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Unique continuation for parabolic operators

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Abstract. It is shown that if a function u satisfies a backward parabolic inequality in an open set $\Omega\subset\mathbf{R}^{n+1}$ and vanishes to infinite order at a point (x_0, t_0) in Ω, then $u(x,t_0)=0$ for all x in the connected component of x_0 in $\Omega\cap(\mathbf{R}^n\times\{t_0\})$.

1. Introduction

This work is devoted to the study of the unique continuation property for second order parabolic operators with *time-dependent* variable coefficients.

For second order linear parabolic operators with *time-independent* coefficients, the strong unique continuation property was reduced by F. H. Lin [13] and independently by Landis and Oleinik [12] to the previously established elliptic counterparts. In particular, F. H. Lin shows that a parabolic operator P of the form

$$Pu=\operatorname{div}(a(x)\nabla u)+\partial_t u+b(x)\cdot\nabla u+c(x)u,$$

where the coefficient matrix $a(x)=(a^{ij}(x))$ is Lipschitz and the lower order coefficients b and c are bounded has the following unique continuation property:

If u satisfies $Pu=0$ in $S_T=\Omega\times(0,T)$ and at some interior point (x_0,t_0) in S_T the function u vanishes to infinite order in the space direction (i.e. $|u(x,t_0)|\le C_k|x-x_0|^k$ for any integer k), then $u(x,t_0)\equiv0$ for all $x\in\Omega$.

The reduction from time-independent parabolic equations to elliptic equations, a basic technique used in [12] and [13], relies on the representation formula for solutions of parabolic equations in terms of the eigenfunctions of the corresponding elliptic operator, and therefore cannot be applied to more general equations with time-dependent coefficients.

Time-dependent parabolic equations with variable coefficients have been treated by Saut and Scheurer in [17] and by Sogge in [18], where a weak unique continuation theorem is proven using a Carleman inequality. In [17] this is established for variable C^1 second order and bounded lower order coefficients, while in [18] unbounded potentials and smooth coefficients are treated, in particular it is shown that if u satisfies

$$|\Delta u+\partial_t u|\leq V(x,t)|u| \quad \text{in } S_T,$$

where $V\in L^{(n+2)/2}_{\mathrm{loc}}(dx\,dt)$ and $u\equiv 0$ in an open set $W\subset S_T$, then $u(\,\cdot\,,s)\equiv 0$ for all times $s\in(0,T)$ such that the hyperplane $t=s$ has a nonempty open intersection with W.

The strong unique continuation property for parabolic equations with time-dependent coefficients is treated by Poon in [16], Chen [4] and Escauriaza and Vega in [5] and [6]. In [16], the author defined a suitable frequency function measuring the space-time vanishing rate of a global solution to the backward heat equation and obtaining the following unique continuation property:

Assume that for some positive constant N a function u satisfies the inequality

$$|\Delta u+\partial_t u|\leq N(|\nabla u|+|u|) \quad \text{in } \mathbf{R}^n\times(0,T).$$

Then, $u\equiv 0$ in $\mathbf{R}^n\times(0,T)$ if u vanishes to infinite order from above in both the space and time variables at $(0,0)$.

By the former we mean that for each $k\geq 1$ there is a constant C_k such that

$$|u(x,t)|\leq C_k\big(|x|+\sqrt{t}\,\big)^k \tag{1.1}$$

for all (x,t), $t\geq 0$, in the domain of definition of u.

In [5] and [6] the authors prove a Carleman inequality arising naturally from the frequency function defined by Poon and obtain strong unique continuation type properties for global (defined in $\mathbf{R}^n\times(0,T)$) and local solutions of the inequality

$$|\Delta u+\partial_t u|\leq V(x,t)|u| \tag{1.2}$$

for some unbounded potentials V. In particular, they show that under certain $L^r_xL^s_t$ type conditions for the potential V, all functions u satisfying (1.1) for all $k\geq 1$ and (1.2) in $B_2\times[0,2)$ must vanish identically in $B_2\times\{0\}$. Moreover, it is shown in [6] that if the potential V is bounded in both variables (weaker conditions on V do work as well), there is a constant N depending on n and $\|V\|_{L^\infty(B_2\times(0,2))}$ such that

$$|u(x,t)|\leq Ne^{-1/Nt}\|u\|_{L^\infty(B_2\times(0,2))} \quad \text{for all } (x,t) \text{ in } B_1\times(0,1). \tag{1.3}$$

Aronszajn, Krzywicki and Szarski [3], and independently Hörmander [9], proved the strong unique continuation property for solutions to elliptic equations with variable Lipschitz second order coefficients using a Carleman inequality derived with methods based on the fundamental theorem of calculus (integration by parts).

In this work and using again only integration by parts to obtain a suitable Carleman inequality, we derive the unique continuation property (1.3) for local solutions to parabolic inequalities with *time-dependent* variable coefficients.

In particular, if B_r denotes an open ball of radius r centered at the origin in $\mathbf{R}^n$ and P the backward-parabolic operator

$$Pu=\sum_{i,j=1}^{n}\partial_i(a^{ij}(x,t)\partial_j u)+\partial_t u, \tag{1.4}$$

where the coefficient matrix $a(x,t)=(a^{ij}(x,t))$ is symmetric and for all $(x,t)\in\mathbf{R}^{n+1}$ and ξ in $\mathbf{R}^n$ satisfies the standard ellipticity condition

$$\lambda|\xi|^2\leq\sum_{i,j=1}^{n}a^{ij}(x,t)\xi_i\xi_j\leq\frac{1}{\lambda}|\xi|^2, \tag{1.5}$$

the following results are shown.

Theorem 1. *Assume that there are constants $M>0$ and $0<\beta<1$ such that one of the following conditions holds for all x and y in $\mathbf{R}^n$ and $0\leq t,s<+\infty$:*

(i) $|a(x,t)-a(y,s)|\leq M(|x-y|^2+|t-s|)^{1/2}$;

(ii) $|a(x,t)-a(y,s)|\leq M(|x-y|^2+|t-s|)^{\beta/2}$ *and also* $|\nabla a(x,t)|\leq M|x|^{\beta-1}$ *and* $|\partial_t a(x,t)|\leq Mt^{\beta/2-1}$.

Then, if u satisfies (1.1) *and*

$$|Pu|\leq M(|\nabla u|+|u|) \tag{1.6}$$

in $B_2\times[0,2)$, it follows that $u(x,0)=0$ for all $x\in B_2$. Moreover, there is a constant N depending on β, M and n such that,

$$|u(x,t)|\leq Ne^{-1/Nt}\|u\|_{L^\infty(B_2\times(0,2))}\quad \textit{when } (x,t)\in B_1\times(0,1). \tag{1.7}$$

The counterexamples by Plis [15] and Miller [14] establishing the existence of elliptic operators with Hölder continuous coefficients and having a nonzero solution vanishing on an open set, show that the Lipschitz regularity in the space variable required in Theorem 1 is sharp. We do not know whether the $\frac{1}{2}$-Hölder regularity in the time variable required in Theorem 1 is the best in order to derive (1.7).

These results can be carried out up to the boundary under proper Dirichlet or Neumann boundary conditions, extending and localizing the results obtained by Escauriaza and Adolfsson in [1] for *time-independent* parabolic operators to the case of *time-dependent* operators.

In what follows $D=\{x=(x',x_n)\in\mathbf{R}^n:x_n>\varphi(x')\}$, where $\varphi\colon\mathbf{R}^{n-1}\to\mathbf{R}$ is a Lipschitz function satisfying $\varphi(0)=0$, $\|\nabla\varphi\|_\infty\leq M$, and $d\sigma$ denotes surface measure on ∂D.

Theorem 2. *Assume that u satisfies* (1.1) *and* (1.6) *in* $(B_2\cap D)\times[0,2]$. *Then, there is a constant N such that*

$$|u(x,t)|\le Ne^{-1/Nt}\|u\|_{L^\infty((B_2\cap D)\times(0,2))}\quad \textit{when } (x,t)\in(B_1\cap D)\times[0,1], \tag{1.8}$$

whenever one of the following conditions hold:

(1) *D is a $C^{1,1}$ domain, the coefficient matrix of P satisfies condition* (i) *in Theorem* 1 *and either $u=0$ or $a\nabla u\cdot n=0$ on $(B_2\cap\partial D)\times[0,2]$;*

(2) *$u=0$ in $(B_2\cap\partial D)\times[0,2]$, the coefficient matrix of P satisfies condition* (ii) *in Theorem* 1 *and for some $\Lambda\ge 0$ and $0<\beta<1$,*

$$x'\cdot\nabla\varphi(x')-\varphi(x')\ge-\Lambda|x'|^{1+\beta}\quad \textit{for } |x'|\le 1; \tag{1.9}$$

(3) *$u=0$ in $(B_2\cap\partial D)\times[0,2]$, the coefficient matrix of P satisfies the first condition in Theorem* 1 *and for some $\Lambda\ge 0$ and $0<\beta<1$,*

$$\varphi+\Lambda|x'|^{1+\beta}\quad \textit{is a convex function for } |x'|\le 1. \tag{1.10}$$

Observe that (1.9) holds when φ can be written as the sum of a convex function and a $C^{1,\beta}$ function, while (1.10) is weaker than being a convex function and implies (1.9).

