$, one has $\varphi=\bar\partial\alpha$ with $\alpha\in\mathcal{C}^\infty_{n-p,n-q-1}(X,\bar\Omega)$ by Theorem 4.2. As $\mathcal{D}^{n-p,n-q-1}(\Omega)$ is dense in $\mathcal{C}^\infty_{n-p,n-q-1}(X,\bar\Omega)$, there exists $(\alpha_j)_{j\in\mathbf{N}}\in\mathcal{D}^{n-p,n-q-1}(\Omega)$ such that $\bar\partial\alpha_j\to\bar\partial\alpha$, as $j\to+\infty$, in $\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega)$.

Hence $\langle T,\varphi\rangle=\langle T,\bar\partial\alpha\rangle=\lim_{j\to+\infty}\langle T,\bar\partial\alpha_j\rangle=0$, because $\bar\partial T=0$.

By Theorem 4.2, $\bar\partial\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega)$ is a closed subspace of $\mathcal{C}^\infty_{n-p,n-q+1}(X,\bar\Omega)$, and thus a Fréchet space. Using Banach's open mapping theorem, L_T is in fact continuous, so by the Hahn–Banach theorem, we can extend L_T to a continuous linear operator $\tilde L_T\colon\mathcal{C}^\infty_{n-p,n-q+1}(X,\bar\Omega)\to\mathbf{C}$, i.e. $\tilde L_T$ is an extensible current on Ω satisfying

$$\langle\bar\partial\tilde L_T,\varphi\rangle=(-1)^{p+q}\langle\tilde L_T,\bar\partial\varphi\rangle=(-1)^{p+q}\langle T,\varphi\rangle$$

for every $\varphi\in\mathcal{C}^\infty_{n-p,n-q}(X,\bar\Omega)$. Therefore $T=(-1)^{p+q}\bar\partial\tilde L_T$. $\square$

For the notion of differential forms admitting distribution boundary values, which is used in the following corollary, we refer the reader to [LT].

Corollary 5.2. *Let $\Omega\Subset X$ be a $\mathcal{C}^\infty$-smooth relatively compact domain in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Let f be a smooth $\bar\partial$-closed $(0,1)$-form on Ω admitting distribution boundary values on $\partial\Omega$.*

Then there exists a smooth function g on Ω admitting a distribution boundary value on $\partial\Omega$ such that $\bar\partial g=f$ on Ω.

Proof. As f admits distribution boundary values, we may view f as an extensible $\bar\partial$-closed current on Ω (see [LT]). Applying Theorem 5.1, there exists an extensible current S of bidegree $(0,0)$ on Ω such that $\bar\partial S = T$.

The hypoellipticity of $\bar\partial$ in bidegree $(0,1)$ yields that S is in fact a $\mathcal{C}^\infty$-smooth function on Ω. But a $\mathcal{C}^\infty$-smooth function S, extensible as a current, such that $\bar\partial S$ admits distribution boundary values, admits itself distribution boundary values (see [Sa, Lemme 4.3]). □

Corollary 5.3. *Let $\Omega \Subset X$ be a $\mathcal{C}^\infty$-smooth domain in a Kähler manifold (X,ω). We assume that Ω is $\log\delta$-pseudoconvex. Then we have*

$$H^q(\Omega, \breve{\mathcal{O}}_\Omega) = 0$$

for every $q \geq 1$, where $\breve{\mathcal{O}}_\Omega$ is the sheaf of germs on $\bar\Omega$ of holomorphic functions admitting a distribution boundary value.

Proof. We have the following exact sequence of sheaves on $\bar\Omega$

$$0 \longrightarrow \breve{\mathcal{O}}_\Omega \longrightarrow \breve{\mathcal{D}}'^{0,0}_{\Omega}(X) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \breve{\mathcal{D}}'^{0,n}_{\Omega}(X) \longrightarrow 0.$$

Indeed, that $\operatorname{Ker}(\bar\partial : \breve{\mathcal{D}}'^{0,0}_{\Omega}(X) \to \breve{\mathcal{D}}'^{0,1}_{\Omega}(X)) = \breve{\mathcal{O}}_\Omega$ was proved in [LT], and the exactness of the rest of the sequence follows from Theorem 5.1.

