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Compactness of operators acting from a Lorentz sequence space to an Orlicz sequence space

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Abstract. Let X and Y be closed subspaces of the Lorentz sequence space $d(v,p)$ and the Orlicz sequence space l_M, respectively. It is proved that every bounded linear operator from X to Y is compact whenever

$$p>\beta_M:=\inf\{q>0:\inf\{M(\lambda t)/M(\lambda)t^q:0<\lambda,t\leq 1\}>0\}.$$

As an application, the reflexivity of the space of bounded linear operators acting from $d(v,p)$ to l_M is characterized.

1. For Banach spaces X and Y, let $L(X,Y)$ be the Banach space of all bounded linear operators from X to Y, and let $K(X,Y)$ denote its subspace of compact operators.

Let $1\leq p,q<\infty$. By the classical *Pitt's theorem* (cf. e.g. [5, p. 76]), $K(l_p,l_q)=L(l_p,l_q)$ *whenever* $p>q$. On the other hand, if $p\leq q$, then $K(l_p,l_q)\neq L(l_p,l_q)$ (because the formal identity map from l_p to l_q is clearly non-compact).

One of the closest analogues of the space l_p is the Lorentz sequence space $d(v,p)$. Recall its definition. Let $v=(v_k)=(v_k)_{k=1}^{\infty}$ be a non-increasing sequence of positive numbers such that $v_1=1$, $\lim_k v_k=0$, and $\sum_{k=1}^{\infty} v_k=\infty$. The *Lorentz sequence space* $d(v,p)$ is the Banach space of all sequences of scalars $x=(\xi_k)$ for which

$$\|x\|=\sup_{\pi}\left(\sum_{k=1}^{\infty} v_k|\xi_{\pi(k)}|^p\right)^{1/p}<\infty,$$

where π ranges over all permutations of the natural numbers $\mathbf{N}$.

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The spaces $d(v,p)$ and l_p are never isomorphic but they have similar properties. For example, every infinite-dimensional closed subspace of l_p or $d(v,p)$ has a subspace which is isomorphic to l_p (cf. e.g. [5, pp. 53, 177]). Background material on Lorentz sequence spaces can be found e.g. in [5].

In [6], E. Oja proved the following analogue of Pitt's theorem for the case of operators acting from l_p to $d(v,q)$.

Theorem 1. (cf. [6]) *Let X and Y be closed subspaces of l_p and $d(v,q)$, respectively. If $p>q$ and $v\notin l_{p/(p-q)}$, then $K(X,Y)=L(X,Y)$. If $p>q$ and $v\in l_{p/(p-q)}$, then $K(l_p,d(v,q))\neq L(l_p,d(v,q))$.*

Here again, if $p\leq q$, then $K(l_p,d(v,q))\neq L(l_p,d(v,q))$ because the formal identity map from l_p to $d(v,q)$ is not compact.

We shall prove the analogue of Pitt's theorem for the case of operators acting from $d(v,p)$ to l_q. However, we shall do it in a much more general context, considering instead of the spaces l_q their well-known generalizations—Orlicz sequence spaces l_M.

Recall the definition of Orlicz sequence spaces. An *Orlicz function* M is a continuous convex function on $[0,\infty)$ such that $M(0)=0$, $M(t)>0$ if $t>0$, and $\lim_{t\to\infty} M(t)=\infty$. The *Orlicz sequence space* l_M is the Banach space of all sequences of scalars $x=(\xi_k)$ such that $\sum_{k=1}^{\infty} M(|\xi_k|/\varrho)<\infty$, for some $\varrho=\varrho(x)>0$, under the norm

$$\|x\|=\inf\left\{\varrho>0:\sum_{k=1}^{\infty} M(|\xi_k|/\varrho)\leq 1\right\}.$$

Denote

$$\alpha_M=\sup\{q>0:\sup\{M(\lambda t)/M(\lambda)t^q:0<\lambda,t\leq 1\}<\infty\},$$
$$\beta_M=\inf\{q>0:\inf\{M(\lambda t)/M(\lambda)t^q:0<\lambda,t\leq 1\}>0\}.$$

It is easily verified that $1\leq\alpha_M\leq\beta_M\leq\infty$, and $\beta_M<\infty$ if and only if M satisfies the Δ_2-condition at zero, i.e. $\limsup_{t\to 0} M(2t)/M(t)<\infty$. This implies that $\limsup_{t\to 0} M(Qt)/M(t)<\infty$ for every positive number Q.