If D is a bounded Lipschitz domain in $\mathbf{R}^n$ with $0\in\overline{D}$ and u satisfies (1.1) and (1.6) in $D\times[0,T]$, and either $u=0$ or $a\nabla u\cdot n=0$ on $\partial D\times[0,T]$ for some $T>0$, an iteration of the results in Theorems 1 and 2 imply that $u(x,0)=0$ for all $x\in D$. But once you know this, the standard backward unique continuation property of parabolic equations ([13, pp. 133–134], [7, Chapter 3, Theorem 11]) implies that $u\equiv 0$ in $D\times[0,T]$. In fact, if

$$\int_0^T \sup_{x\in D}|\partial_t a(x,t)|\,dt\le M<+\infty,$$

it is well known that the function

$$e^{\theta(t)+M^2t}\partial_t\log\int_D u^2(x,t)\,dx+M^2t,\quad \lambda\theta(t)=\int_0^t\sup_{x\in D}|\partial_s a(x,s)|\,ds,$$

is essentially nondecreasing, which implies, when $0<t<T$, that

$$\int_D u^2(x,t)\,dx\le C(M,\lambda,T)\left(\int_D u^2(x,0)\,dx\right)^{\alpha(t)}\left(\int_D u^2(x,T)\,dx\right)^{1-\alpha(t)},$$

where

$$\alpha(t)=\frac{\int_t^T e^{-\theta(s)-M^2s}\,ds}{\int_0^T e^{-\theta(s)-M^2s}\,ds}.$$

In relation to the backward unique continuation property, Miller [14] has a counterexample of a parabolic operator P whose coefficient matrix $a(x,t)$ satisfies $a(x,0)\equiv\mathcal{I}$ the identity matrix, $a(\,\cdot\,,t)\in C^\infty(\mathbf{R})$ for all $t\geq 0$ and

$$|a(x,t)-a(x,0)|\leq Mt^{1/6}.$$

This operator has a solution u in $D\times(-\infty,T]$, $D=(0,\pi)\times(0,\pi)$, with zero conormal derivative on $\partial D\times(-\infty,T]$, u is never identically zero on open sets contained in $D\times(0,T)$ and $u\equiv 0$ for $t\leq 0$.

The proofs of the results in Theorems 1 and 2 are based on a perturbation of the following identity for the backward heat operator.

Theorem 3. *Assume that G is a positive caloric function in $\mathbf{R}^{n+1}_+$. Then, the following identity holds for all u in $C_0^\infty(\mathbf{R}^{n+1}_+)$ and $\alpha\in\mathbf{R}$,*

$$\int_{\mathbf{R}^{n+1}_+} t^{1-\alpha}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha u}{2t}\Big)^2 G\,dX+\int_{\mathbf{R}^{n+1}_+} t^{1-\alpha}\mathcal{D}_G\nabla u\cdot\nabla u\,dX$$
$$=\int_{\mathbf{R}^{n+1}_+} t^{1-\alpha}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha u}{2t}\Big)(\Delta u+\partial_t u)G\,dX,$$

where $dX=dx\,dt$ and $\mathcal{D}_G$ is the nonnegative $n\times n$ matrix,

$$\mathcal{D}_G=\frac{G}{2t}\mathcal{I}+D^2G-\frac{\nabla G\otimes\nabla G}{G}. \tag{1.11}$$

In Section 2 we prove some auxiliary lemmas and the generalization of the identity in Theorem 3 which appears when one replaces the backward heat operator by a general operator P. In Section 3 we show how to use this identity to find a suitable Carleman inequality implying Theorem 1 and in the fourth section we prove Theorem 2.

2. Some auxiliary lemmas

Letting P denote the operator in (1.4) and to simplify the writing and calculations we shall use some of the standard notation in Riemannian geometry, but always dropping the corresponding volume element in the definition of the Laplace–Beltrami operator associated to a Riemannian metric. We do this, because it simplifies the formulae appearing in the proofs of the following lemmas, and especially when the metric is allowed to depend on the time variable and we make use of partial integration with respect to this variable.

In particular, letting $g(x,t)=(g_{ij}(x,t))$ denote the inverse matrix of the coefficient matrix $a(x,t)$ of P and $g^{-1}=(g^{ij}(x,t))$ the inverse matrix of g, we use the following notation when considering either a function f or two variable vector fields ξ and η:

(1) $\xi\cdot\eta=\sum_{i,j=1}^{n} g_{ij}(x,t)\xi_i\eta_j$, $|\xi|^2=\xi\cdot\xi$;

(2) $\partial_i f=\partial f/\partial x_i$, $\partial_t f=\partial f/\partial t$, $\partial_{ij} f=\partial_i\partial_j f$, $\operatorname{div}\xi=\sum_{i=1}^{n}\partial_i\xi_i$, $\nabla f=g^{-1}\nabla_n f$, where ∇_n denotes the usual gradient in $\mathbf{R}^n$ and $\Delta f=\operatorname{div}(\nabla f)$.

With this notation the following formulae hold when u, f and h are smooth functions,

$$Pu=\Delta u+\partial_t u,$$
$$\Delta f^2=2f\Delta f+2|\nabla f|^2,$$
$$\int_{\mathbf{R}^n} h\Delta f\,dx=\int_{\mathbf{R}^n} f\Delta h\,dx=-\int_{\mathbf{R}^n}\nabla f\cdot\nabla h\,dx.$$

By $A\lesssim B$ we mean $A\le NB$, where N depends at most on n and the constants λ, M, Λ and β appearing in Theorems 1 and 2.

Lemma 1. *Let $\sigma=\sigma(t)$ be a nondecreasing function satisfying $\sigma(0)=0$, $\alpha\in\mathbf{R}$, and F and G denote two functions in $\mathbf{R}^{n+1}_+$, G nonnegative. Then, the following identity holds for all $u\in C_0^\infty(\mathbf{R}^{n+1}_+)$,*

$$\begin{aligned}
&2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\left(\partial_t u-\nabla\log G\cdot\nabla u+\frac12 Fu-\frac{\alpha\dot\sigma}{2\sigma}u\right)^2 G\,dX\\
&\quad+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\boldsymbol\cdot}|\nabla u|^2 G\,dX\\
&=2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta u+\partial_t u)\left(\partial_t u-\nabla\log G\cdot\nabla u+\frac12 Fu-\frac{\alpha\dot\sigma}{2\sigma}u\right)G\,dX\\
&\quad+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u\nabla u\cdot\nabla FG\,dX-\frac12\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2M\,dX\\
&\quad+\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\sigma^{-\alpha}u^2(\partial_t G-\Delta G-FG)\,dX\\
&\quad-\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}|\nabla u|^2(\partial_t G-\Delta G-FG)\,dX-2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\mathcal{D}_G\nabla u\cdot\nabla u\,dX,
\end{aligned}$$

where

$$M=\widehat{\log\frac{\sigma}{\dot\sigma}}^{\boldsymbol\cdot}FG+\partial_t FG+F(\partial_t G-\Delta G-FG)-\nabla G\cdot\nabla F,$$

$\mathcal{D}_G=\mathcal{J}g$ and $\mathcal{J}=(\mathcal{J}^{ij})$ is the $n\times n$ symmetric matrix defined as

$$\begin{aligned}
\mathcal{J}^{ij}=\frac{g^{ij}}{2t}G+g^{il}\partial_{kl}Gg^{kj}-\frac{g^{ik}\partial_k Gg^{jl}\partial_l G}{G}+\frac12\partial_k g^{il}\partial_l Gg^{kj}+\frac12\partial_k g^{jl}\partial_l Gg^{ki}\\
-\tfrac12 g^{kl}\partial_l G\partial_k g^{ij}+\tfrac12\partial_t g^{ij}G.
\end{aligned}$$

Observe that this identity contains the one in Theorem 3 when $g(x,t)=\mathcal{I}$ is the identity matrix, $\sigma(t)\equiv t$ and $F\equiv 0$. In this case $\mathcal{D}_G$ is given by (1.11), and since every nonnegative caloric functions can be represented as the Gaussian extension of some positive measure μ on $\mathbf{R}^n$,

$$\begin{aligned}G(x,t)&=\frac{1}{t^{n/2}}\int_{\mathbf{R}^n} e^{-|x-y|^2/4t}\,d\mu,\\ \nabla G&=\frac{1}{t^{n/2}}\int_{\mathbf{R}^n}\frac{y-x}{2t}e^{-|x-y|^2/4t}\,d\mu,\\ D^2G&=-\frac{G}{2t}\mathcal{I}+\frac{1}{t^{n/2}}\int_{\mathbf{R}^n}\frac{(y-x)\otimes(y-x)}{4t^2}e^{-|x-y|^2/4t}\,d\mu,\end{aligned}$$

and if $\xi\in\mathbf{R}^n$, then $t^nG\mathcal{D}_G\xi\cdot\xi$ is equal to

$$\int_{\mathbf{R}^n}\frac{((y-x)\cdot\xi)^2}{4t^2}e^{-|x-y|^2/4t}\,d\mu\int_{\mathbf{R}^n}e^{-|x-y|^2/4t}\,d\mu-\left(\int_{\mathbf{R}^n}\frac{(y-x)\cdot\xi}{2t}e^{-|x-y|^2/4t}\,d\mu\right)^2,$$

which always remains positive due to the Cauchy–Schwarz inequality.

