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Global moduli for contacts

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Suppose we have two smooth subvarieties, S and T, of a smooth algebraic variety X. Following [S1], we say that S and T have a *contact* at a point p of X if S and T meet at p but not transversally. Two smooth plane curves therefore have a contact at p if they are tangent there; their local Taylor expansions agree through degree one. Higher-order contacts are defined by requiring agreement through given higher degrees, but the order of a contact does not depend on the choice of local coordinates. More generally, for higher-dimensional embedded varieties, we can replace local Taylor expansions by appropriate derivatives of the Gauss map. Two surfaces in $\mathbf{P}^3$, for example, have a contact of order two at p if, in addition to a tangency at p, they have the same second fundamental form there. In local coordinates, the last condition means that the Hessians of the two surfaces will coincide.

This paper is about what two embedded varieties—smooth or not, whatever their dimensions—share when they make contact of a given but arbitrarily high order. In our view, they share *data*, represented by points of a suitable *data scheme* canonically associated to the ambient space X. A main goal of this paper is to construct such data schemes and launch the study of their geometry, especially their intersection theory. When the intersecting varieties S and T are smooth at p, their rth order data at p can be given locally by suitable multilinear forms, which represent the higher derivatives of their Gauss maps; on the other hand, if p is singular on S or T, new objects, on the boundary of these forms, appear. For each given order $r\geq 1$ we obtain a global moduli space, denoted $D_k^r X$, *proper over* X, which parametrizes all the data for all possible rth order contacts between k-dimensional embedded smooth subvarieties of X, together with their limits, in a natural sense, at singularities.

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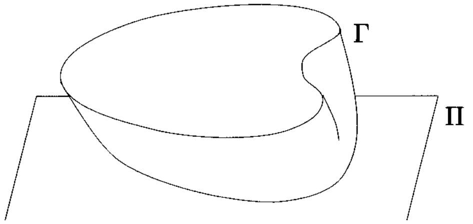
Griffiths and Harris [GH] have used local fundamental forms to draw striking conclusions about embeddings in projective space. Our theory, completely global and intrinsic, runs somewhat parallel to theirs, in that our data give canonical descriptions of all the derivatives of the Gauss map, in particular its first derivative, which localizes to their second fundamental form.

When $X \subset \mathbf{P}^n$, it is easy, for example, to write down the intersection calculus of the data space, denoted $D_k^r X$. While this paper emphasizes laying groundwork, we will illustrate our approach with several basic applications. For a first illustration, we consider curves in $\mathbf{P}^3$. Here we recover the classical results about the number of second-order contacts between a fixed curve and the members of a two-parameter family of surfaces obtained by Schubert [Sch], recently verified by Rosselló ([R1], [R2]). In Rosselló's approach, Schubert's results follow from the intersection calculus on $\mathrm{Hilb}_3 \mathbf{P}^3$, but apply only to curves without flexes. Our methods give a new generalization of Schubert's conclusions to curves with flexes, in just a few lines, as corollaries of the richer intersection calculus on a simple blowup of the data scheme $D_1^2 \mathbf{P}^3$. Turning next to surfaces in $\mathbf{P}^3$, we immediately obtain formulas, for example, for the classes of the flat and parabolic loci on a general surface.

To explain our constructions, fix a smooth ambient variety X; the theory actually works for any scheme X which is smooth over a given base scheme S. Fix any k with $1 \leq k \leq n$; we will construct data for embedded k-folds in X. Denote by I the Grassmannian $G_k(TX)$ of k-planes in the tangent bundle of X, and denote by p the projection $I \to X$. In the special case $X = \mathbf{P}^n$, we know that I is canonically isomorphic to the incidence correspondence of points and k-planes in $\mathbf{P}^n$. We regard I as the space of *first-order data* on X; it parametrizes all possible pairs (x, Π) consisting of a point and a tangent k-plane at that point.

We obtain a space $D_k^2 X$ of *second-order data*, fibered over I, by differentiating the motion of the pair (x, Π). To be precise, we take $D_k^2 X$ to be the collection of all k-parameter linear first order-deformations Δ of first-order data (x, Π), such that, under the motion defined by Δ, the point x traces the germ of a k-fold tangent to Π.



In the illustration, Γ indicates a smooth k-fold which represents the germ given by Δ. Denote by $dp\colon TI \to TX$ the derivative of p, and view Δ as a k-plane in TI. Then the condition on Δ means that we have $dp(\Delta) \subset \Pi$. Inclusion here, rather than equality, permits ramification; this provides the boundary which completes the data space. To construct spaces of higher-order data, we begin with the fundamental observation that I plays two essentially different roles here: it parametrizes first-order data (x, Π); it is also the space of k-planes in the tangent bundle, hence a target for derivatives. To accomplish the construction, we will need to separate these roles, and play them on a larger stage.