It is also easily checked that $l_M=l_q$ whenever $M(t)=t^q$, and, in this case, $\alpha_M=\beta_M=q$.

These and other necessary facts on Orlicz sequence spaces can be found e.g. in [5].

2. Let us state *the main result* of the present note.

Theorem 2. *Let X and Y be closed subspaces of $d(v,p)$ and l_M, respectively. If $p>\beta_M$, then $K(X,Y)=L(X,Y)$.*

The proof of Theorem 2 is based on the following result from the paper [1] by J. Ausekle and E. Oja. It uses the definition of α-domination of sequences. Let $\alpha=(a_k)$ be a sequence of numbers, and let (x_k) and (y_k) be two sequences in some Banach spaces. We say that (x_k) *α-dominates* (y_k) if there exists $C>0$ such that

$$\left\|\sum_{k=1}^{n} a_k y_k\right\| \le C\left\|\sum_{k=1}^{n} a_k x_k\right\| \quad \text{for all } n\in\mathbf{N}.$$

In this case, we write $(x_k)>_\alpha(y_k)$.

Proposition 3. (cf. [1]) *Let $\alpha=(a_k)$ be a sequence of numbers. Let (ε_k) and (φ_k) be two sequences in some Banach spaces. Suppose that (ε_k) does not α-dominate (φ_k). Let (e_k) and (f_k) be bases in Banach spaces E and F, respectively. Suppose that (ε_k) α-dominates any normalized block-basis (u_k) of (e_k) and any normalized block-basis (v_k) of (f_k) has a subsequence $(v_{n_k})>_\alpha(\varphi_k)$. If X and Y are closed subspaces of E and F, respectively, with X^* being separable, then $K(X,Y)=L(X,Y)$.*

Proof of Theorem 2. Let q be such that $p>q\ge\beta_M$ and, for some $k>0$,

$$kt^q M(\lambda)\le M(\lambda t), \quad 0<\lambda, t\le 1. \tag{1}$$

Put $\alpha=(1,1,...)$. Denote by (ε_k) and (φ_k) the unit vector bases in l_p and l_q, respectively. First of all, notice that (ε_k) does not α-dominate (φ_k), because

$$\left\|\sum_{k=1}^{n}\varphi_k\right\|_{l_q}=n^{1/q}, \quad \left\|\sum_{k=1}^{n}\varepsilon_k\right\|_{l_p}=n^{1/p},$$

and $n^{1/q-1/p}\to\infty$.

Since $d(v,p)$ is reflexive and separable, X is also reflexive and separable, and therefore X^* is separable.

For completing the proof of the theorem, it remains to show that, in Proposition 3, one can take $E=d(v,p)$ and $F=l_M$ with their unit vector bases (e_k) and (f_k), respectively.

Let (u_k) be a normalized block-basis of the unit vector basis (e_k) of $d(v,p)$. It is easily checked (cf. e.g. [5, p. 177]) that

$$\left\|\sum_{k=1}^{n} u_k\right\|_{d(v,p)}\le n^{1/p}=\left\|\sum_{k=1}^{n}\varepsilon_k\right\|_{l_p} \quad \text{for all } n\in\mathbf{N}.$$

Hence $(\varepsilon_k)>_\alpha(u_k)$.