Proof. For $u\in C_0^\infty(\mathbf{R}^{n+1}_+)$ set

$$H(t)=\int_{\mathbf{R}^n}u^2G\,dx\quad\text{and}\quad D(t)=\int_{\mathbf{R}^n}|\nabla u|^2G\,dx.$$

The identities,

$$\begin{aligned}\partial_t(u^2G)&=2u(\Delta u+\partial_t u)G+2|\nabla u|^2G+u^2(\partial_t G-\Delta G)+[\operatorname{div}(u^2\nabla G)-\operatorname{div}(G\nabla u^2)],\\ \partial_t(|\nabla u|^2G)&=-2(\Delta u+\partial_t u)\partial_t uG+2(\partial_t u)^2G-2\nabla u\cdot\nabla G\partial_t u+|\nabla u|^2(\partial_t G-\Delta G)\\ &\quad+|\nabla u|^2\Delta G+2\operatorname{div}((\partial_t uG)\nabla u)+\partial_t g^{ij}\partial_i u\partial_j uG,\end{aligned}$$

imply, respectively, together with the divergence theorem that

$$\dot H(t)=2\int_{\mathbf{R}^n}u(\Delta u+\partial_t u)G\,dx+2D(t)+\int_{\mathbf{R}^n}u^2(\partial_t G-\Delta G)\,dx,\tag{2.1}$$

and

$$\begin{aligned}\dot D(t)&=-2\int_{\mathbf{R}^n}(\Delta u+\partial_t u)\partial_t uG\,dx+2\int_{\mathbf{R}^n}[(\partial_t u)^2G-\nabla u\cdot\nabla G\partial_t u]\,dx\\ &\quad+\int_{\mathbf{R}^n}|\nabla u|^2(\partial_t G-\Delta G)\,dx+\int_{\mathbf{R}^n}|\nabla u|^2\Delta G\,dx+\int_{\mathbf{R}^n}\partial_t g^{ij}\partial_i u\partial_j uG\,dx.\end{aligned}\tag{2.2}$$

The following Rellich–Nečas identity

$$\begin{aligned}\operatorname{div}[(\nabla G)|\nabla u|^2]-2\operatorname{div}[(\nabla u\cdot\nabla G)\nabla u] &= |\nabla u|^2\Delta G-2\nabla G\cdot\nabla u\Delta u\\ &\quad+g^{kl}\partial_l G\partial_k g^{ij}\partial_i u\partial_j u-2\partial_k g^{il}\partial_l G g^{kj}\partial_i u\partial_j u\\ &\quad-2g^{il}\partial_{kl}G g^{kj}\partial_i u\partial_j u,\end{aligned}$$

and the divergence theorem gives that

$$\begin{aligned}\int_{\mathbf{R}^n}|\nabla u|^2\Delta G\,dx &= 2\int_{\mathbf{R}^n}\nabla G\cdot\nabla u(\Delta u+\partial_t u)\,dx-2\int_{\mathbf{R}^n}\nabla G\cdot\nabla u\partial_t u\,dx\\ &\quad+2\int_{\mathbf{R}^n}\mathcal{M}^{ij}\partial_i u\partial_j u\,dx,\end{aligned}$$

where

$$\mathcal{M}^{ij}=g^{il}\partial_{kl}G g^{kj}+\tfrac{1}{2}\partial_k g^{il}\partial_l G g^{kj}+\tfrac{1}{2}\partial_k g^{jl}\partial_l G g^{ki}-\tfrac{1}{2}g^{kl}\partial_l G\partial_k g^{ij}+\tfrac{1}{2}\partial_t g^{ij}G,$$

and substituting the last identity for the fourth term on the right-hand side of (2.2) it follows that

$$\begin{aligned}\dot D(t) &= -2\int_{\mathbf{R}^n}(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dx+\int_{\mathbf{R}^n}|\nabla u|^2(\partial_t G-\Delta G)\,dx\\ &\quad+2\int_{\mathbf{R}^n}[(\partial_t u)^2-2\nabla\log G\cdot\nabla u\partial_t u]G\,dx+2\int_{\mathbf{R}^n}\mathcal{M}^{ij}\partial_i u\partial_j u\,dx.\end{aligned}\tag{2.3}$$

Next we do the following, first we complete the square in the third integral on the right-hand side of (2.3) by adding and subtracting the integral

$$2\int_{\mathbf{R}^n}(\nabla\log G\cdot\nabla u)^2 G\,dx,$$

and then we subtract and add the term $D(t)/t$ on the right-hand side of (2.3), obtaining the formula

$$\begin{aligned}\dot D(t) &= -2\int_{\mathbf{R}^n}(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dx+\int_{\mathbf{R}^n}|\nabla u|^2(\partial_t G-\Delta G)\,dx\\ &\quad+2\int_{\mathbf{R}^n}(\partial_t u-\nabla\log G\cdot\nabla u)^2 G\,dx-\frac{D(t)}{t}+2\int_{\mathbf{R}^n}\mathcal{J}^{ij}\partial_i u\partial_j u\,dx,\end{aligned}\tag{2.4}$$

where

$$\mathcal{J}^{ij}=\frac{g^{ij}}{2t}G-\frac{g^{ik}\partial_k G g^{jl}\partial_l G}{G}+\mathcal{M}^{ij}\quad\text{for } i,j=1,\dots,n,$$

is the ij-entry of the $n\times n$ matrix $\mathcal{J}$ defined in Lemma 1. Then, defining $\mathcal{D}_G$ as in Lemma 1 (i.e. $\mathcal{D}_G=\mathcal{J}g$), it follows from (2.4) and the definition of the $\cdot$ inner product that the following identity holds

$$
\begin{aligned}
\dot{D}(t) = & -2\int_{\mathbf{R}^n}(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dx+\int_{\mathbf{R}^n}|\nabla u|^2(\partial_t G-\Delta G)\,dx \\
& +2\int_{\mathbf{R}^n}(\partial_t u-\nabla\log G\cdot\nabla u)^2 G\,dx-\frac{D(t)}{t}+2\int_{\mathbf{R}^n}\mathcal{D}_G\nabla u\cdot\nabla u\,dx.
\end{aligned}
\tag{2.5}
$$

Now, given $\alpha\in\mathbf{R}$ and $\sigma=\sigma(t)$ rewrite the term $\partial_t u-\nabla\log G\cdot\nabla u$ appearing in the third integral on the right-hand side of (2.5) as

$$
\partial_t u-\nabla\log G\cdot\nabla u=\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot{\sigma}}{2\sigma}u\Big)+\frac{\alpha\dot{\sigma}}{2\sigma}u,
$$

and expand the square in the same integral. These two calculations yield the identity

$$
\begin{aligned}
\dot{D}(t) = & -2\int_{\mathbf{R}^n}(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dx+\int_{\mathbf{R}^n}|\nabla u|^2(\partial_t G-\Delta G)\,dx \\
& +2\int_{\mathbf{R}^n}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot{\sigma}}{2\sigma}u\Big)^2 G\,dx \\
& +\frac{2\alpha\dot{\sigma}}{\sigma}\int_{\mathbf{R}^n}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot{\sigma}}{2\sigma}u\Big)uG\,dx \\
& +\frac{\alpha^2\dot{\sigma}^2}{2\sigma^2}H(t)-\frac{D(t)}{t}+2\int_{\mathbf{R}^n}\mathcal{D}_G\nabla u\cdot\nabla u\,dx.
\end{aligned}
\tag{2.6}
$$

On the other hand, using calculus and integration by parts we have the following facts

$$
\frac{d}{dt}\log\frac{\sigma}{\dot{\sigma}t}=\frac{\dot{\sigma}}{\sigma}-\frac{1}{t}-\frac{\ddot{\sigma}}{\dot{\sigma}},
\tag{2.7}
$$

$$
\int_0^\infty\frac{\sigma^{1-\alpha}}{\dot{\sigma}}\dot{D}(t)\,dt=\int_0^\infty\frac{1}{\sigma^\alpha}\Big(\alpha+\frac{\sigma\ddot{\sigma}}{\dot{\sigma}^2}-1\Big)D(t)\,dt,
\tag{2.8}
$$

$$
2\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot{\sigma}}{2\sigma}u\Big)uG\,dX=-\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2(\partial_t G-\Delta G)\,dX.
\tag{2.9}
$$

Then, multiplying the formula (2.6) by $\sigma^{1-\alpha}/\dot{\sigma}$ and integrating the outcome over $(0,+\infty)$ with respect to dt, and using the identities (2.8) and (2.9), respectively, in the terms arising after the multiplication by $\sigma^{1-\alpha}/\dot\sigma$ on the left- and right-hand sides of (2.6) (the fourth term on the right-hand side), it follows from (2.7) that

$$
\begin{aligned}
2\int_{\mathbf{R}^{n+1}_+} &\frac{\sigma^{1-\alpha}}{\dot\sigma}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot\sigma}{2\sigma}u\Big)^2 G\,dX\\
&+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\,\cdot}|\nabla u|^2G\,dX+2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\mathcal{D}_G\nabla u\cdot\nabla u\,dX\\
&=\alpha\int_0^\infty\frac{1}{\sigma^\alpha}D(t)\,dt-\frac{\alpha^2}{2}\int_0^\infty\frac{1}{\sigma^{\alpha+1}}\dot\sigma H(t)\,dt\\
&+2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dX\\
&+\alpha\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2(\partial_t G-\Delta G)\,dX-\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}|\nabla u|^2(\partial_t G-\Delta G)\,dX.
\end{aligned}\tag{2.10}
$$