Quite generally, consider a commutative triangle of smooth schemes of the form

(1)
$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"] & G_k(TY) \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

where b denotes a *smooth* morphism. We shall now construct a *derived triangle*

(2)
$$\begin{tikzcd} DX \arrow[r, "Da"] \arrow[d, "Db"] & G_k(TX) \arrow[dl, "p"] \\ X & \end{tikzcd}$$

of the same form. To do so, denote by B the blowup of $G_k(TX)$ along the Schubert variety Z consisting of k-planes which meet the kernel of db nontrivially. The rational map $G_k(TX) \to X \times_Y G_k(TY)$ induced by b gives a morphism $B \to X \times_Y G_k(TY)$. On the other hand we obtain a section $X \to X \times_Y G_k(TY)$ via the assignment $x \mapsto (x, a(x))$. Then we define DX to be the fiber product of B and X over $X \times_Y G_k(TY)$, with the obvious maps to X and to $G_k(TX)$. We call DX the *derived scheme.* We shall see in a moment that the map Db is smooth, so we can iterate.

Coming back to our original ambient space X, the triangle

(3)
$$\begin{tikzcd} I \arrow[r, "1"] \arrow[d, "p"] & I \arrow[dl, "p"] \\ X & \end{tikzcd}$$

gives $D_k^2 X$ as the derived scheme. Iterating, we obtain the higher-order data schemes, denoted $D_k^r X$.

To see that the map Db in diagram (2) is smooth, as well as to work out the intersection calculus of a data scheme, it helps to think more analytically about the derived scheme. To motivate the key point, consider a real C^∞ manifold M, write p for the projection $TM \to M$ and write dp for its derivative. A second-order ordinary differential equation on M is defined [D4, 18.3.2, p. 18] to be a smooth vector field α on TM, such that we have $dp(\alpha(\mathsf{v}))=\mathsf{v}$ for all $\mathsf{v} \in TM$. Clearly not every vector field on TM can have this property! If we view α as a section of $T(TX) \to TX$, then those α which do give second-order differential equations fill out a nontrivial subbundle of the second tangent bundle $T(TX)$, defined by the condition $dp \circ \alpha = 1_{TM}$. This idea inspires a useful alternative construction of the scheme DX of (2).

Returning to algebraic geometry, a triangle (1) as above can be viewed as giving a *distribution of k-planes on Y, parametrized by X.* Write Σ for the pullback to X of the universal k-subbundle in $b^*TY = X \times_Y TY$, and form the fiber square of bundles on X given by Σ and the derivative ∂b:

$$
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & TX \\
\downarrow & & \downarrow{\scriptstyle \partial b} \\
\Sigma & \xrightarrow{\iota} & b^*TY
\end{array}
\tag{4}
$$

Then $\mathcal{F}$ consists precisely of those tangent vectors on X which map into the k-planes on Y which are given by the distribution. For a general triangle (1), we show (Proposition 2.1) that there is a canonical X-isomorphism $DX \cong G_k(\mathcal{F})$. Since $\mathcal{F}$ is a bundle, it follows that Db, the structure map of $G_k(\mathcal{F})$, is smooth. In addition, the fiber square defining $\mathcal{F}$ gives a simple description (2.2) of the functor of points of the derived scheme, and, since $\ker(db) = T_{X/Y}$, the intersection calculus for DX, relative to X, is also clear.

We now describe the second fundamental form, and indicate how the higher derivatives of the Gauss map enter. In diagram (4), for any triangle (1), the set of k-planes Π in $\mathcal{F}$ which do not map isomorphically under ∂b represents the pullback to DX of the exceptional divisor on B. We denote this divisor on DX by C, and call it the *cuspidal divisor.* Equivalently, for points of C, we have a strict inclusion $dp(\Delta) \subset \Pi$ above. The *noncuspidal subscheme* of DX is defined to be the complement $DX \setminus C$, denoted DX_{nc}. We then show (Proposition 7.1) that DX_{nc} is a principal homogeneous space for the bundle $\mathrm{Hom}(\Sigma, T_{X/Y})$. Hence, given a section σ of $Db\colon DX_{\mathrm{nc}} \to X$, we have an isomorphism

$$DX_{\mathrm{nc}} \cong \mathrm{Hom}(\Sigma, T_{X/Y}),$$

where σ gives the zero-section on the right. In triangle (3), we have $TG_k(TX)_X = \mathrm{Hom}(\Sigma, \Lambda)$, where Λ denotes the universal quotient on $G_k(TX)$. Then, from any given section

$$\sigma\colon D_k^1X = G_k(TX) \longrightarrow D_k^2X_{\mathrm{nc}},$$

we obtain isomorphisms

$$D_k^2X_{\mathrm{nc}} \cong \mathrm{Hom}(\Sigma, \mathrm{Hom}(\Sigma, \Lambda)) \cong \mathrm{Hom}(\Sigma\otimes\Sigma, \Lambda). \tag{5}$$

When we constrain first-order data on X to move along a given smooth k-fold S in X, the universal subbundle Σ restricts to the tangent bundle TS, while Λ pulls back to the normal bundle. Therefore the isomorphisms (5) will give the second fundamental form and hence the Hessian, universally. To study second-order contacts of not necessarily smooth k-folds in X, it is convenient technically to pass to the closure, denoted S_k^2X, of the subscheme of D_k^2X which corresponds to the *symmetric* forms in (5), due to the symmetry of the second derivative.(3) From each higher derived triangle, in a similar way, we obtain a universal multilinear form, corresponding to a higher derivative of the Gauss map, together with a boundary divisor consisting of singular limit data. Here again, when we restrict to k-fold subschemes, we obtain symmetric forms.(4)