Finally, we show that $(v_k)>_\alpha(\varphi_k)$ for any normalized block-basis (v_k) of the unit vector basis (f_k) of l_M, i.e. there exists $C>0$ such that

$$\nu_n := \Big\|\sum_{k=1}^{n} v_k\Big\|_{l_M} \ge C\Big\|\sum_{k=1}^{n} \varphi_k\Big\|_{l_q} = Cn^{1/q} \quad \text{for all } n\in\mathbf{N}. \tag{2}$$

Let

$$v_k = \sum_{j=m_k+1}^{m_{k+1}} c_j f_j, \quad k\in\mathbf{N}.$$

Since

$$\nu_n = \inf\Big\{\varrho>0 : \sum_{k=1}^{n}\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|/\varrho)\le 1\Big\},$$

we have $\nu_1\le\nu_2\le\ldots$ and

$$\sum_{k=1}^{n}\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|/\nu_n)=1 \quad \text{for all } n\in\mathbf{N}. \tag{3}$$

We also have that

$$\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|)=1 \quad \text{for all } k\in\mathbf{N}, \tag{4}$$

because $\|v_k\|_{l_M}=1$. It follows from (4) that $|c_j|\le\gamma$, $j\in\mathbf{N}$, for some $\gamma\ge 1$.

Note that $\nu_n\to\infty$. In fact, if $\nu_n\le Q$, $n\in\mathbf{N}$, for some $Q>0$, then we also could assume that $|c_j|\le Q$, $j\in\mathbf{N}$. Since $\beta_M<\infty$, the function M satisfies the Δ_2-condition at zero. Hence, for some $K>0$,

$$M(Qt)\le KM(t), \quad 0\le t\le 1.$$

Consequently, by (3) and (4), we would have that

$$\begin{aligned} 1 &= \sum_{k=1}^{n}\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|/\nu_n) \ge \sum_{k=1}^{n}\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|/Q) \\ &\ge \frac{1}{K}\sum_{k=1}^{n}\sum_{j=m_k+1}^{m_{k+1}} M(|c_j|) = \frac{n}{K} \quad \text{for all } n\in\mathbf{N}, \end{aligned}$$

a contradiction.

Since $\nu_1 \le \nu_2 \le ...$ and $\nu_n \to \infty$, for the proof of (2), it is sufficient to consider those $n \in \mathbf{N}$ for which $\nu_n \ge \gamma$. In this case, by (1) and the Δ_2-condition at zero,

$$M(|c_j|/\nu_n) = M((|c_j|/\gamma)(\gamma/\nu_n)) \ge kM(|c_j|/\gamma)\frac{\gamma^q}{\nu_n^q} \ge \frac{k\gamma^q}{K\nu_n^q}M(|c_j|) \quad \text{for all } j \in \mathbf{N}$$

for some $K > 0$. It follows from (3) and (4) that

$$1 \ge \sum_{k=1}^{n} \frac{k\gamma^q}{K\nu_n^q} \sum_{j=m_k+1}^{m_{k+1}} M(|c_j|) = \frac{k\gamma^q}{K\nu_n^q} n.$$

This proves (2). The proof is complete. □

Remark. In [1], we proved the equality $K(X,Y) = L(X,Y)$ for closed subspaces $X \subset d(v,p)$, $Y \subset d(w,q)$ with $p > q$, $w \notin l_{p/(p-q)}$ and also for closed subspaces $X \subset h_M$, $Y \subset l_N$ with $\alpha_M > \beta_N$.

3. The next result shows that *the condition $p > \beta_M$ is essential* in Theorem 2.

Theorem 4. *Let X be an infinite-dimensional closed subspace of $d(v,p)$. If $p \le \beta_M$, then $K(X, l_M) \ne L(X, l_M)$.*

The proof of Theorem 4 uses the following easy observation whose proof is straightforward.

Proposition 5. *Let X, Y, Z, W be Banach spaces and $K(X,Y) = L(X,Y)$. Suppose that Z is isomorphic to a complemented subspace of X and W is isomorphic to a subspace of Y. Then $K(Z,W) = L(Z,W)$.*

Proof of Theorem 4. Assume for contradiction that $K(X, l_M) = L(X, l_M)$. Set $q = \beta_M$. Since $q \in [\alpha_M, \beta_M]$, l_M contains a subspace isomorphic to l_q (see [5, p. 143]). It is also known (see e.g. [5, p. 177]) that every infinite-dimensional closed subspace of $d(v,p)$ contains a complemented subspace isomorphic to l_p. Therefore, we get from Proposition 5 that $K(l_p, l_q) = L(l_p, l_q)$. Since $p \le q$, this is a contradiction and we have $K(X, l_M) \ne L(X, l_M)$. □

Since every infinite-dimensional closed subspace of l_q contains a subspace isomorphic to l_q (cf. e.g. [5, p. 53]), the following is clear from the proof of Theorem 4.