Then the de Rham–Weil theorem yields

$$H^q(\Omega, \breve{\mathcal{O}}_\Omega) \cong \frac{\operatorname{Ker}(\bar\partial : \breve{\mathcal{D}}'^{0,q}_{\Omega}(X) \longrightarrow \breve{\mathcal{D}}'^{0,q+1}_{\Omega}(X))}{\operatorname{Im}(\bar\partial : \breve{\mathcal{D}}'^{0,q-1}_{\Omega}(X) \longrightarrow \breve{\mathcal{D}}'^{0,q}_{\Omega}(X))}$$

and by Theorem 5.1, the right-hand side of the above isomorphism is 0 for $q \geq 1$. □

6. Boundaries of weakly pseudoconvex domains

Let $\Omega \Subset X$ be a domain with smooth boundary M in an n-dimensional complex manifold X. We then denote by $H^{p,q}(M)$ and $H^{p,q}_{\mathrm{cur}}(M)$ the cohomology groups of the tangential Cauchy–Riemann operator on smooth forms and currents on M, $0 \leq p \leq n$, $0 \leq q \leq n-1$.

Theorem 6.1. *Let X be an n-dimensional Stein manifold and $\Omega \Subset X$ a domain with smooth boundary M. We assume that Ω is weakly pseudoconvex. Then $H^{p,q}(M) = H^{p,q}_{\mathrm{cur}}(M) = 0$ for $0 \leq p \leq n$ and $1 \leq q \leq n-2$. Moreover, $H^{p,0}(M)$, $H^{p,0}_{\mathrm{cur}}(M)$, $H^{p,n-1}(M)$ and $H^{p,n-1}_{\mathrm{cur}}(M)$ are infinite-dimensional and, if $n \geq 3$, separated.*

In the case $X = \mathbf{C}^n$, the statement about the cohomology groups $H^{p,q}(M)$ was already proved in [Ro].

Proof. It follows from [HN1] and [HN3] that $H^{p,0}(M)$, $H^{p,n-1}(M)$, $H^{p,0}_{\mathrm{cur}}(M)$ and $H^{p,n-1}_{\mathrm{cur}}(M)$ are infinite-dimensional.

Now let $f\in\mathcal{C}^\infty_{p,q}(M)$ satisfy the tangential Cauchy–Riemann equations, $1\le q\le n-2$. It follows from Corollary 4.3 that there exists $F\in\mathcal{C}^\infty_{p,q}(\bar\Omega)$ satisfying $F|_M=f$ and $\bar\partial F=0$. Using Kohn's result on the solvability of the $\bar\partial$-equation with regularity up to the boundary in weakly pseudoconvex domains [K1], [K2], there exists $U\in\mathcal{C}^\infty_{p,q-1}(\bar\Omega)$ satisfying $\bar\partial U=F$ in Ω. Then $u=U|_M$ satisfies $\bar\partial_M u=f$. Hence $H^{p,q}(M)=0$ for $1\le q\le n-2$.

Moreover, we know from abstract duality arguments (see [Se]) that $H^{p,q}_{\mathrm{cur}}(M)$ is separated if and only if $H^{n-p,n-q}(M)$ is separated. Furthermore, if any one of these equivalent conditions is satisfied, we have $H^{p,q}_{\mathrm{cur}}(M)\simeq H^{n-p,n-q-1}(M)'$.

Therefore we have $H^{p,q}_{\mathrm{cur}}(M)=0$ for $2\le q\le n-2$ and $H^{p,n-1}_{\mathrm{cur}}(M)$ is separated for $n\ge3$; to complete the proof of the theorem, it remains to show that $H^{p,1}_{\mathrm{cur}}(M)=0$ if $n\ge3$, $0\le p\le n$.