For $X = \mathbf{P}^n$, simple motions of linear subspaces give natural sections σ; these define the data points for the usual *flat* or *inflectional* points on projective varieties. Further, for any X over the complex numbers, Proposition 9.2 asserts that a C^∞ *connection* on TX gives a natural C^∞ section $\sigma\colon D_k^1X \to D_k^2X_{\mathrm{nc}}$, for any k, induced by parallel transport. In the special case $X = \mathbf{P}_{\mathbf{C}}^n$ with the standard connection induced by the Fubini–Study metric, which is C^∞, we obtain the previous section σ, which we already know is algebraic.

Classically, spaces for first-order data appeared naturally in the study of contact problems; for a survey, including modern results, see [S1]. One way to define higher-order data is to generalize the construction of the tangent line as a limit of secants. This approach leads naturally to the study of configuration spaces. For example, to count second-order contacts, that is, osculations, of plane curves, Schubert [Sch] considered triangles equipped with conics through their vertices, degenerating along suitable curves. Passing to the limit, Schubert obtained second-order data. More recently, in 1954, Semple [Se1] following Schubert's lead, constructed a parameter variety for triangles in $\mathbf{P}^2$, each equipped with a 2-parameter family of conics through its vertices. This triangle variety naturally contains a smooth

(3) Using exterior differential systems, Speiser [S2] generalizes the construction of symmetric data to arbitrary triangles, through a fully global treatment which eliminates the passage to the closure.

(4) Based on [S2] Speiser and Laksov [LS] generalize further, and obtain explicit equations for S_k^rX in D_k^rX as special cases of their general approach.

subvariety of second-order data; going further, Joel Roberts and the third author of this article gave its intersection calculus in full, and verified Schubert's enumeration of triple contacts [RS1]–[RS3]. Given the fundamental use of linear systems through the points of a configuration, it is very natural to consider Hilbert schemes in this context. For comparisons between the Semple's triangle variety and $\mathrm{Hilb}^3\mathbf{P}^2$, see [S1]. More generally, Göttsche's thesis [Gö] independently employs a version of our Grassmannian construction to study more general zero-dimensional subschemes. The study of configuration spaces for their own sake continues, for example, in Fulton and MacPherson's recent work [FM].

Another 1954 paper by Semple, [Se2], inspired in part by Gherardelli [Gh], takes a different point of departure, which inspired, in turn, our own work: to define rth order data (for $r>1$) by directly differentiating the motion of data of order $r-1$. Related papers by Longo [L] in and Du Val [DuV] soon followed, mainly about curvilinear data in $\mathbf{P}^n$. After more than 30 years, Semple's ideas were taken up, this time in the language of schemes, first by Collino [C], who constructed data varieties of all orders for curves in $\mathbf{P}^2$, and then by Colley and Kennedy, who applied Collino's construction in a penetrating study [CK1], [CK2] of curve contacts in the plane. The resulting variety of second-order data, for example, is canonically isomorphic to that of [Se1] and [RS1]–[RS3], and the varieties of higher-order data, obtained by iteration, inspired the construction of our data schemes. Semple, however, had gone significantly beyond these later papers in two respects: first, he considered curvilinear data in projective *space*, as Schubert did at the end of [Sch]; second, he gave the challenge to construct appropriate parameter spaces for higher-dimensional data.

Our work, which takes up Semple's challenge, begins by generalizing Collino's construction. We follow a design conditioned by the isomorphism (5) and its analogues for higher-order data, which demand a thoroughly functorial technique. In some respects, our data spaces parallel, but in greater generality, the spaces of jets of submanifolds constructed in the C^∞ setting, which have been helpful in the study of differential equations, as described, for example, in [G, Chapter 5]. Over $\mathrm{Spec}(\mathbf{C})$, the symmetric data spaces $S_k^r X$ do parametrize the algebraic jets, but they also give, in contrast, the singular *boundary*, which compactifies the jet space in a very natural way.

Our exposition interweaves the abstract theory with its applications. After preliminaries on projections and blowups, we define derived triangles in §2, and give their basic properties. We introduce the cuspidal divisors and describe sections of DX_{nc} in §3, then iterate in §4 to construct the data schemes $D_k^r X$. In §5 we describe noncuspidal data in local coordinates, and discuss the related concept of flat data in $\mathbf{P}^n$. As a first example, we write down the intersection calculus for second-order curvilinear data in $\mathbf{P}^3$, and extend Schubert's results, in §6. Then in §7 we introduce the second fundamental form and its higher-order analogues. As an application, we compute the intersection ring of the second-order data scheme for surfaces in $\mathbf{P}^3$ in §8, together with its subscheme of symmetric data. This implies, for example, that the curve of parabolic points, on a general surface of degree $d>1$ in $\mathbf{P}^3$, has degree $4d(d-2)$. Finally, in §9, working over $\mathbf{C}$ we introduce a C^∞ connection on X, and obtain C^∞ sections σ of D_k^2X, for all k, via moving frames.

