Ark. Mat., **38** (2000), 201–208
© 2000 by Institut Mittag-Leffler. All rights reserved

The pluripolar hull of $\{w=e^{-1/z}\}$

Jan Wiegerinck

Abstract. In this paper we show that the pluripolar hull of $E=\{(z,w)\in\mathbf{C}^2:w=e^{-1/z},\ z\neq0\}$ is equal to E. This implies that E is plurithin at 0, which answers a question of Sadullaev. The result remains valid if $e^{-1/z}$ is replaced by certain other holomorphic functions with an essential singularity at 0.

1. Introduction

In [5] Sadullaev poses the following question. Consider the sets

$$E_1=\{(x,y)\in\mathbf{C}^2:y=x^\alpha,\ x\in(0,1)\}$$

with α irrational, and

$$E_2=\{(x,y)\in\mathbf{C}^2:y=e^{-1/x},\ x\in(0,1)\}.$$

Are the sets E_j plurithin at the origin? That is, does there exist a plurisubharmonic function h on a neighborhood V of $\overline{E}_j$ such that

$$h(0)>\limsup_{\substack{x\to0\\ x\in E_j}} h(x).$$

See also Bedford's survey [1]. In [3] Levenberg and Poletsky show that E_1 is plurithin at 0. In fact, they show something stronger: there exists a negative plurisubharmonic function on the polydisk such that $h|_{E_1}=-\infty$, while $h(0)>-1$.

In this note we will prove the same result for E_2 and a related class of pluripolar sets. Our proof follows the line of argument of [3]. For convenience of the reader we summarize in the next section the relevant parts of [3], where all proofs may be found.

2. Preliminaries

2.1. Pluripolar hulls

A set $E \subset \mathbf{C}^N$ is called *pluripolar* if there exists a plurisubharmonic function $u \not\equiv -\infty$, defined on a neighborhood of E such that $E \subset \{z : u(z) = -\infty\}$.

There are two kinds of *pluripolar hulls* of a pluripolar set E relative to a neighborhood D of E. One defines

$$E_D^* = \bigcap_{u \in \mathcal{F}} \{z \in D : u(z) = -\infty\},$$

where $\mathcal{F}$ is the set of all plurisubharmonic functions on D which are $-\infty$ on E; next one defines

$$E_D^- = \bigcap_{u \in \mathcal{F}^-} \{z \in D : u(z) = -\infty\},$$

where $\mathcal{F}^-$ stands for the set of all negative plurisubharmonic functions on D which are $-\infty$ on E.

Of course, $E_D^* \subset E_D^-$. Moreover, these hulls are related as follows, see [3].

Theorem 1. *Let D be pseudoconvex in $\mathbf{C}^N$ and $E \subset D$ pluripolar. Suppose $D = \bigcup_{j=1}^{\infty} D_j$, where D_j form an increasing sequence of relatively compact subdomains of D. Then*

$$E_D^* = \bigcup_{j=1}^{\infty} (E \cap D_j)_{D_j}^-.$$

Moreover, if D is hyperconvex, that is, D admits a bounded plurisubharmonic exhaustion function, then

$$E_D^- = \bigcup_{j=1}^{\infty} (E \cap D_j)_{D_j}^-.$$

2.2. Harmonic measure

Let E be a subset of a domain $D \subset \mathbf{C}^n$. The *harmonic measure* at $z \in D$ of E relative to D is the number

$$\omega(z, E, D) = -\sup\{u(z) : u \text{ plurisubharmonic on } D \text{ and } u \le -\chi_E\}.$$

Here χ_E is the characteristic function of E on D.

Notice that this boils down to the usual concept of harmonic measure if $N=1$. The harmonic measure has the properties

$$\omega(z,E,D)=\inf_{\substack{V \text{open} \\ E\subset V\subset D}} \omega(z,V,D)$$

and

$$\omega(z,E_1\cup E_2,D)\leq \omega(z,E_1,D)+\omega(z,E_2,D). \tag{2.1}$$

Moreover, for open V there is the following important duality result due to Poletsky [4],

$$\omega(z,V,D)=\frac{1}{2\pi}\sup_f\{m(\{t\in(0,2\pi):f(e^{it})\in V\})\},$$

where f runs over all analytic disks $f:\{|\zeta|\leq 1\}\to D$ with $f(0)=z$, while m denotes Lebesgue measure.

The harmonic measure is connected to pluripolar hulls.

Proposition 2. *Let D be a hyperconvex domain in $\mathbf{C}^N$ and let $E\subset D$ be pluripolar. Then*

$$E_D^- = \{z\in D:\omega(z,E,D)>0\}.$$

To compute $\omega(z,E,D)$ we will use some more results from [3].