To prove this, we note that we have a direct splitting

$$H^{p,q}_{\mathrm{cur}}(M)\simeq H^{p,q}(\breve{\mathcal{D}}'_\Omega(X))\oplus H^{p,q}(\breve{\mathcal{D}}'_{X\setminus\bar\Omega}(X)),$$

$q\ge1$. Here $H^{p,q}(\breve{\mathcal{D}}'_\Omega(X))$ (resp. $H^{p,q}(\breve{\mathcal{D}}'_{X\setminus\bar\Omega}(X))$) denote the $\bar\partial$-cohomology groups for currents on Ω (resp. on $X\setminus\bar\Omega$) which are extendable to X. Indeed, it is a well-known fact that we have the following long exact sequence (cf. [HN2] and [NV])

$$\ldots\longrightarrow H^{p,q}(X)\longrightarrow H^{p,q}(\breve{\mathcal{D}}'_{X\setminus M}(X))\longrightarrow H^{p,q+1}(\mathcal{D}'_M(X))\longrightarrow H^{p,q+1}(X)\longrightarrow\ldots,$$

where $H^{p,q}(\breve{\mathcal{D}}'_{X\setminus M}(X))$ are the $\bar\partial$-cohomology groups of currents on $X\setminus M$, which are extendable across M. Since X is Stein, it follows that $H^{p,q}(X)=0$ for $q\ge1$. Together with the well-known isomorphism $H^{p,q}_{\mathrm{cur}}(M)\simeq H^{p,q+1}(\mathcal{D}'_M(X))$ (see [HN2] and [NV]), this yields

$$H^{p,q}_{\mathrm{cur}}(M)\simeq H^{p,q}(\breve{\mathcal{D}}'_{X\setminus M}(X))\simeq H^{p,q}(\breve{\mathcal{D}}'_\Omega(X))\oplus H^{p,q}(\breve{\mathcal{D}}'_{X\setminus\bar\Omega}(X)),\quad q\ge1.$$

Theorem 5.1 implies that $H^{p,q}(\breve{\mathcal{D}}'_\Omega(X))=0$ for $q\ge1$. That $H^{p,1}_{\mathrm{cur}}(M)=0$ for $n\ge3$ is now an immediate consequence of the following lemma. □

Lemma 6.2. *Let X be an n-dimensional Stein manifold and $\Omega\Subset X$ a domain with smooth boundary M. We assume that Ω is weakly pseudoconvex. Then $H^{p,q}(\breve{\mathcal{D}}'_{X\setminus\bar\Omega}(X))=0$ for $1\le q\le n-2$.*

Proof. We first prove the following claim:

Let Ω_1 and Ω_2 be two weakly pseudoconvex domains with smooth boundary such that $\Omega_1 \Subset \Omega_2 \Subset X$. Then $H^{p,q}(X, \bar\Omega_2 \setminus \Omega_1) = 0$ for $2 \le q \le n-1$ and $H^{p,n}(X, \bar\Omega_2 \setminus \Omega_1)$ is separated, $0 \le p \le n$.

Indeed, let $f \in \mathcal{C}^\infty_{p,q}(X, \bar\Omega_2 \setminus \Omega_1) \cap \operatorname{Ker} \bar\partial$, $2 \le q \le n-1$. Then, since Ω_2 satisfies the assumptions of Theorem 4.2, there exists $u \in \mathcal{C}^\infty_{p,q-1}(X, \bar\Omega_2)$ satisfying $\bar\partial u = f$ in X. This implies that $\bar\partial u = 0$ in Ω_1. Hence, since $q-1 \ge 1$, there exists $h \in \mathcal{C}^\infty_{p,q-1}(\bar\Omega_1)$ satisfying $\bar\partial h = u$ in Ω_1 (see [K1] and [K2]). Let $\tilde h$ be a smooth extension of h to X with compact support in Ω_2 and set $g = u - \bar\partial \tilde h$. Then g satisfies $\bar\partial g = f$ and $\operatorname{supp} g \subset \bar\Omega_2 \setminus \Omega_1$. The separation statement is proved similarly, using the separation statement of Theorem 4.2.

With the same proof as the proof of Theorem 5.1, it follows from the above claim that $H^{p,q}(\breve{\mathcal{D}}'_{\Omega_2 \setminus \bar\Omega_1}(X)) = 0$ for $0 \le p \le n$ and $1 \le q \le n-2$.

Now let $\psi \in \mathcal{C}^\infty(X)$ be a strictly plurisubharmonic exhaustion function such that $\psi < 0$ on Ω and set $\Omega_j = \{z \in X \mid \psi(z) < j\}$, where we may suppose that $\partial\Omega_j$ is of class $\mathcal{C}^\infty$.