On the other hand,

$$
\int_0^\infty\frac{1}{\sigma^\alpha}\dot H(t)\,dt=\alpha\int_0^\infty\frac{1}{\sigma^{1+\alpha}}\dot\sigma H(t)\,dt,\tag{2.11}
$$

and multiplying (2.1) by $\frac12\alpha\sigma^{-\alpha}$, integrating the outcome with respect to dt over $(0,+\infty)$ and using (2.11) it follows that

$$
\begin{aligned}
\alpha\int_0^\infty\frac{1}{\sigma^\alpha}D(t)\,dt-\frac{\alpha^2}{2}\int_0^\infty\frac{1}{\sigma^{\alpha+1}}\dot\sigma H(t)\,dt&=-\alpha\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u(\Delta u+\partial_t u)G\,dX\\
&\quad-\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2(\partial_t G-\Delta G)\,dX.
\end{aligned}
$$

Then, replacing the two first integrals on the right-hand side of (2.10) by the right-hand side of the previous identity we obtain the formula

$$
\begin{aligned}
2\int_{\mathbf{R}^{n+1}_+}&\frac{\sigma^{1-\alpha}}{\dot\sigma}\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot\sigma}{2\sigma}u\Big)^2G\,dX+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\,\cdot}|\nabla u|^2G\,dX\\
&=2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta u+\partial_t u)\Big(\partial_t u-\nabla\log G\cdot\nabla u-\frac{\alpha\dot\sigma}{2\sigma}u\Big)G\,dX\\
&+\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2(\partial_t G-\Delta G)\,dX-\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}|\nabla u|^2(\partial_t G-\Delta G)\,dX\\
&-2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\mathcal{D}_G\nabla u\cdot\nabla u\,dX.
\end{aligned}\tag{2.12}
$$

Now, if $F = F(x,t)$ rewrite the term

$$\partial_t u - \nabla \log G \cdot \nabla u - \frac{\alpha\dot{\sigma}}{2\sigma} u$$

on the left-hand side of (2.12) as

$$\left(\partial_t u - \nabla \log G \cdot \nabla u + \frac{1}{2} F u - \frac{\alpha\dot{\sigma}}{2\sigma} u\right) - \frac{1}{2} F u,$$

and after expanding the corresponding square, the left-hand side of (2.12) is equal to

$$\begin{aligned}
&2\int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} \left(\partial_t u - \nabla \log G \cdot \nabla u + \frac{1}{2} F u - \frac{\alpha\dot{\sigma}}{2\sigma} u\right)^2 G\, dX \\
&\quad + \int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} \widehat{\log \frac{\sigma}{\dot{\sigma} t}}^{\,\cdot} |\nabla u|^2 G\, dX + \frac{1}{2} \int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} u^2 F^2 G\, dX \\
&\quad - 2\int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} \left(\partial_t u - \nabla \log G \cdot \nabla u + \frac{1}{2} F u - \frac{\alpha\dot{\sigma}}{2\sigma} u\right) u F G\, dX.
\end{aligned} \tag{2.13}$$

Proceeding in the same way with the term $\partial_t u - \nabla \log G \cdot \nabla u - \alpha\dot{\sigma}/2\sigma u$ appearing in the first integral on the right-hand side of (2.12) and rewriting the two terms $\partial_t G - \Delta G$ in the second and third integrals in the same right-hand side as

$$(\partial_t G - \Delta G - FG) + FG,$$

it follows from the identity $\frac{1}{2}(\Delta + \partial_t)(u^2) = u(\Delta u + \partial_t u) + |\nabla u|^2$ that this right-hand side is equal to

$$\begin{aligned}
&2\int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} (\Delta u + \partial_t u) \left(\partial_t u - \nabla \log G \cdot \nabla u + \frac{1}{2} F u - \frac{\alpha\dot{\sigma}}{2\sigma} u\right) G\, dX \\
&\quad + \frac{\alpha}{2} \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} u^2 (\partial_t G - \Delta G - FG)\, dX \\
&\quad - \int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} |\nabla u|^2 (\partial_t G - \Delta G - FG)\, dX \\
&\quad + \frac{\alpha}{2} \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} u^2 F G\, dX - \frac{1}{2} \int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} (\Delta + \partial_t)(u^2) F G\, dX \\
&\quad - 2\int_{\mathbf{R}^{n+1}_+} \frac{\sigma^{1-\alpha}}{\dot{\sigma}} \mathcal{D}_G \nabla u \cdot \nabla u\, dX
\end{aligned} \tag{2.14}$$

and from (2.12), (2.13) and (2.14) we have,

$$\begin{aligned}
&2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\left(\partial_t u-\nabla\log G\cdot\nabla u+\frac{1}{2}Fu-\frac{\alpha\dot\sigma}{2\sigma}u\right)^2 G\,dX\\
&\quad+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\,\cdot}|\nabla u|^2 G\,dX\\
&=2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta u+\partial_t u)\left(\partial_t u-\nabla\log G\cdot\nabla u+\frac{1}{2}Fu-\frac{\alpha\dot\sigma}{2\sigma}u\right)G\,dX\\
&\quad+I+II+\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2(\partial_t G-\Delta G-FG)\,dX\\
&\quad-\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}|\nabla u|^2(\partial_t G-\Delta G-FG)\,dX-2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\mathcal{D}_G\nabla u\cdot\nabla u\,dX,
\end{aligned}\tag{2.15}$$

where

$$\begin{aligned}
I=&-\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2F^2G\,dX+\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2FG\,dX\\
&-\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta+\partial_t)(u^2)FG\,dX
\end{aligned}$$

and

$$II=2\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}\left(\partial_t u-\nabla\log G\cdot\nabla u+\frac{1}{2}Fu-\frac{\alpha\dot\sigma}{2\sigma}u\right)uFG\,dX.$$

In the final application of these formulae, F will be a function which can be differentiated only one time, for this reason we integrate by parts the operator $P=\Delta+\partial_t$ which is acting over u^2 in the third integral of I over FG, but only using one derivative with respect to the space variables of F. In particular,

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}(\Delta+\partial_t)(u^2)FG\,dX\\
&\quad=\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}u^2\partial_t\left(\frac{\sigma^{1-\alpha}}{\dot\sigma}FG\right)dX+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u\nabla u\cdot\nabla FG\,dX\\
&\quad-\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2\operatorname{div}(\nabla GF)\,dX\\
&\quad=-\frac{\alpha}{2}\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}u^2FG\,dX+\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u\nabla u\cdot\nabla FG\,dX\\
&\quad+\frac{1}{2}\int_{\mathbf{R}^{n+1}_+}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2\left(\widehat{\log\frac{\sigma}{\dot\sigma}}^{\,\cdot}FG+\partial_tFG+F(\partial_tG-\Delta G)-\nabla G\cdot\nabla F\right)dX,
\end{aligned}$$

and replacing the right-hand side of this identity by the third integral in the definition of I it follows that

$$I=\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u\nabla u\cdot\nabla FG\,dX+\frac{1}{2}\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2M\,dX, \tag{2.16}$$

where M is given in Lemma 1.

Now,

$$\begin{aligned}II=&\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}[\partial_t(u^2)FG-\nabla G\cdot\nabla(u^2)F]\,dX\\&+\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2F^2G\,dX-\alpha\int_{\mathbf{R}_+^{n+1}}\frac{1}{\sigma^\alpha}u^2FG\,dX\end{aligned}$$

and integrating by parts all the derivatives acting over u^2 in the first integral,

$$II=-\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2M\,dX.$$

We obtain from (2.16) that

$$I+II=\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u\nabla u\cdot\nabla FG\,dX-\frac{1}{2}\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}u^2M\,dX.$$

Finally, plugging the last identity into (2.15) yields the formula in Lemma 1. □

Lemma 2. *Assume that σ and G are as before. Then, the following identity holds when $u\in C_0^\infty(\mathbf{R}_+^{n+1})$ and $\alpha\in\mathbf{R}$,*

$$\begin{aligned}(\alpha-1)\int_{\mathbf{R}_+^{n+1}}\frac{1}{\sigma^\alpha}\widehat{\log\frac{\sigma}{\dot\sigma t}}\,u^2G\,dX=&\,2\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}[u(\Delta u+\partial_t u)+|\nabla u|^2]G\,dX\\&+\int_{\mathbf{R}_+^{n+1}}\frac{\sigma^{1-\alpha}}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}u^2(\partial_tG-\Delta G)\,dX\\&+\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}\partial_t\left(\frac{1}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}\right)u^2G\,dX.\end{aligned}$$

Proof. This follows upon multiplying the identity

$$(\Delta+\partial_t)u^2=2u(\Delta u+\partial_t u)+2|\nabla u|^2$$

by $(\sigma^{1-\alpha}/\dot\sigma)\widehat{\log(\sigma/\dot\sigma t)}G$ and integrating by parts the operator $\Delta+\partial_t$ acting on u^2 over the other terms in the corresponding integral. □