Lemma 3. *Let $D\subset\mathbf{C}^N$ be a domain, $E\subset D$ and let $A\subset D\setminus E$ be closed and pluripolar. Then for $z\in D\setminus A$ we have $\omega(z,E,D)=\omega(z,E,D\setminus A)$.*

Proposition 4. *Let $D\subset\mathbf{C}^N$ and $G\subset\mathbf{C}^M$ be domains and let $h:D\to G$ be holomorphic. Then for $z\in D$ and $E\subset G$*

$$\omega(z,h^{-1}(E),D)\leq \omega(h(z),E,G). \tag{2.2}$$

Moreover, equality holds in (2.2) if the following additional requirements are met: The map h is a covering map; E admits a simply connected open neighborhood V such that $h^{-1}(V)$ is a disjoint union of connected open V_j; $z\in D$ has the property

$$\lim_{j\to\infty}\omega\left(z,\bigcup_{k=j}^{\infty}V_k,D\right)=0. \tag{2.3}$$

Lemma 5. *Let D be a relatively compact subdomain of a domain $G\subset\mathbf{C}^N$. Suppose that $E\subset D$ is compact, $V\subset G\setminus E$ is a domain and $z\in V\cap D$. Put $K=\overline{\partial V\cap D}$. If $\omega(z,E,D)=a$ then $\omega(w,E,G)\geq a$ for some $w\in K$.*

3. Hulls of graphs at essential singularities

In this section we prove the main result. We start with a lemma which supplements Proposition 4 and is a variation of Lemma 3.2 in [2].

Lemma 6. *Let $D\subset\mathbf{C}^N$ and $G\subset\mathbf{C}^M$ be domains and let $h\colon D\to G$ be a finite branched holomorphic covering. Then for $z\in D$ and $E\subset G$,*

$$\omega(z,h^{-1}(E),D)=\omega(h(z),E,G). \tag{3.1}$$

Proof. Let $z\in D$. In view of Proposition 4, we only have to prove the inequality $\omega(z,h^{-1}(E),D)\geq\omega(h(z),E,G)$. Let $\varepsilon>0$ and let u be plurisubharmonic on D, $u\leq-\chi_{h^{-1}(E)}$ and $u(z)\geq-\omega(z,h^{-1}(E),D)-\varepsilon$. Define the function $\tilde u$ on G by

$$\tilde u(w)=\max_{z\in h^{-1}(w)} u(z).$$

Obviously, $\tilde u$ is plurisubharmonic outside the branch locus B of h. The set B is pluripolar and $\tilde u$ is bounded from above, therefore the function u^* defined by

$$u^*(w)=\limsup_{\substack{v\to w\\ v\notin B}}\tilde u(v)$$

is a plurisubharmonic extension of $\tilde u|_{G\setminus B}$ to G. Clearly $\tilde u=u^*$ on all of G. Moreover $\tilde u\leq-\chi_E$. Hence

$$\omega(h(z),E,G)\leq-\tilde u(h(z))\leq-u(z)\leq\omega(z,h^{-1}(E),D)+\varepsilon.$$

It follows that $\omega(z,h^{-1}(E),D)\geq\omega(h(z),E,G)$ and we are done. □

In order to keep close to [3], where a covering map in the first coordinate is used, we formulate our theorem with w as independent variable. The symbol M will denote a positive number or ∞. Let φ denote a branched cover of the Riemann sphere of the form

$$\varphi(z)=\alpha z+\frac{\beta}{z},$$

where α, $\beta\in\mathbf{C}$ are not both equal to 0.

Theorem 7. *Suppose that f is holomorphic on $\{|w|<M\}\subset\mathbf{C}$, $f(0)\neq0$ and let n be a positive integer. Let $E=\{(z,w)\in\mathbf{C}^2:z=\varphi(e^{f(w)/w^n}),\ 0<|w|<M\}$ and let Ω be the domain $\{(z,w)\in\mathbf{C}^2:|w|<M\}$. Then $E^*_\Omega=E$.*

Proof. In view of Theorem 1, it suffices to show that $(E\cap D)^-_D=E\cap D$ for every bidisk D compactly contained in Ω. We may assume that $M>1$ and that D equals

$kU\times U$, where U is the unit disk in $\mathbf{C}$ and $k\geq 1$. Observing that $E\cap D$ is never empty, let $\Delta=\{z:|z-a|\leq b\}\subset U$, with $b<|a|$, be contained in the image of U under the map $w\mapsto\varphi(e^{f(w)/w^n})$. Write $E^-=(E\cap D)^-_D$. Let η be so small that the set

$$F=\{(z,w):z=\varphi(e^{f(w)/w^n}),\ z\in\Delta,\ \eta\leq|w|\leq 1-\eta\}$$

is compact and contains a relatively open subset of the analytic variety E. Then we have $F^-_D=E^-$.