Let $T \in \breve{\mathcal{D}}'^{p,q}_{X \setminus \bar\Omega}(X)$ be $\bar\partial$-closed in $X \setminus \bar\Omega$, $1 \le q \le n-2$. As shown before, there exists $S_j \in \breve{\mathcal{D}}'^{p,q-1}_{X \setminus \bar\Omega}(X)$ satisfying $\bar\partial S_j = T$ in $\Omega_j \setminus \bar\Omega$. We then have $\bar\partial(S_{j+1} - S_j) = 0$ in $\Omega_j \setminus \bar\Omega$.

First assume $q \ge 2$. Then there exists $H \in \breve{\mathcal{D}}'^{p,q-2}_{X \setminus \bar\Omega}(X)$ such that $\bar\partial H = S_{j+1} - S_j$ in $\Omega_j \setminus \bar\Omega$. Setting $\tilde S_{j+1} = S_{j+1} - \bar\partial H$, we have $\bar\partial \tilde S_{j+1} = T$ in $\Omega_{j+1} \setminus \bar\Omega$ and $\tilde S_{j+1} = S_j$ in $\Omega_j \setminus \bar\Omega$. We can thus find a sequence $(G_j)_{j \in \mathbf{N}}$, $G_j \in \breve{\mathcal{D}}'^{p,q-1}_{X \setminus \bar\Omega}(X)$, satisfying $\bar\partial G_j = T$ in $\Omega_j \setminus \bar\Omega$ and $G_{j+1} = G_j$ in $\Omega_j \setminus \bar\Omega$. Then $(G_j)_j$ converges to $G \in \breve{\mathcal{D}}'^{p,q-1}_{X \setminus \bar\Omega}(X)$ such that $\bar\partial G = T$ in $X \setminus \bar\Omega$.

Now suppose $q = 1$. Then $S_{j+1} - S_j$ is a holomorphic p-form on $\Omega_j \setminus \bar\Omega$. By the Hartogs phenomenon on Stein manifolds, $S_{j+1} - S_j$ extends to a holomorphic p-form on Ω_j. Moreover, we may approximate holomorphic p-forms on Ω_j uniformly on $\bar\Omega_{j-1}$ by holomorphic p-forms on Ω_{j+2}, hence there is a holomorphic p-form H on Ω_{j+2} satisfying $|\langle H - (S_{j+1} - S_j), \varphi \rangle| \le 2^{-j} |\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \bar\Omega_{j-1} \setminus \Omega)$. Let $\chi \in \mathcal{C}^\infty(X \setminus \bar\Omega)$ satisfy $\chi \equiv 1$ on $\bar\Omega_{j+1}$ and $\operatorname{supp} \chi \subset \Omega_{j+2}$. Setting $\tilde S_{j+1} = S_{j+1} - \chi H$, we have $\tilde S_{j+1} \in \breve{\mathcal{D}}'^{p,q-1}_{X \setminus \bar\Omega}(X)$, $\bar\partial \tilde S_{j+1} = T$ on $\Omega_{j+1} \setminus \bar\Omega$ and $|\langle \tilde S_{j+1} - S_j, \varphi \rangle| \le 2^{-j} |\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \bar\Omega_{j-1} \setminus \Omega)$. Thus there is a sequence $(G_j)_{j \in \mathbf{N}}$, $G_j \in \breve{\mathcal{D}}'^{p,q-1}_{X \setminus \bar\Omega}(X)$, so that $\bar\partial G_j = T$ in $\Omega_j \setminus \bar\Omega$ and $|\langle G_{j+1} - G_j, \varphi \rangle| \le 2^{-j} |\varphi|$ for every $\varphi \in \mathcal{C}^\infty_{n-p,n}(X, \bar\Omega_{j-1} \setminus \Omega)$. It follows that $(G_j)_{j \in \mathbf{N}}$ is a Cauchy sequence in the weak topology. Thus $(G_j)_{j \in \mathbf{N}}$ converges weakly to G, where G in fact is an extensible current on $X \setminus \bar\Omega$ satisfying $\bar\partial G = T$ in $X \setminus \bar\Omega$. $\square$

7. Applications to Levi flat CR manifolds

Here we want to solve the tangential Cauchy–Riemann equation for currents on certain domains in Levi flat submanifolds of a Stein manifold. The submanifolds will be of any codimension where the problem makes sense. More precisely, we consider the following set-up.