Lemma 3. *Given $m>0$ there is a constant C_m such that for all $y\geq 0$ and $0<\varepsilon<1$,*

$$y^m e^{-y}\leq C_m\left[\varepsilon+\left(\log\frac{1}{\varepsilon}\right)^m e^{-y}\right].$$

Proof. The maximum value of the function $g(y)=y-\varepsilon e^y$ on $\mathbf{R}_+$ is bounded by $\log(1/\varepsilon)$, proving the case $m=1$. The other cases follow from this, the convexity of y^m when $m>1$, and the fact that $(a+b)^m\leq a^m+b^m$ when $m<1$. □

Lemma 4. *Assume that $\theta\colon(0,1)\to\mathbf{R}_+$ satisfies*

$$0\leq\theta\leq N,\quad |t\dot\theta(t)|\leq N\theta(t)\quad\text{and}\quad\int_0^1\left(1+\log\frac{1}{t}\right)\frac{\theta(t)}{t}\,dt\leq N$$

for some constant N. Then, the solution to the ordinary differential equation

$$\frac{d}{dt}\log\left(\frac{\sigma}{t\dot\sigma}\right)=\frac{\theta(\gamma t)}{t},\quad\sigma(0)=0,\ \dot\sigma(0)=1,$$

where $\gamma>0$, has the following properties when $0\leq\gamma t\leq 1$,

$$\begin{aligned}te^{-N}&\leq\sigma(t)\leq t,\\ e^{-N}&\leq\dot\sigma(t)\leq 1,\\ \left|\sigma\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\,\cdot}\right|+\left|\sigma\widehat{\log\frac{\sigma}{\dot\sigma}}^{\,\cdot}\right|&\leq 3N,\\ \left|\sigma\partial_t\left(\frac{1}{\dot\sigma}\widehat{\log\frac{\sigma}{\dot\sigma t}}^{\,\cdot}\right)\right|&\leq 3Ne^N\frac{\theta(\gamma t)}{t}.\end{aligned}$$

Proof. The solution of the ordinary differential equation is

$$\sigma(t)=t\exp\left[-\int_0^{\gamma t}\left(1-\exp\left(-\int_0^s\frac{\theta(u)}{u}\,du\right)\right)\frac{ds}{s}\right],$$

and the verification of the properties is straightforward. □

From now on $0<\delta<1$ denotes a small number to be chosen later, and α and β two numbers satisfying $\alpha\geq 1$ and $0<\beta\leq 1$.

Lemma 5. *Let $G(x,t)=t^{-n/2}e^{-|x|^2/4t}$ and σ denote the function defined in Lemma 4 for $\gamma=\alpha/\delta^2$ and*

$$\theta(t)=t^{\beta/2}\left(\log\frac{1}{t}\right)^{1+\beta/2}.$$

Then, there is a constant N depending on β and n such that the following inequalities hold for all functions $u\in C_0^\infty(\mathbf{R}^n\times[0,1/2\gamma))$,

$$\int_{\mathbf{R}_+^{n+1}} \frac{1}{\sigma^\alpha}u^2\left(\frac{|x|^\beta}{t}+\frac{|x|^{2+\beta}}{\alpha t^2}+t^{\beta/2-1}\right)G\,dX\le Ne^{N\alpha}\gamma^{\alpha+N}\int_{\mathbf{R}_+^{n+1}} u^2\,dX + N\delta^\beta\int_{\mathbf{R}_+^{n+1}}\frac{1}{\sigma^\alpha}\frac{\theta(\gamma t)}{t}u^2G\,dX,$$

$$\int_{\mathbf{R}_+^{n+1}} \sigma^{1-\alpha}|\nabla u|^2\left(\frac{|x|^\beta}{t}+\frac{|x|^{2+\beta}}{\alpha t^2}+\frac{|x|^{1+\beta}}{t\delta}+t^{\beta/2-1}\right)G\,dX \le Ne^{N\alpha}\gamma^{\alpha+N}\int_{\mathbf{R}_+^{n+1}} t|\nabla u|^2\,dX+N\delta^\beta\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX.$$

Proof. From Lemma 3 with $m=\frac12\beta$ and taking $\varepsilon=(\gamma t)^{n/2+1-\beta/2+\alpha}$ and $y=|x|^2/4t$ we have that for $x\in\mathbf{R}^n$ and $0<2\gamma t<1$,

$$\begin{aligned}\frac{|x|^\beta}{t}G&=2^\beta t^{\beta/2-1-n/2}\left(\frac{|x|^2}{4t}\right)^{\beta/2}e^{-|x|^2/4t}\\&\le N\left(\gamma^{n/2+1-\beta/2+\alpha}t^\alpha+\delta^\beta\left(\gamma t\log\frac{1}{\gamma t}\right)^{\beta/2}\frac{1}{t}G\right).\end{aligned}$$

Thus,

$$\frac{|x|^\beta}{t}G\lesssim\gamma^{\alpha+N}t^\alpha+\delta^\beta\frac{\theta(\gamma t)}{t}G. \tag{2.17}$$

Using again Lemma 3 with $m=1+\frac12\beta$ and the same value of ε,

$$\frac{|x|^{2+\beta}}{t^2}G\le N\left(\gamma^{n/2+1-\beta/2+\alpha}t^\alpha+\left(\alpha\log\frac{1}{\gamma t}\right)^{1+\beta/2}t^{\beta/2-1}G\right),$$

and in particular

$$\frac{|x|^{2+\beta}}{t^2}G\lesssim\gamma^{\alpha+N}t^\alpha+\alpha\delta^\beta\frac{\theta(\gamma t)}{t}G. \tag{2.18}$$

Multiplying (2.17) and (2.18) by $\sigma^{-\alpha}u^2$ and recalling that $\sigma(t)\ge e^{-N}t$ when $0<\gamma t<1$, the first inequality follows. The proof of the second inequality is similar. □

Lemma 6. *Let D be as in Theorem 2 and $0<\beta<1$. Then, there is a constant N depending on β, n and $\|\nabla\varphi\|_\infty$ such that the inequality*

$$\int_D |x|^{\beta-2} f^2\,dx \le N \int_D |x|^\beta |\nabla f|^2\,dx$$

holds for all $f\in C_0^\infty(\overline{D})$.

Proof. When $D=\{x:x_n>0\}$ the lemma follows from the identity $\operatorname{div}(x|x|^{\beta-2})=(n+\beta-2)|x|^{\beta-2}$ and the observation that the boundary term arising from an application of the divergence theorem is identically zero. In general, flattening the boundary of D using the change of variables $y'=x'$, $y_n=x_n+\varphi(x')$ and undoing the change of variables we find that

$$\int_D [|x'|^2+(x_n-\varphi(x'))^2]^{(\beta-2)/2} f^2\,dx \le N \int_D [|x'|^2+(x_n-\varphi(x'))^2]^{\beta/2}|\nabla f|^2\,dx,$$

and the lemma follows because $\sqrt{|x'|^2+(x_n-\varphi(x'))^2}\le N|x|$ in D and $0<\beta<1$. □

3. In the interior

To prove Theorem 1 we use the following Carleman inequality.

Theorem 4. *Define G, θ and σ as in Lemma 5 and take $\beta=1$ when the operator P satisfies the first condition in Theorem 1. Then, there are numbers δ_0 and N depending on λ, M, n and β such that if $\alpha\ge2$, $\gamma=\alpha/\delta^2$ and $\delta\le\delta_0$, the following inequality holds for all $u\in C_0^\infty(\mathbf{R}^n\times(0,1/2\gamma))$,*

$$\alpha\int_{\mathbf{R}^{n+1}_+}\frac{1}{\sigma^\alpha}\frac{\theta(\gamma t)}{t}u^2G\,dX+\int_{\mathbf{R}^{n+1}_+}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX\le N\int_{\mathbf{R}^{n+1}_+}\sigma^{1-\alpha}|Pu|^2G\,dX$$
$$+e^{N\alpha}\gamma^{\alpha+N}\int_{\mathbf{R}^{n+1}_+}(u^2+t|\nabla u|^2)\,dX.$$

Proof. We begin by assuming that the coefficient matrix of P satisfies the second condition in Theorem 1 for some $0<\beta\le1$. Without loss of generality we may assume that $a(0,0)=\mathcal{I}$. Defining $r(x)=|x|$ and $F=r^2(1-|\nabla r|^2)/4t^2$, a calculation shows that

$$\partial_t G-\Delta G=\left(\frac{r^2(1-|\nabla r|^2)}{4t^2}+\frac{|\nabla r|^2-1}{2t}+\frac{r\Delta r-(n-1)}{2t}\right)G$$

and

$$|F| \lesssim \min\left\{\frac{|x|^2}{t^2}, \frac{(|x|^2+t)^{1+\beta/2}}{t^2}\right\}, \quad |\partial_t G - \Delta G - FG| \lesssim \frac{(|x|^2+t)^{\beta/2}}{t} G. \tag{3.1}$$