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Notation. We shall tacitly assume that all schemes are separated and of finite type over the given base. In general, our notation follows [F]. In particular, we shall write $\mathrm{Hom}(E,F)$ for the bundle of homomorphisms from a vector bundle E to a vector bundle F. Also, given a morphism $b\colon X\to Y$, we distinguish carefully between $\partial b\colon TX\to b^*TY$, a morphism of bundles on X, and $db\colon TX\to TY$, which maps a bundle over X to a bundle over Y.

1. Projections of Grassmannians

Throughout this section we shall work over a fixed base scheme X. For any bundle F, we shall write $G_k(F)$ for the Grassmannian of k-planes in F, and, when confusion seems unlikely, we shall denote by p the structure map $G_k(F)\to X$, regardless of F. Write r for the rank of F; we shall tacitly assume that $r\geq k$, so the Grassmannian will be nonempty. Then the relative dimension of $G_k(F)$ over X is

$$n\dim(p)=\dim(G_k(F)/X)=k(r-k).$$

We begin with a surjection

$$D\xrightarrow{\varphi}E,$$

of vector bundles on X, and we denote by K its kernel. We shall suppose that E has positive rank, so that $D\neq K$. The surjection φ defines a rational map, denoted

ϱ, from $G_k(D)$ to $G_k(E)$, whose domain of definition is the complement, denoted U, of the closed subset

$$Z = \{k\text{-planes which meet } K \text{ nontrivially}\}$$

of $G_k(D)$. More explicitly, write $\mathcal{S}$ for the universal k-subbundle of the pullback $D_{G_k(D)}$, and denote by λ the composite induced map

$$\bigwedge^k \mathcal{S} \longrightarrow \bigwedge^k D_{G_k(D)} \longrightarrow \bigwedge^k E_{G_k(D)}.$$

Here $\bigwedge^k \mathcal{S}$ is a line bundle on $G_k(D)$, and the arrow $\bigwedge^k \mathcal{S} \to \bigwedge^k D_{G_k(D)}$, an injection, induces the Plücker embedding $G_k(D) = P(\bigwedge^k \mathcal{S}) \hookrightarrow P(\bigwedge^k D_{G_k(D)})$. Because λ expresses the rational map ϱ in Plücker coordinates, hence vanishes exactly on the points of Z, we shall give Z a scheme structure as the locus of zeros of λ. The morphism

$$U = G_k(D) \setminus Z \xrightarrow{\pi} G_k(E)$$

will be called the *projection* induced by φ, with *center* Z.

To obtain a morphism which, in effect, extends π, denote by B the blowup of $G_k(D)$ along Z. Locally, the ideal defining Z can be generated by the $k \times k$ minors of any matrix representing φ. It follows directly from the construction that the embedding of U in $G_k(D)$ induces an embedding of B in $G_k(D) \times_X G_k(E)$, whose image is the closure of the graph of π. The second projection of $G_k(D) \times_X G_k(E)$ defines a morphism $\tau: B \to G_k(E)$, and as soon as we identify the complement of the exceptional divisor of B with U, it is clear that τ extends π.

Now fix a subbundle

$$\Sigma \overset{\iota}{\hookrightarrow} E,$$

of rank k. Because Σ has rank k, we know that $G_k(\Sigma)$ identifies canonically with X. Denote by P the fiber product scheme defined by the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\tau'"] & B \arrow[d, "\tau"] \\ X = G_k(\Sigma) \arrow[r, "\sigma"] & G_k(E). \end{tikzcd}$$

Here the horizontal arrows are embeddings, because σ, induced by the given inclusion $\iota: \Sigma \to E$, is a section of the structure map $G_k(E) \to X$. To describe P more explicitly, consider the fiber product bundle $\mathcal{F}$ defined by the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "\varphi'"] & D \arrow[d, "\varphi"] \\ \Sigma \arrow[r, "\iota"] & E \end{tikzcd}$$

of bundles on X. Under the identification of X with $G_k(\Sigma)$, the left vertical arrow φ' induces the structure map $G_k(\mathcal{F})\to X$, while ι induces the section σ in the square above. Both these maps are morphisms, while φ, of course, induces π, which is only a rational map.

Proposition 1.1. *There is a canonical isomorphism of X-schemes*

$$P \xrightarrow{\sim} G_k(\mathcal{F}).$$

Proof. The X-scheme P embeds in B by definition, hence in $G_k(D)\times_X G_k(E)$ because B does. The inclusion $G_k(\mathcal{F})\to G_k(D)$ and the composite map

$$G_k(\mathcal{F}) \longrightarrow X = G_k(\Sigma) \longrightarrow G_k(E)$$

embed $G_k(\mathcal{F})$ in $G_k(D)\times_X G_k(E)$ as well. Over the dense open subscheme U, it is clear from the construction that the images of both embeddings coincide as sets. Moreover, on U, both images restrict to the same scheme, the graph of π. Now B, as we observed above, is the scheme closure of this graph, so B is a subscheme of $G_k(\mathcal{F})$. To check that the resulting inclusion $B\hookrightarrow G_k(\mathcal{F})$ is an isomorphism, we need only show that it is étale. By [GD, 17.8.2], we may check this on the fibers over scheme points $x\in X$. But, at any x, the potentially thicker fiber $G_k(\mathcal{F})_x = G_k(\mathcal{F}\otimes k(x))$ is reduced, so both fibers agree. This proves the proposition.