Corollary 6. *Let X and Y be infinite-dimensional closed subspaces of $d(v,p)$ and l_q, respectively. If $p \le q$, then $K(X,Y) \ne L(X,Y)$.*

Remark. In Theorem 4, the space l_M cannot be replaced by its infinite-dimensional closed subspace (cf. Theorem 2 and Corollary 6). For example, let l_M be an Orlicz space such that $\alpha_M<\beta_M$, and let $p\in(\alpha_M,\beta_M)$. Putting $q=\alpha_M$, we get that l_M contains a subspace Y isomorphic to l_q. We also know that $d(v,p)$ contains a complemented subspace X isomorphic to l_p. Since $q<p$, by Pitt's theorem, $K(l_p,l_q)=L(l_p,l_q)$. Hence $K(X,Y)=L(X,Y)$.

4. We conclude with *some applications to the reflexivity* of spaces of operators acting from $d(v,p)$ to l_M. Recall that $d(v,p)$ is reflexive if and only if $p>1$. Recall also that l_M is reflexive if and only if both M and its complementary Orlicz function M^* satisfy the Δ_2-condition at zero. This means that $\beta_M<\infty$ and $\alpha_M>1$.

We shall apply the following result proved by S. Heinrich [3] and independently by N. J. Kalton [4]: *if X and Y are reflexive, and $K(X,Y)=L(X,Y)$, then $L(X,Y)$ is reflexive.* This result, together with Theorem 2, yields Corollaries 7 and 8 below.

Corollary 7. *Let X be a closed subspace of $d(v,p)$, and let Y be a reflexive subspace of l_M. If $p>\beta_M$, then $L(X,Y)$ is reflexive.*

Corollary 8. *Let X and Y be closed subspaces of $d(v,p)$ and l_q, respectively. If $p>q>1$, then $L(X,Y)$ is reflexive.*

We now come to the main application of this note.

Theorem 9. *The following assertions are equivalent:*
(a) *$L(d(v,p),l_M)$ is reflexive,*
(b) *$K(d(v,p),l_M)$ is reflexive,*
(c) $1<\alpha_M\le\beta_M<p$.

Proof. (a) $\Rightarrow$ (b) This is true because the reflexivity passes to closed subspaces.

(b) $\Rightarrow$ (c) Since $K(d(v,p),l_M)$ is reflexive, its subspace l_M is also reflexive. Hence $\alpha_M>1$. It is well known (cf. e.g. [2, p. 247]) that if X and Y are Banach spaces, one of them having the approximation property, and $K(X,Y)$ is reflexive, then $K(X,Y)=L(X,Y)$. This implies $K(d(v,p),l_M)=L(d(v,p),l_M)$. Therefore, $\beta_M<p$ by Theorem 4.

(c) $\Rightarrow$ (a) This is clear from Corollary 7 because $\alpha_M>1$ and $\beta_M<p$ imply the reflexivity of l_M. □

The next corollary is immediate from Theorem 9.

Corollary 10. *The following assertions are equivalent:*
(a) *$L(d(v,p),l_q)$ is reflexive,*
(b) *$K(d(v,p),l_q)$ is reflexive,*
(c) $1<q<p$.

The last result can be derived from Theorem 1 similarly to the proof of Theorem 9. We include it for comparison with Corollary 10.

Theorem 11. *The following assertions are equivalent:*

(a) $L(l_p, d(w,q))$ *is reflexive,*

(b) $K(l_p, d(w,q))$ *is reflexive,*

(c) $1<q<p$ *and* $w \notin l_{p/(p-q)}$.

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