Fixing $\alpha \geq 2$ and with $\gamma = \alpha/\delta^2$, θ being defined as in Lemma 5 and σ denoting the corresponding solution in Lemma 4, we have from the identity in Lemma 2, (3.1), the bounds for σ in Lemma 4 and the first inequality in Lemma 5 that for $u \in C_0^\infty(\mathbf{R}^n \times (0, 1/2\gamma))$,

$$\begin{aligned} \alpha \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} \frac{\theta(\gamma t)}{t} u^2 G\, dX \lesssim{}& \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} \frac{\theta(\gamma t)}{t} |\nabla u|^2 G\, dX + \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} |Pu|^2 G\, dX \\ &+ e^{N\alpha} \gamma^{\alpha+N} \int_{\mathbf{R}^{n+1}_+} u^2\, dX + \alpha\delta^3 \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} \frac{\theta(\gamma t)}{t} u^2 G\, dX, \end{aligned}$$

and choosing δ sufficiently small

$$\begin{aligned} \alpha \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} \frac{\theta(\gamma t)}{t} u^2 G\, dX \lesssim{}& \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} \frac{\theta(\gamma t)}{t} |\nabla u|^2 G\, dX \\ &+ \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} |Pu|^2 G\, dX + e^{N\alpha} \gamma^{\alpha+N} \int_{\mathbf{R}^{n+1}_+} u^2\, dX. \end{aligned} \tag{3.2}$$

Because,

$$\frac{\delta_{ij}}{2t} G + \partial_{ij} G - \frac{\partial_i G \partial_j G}{G} = 0 \quad \text{for all } i, j = 1, \dots, n$$

and $|\nabla G| \lesssim |x| G/t$ it is simple to verify that

$$|\mathcal{D}_G \nabla u \cdot \nabla u| \lesssim \frac{(|x|^2+t)^{\beta/2}}{t} |\nabla u|^2 G, \tag{3.3}$$

and from Lemma 4 and (3.1),

$$|\sigma \nabla F| \lesssim \frac{(|x|^2+t)^{(1+\beta)/2}}{t} \quad \text{and} \quad |\sigma M| \lesssim \frac{(|x|^2+t)^{1+\beta/2}}{t^2} G. \tag{3.4}$$

Now, using the Cauchy–Schwarz inequality to handle the first term on the right-hand side of the identity in Lemma 1, it follows from the identity in Lemma 1, the bounds (3.1), (3.3), (3.4) and Lemma 5, that

$$\begin{aligned} \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} \frac{\theta(\gamma t)}{t} |\nabla u|^2 G\, dX \lesssim{}& \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} |Pu|^2 G\, dX + \alpha\delta^3 \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} \frac{\theta(\gamma t)}{t} u^2 G\, dX \\ &+ \delta^\beta \int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha} \frac{\theta(\gamma t)}{t} |\nabla u|^2 G\, dX \\ &+ \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha} |u|\, |\nabla u| \frac{(|x|^2+t)^{(1+\beta)/2}}{t} G\, dX \\ &+ e^{N\alpha} \gamma^{\alpha+N} \int_{\mathbf{R}^{n+1}_+} (u^2 + t|\nabla u|^2)\, dX. \end{aligned} \tag{3.5}$$

Using again the Cauchy–Schwarz inequality, we have

$$\int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha}|u|\,|\nabla u|\frac{(|x|^2+t)^{(1+\beta)/2}}{t}G\,dX \lesssim \int_{\mathbf{R}^{n+1}_+} \frac{1}{\sigma^\alpha}|u|^2\frac{(|x|^2+t)^{1+\beta/2}}{t^2}G\,dX$$
$$+\int_{\mathbf{R}^{n+1}_+} \sigma^{1-\alpha}|\nabla u|^2\frac{(|x|^2+t)^{\beta/2}}{t}G\,dX, \tag{3.6}$$

and thus (3.2), (3.5), (3.6) and Lemma 5 imply that if δ is sufficiently small the inequality in Theorem 4 holds for all functions $u\in C_0^\infty(\mathbf{R}^n\times(0,1/2\gamma))$.

When the operator P satisfies the first condition in Theorem 1, the operator

$$Qu=\sum_{i,j=1}^n \partial_i(a^{ij}(x,0)\partial_j u)+\partial_t u$$

satisfies the second condition in Theorem 1 with $\beta=1$. Thus, the inequality in Theorem 4 holds if we replace P by Q and consider this value of β in the definition of θ and σ. On the other hand,

$$Q((\partial_k u)^2)=2\partial_k(\partial_k u Q(u))+2\partial_i(\partial_k u\partial_k a^{ij}\partial_j u)-2\partial_{kk}uQ(u)$$
$$-2\partial_k a^{ij}\partial_j u\partial_{ik}u+2a^{ij}\partial_{ik}u\partial_{jk}u,$$

for $k=1,\dots,n$. Then, multiplying this identity by $\sigma^{2-\alpha}G$ and using again partial integration we have that

$$\sum_{k=1}^n\int_{\mathbf{R}^{n+1}_+}\sigma^{2-\alpha}|\nabla\partial_k u|^2G\,dX \lesssim \alpha\int_{\mathbf{R}^{n+1}_+}\sigma^{1-\alpha}|\nabla u|^2G\,dX$$
$$+\int_{\mathbf{R}^{n+1}_+}\sigma^{2-\alpha}|\nabla u|^2|Q^*(G)|\,dX$$
$$+\int_{\mathbf{R}^{n+1}_+}\sigma^{2-\alpha}(|\nabla u|\,|\nabla G|\,|Q(u)|+|\nabla u|^2|\nabla G|)\,dX \tag{3.7}$$
$$+\int_{\mathbf{R}^{n+1}_+}\sigma^{2-\alpha}(|\nabla\partial_k u|\,|Q(u)|+|\nabla u|\,|\nabla\partial_k u|)G\,dX,$$

where $Q^*(G)=\sum_{i,j=1}^n\partial_i(a^{ij}(x,0)\partial_j G)-\partial_t G$. Using that $\gamma\lesssim\theta(\gamma t)/t$ to handle the first integral on the right-hand side of (3.7), that $\sigma\le 1/\gamma$ in the support of u and the bounds

$$|Q^*(G)|\lesssim\left(\frac{|x|^3}{t^2}+\frac{|x|}{t}\right)G,\quad |\nabla G|\lesssim\frac{|x|}{t}G$$

to handle the other terms, it is simple to derive from (3.7), the second inequality in Lemma 5 with $\beta=1$ and the Cauchy–Schwarz inequality that

$$\frac{1}{\delta^2}\int_{\mathbf{R}_+^{n+1}}\sigma^{2-\alpha}|D^2u|^2G\,dX\lesssim\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX$$
$$+\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}|Q(u)|^2G\,dX+e^{N\alpha}\gamma^{\alpha+N}\int_{\mathbf{R}_+^{n+1}}t|\nabla u|^2\,dX. \tag{3.8}$$

This inequality and the fact that the inequality in Theorem 4 holds for the operator Q give that

$$\frac{1}{\delta^2}\int_{\mathbf{R}_+^{n+1}}\sigma^{2-\alpha}|D^2u|^2G\,dX+\alpha\int_{\mathbf{R}_+^{n+1}}\frac{1}{\sigma^\alpha}\frac{\theta(\gamma t)}{t}u^2G\,dX$$
$$+\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX$$
$$\lesssim\int_{\mathbf{R}_+^{n+1}}\sigma^{1-\alpha}|Q(u)|^2G\,dX+e^{N\alpha}\gamma^{\alpha+N}\int_{\mathbf{R}_+^{n+1}}(u^2+t|\nabla u|^2)\,dX. \tag{3.9}$$

Finally, since $|a(x,t)-a(x,0)|\le M\sqrt{t}$, it is possible to replace P by Q in the first term on the right-hand side of the last inequality choosing $\delta>0$ sufficiently small and hiding the corresponding error terms on the left-hand side of the inequality, which proves Theorem 4. □

Proof of Theorem 1. Proceeding as in [5] and [6], if u satisfies the conditions in Theorem 1 in $B_2\times[0,2]$, given an integer $k\ge1$ we apply the inequality in Theorem 4 with $\alpha=2k$ to $u_\varepsilon=u\varphi_\varepsilon(t)\psi(x)$, where $\psi\in C_0^\infty(\mathbf{R}^n)$ and $\varphi_\varepsilon\in C_0^\infty(\mathbf{R})$ satisfy $\psi=1$ for $|x|\le1$, $\psi=0$ for $|x|\ge\frac{3}{2}$, $\varphi_\varepsilon=1$ when $\varepsilon\le t\le1/4\gamma$ and $\varphi_\varepsilon=0$ when $t\le\frac{1}{2}\varepsilon$ or $t\ge1/2\gamma$. Because $\delta^{-\beta}\lesssim\theta(\gamma t)/t$ when $0<2\gamma t<1$, choosing δ sufficiently small it is possible to hide in the standard way the term $M(|\nabla u_\varepsilon|+|u_\varepsilon|)$ arising on the right-hand side of the inequality

$$|Pu_\varepsilon|\le M(|\nabla u_\varepsilon|+|u_\varepsilon|)+M|u\varphi_\varepsilon\nabla\psi|+|u\varphi_\varepsilon\Delta\psi|+|u\psi\partial_t\varphi_\varepsilon|+|\varphi_\varepsilon\nabla u\cdot\nabla\psi|,$$

and since u satisfies (1.1) and $e^{-N}t\le\sigma\le t$ when $0<\gamma t<1$, after letting ε tend to zero it follows that

$$\|t^{-k}G^{1/2}u\|_{L^2(B_1\times(0,1/4\gamma))}\lesssim e^{Nk}k^k(\|t^{1/2}\nabla u\|_{L^2(B_{3/2}\times(0,1))}+\|u\|_{L^2(B_2\times(0,2))})$$
$$+e^{Nk}(\|t^{-k}G^{1/2}u\|_{L^2((B_2\setminus B_1)\times(0,1/2\gamma))}$$
$$+\|t^{-k}G^{1/2}u\|_{L^2(B_2\times(1/4\gamma,1/2\gamma))}). \tag{3.10}$$