In view of Proposition 2, the proof is complete once we have shown that

$$\omega((z,w),F,D)>0 \iff (z,w)\in E\cap D.$$

Now the proof divides into two cases, namely $\alpha\cdot\beta=0$ and $\alpha\cdot\beta\neq 0$. In the first case we may assume $\alpha=1$, $\beta=0$ and proceed as follows.

First let $(z,w)\in D\setminus E$, $z\neq 0$. Let $A=D\cap\{z=0\}$. Then by Lemma 3

$$\omega((z,w),F,D)=\omega((z,w),F,D\setminus A). \tag{3.2}$$

Let $H=\{\zeta:\operatorname{Re}\zeta<0\}\subset\mathbf{C}$, $G=H\times U$ and $h:G\to D\setminus A$, $(\zeta,w)\mapsto(ke^{\zeta},w)$. The map h is a holomorphic covering map. Let also $T=\{z:|z-a|<b+\varepsilon\}\subset U\setminus\{0\}$, so that $|\arg z-\arg a|<\frac{1}{2}\pi$ on T. The set $V=T\times U$ is a simply connected neighborhood of F. We have

$$h^{-1}(V)=\bigcup_{j=-\infty}^{\infty}(T'_j\times U),$$

where $T'_j\subset\{\zeta:\operatorname{Re}\zeta<0,\ (2j-\frac{1}{2})\pi<\operatorname{Im}\zeta-\arg a<(2j+\frac{1}{2})\pi\}$. Then for the (ordinary) harmonic measure we have

$$\lim_{j\to\infty}\omega\biggl(\zeta,\bigcup_{|l|\geq j}T'_l,H\biggr)=0,$$

and hence by Proposition 4 applied to the map $(\zeta,w)\mapsto\zeta$, also

$$\lim_{j\to\infty}\omega\biggl((\zeta,w),\biggl(\bigcup_{|l|\geq j}T'_l\biggr)\times U,G\biggr)=0.$$

Again by Proposition 4 and (3.2)

$$\omega((\zeta,w),h^{-1}(F),G)=\omega((ke^{\zeta},w),F,D). \tag{3.3}$$

Now $h^{-1}(E)=\bigcup_{j=-\infty}^{\infty}E_j$ with

$$E_j=\{(\zeta,w)\in G:\zeta+\log k+2\pi ij=f(w)/w^n\}.$$

For each $j\in\mathbf{Z}$ the function $u_j(\zeta,w)=\log|(\zeta+\log k+2\pi ij)w^n-f(w)|-\log|\zeta-1|$ is bounded on G and equals $-\infty$ precisely on E_j. It follows that

$$\omega((\zeta,w),E_j,G)=0, \tag{3.4}$$

if $(\zeta,w)\in G\setminus E_j$. By considering $u(\zeta,w)=\sum_{j=-\infty}^{\infty}2^{-|j|}u_j(\zeta,w)$, we conclude that for $(\zeta,w)\in G\setminus h^{-1}(E)$

$$\omega((\zeta,w),h^{-1}(E),G)=0,$$

hence by (3.3), if $(z,w)\in D\setminus(E\cup A)$, then

$$\omega((z,w),F,D)=0. \tag{3.5}$$

Next, let $(z,w)\in E\cap D$, $z\neq 0$. Let $(\zeta,w)\in E_0$ be such that $h(\zeta,w)=(z,w)$. From (3.4) we obtain

$$\omega((\zeta,w),h^{-1}(F),G)=\omega((\zeta,w),F_0,G),$$

where $F_0=\{(\zeta,w)\in G:\zeta w^n-f(w)=0,\ \zeta\in\Delta_0\}$ and

$$\Delta_0=\left\{\zeta\in h^{-1}(\Delta):-\tfrac{1}{2}\pi<\operatorname{Im}\zeta-\arg a<\tfrac{1}{2}\pi\right\}.$$

Now $\omega(\zeta,\Delta_0,H)$ is positive, but tends to 0, as $\operatorname{Re}\zeta\to-\infty$. From Proposition 4 we see again that $\omega((\zeta,w),F_0,G)\to 0$, as $\operatorname{Re}\zeta\to\infty$, and therefore keeping in mind (3.5)

$$\omega((z,w),F,D)\to 0, \quad \text{as } |z|\to 0. \tag{3.6}$$

Thus far we have proved that for *every* polydisk D compactly contained in Ω, $\omega((z,w),F,D)\to 0$, as $|z|\to 0$. Now we apply Lemma 5 to deal with $z=0$. Let $\omega((0,w),F,D)=c\geq 0$. Let D' be a polydisk with $D\subset D'\subset\Omega$ and let $V=\{(z,w)\in D': |z|<r\}$, where r is so small that $\omega((z,w),F,D')\leq\frac{1}{2}c$ if $|z|=r$. From Lemma 5 we see that there is a point P in $\partial V\cap D'$ with $\omega(P,F,D')\geq c$. It follows that $c=0$. The conclusion is that $\omega((z,w),F,D)>0$ if and only if $(z,w)\in E$ and the proof of the first case is finished.