Write r for $\operatorname{rank}(\mathcal{F})$. Then we have

$$r = k + \dim(\varphi),$$

where $\dim(\varphi)$ is the relative dimension, $\operatorname{rank}(E) - \operatorname{rank}(F)$.

For any k and n, we write $G_k(\mathbf{P}^n)$ for the Grassmannian of projective k-planes in $\mathbf{P}^n$.

Corollary 1.2. *The scheme P is a locally trivial $G_{k-1}(\mathbf{P}^{r-1})$-bundle over X. In particular, P is smooth over X, of relative dimension $k\dim(\varphi)$.*

Proof. Clearly $G_k(\mathcal{F})$ is a locally trivial $G_{k-1}(\mathbf{P}^{r-1})$-bundle over X, so the corollary follows immediately from the proposition.

2. Derived triangles

We shall work, from now on, over a fixed base scheme S, and we shall fix a positive integer k. When there is little danger of confusion, we shall write p for the structure map of any Grassmannian.

We now introduce the main construction of this paper. To each commutative triangle of smooth, equidimensional S-schemes

$$
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "b"] & G_k(TY) \arrow[dl, "p"] \\
Y
\end{tikzcd}
\tag{1}
$$

with a *smooth* map b, we shall associate a *derived triangle*

$$
\begin{tikzcd}
DX \arrow[r, "Da"] \arrow[d, "Db"] & G_k(TX) \arrow[dl, "p"] \\
X
\end{tikzcd}
\tag{2}
$$

as follows. Write $\mathcal{S}$ for the universal k-subbundle of p^*TY, pull the natural inclusion $\mathcal{S}\subset p^*TY$ back to X via a, and set $\Sigma = a^*\mathcal{S}$. By the commutativity of (1), we have an inclusion $\Sigma \subset b^*TY$, which exhibits Σ as a subbundle of b^*TY. Denote by ∂b the derivative of b, and write B for the blowup of $G_k(TX)$ along the center of the projection

$$G_k(TX) \longrightarrow G_k(b^*TY) = X \times_Y G_k(TY)$$

induced by ∂b. This projection induces a morphism $f\colon B \to X \times_Y G_k(TY)$. We define the scheme DX and maps Db and σ' by the fiber product diagram

$$
\begin{tikzcd}
DX \arrow[r, "\sigma'"] \arrow[d, "Db"] & B \arrow[d, "\tau"] \\
X = G_k(\Sigma) \arrow[r, "\sigma"] & X \times_Y G_k(TY),
\end{tikzcd}
$$

where the bottom map σ is induced by the inclusion $\Sigma \subset b^*TY$. Then we define $Da\colon DX \to G_k(TX)$ to be the composite of σ' and the structure map $B \to G_k(TX)$.

As in §1, we write C for the pullback to DX of the exceptional divisor on B. Because C parametrizes k-planes which contain nonzero tangent vectors which map to zero on Y, we call C the *cuspidal divisor* of DX.

For a more explicit description of DX, write $\mathcal{F}$ for the fiber product bundle defined by the diagram

$$
\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & TX \arrow[d, "\partial b"] \\
\Sigma \arrow[r, "\iota"] & b^*TY
\end{tikzcd}
$$

of vector bundles on X. We shall call $\mathcal{F}$ the *Semple bundle* associated to the triangle (1), and denote by r its rank. We have

$$r = k + \dim(b),$$

where $\dim(b)$ denotes the relative dimension of X over Y.

In other words, the triangle (1) determines a distribution of k-planes in the tangent bundle of Y, and the bundle $\mathcal{F}$ consists exactly of those tangent vectors on TX which ∂b carries into k-planes of the distribution.

Proposition 2.1. *Under the assumptions above, we have a canonical isomorphism of X-schemes*

$$DX \xrightarrow{\sim} G_k(\mathcal{F}).$$

Proof. This follows immediately from Proposition 1.1.

Corollary 2.2. *The scheme DX is a locally trivial $G_{k-1}(\mathbf{P}^{r-1})$-bundle over X. In particular, DX is smooth and equidimensional over S, and Db is a smooth S-morphism, of relative dimension $k\dim(b)$.*

Proof. This follows from Corollary 1.2.

By the last assertion of Proposition 2.1, the derived triangle (2) satisfies the same hypotheses as the original triangle (1), and hence gives rise to its own derived triangle. Therefore we can repeat the derivation process as often as we like.

The functor of points

To describe the points of DX, define a contravariant functor $\mathcal{D}$ from X-schemes to sets by the assignment

$$\mathcal{D}(T \xrightarrow{f} X) = \{k\text{-subbundles } S \text{ of } (TX)_T \mid \partial b(S) \subset \iota_T(\Sigma_T)\},$$

where the subscripts, as usual, indicate pullbacks to T via f.