On the other hand, $t^{-k}G^{1/2}\leq e^{Nk}k^k$ when either $t>0$ and $|x|\geq 1$ or $t\geq 1/4\gamma$. Also, from Stirling's formula $k^k\leq e^{Nk}k!$ for all $k\geq 1$, [2]. These two facts, (3.10) and standard estimates for subsolutions of parabolic inequalities imply that there is a constant N depending on n, λ, β and M such that u satisfies

$$\|t^{-k}uG^{1/2}\|_{L^2(B_1\times(0,2))}\leq \left(\tfrac{1}{4}N\right)^{1+k}k!\|u\|_{L^\infty(B_2\times(0,2))} \quad \text{for all } k\geq 0.$$

Then, multiplying this inequality by $2^k/N^kk!$ and summing over k,

$$\|e^{2/Nt}t^{-n/4}e^{-|x|^2/8t}u\|_{L^2(B_1\times(0,2))}\leq N\|u\|_{L^\infty(B_2\times(0,2))}.$$

Using this inequality and the observation that $1/Nt\geq |x|^2/8t-|x-y|^2/8t$ when $t>0$, $|x|\leq\frac{1}{2}$ and $|y|\leq 8/N$ one gets that

$$\|t^{-n/4}e^{-|x-y|^2/8t}u\|_{L^2(B_{1/2}\times(0,s))}\lesssim e^{-1/Ns}\|u\|_{L^\infty(B_2\times(0,2))} \tag{3.11}$$

when $|y|\leq 8/N$, $0<s<1$. Finally, standard estimates for subsolutions of parabolic inequalities [11] imply that with constants depending on λ, n and M,

$$|u(y,s)|\lesssim \frac{1}{s^{n/2+1}}\int_s^{2s}\int_{B_{\sqrt{s}}(y)}|u|\,dX \quad \text{when } |y|\leq\tfrac{1}{2} \text{ and } 0<s<\tfrac{1}{2}, \tag{3.12}$$

and from (3.11) and (3.12),

$$|u(x,t)|\leq Ne^{-1/Nt}\|u\|_{L^\infty(B_2\times(0,2))} \quad \text{when } |x|\leq\frac{8}{N} \text{ and } 0<t<1,$$

proving Theorem 1. □

Remark. If one replaces the condition (i) in Theorem 1 by

$$|a(x,0)-a(0,0)|\leq M|x|^\beta, \quad |\nabla a(x,0)|\leq M|x|^{\beta-1} \quad \text{and} \quad |a(x,t)-a(x,0)|\leq M\sqrt{t}$$

for some $0<\beta<1$, the result in Theorem 1 still holds. This follows because the inequality

$$\int_{\mathbf{R}^n}|x|^{l-2}f^2\,dx\lesssim\int_{\mathbf{R}^n}|x|^l|\nabla f|^2\,dx,$$

holds for all $l>0$ and $f\in C_0^\infty(\mathbf{R}^n)$ $(\operatorname{div}(x|x|^{l-2})=(n+l-2)|x|^{l-2})$. With this it is possible to handle the corresponding terms arising in (3.7), though in this case, (3.8) and (3.9) hold for functions $u\in C_0^\infty(B_{r_0}\times(0,1/2\gamma))$, where r_0 is sufficiently small depending on n, λ, M and β, and this suffices to prove the unique continuation property.

4. At the boundary

Proof of Theorem 2. Assume that $D=\{(x',x_n):x_n>\varphi(x')\}$, where $\varphi:\mathbf{R}^{n-1}\to\mathbf{R}$ is a $C^{1,1}$ function satisfying $\varphi(0)=0$ and $\nabla\varphi(0)=0$. If u satisfies (1.6) and (1.1) in $(B_2\cap D)\times[0,2]$ and either u or its conormal derivative is zero on $(B_2\cap\partial D)\times[0,2]$, then using standard methods it is possible to flatten the boundary of D by means of a $C^{1,1}$ change of variables, in such a way that after the transformation the composition, also denoted u, satisfies (1.1) and (1.6) in $(B_{r_0}\cap\mathbf{R}^n_+)\times[0,2]$ for some $r_0>0$ and with a new parabolic operator P_2 whose coefficient matrix $a(x,t)$ satisfies the first condition in Theorem 1 for $x,y\in B_{r_0}\cap\mathbf{R}^n_+$, $t\in[0,2]$ and

$$a^{in}(x',0,t)=0 \quad \text{for } i=1,\dots,n-1,\ |x'|\le r_0 \text{ and } t\ge 0. \tag{4.1}$$

Extending u for $x_n<0$ as an odd function in the x_n variable when u vanishes on the lateral boundary or as an even function when its conormal derivative is zero, one gets a function u satisfying (1.1) and (1.6) in a neighborhood of $(0,0)$ in $\mathbf{R}^{n+1}$ with a new parabolic operator P_3, which due to (4.1) satisfies the first condition in Theorem 1. These standard arguments reduce the proof of the first case in Theorem 2 to Theorem 1.

When D is a Lipschitz domain in $\mathbf{R}^n$ and if we carry out the calculations in the proof in Lemma 1 over $D\times[0,+\infty)$ with a function $u\in C^\infty(\bar{D}\times(0,+\infty))$ satisfying $u=0$ on $\partial D\times[0,+\infty)$, one gets exactly the same identity except for a boundary term arising on the right-hand side and given by

$$\int_{\partial D\times(0,+\infty)}\frac{\sigma^{1-\alpha}}{\dot\sigma}\nabla G\cdot N|\nabla u|^2\,dS, \tag{4.2}$$

where n denotes the unit exterior normal to ∂D, $N=g^{-1}(x,t)n$ and $dS=d\sigma\,dt$. This is because $u=0$ on ∂D and the fact that from all the integration by parts carried out in the proof of Lemma 1, there is only one which generates a nonzero boundary term. This occurs in applying the Rellich–Nečas identity in order to find the value of $\dot D(t)$ (see (2.2), (2.3) and (2.4)). In particular, in this case

$$\begin{aligned}\dot D(t)&=-2\int_D(\Delta u+\partial_t u)(\partial_t u-\nabla\log G\cdot\nabla u)G\,dx+\int_D|\nabla u|^2(\partial_t G-\Delta G)\,dx\\&\quad+2\int_D(\partial_t u-\nabla\log G\cdot\nabla u)^2G\,dx-\frac{D(t)}{t}\\&\quad+2\int_D\mathcal{D}_G\nabla u\cdot\nabla u\,dx-\int_{\partial D}\nabla G\cdot N|\nabla u|^2\,d\sigma.\end{aligned}$$

To derive this formula, we have used

$$(\nabla Gn)|\nabla u|^2-2\nabla G\cdot\nabla u(\nabla un)=-\nabla G\cdot N|\nabla u|^2$$

whenever $u(\cdot,t)\equiv 0$ on ∂D. Here $(\xi\eta)$ denotes the usual inner product on $\mathbf{R}^n$ of ξ with η, while the $\cdot$ product of two vector fields and the gradient of a function was defined at the beginning of Section 2.

In this case, Lemma 2 also remains invariant. For these reasons, to prove Theorem 2 when ∂D satisfies (1.9) it suffices to find a function G so that the bounds (3.1), (3.3) and (3.4) hold on $(B_{r_0}\cap D)\times[0,r_0^2]$ and $\nabla G\cdot N\leq 0$ on $(B_{r_0}\cap\partial D)\times[0,r_0^2]$ for some $r_0>0$. Here we take $G=t^{-n/2}e^{-r(x,t)^2/4t}$, where

$$r(x,t)=|x|-2(\Lambda+M)x_n(|x|^2+t)^{3/2},$$

and $F=r^2(1-|\nabla r|^2)/4t$, and with these choices it is easy to verify that provided r_0 is sufficiently small this G satisfies the previous requirements and

$$\frac{1}{N}|x|\leq r(x,t)\leq N|x| \quad \text{for all } (x,t)\in(B_{r_0}\cap D)\times[0,r_0^2].$$

It is also well known that the standard estimates for subsolutions to parabolic inequalities hold near the boundary when u has either zero Dirichlet or Neumann data on the lateral boundary [11]. In particular, if u satisfies (1.6) in $(B_2\cap D)\times[0,2]$ and either $u\equiv 0$ or $\nabla u\cdot N\equiv 0$ on $(B_2\cap\partial D)\times[0,2]$, then

$$|u(y,s)|\lesssim\frac{1}{s^{n/2+1}}\int_s^{2s}\int_{B_{\sqrt{s}}(y)\cap D}|u|\,dX \quad \text{when } (y,s)\in(B_{1/2}\cap D)\times\left(0,\tfrac{1}{2}\right),$$

and with a constant depending on λ, n, M and $\|\nabla\varphi\|_\infty$.