Let $M\subset X$ be a smooth generic CR manifold of real codimension k in an n-dimensional Stein manifold X. We moreover assume that M is globally defined by

$$M=\{z\in X \mid \varrho_1(z)=\ldots=\varrho_k(z)=0\},$$

where the ϱ_ν's, $1\le\nu\le k$, are real $\mathcal{C}^\infty$ functions in X satisfying $\partial\varrho_1\wedge\ldots\wedge\partial\varrho_k\neq 0$ on M. Our most important assumption is that M should be Levi flat, i.e.

$$i\partial\bar\partial\varrho_\nu(z)(\xi,\bar\xi)=0$$

for $\nu=1,\ldots,k$, $z\in M$ and every $\xi\in\mathbf{C}^n$ satisfying

$$\sum_{j=1}^{n}\frac{\partial\varrho_\mu}{\partial z_j}(z)\xi_j=0$$

for $\mu=1,\ldots,k$.

For $\nu=1,\ldots,k$, we set $\varphi_\nu=\varrho_\nu+\psi\sum_{j=1}^k\varrho_j^2$ and $\varphi_0=-\sum_{j=1}^k\varrho_j+\psi\sum_{j=1}^k\varrho_j^2$, where ψ is a strictly plurisubharmonic positive function of class $\mathcal{C}^\infty$ on X. Then for every ordered collection of k integers $0\le i_1<\ldots<i_k\le k$ we have $\partial\varrho_{i_1}\wedge\ldots\wedge\partial\varrho_{i_k}\neq 0$ on M. For an adequate choice of ψ we can then arrange it so that if we set $\Omega_\nu=\{z\in X\,|\,\varphi_\nu(z)<0\}$, $\nu=0,\ldots,k$, then each Ω_ν is a weakly pseudoconvex set and

$$M=\bigcap_{\nu=0}^{k}\bar\Omega_\nu,\quad X\setminus M=\bigcup_{\nu=0}^{k}\Omega_\nu,\quad X=\bigcup_{\nu=0}^{k}\bar\Omega_\nu\quad\text{and}\quad\bigcap_{\nu=0}^{k}\Omega_\nu=\emptyset.$$

Let Ω be a piecewise $\mathcal{C}^\infty$ bounded weakly pseudoconvex domain such that Ω intersects each Ω_ν transversally.

If U is any open set in M, we denote by $H^{p,q}_{\mathrm{cur}}(U)$ the cohomology groups of the tangential Cauchy–Riemann operator acting on currents in U, $0\le p\le n$, $0\le q\le n-k$.

Theorem 7.1. *Let M and Ω be as above, $0\le p\le n$ and $2\le q\le n-k$.*

Then $H^{p,q}_{\mathrm{cur}}(M\cap\Omega)=0$.

Moreover, let Ω' be any open set which is relatively compact in Ω. Then the restriction mapping

$$H^{p,1}_{\mathrm{cur}}(M\cap\Omega)\longrightarrow H^{p,1}_{\mathrm{cur}}(M\cap\Omega')$$

is the zero mapping. In other words, let $T\in[\mathcal{D}'^{p,1}](M\cap\Omega)$ such that $\bar\partial_M T=0$ in $M\cap\Omega$. Then there exists $S\in[\mathcal{D}'^{p,0}](M\cap\Omega')$ satisfying $\bar\partial_M S=T$ in $M\cap\Omega'$.