Next we assume that $\alpha\cdot\beta\neq 0$. We consider the map $h_1\colon(\zeta,w)\mapsto(\varphi(\zeta),w)$ which defines a 2-sheeted branched covering $h_1^{-1}(D)\to D$. Let $(z,w)\in D$ and $z=\varphi(\zeta)$ for some ζ. By Lemma 6 we have

$$\omega((z,w),F,D)=\omega((\zeta,w),h_1^{-1}(F),h_1^{-1}(D)).$$

Now $h_1^{-1}(F)=F_1\cup F_2$, where

$$F_1=\{(\zeta,w):\zeta=e^{f(w)/w^n},\ \varphi(\zeta)\in\Delta\}$$

and

$$F_2=\{(\zeta,w):\zeta=e^{(-b/a)f(w)/w^n},\ \varphi(\zeta)\in\Delta\}.$$

Let $l<M$. From the first part of the proof it follows that $\omega((\zeta,w),F_1,kU\times lU)=0$ if and only if $\zeta\neq e^{f(w)/w^n}$, hence also $\omega((\zeta,w),F_1,h_1^{-1}(D))=0$ if and only if $\zeta\neq e^{f(w)/w^n}$. Similarly, $\omega((\zeta,w),F_2,h_1^{-1}(D))=0$ if and only if $\zeta\neq e^{(-b/a)f(w)/w^n}$. Using (2.1) we obtain

$$\omega((z,w),F,D)\leq\omega((\zeta,w),F_1,h_1^{-1}(D))+\omega((\zeta,w),F_2,h_1^{-1}(D))=0$$

if and only if $z\neq\varphi(e^{f(w)/w^n})$, i.e. $(z,w)\notin E\cap D$. This completes the proof. □

Theorem 7 implies immediately that the graph

$$\{(x,y)\in\mathbf{C}^2:y=\varphi(e^{f(x)/x^n}),\ 0<x<M\}$$

is plurithin at the origin. Typical graphs that can be handled by the theorem are those of $y=e^{-1/x}$ and $y=\sin(1/x)$.

Note added in proof. I recently became aware of the following result of Zeriahi, [6], Proposition 2.1: Let E be a pluripolar subset of a pseudoconvex domain Ω in $\mathbf{C}^n$. If $E_\Omega^*=E^*$ and E is a G_δ as well as an F_σ set, then E is complete pluripolar in Ω.

It is easy to see that the graphs E that we consider in Theorem 7 are G_δ and F_σ. Theorem 7 gives $E=E_\Omega^*$, therefore E is complete pluripolar in Ω, that is, there exists a plurisubharmonic function u on Ω such that

$$E=\{p\in\Omega:u(p)=-\infty\}.$$

References

1. BEDFORD, E., Survey of pluripotential theory, in *Several Complex Variables, Proceedings of the Mittag-Leffler Institute 1987–1988* (Fornæss, J. E., ed.), Math. Notes **38**, Princeton Univ. Press, Princeton, N. J., 1993.
2. LÁRUSSON, F. and SIGURDSSON, R., Plurisubharmonic functions and analytic discs on manifolds, *J. Reine Angew. Math.* **501** (1998), 1–39.
3. LEVENBERG, N. and POLETSKY, E. A., Pluripolar hulls, *Michigan Math. J.* **46** (1999), 151–162.
4. POLETSKY, E. A., Holomorphic currents, *Indiana Univ. Math. J.* **42** (1993), 85–144.
5. SADULLAEV, A., Plurisubharmonic measures and capacities on complex manifolds, *Uspekhi Mat. Nauk* **36** (1981), 53–105. English transl.: *Russian Math. Surveys* **36** (1981), 61–119.

6. ZERIAHI, A., Ensembles pluripolaires exceptionnelles pour la croissance partielle des fonctions holomorphes, *Ann. Polon. Math.* **50** (1989), 81–91.

Received September 28, 1998

Jan Wiegerinck
University of Amsterdam
Faculty of Mathematics
Plantage Muidergracht 24
NL-1018 TV Amsterdam
The Netherlands
email: janwieg@wins.uva.nl