From the above discussion, it is clear that the same argument can be repeated again to obtain the second case in Theorem 2 when the boundary of D satisfies the condition (1.9) for some $\Lambda\geq 0$ and the coefficients of P satisfy the second condition in Theorem 1. Observe that under the condition (1.9) we cannot expect to control the second derivatives of u near the boundary when u has zero Dirichlet data on the lateral boundary, and this forces us to have to work out the proof with the full operator P when its coefficients depend on the time variable.

To prove Theorem 2 when the convexity condition (1.10) holds and in order to simplify and make the arguments more clear we assume that $a(x,0)\equiv\mathcal{I}$ ($Q=\Delta+\partial_t$ is the backward heat operator on $\mathbf{R}^n$). Under this assumption, (1.10) implies that with $r(x)=|x|-4\beta\Lambda x_n|x|^\beta$ and $G=t^{-n/2}e^{-r(x)^2/4t}$ we have

$$\nabla G\cdot N=-\frac{1}{2t}\frac{x'\cdot\nabla\varphi(x')-\varphi(x')}{\sqrt{1+|\nabla\varphi|^2}}G\lesssim-|x|^{1+\beta}\frac{1}{t}G \quad \text{on } (B_{r_0}\cap\partial D)\times[0,2],$$

and the arguments up to (4.2) imply that the following inequality holds with constants independent of $\alpha\geq 2$ and $\delta\leq\delta_0$ for all $u\in C_0^\infty((B_{r_0}\cap\bar{D})\times(0,1/2\gamma))$ satisfying $u=0$ on $(B_{r_0}\cap\partial D)\times(0,2)$,

$$\begin{aligned}
-\int_{\partial D_+}\sigma^{1-\alpha}\nabla G\cdot N|\nabla u|^2\,d\mathcal{S}+\alpha\int_{D_+}\frac{1}{\sigma^\alpha}\frac{\theta(\gamma t)}{t}u^2G\,dX+\int_{D_+}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX\\
\lesssim\int_{D_+}\sigma^{1-\alpha}|Q(u)|^2G\,dX+e^{N\alpha}\gamma^{\alpha+N}\int_{D_+}(u^2+t|\nabla u|^2)\,dX, \qquad (4.3)
\end{aligned}$$

where $\partial D_+=\partial D\times(0,+\infty)$ and $D_+=D\times(0,+\infty)$.

When the integration by parts carried out in the derivation of (3.7) is done over D with $u\in C_0^\infty((B_{r_0}\cap\bar{D})\times(0,1/2\gamma))$, $u\equiv 0$ on $\partial D\times[0,2]$ and G defined as above, one gets the following boundary terms on the right-hand side of (3.7),

$$-\int_{\partial D_+}\sigma^{2-\alpha}|\nabla u|^2\nabla G\cdot N\,d\mathcal{S}+\int_{\partial D_+}\sigma^{2-\alpha}[\nabla|\nabla u|^2\cdot N-2\nabla u\cdot NQ(u)]G\,d\mathcal{S}, \qquad (4.4)$$

where now the symbols $\cdot$ and ∇ denote, respectively, the usual product and gradient in $\mathbf{R}^n$. Observe that the first term in (4.4) can be controlled by $1/\gamma$ times the "good" boundary term appearing in (4.3), and from (3.7), (4.3) and (4.4) we get that with constants independent of $\alpha\geq 2$ and $\delta\leq\delta_0$ the following inequality holds for all $u\in C_0^\infty((B_{r_0}\cap\bar{D})\times(0,1/2\gamma))$, $u=0$ on $(B_{r_0}\cap\partial D)\times(0,2)$,

$$\begin{aligned}
\int_{D_+}\sigma^{2-\alpha}|D^2u|^2G\,dX\lesssim\delta^2\int_{D_+}&\sigma^{1-\alpha}|Q(u)|^2G\,dX\\
&+\delta^2e^{N\alpha}\gamma^{\alpha+N}\int_{D_+}(u^2+t|\nabla u|^2)\,dX\\
&+\int_{\partial D_+}\sigma^{2-\alpha}[\nabla|\nabla u|^2\cdot N-2\nabla u\cdot NQ(u)]G\,d\mathcal{S}. \qquad (4.5)
\end{aligned}$$

Well known calculations [8, Theorems 3.1.1.1, 3.1.1.2 and 3.1.2.1] show that the following identity holds when $u(\cdot,t)=0$ on ∂D,

$$\nabla|\nabla u|^2\cdot N-2\nabla u\cdot NQ(u)=-(\partial_n u)^2\left(\Delta\varphi-\frac{D^2\varphi\nabla\varphi\cdot\nabla\varphi}{1+|\nabla\varphi|^2}\right)\sqrt{1+|\nabla\varphi|^2}. \qquad (4.6)$$

The convexity condition (1.10) implies that $D^2\varphi\geq-\beta(1+\beta)\Lambda|x'|^{\beta-1}\mathcal{I}$, and since the matrix $\mathcal{I}-\nabla\varphi\otimes\nabla\varphi/(1+|\nabla\varphi|^2)$ is positive and the identity

$$\Delta\varphi-\frac{D^2\varphi\nabla\varphi\cdot\nabla\varphi}{1+|\nabla\varphi|^2}=\operatorname{trace}\left[D^2\varphi\left(\mathcal{I}-\frac{\nabla\varphi\otimes\nabla\varphi}{1+|\nabla\varphi|^2}\right)\right],$$

holds, it follows that

$$\int_{\partial D_+} \sigma^{2-\alpha}[\nabla|\nabla u|^2\cdot N-2\nabla u\cdot NQ(u)]G\,dS \lesssim \int_{\partial D_+} \sigma^{2-\alpha}|x|^{\beta-1}|\nabla u|^2G\,dS. \tag{4.7}$$

Integrating over D the Rellich–Nečas identity

$$\operatorname{div}(Y|\nabla u|^2)-2\operatorname{div}[(Y\cdot\nabla u)\nabla u]=|\nabla u|^2\operatorname{div}Y-2Y\cdot\nabla u\Delta u-2\partial_i y_j\partial_i u\partial_j u,$$

with vector field $Y=|x|^{\beta-1}Ge_n$ and using the Cauchy–Schwarz inequality we get

$$\begin{aligned}\int_{\partial D_+} \sigma^{2-\alpha}|x|^{\beta-1}|\nabla u|^2G\,dS \lesssim &\int_{D_+} \sigma^{2-\alpha}|x|^{\beta-2}|\nabla u|^2G\,dX\\ &+\int_{D_+} \sigma^{2-\alpha}\frac{|x|^{\beta}}{t}|\nabla u|^2G\,dX\\ &+\int_{D_+} \sigma^{2-\alpha}|x|^{\beta}|D^2u|G\,dX,\end{aligned}$$

and using Lemma 6 to handle the first term on the right-hand side of the previous inequality it follows that

$$\begin{aligned}\int_{\partial D_+} \sigma^{2-\alpha}|x|^{\beta-1}|\nabla u|^2G\,dS \lesssim &\int_{D_+} \sigma^{2-\alpha}|x|^{\beta}|D^2u|G\,dX\\ &+\int_{D_+} \sigma^{2-\alpha}\left(\frac{|x|^{\beta}}{t}+\frac{|x|^{2+\beta}}{t^2}\right)|\nabla u|^2G\,dX.\end{aligned} \tag{4.8}$$

The second inequality in Lemma 5 gives that the second term in the previous right-hand side is bounded by

$$\delta^2\int_{D_+} \sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX+\delta^2e^{N\alpha}\gamma^{\alpha+N}\int_{D_+} t|\nabla u|^2\,dX, \tag{4.9}$$

and from (4.3), (4.5), (4.7), (4.8) and (4.9) it follows that if r_0 is sufficiently small, there is a constant independent of $\alpha\geq 2$ and $\delta\leq\delta_0$ such that the inequality

$$\begin{aligned}\frac{1}{\delta^2}\int_{D_+} \sigma^{2-\alpha}|D^2u|^2G\,dX+\alpha\int_{D_+}\frac{1}{\sigma^{\alpha}}\frac{\theta(\gamma t)}{t}u^2G\,dX+\int_{D_+}\sigma^{1-\alpha}\frac{\theta(\gamma t)}{t}|\nabla u|^2G\,dX\\ \lesssim \int_{D_+}\sigma^{1-\alpha}|Q(u)|^2G\,dX+e^{N\alpha}\gamma^{\alpha+N}\int_{D_+}(u^2+t|\nabla u|^2)\,dX\end{aligned}$$

holds for all functions $u\in C_0^\infty((B_{r_0}\cap\overline{D})\times(0,1/2\gamma))$ satisfying $u=0$ on $(B_{r_0}\cap\partial D)\times(0,2)$.

In general, when Q is a backward parabolic operator with time-independent Lipschitz coefficients, the same calculations can be carried out [8, Theorem 3.1.3.1], and the analogous boundary terms to those appearing in (4.4) and arising in the calculation (3.7) can be handled in a similar way. These arguments finish the proof of Theorem 2. □

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