Proposition 2.3. *The X-scheme DX represents the functor $\mathcal{D}$. In particular, there is a k-subbundle $\mathcal{U} \subset TX_{DX}$ such that, for any T-point $S \subset TX_T$ of DX, over $f: T \to X$, we have $S = f^*\mathcal{U}$.*

Proof. Start with a T-point $f: T \to X$. By Proposition 2.1, a T-point of DX over f is a T-point of $G_k(\mathcal{F})$ over f, that is, a k-subbundle $S \subset \mathcal{F}_T$. Because ι is the inclusion of a subbundle, it follows, by base change, that $\mathcal{F}_T \subset TX_T$ is a subbundle. Therefore, by the fiber square defining $\mathcal{F}$, every S as above is a subbundle of TX_T, carried by ∂b into the image of Σ_T. Conversely, it is clear that any element S of $\mathcal{D}(T \xrightarrow{f} X)$ appears in this way, which proves the first assertion. For the second, we take $\mathcal{U}$ to be the universal k-subbundle of $\mathcal{F}_{DX} = \mathcal{F}_{G_k(\mathcal{F})}$, and the proposition follows.

Given a geometric point x of X, with image $y = b(x) \in Y$, denote by Π the k-plane in the tangent space $T_y Y$ assigned by triangle (1). By Proposition 2.3, each point of DX, over $x \in X$, consists of a k-plane Π' in $T_x X$, subject to the condition that the derivative of $b\colon X \to Y$ carries Π' into Π. Recall that the kernel of $\partial b\colon TX \to b^* TY$ is canonically isomorphic to the relative tangent bundle $T_{X/Y}$. Hence the projection $\Pi' \to \Pi$ is injective, hence an isomorphism, precisely when $\Pi' \cap T_{X/Y} = 0$, and this happens for a general Π'.

Direct images

Now suppose we have two triangles and a map between them, that is, a commutative diagram of the following form.

(3)
$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd] & & X' \arrow[dr] \arrow[dd] & \\
& G_k(TY) \arrow[rr] \arrow[dl] & & G_k(TY') \arrow[dl] \\
Y \arrow[rr] & & Y' &
\end{tikzcd}$$

Here we follow the previous notation for the maps in our triangles, indicating parts of the right-hand triangle with primes, and continue under our previous assumptions, too. In particular, the two vertical arrows, denoted b and b', represent smooth maps. We shall refer to the three rectangles in the diagram as the *top*, the *bottom* and the *back*, in the obvious sense. A map of triangles will be called an *embedding* if all three horizontal arrows are embeddings over S.

Proposition 2.4. *An embedding of triangles induces an embedding*

(4)
$$\begin{tikzcd}
DX \arrow[rr] \arrow[dr] \arrow[dd] & & DX' \arrow[dr] \arrow[dd] & \\
& G_k(TX) \arrow[rr] \arrow[dl] & & G_k(TX') \arrow[dl] \\
X \arrow[rr] & & X' &
\end{tikzcd}$$

of derived triangles.

Proof. The bottom arrow $X \to X'$ comes from the top of diagram (3). We obtain the middle arrow $G_k(TX) \to G_k(TX')$ by differentiating the embedding $X \to X'$ to obtain an embedding $TX \to TX'$, and then taking Grassmannians. Finally, we construct the top arrow $DX \to DX'$, using the functors of points, as follows. A T-point of DX, by Corollary 2.2, is a map $f: T \to X$, together with a subbundle $S \subset TX_T$ which maps into $\Sigma_T \subset b^* TY_T$. (Here the subbundle $\Sigma_T \subset b^* TY_T$ determines the T-point $fa: T \to G_k(TY)$ uniquely, by the way.) Note that $(TX'|_X)_T = TX'_T$, because f maps into $X \subset X'$. Hence, given a subbundle $S \subset TX_T$ as above, it follows that the composite inclusion

$$S \subset TX_T \subset TX'|_{X_T} = TX'_T$$

gives a subbundle, too, because X is regularly embedded in the smooth X'. Now, by definition, Σ_T is the pullback to T of the universal k-subbundle $\mathcal{S} \subset p^* TY$ on $G_k(TY)$. But consider also the universal subbundle $\mathcal{S}' \subset (p')^* TY$ on $G_k(TY')$, and write f' for the induced T-point $T \to X'$. Commutativity of the top square of diagram (3) gives

$$(f')^*(a')^*(p')^* TY' = (f')^*(b') * TY' = f^* b^* (TY'|_Y) = TY_T,$$

while commutativity of the bottom square of diagram (3) guarantees that the restriction $\mathcal{S}'|_{G_k(TY)}$ is $\mathcal{S}$. Denote by Σ' the universal subbundle on $G_k(TY')$. Then, pulling back to T, we obtain the equality

$$\Sigma_T = \Sigma'_T. \tag{5}$$

It follows at once that our subbundle $S \subset TX'_T$ satisfies the compatibility condition for a T-point of DX'. In other words, the passage from $S \subset TX_T$ to $S \subset TX'_T$ defines a morphism of functors $DX \to DX'$, hence a morphism of schemes, so we have a diagram of the form (4). Because the morphism we have just defined is clearly injective on T-points, it is an embedding. Finally, a direct inspection shows that (4) commutes, and the proposition follows.