Proof. Since Ω is Stein, we have the following well-known isomorphisms (cf. [HN2], [NV] and the proof of Theorem 6.1)

$$H^{p,q}_{\mathrm{cur}}(M\cap\Omega)\simeq H^{p,q+k-1}(\breve{\mathcal{D}}'_{\Omega\setminus M}(\Omega))$$

for $q\geq 1$. Without loss of generality, we may assume that Ω' is also weakly pseudoconvex. Then the above isomorphisms also hold with Ω replaced by Ω'. The theorem is then an immediate consequence of the following lemma. For the case $q=1$, note that all diagrams induced by the restriction mapping are commutative. $\square$

Lemma 7.2. *For $0\leq p\leq n$ and $q\geq k+1$ we have $H^{p,q}(\breve{\mathcal{D}}'_{\Omega\setminus M}(\Omega))=0$.*

Moreover, let Ω' be any open set which is relatively compact in Ω. Then the restriction mapping

$$H^{p,k}(\breve{\mathcal{D}}'_{\Omega\setminus M}(\Omega))\longrightarrow H^{p,k}(\breve{\mathcal{D}}'_{\Omega'\setminus M}(\Omega'))$$

is the zero mapping.

Proof. The proof follows an induction argument of [NV]. By induction on l, we show the following claim:

Let $\Omega, D_0, ..., D_l$ be piecewise smooth domains in X which are locally Stein and which intersect pairwise transversally. Set $D=\Omega\cap\bigcup_{j=0}^{l} D_j$ and let Ω' be any relatively compact open set of Ω.

If $T\in\breve{\mathcal{D}}'^{p,q}_{D}(\Omega)$ satisfies $\bar{\partial}T=0$ in $D\cap\Omega'$, and $q\geq l+1$, then there exists $S\in\breve{\mathcal{D}}'^{p,q-1}_{D}(\Omega)$ satisfying $\bar{\partial}S=T$ in $D\cap\Omega'$.

First assume that $l=0$ and let $T\in\breve{\mathcal{D}}'^{p,q}_{D_0\cap\Omega'}(\Omega)$ satisfy $\bar{\partial}T=0$ in $D_0\cap\Omega'$, $q\geq 1$. Without loss of generality, we may assume that Ω' is a piecewise smooth domain which is locally Stein and which intersects D_0 transversally. Then $\Omega'\cap D_0$ has Lipschitz boundary and is relatively compact in Ω. Moreover, since X is a Stein manifold and since $\Omega'\cap D_0$ is locally Stein, it follows from [E] that $\Omega'\cap D_0$ is $\log\delta$-pseudoconvex for some Kähler metric on X. We apply Theorem 5.1 and conclude that there exists $S\in\breve{\mathcal{D}}'^{p,q-1}_{D_0\cap\Omega}(\Omega)$ satisfying $\bar{\partial}S=T$ in $\Omega'\cap D_0$. This proves the claim for $l=0$.

Now assume the claim is true for $l-1$ and let us prove it for $l\geq 1$. Set $U_1=\Omega\cap\bigcup_{j=0}^{l-1} D_j$ and $U_2=\Omega\cap D_l$. Let $T\in\breve{\mathcal{D}}'^{p,q}_{D}(\Omega)$ satisfy $\bar{\partial}T=0$ in $D\cap\Omega'$, $q\geq l+1$. Then, by the induction hypothesis, there exist $S_1, S_2\in\breve{\mathcal{D}}'^{p,q-1}_{D}(\Omega)$ such that $\bar{\partial}S_1=T$ in $U_1\cap\Omega'$ and $\bar{\partial}S_2=T$ in $U_2\cap\Omega'$. Then we have $\bar{\partial}(S_1-S_2)=0$ in $U_1\cap U_2\cap\Omega'$. Again, since $q-1\geq 1$, we may apply Theorem 5.1 to the domain $U_1\cap U_2\cap\Omega'$; note that $U_1\cap U_2\cap\Omega'$ is relatively compact in Ω and locally Stein with Lipschitz boundary.

We conclude that there exists $H\in\check{\mathcal{D}}_D'^{p,q-2}(\Omega)$ satisfying $\bar\partial H=S_1-S_2$ in $U_1\cap U_2\cap\Omega'$. We define the current S of bidegree $(p,q-1)$ on D by $S=S_1$ in U_1 and $S=S_2+\bar\partial H$ in U_2. Then S is well defined and $\bar\partial S=T$ in $\Omega'\cap\bigcup_{j=0}^{l}D_j$. Moreover, S is extendable to a current on Ω. This proves the claim.

Let us now prove that for every relatively compact subset Ω' of Ω and every $T\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q}(\Omega)$ satisfying $\bar\partial T=0$ in $\Omega\setminus M$, $q\geq k$, there exists $S\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$ such that $\bar\partial S=T$ in $\Omega'\setminus M$.