For a given embedding i of triangles, denote by Di the embedding of derived triangles provided by the construction in Proposition 2.4. We call the image of Di the *direct image triangle* given by i.

Corollary 2.5. (Functoriality) *Given a composition*

$$X \xrightarrow{i} X' \xrightarrow{j} X''$$

of embeddings as in Proposition 2.4, *we have*

$$D(j \circ i) = Dj \circ Di.$$

Proof. This is clear from the construction.

Base change

Denote by Θ a triangle of S-schemes of the form (1). Given a flat map $T \to S$, we can raise the base to T. Because taking Grassmannians commutes with base change, we obtain a new triangle, denoted Θ_T, of the following form.

$$\begin{tikzcd} X_T \arrow[r, "a"] \arrow[d, "b_T"'] & G_k(TY_T) \arrow[dl, "p"] \\ Y_T & \end{tikzcd}$$

Applying the derived triangle construction to Θ_T, we obtain a derived triangle, denoted $D(\Theta_T)$. On the other hand, we shall denote by $(D\Theta)_T$ the triangle of T-schemes obtained by base change from the derived triangle $D\Theta$ of S-schemes.

Proposition 2.6. (Base change) *For a flat map $T \to S$ as above, the derived triangle construction commutes with base change to T, that is, we have an isomorphism of triangles*

$$(D\Theta)_T \longrightarrow D(\Theta_T),$$

which is functorial for S-maps $T' \to T$.

Proof. The construction of the derived triangle via the Semple bundle commutes with flat base change and is functorial in T; we need flatness only to ensure that the inclusions of subbundles remain inclusions after the base is raised.

3. Cusps and sections

Here we describe some of the geometry of a derived scheme DX, to prepare for later applications.

The universal bundle

By construction, $DX = G_k(\mathcal{F})$; as in Proposition 2.3 we shall write $\mathcal{U}$ for the universal k-subbundle of the pullback $\mathcal{F}_{DX}$ of the Semple bundle to DX. Repeating the derived triangle construction, we obtain $D(DX)$, the next derived scheme, as follows. First, write $\mathcal{S}'$ for the universal k-subbundle of $TX_{G_k(TX)}$, and set

$$\Sigma' = (Da)^* \mathcal{S}',$$

a k-subbundle of $(Db)^*TX$. Second, define $\mathcal{F}'$ to be the fiber product in the diagram

$$\begin{tikzcd} \mathcal{F}' \arrow[r] \arrow[d] & T(DX) \arrow[d, "\partial(Db)"] \\ \Sigma' \arrow[r, hook] & (Db)*TX \end{tikzcd}$$

of bundles on DX. Then we have $D(DX)=G_k(\mathcal{F}')$.

Proposition 3.1. *We have*

$$\mathcal{U}=\Sigma'$$

as subbundles of $(Db)^*TX$.

Proof. The inclusion $DX=G_k(\mathcal{F})\subset G_k(TX)$ corresponds, by the functor of points, to the composite inclusion $\mathcal{U}\subset\mathcal{F}_{DX}\subset TX_{DX}$. Hence $\mathcal{U}$ is the pullback of the universal subbundle $\mathcal{S}'$ on $G_k(TX)$, as we needed to show.

The cuspidal subschemes

In the triangle (1) of §2, under our standing assumptions, denote by Λ the pullback to X of the universal quotient on $G_k(TY)$, so we have a short exact sequence

$$0\longrightarrow\Sigma\longrightarrow b^*TY\longrightarrow\Lambda\longrightarrow 0.$$

Because the kernel of $\partial b\colon TX\to b^*TY$ is the relative tangent bundle $T_{X/Y}$, it follows that the fiber square which defines the Semple bundle $\mathcal{F}$ extends to give the following commutative diagram with exact rows and columns.

(1)
$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & T_{X/Y} \arrow[r, equal] \arrow[d] & T_{X/Y} \arrow[d] & & \\ 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & TX \arrow[r] \arrow[d, "\partial b"] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Sigma \arrow[r, "\iota"] \arrow[d] & b^*TY \arrow[r] \arrow[d] & \Lambda \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

Write C for the cuspidal divisor of DX. The image of $G_k(T_{X/Y})$ in $DX = G_k(\mathcal{F})$ parametrizes the k-planes in $\mathcal{F}$ which ∂b maps to zero in b^*TY, so we have an inclusion $G_k(T_{X/Y}) \subset C$. In the special case $k=1$, it is clear that $G_k(T_{X/Y}) = C$; on the other hand, for $k>1$ we have a proper inclusion.