We recall that $\Omega\setminus M=\bigcup_{\nu=0}^{k}(\Omega_\nu\cap\Omega)$ and $\bigcap_{\nu=0}^{k}\Omega_\nu=\emptyset$.

Let $T\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q}(\Omega)$ satisfy $\bar\partial T=0$ in $\Omega\setminus M$, $q\geq k$. From the above claim, there exist $S_1,S_2\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$ such that $\bar\partial S_1=T$ in $\bigcup_{\nu=0}^{k-1}(\Omega_\nu\cap\Omega')$ and $\bar\partial S_2=T$ in $\Omega_k\cap\Omega'$. This settles the case $k=1$, since in this situation, $\Omega_0\cap\Omega'$ and $\Omega_1\cap\Omega'$ are disjoint sets.

Now we assume that $k\geq 2$. Then $\bar\partial(S_1-S_2)=0$ in

$$\bigcup_{\nu=0}^{k-1}\Omega_\nu\cap\Omega_k\cap\Omega'=\bigcup_{\nu=0}^{k-2}(\Omega_\nu\cap\Omega_k\cap\Omega')\cup(\Omega_{k-1}\cap\Omega_k\cap\Omega').$$

We set $W_1=\bigcup_{\nu=0}^{k-2}(\Omega_\nu\cap\Omega_k\cap\Omega')$ and $W_2=\Omega_{k-1}\cap\Omega_k\cap\Omega'$. Then, since $\bigcap_{\nu=0}^{k}\Omega_\nu=\emptyset$, W_1 and W_2 are disjoint sets. Thus, in order to solve the $\bar\partial$-equation for extensible currents on $W_1\cup W_2$, it suffices to solve the $\bar\partial$-equation for extensible currents separately on W_1 and W_2. Since $q-1\geq k-1$, it then follows from the above claim that there exists $G\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-2}(\Omega)$ satisfying $\bar\partial G=S_1-S_2$ in $(W_1\cup W_2)\cap\Omega'$. It follows that the current S defined by

$$S=\begin{cases} S_1 & \text{in } \bigcup_{\nu=0}^{k-1}(\Omega_\nu\cap\Omega'),\\ S_2+\bar\partial G & \text{in } \Omega_k\cap\Omega'\end{cases}$$

is well defined, extendable to Ω and satisfies $\bar\partial S=T$ in $\Omega'\setminus M$.

We have thus proved the last assertion of the lemma.

Now suppose that $q\geq k+1$ and let $T\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q}(\Omega)$ satisfy $\bar\partial T=0$ in $\Omega\setminus M$. Let $(\Omega_i')_{i\in\mathbf{N}}$ be an exhaustion of Ω by smooth pseudoconvex domains. We have just proved that for every $i\in\mathbf{N}$, there is $S_i\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$ satisfying $\bar\partial S_i=T$ in $\Omega_i'\setminus M$. Then $\bar\partial(S_{i+1}-S_i)=0$ in $\Omega_i'\setminus M$. Thus, since $q-1\geq k$, there is $H_i\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-2}(\Omega)$ satisfying $S_{i+1}-S_i=\bar\partial H_i$ in $\Omega_i'\setminus M$. We set $\widetilde{S}_{i+1}=S_{i+1}-\bar\partial H_i$. Then $\widetilde{S}_{i+1}\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$, $\bar\partial\widetilde{S}_{i+1}=T$ in $\Omega_{i+1}'\setminus M$ and $\widetilde{S}_{i+1}=S_i$ in $\Omega_i'\setminus M$. Thus it is possible to construct a sequence $(S_j)_{j\in\mathbf{N}}$, $S_j\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$ satisfying $\bar\partial S_j=T$ in $\Omega_j'\setminus M$ and $S_{j+1}=S_j$ in $\Omega_j'\setminus M$. Then $(S_j)_{j\in\mathbf{N}}$ converges to a current $S\in\check{\mathcal{D}}_{\Omega\setminus M}'^{p,q-1}(\Omega)$ satisfying $\bar\partial S=T$ in $\Omega\setminus M$. $\square$

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