We define the *noncuspidal subscheme* of DX to be $DX \setminus C$, denoted DX_{nc}. It is an X-scheme via the restriction of Db. Next, we construct a chain of subschemes between $G_k(T_{X/Y})$ and C. Write α for the composite map

$$\mathcal{U} \hookrightarrow \mathcal{F}_{DX} \longrightarrow \Sigma_{DX}$$

of bundles of rank k on DX, where the second arrow comes from the left-hand column of diagram (1). Then, for $i=1, \dots, k$, we define C_i to be the ith degeneracy locus

$$C_i = \{x \in DX \mid \operatorname{rank}(\alpha(x)) \le i\},$$

by definition [F, p. 243] the zero scheme of $\bigwedge^{i+1} \alpha$. Clearly the C_i define a decreasing sequence of subschemes of DX, with $C_k = DX$, $C_{k-1} = C$, and $C_0 = G_k(T_{X/Y})$, as a glance at the defining ideals will show.

For the next statement, write $\mathcal{V}$ for the universal quotient of $\mathcal{F}_{DX}$, so we have the short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{F}_{DX} \longrightarrow \mathcal{V} \longrightarrow 0 \tag{2}$$

of bundles on DX.

Proposition 3.2. *The map $\alpha: \mathcal{U} \to \Sigma_{DX}$ is an isomorphism precisely on the open set DX_{nc}. Further, the exact sequence (2) splits over DX_{nc}, to give an isomorphism*

$$\mathcal{V}|_{DX_{\mathrm{nc}}} \xrightarrow{\sim} (T_{X/Y})_{DX_{\mathrm{nc}}}.$$

Proof. The first assertion is clear. For the second, restrict (2) to DX_{nc} and then use this assertion: we obtain an embedding $\Sigma_{DX_{\mathrm{nc}}} \to \mathcal{F}_{DX_{\mathrm{nc}}}$ which splits the projection $\mathcal{F} \to \Sigma$ over DX_{nc}. The other summand is at the same time $\mathcal{V}|_{DX_{\mathrm{nc}}}$ and $(T_{X/Y})_{DX_{\mathrm{nc}}}$ by diagram (1), and this completes the proof.

Set $n = \dim(DX)$ and set $m = n - (k-i)^2$, the expected dimension of C_i. Then denote by $\mathbf{D}_i$ the corresponding degeneracy class $\mathbf{D}_i(\sigma) \in A_m(C_i)$, and by $\mathbf{C}_i$ its image in $A_m(X)$. By the Thom–Porteous formula [F, p. 254], we have

$$\mathbf{C}_i = \Delta_{k-i}^{(k-i)} c(\mathcal{U} - \Sigma_{DX}) \cap DX, \tag{3}$$

where $\Delta_q^{(p)}(c)$ denotes the determinant of the $p \times p$ matrix $(c_{q+\beta-\alpha})_{1\leq\alpha,\beta\leq p}$. In particular, for $i=k-1$ we obtain

$$c_1(\mathcal{O}_X(C)) = c_1(\mathcal{U}) - c_1(\Sigma) = c_1\Big(\bigwedge^k \mathcal{U}\Big) - c_1\Big(\bigwedge^k \Sigma\Big).$$

Denote by $\mathcal{O}_{DX}(1)$ the Plücker class on $DX = G_k(\mathcal{F})$; then $c_1\big(\bigwedge^k \mathcal{U}\big) = \mathcal{O}_{DX}(-1)$. Similarly, denote by $\mathcal{O}_X(1)$ the dual of $\bigwedge^k \Sigma$. Then the last relation gives

$$\mathcal{O}_{DX}(1) = \mathcal{O}_{DX}(-C) \otimes (Db)^* \mathcal{O}_X(1) \tag{4}$$

in $\operatorname{Pic}(DX)$. For a simpler proof of (4), we can apply the standard formula for the exceptional class [RS1, Lemma 4.5, p. 1250] to the blowup P in the first construction of §2, and then restrict to DX.

Sections

By Proposition 2.3, sections σ of the structure map $Db\colon DX \to X$ correspond canonically to k-subbundles $\mathcal{S} \subset \mathcal{F}$ over X. For a given subbundle $\mathcal{S}$, corresponding to a section σ, denote by α the composite map $\mathcal{S} \hookrightarrow \mathcal{F} \to \Sigma$. Then σ maps into DX_{nc} exactly when α is bijective, so we obtain a natural 1-1 correspondence

$$\{\text{sections } \sigma \text{ of } DX_{\mathrm{nc}} \to X\} \longleftrightarrow \{\text{sections } s \text{ of } \mathcal{F} \to \Sigma\}. \tag{5}$$

In particular, any section σ with image in DX_{nc} corresponds uniquely to a splitting $\mathcal{F} = s(\Sigma) \oplus T_{X/Y}$, where the embedding of $T_{X/Y}$ in $\mathcal{F}$ is canonical, via diagram (1) above, and we recover σ from the splitting as the induced map

$$X = G_k(\Sigma) \xrightarrow{G_k(s)} G_k(\mathcal{F}) = DX.$$

Return now to the general case. In our original triangle, with vertical map $b\colon X \to Y$, let Z denote a locally closed subscheme of Y. By a *section of b supported on Z* we mean a map $\sigma\colon Z \to X$ such that $b\sigma$ is the the inclusion of Z into Y.

Proposition 3.3. *For a subscheme $Z \subset Y$ in the situation above, each section σ of b supported on Z induces, by differentiation, a map $D\sigma\colon Z \to DX_{\mathrm{nc}}$, such that we have*

$$Db \circ D\sigma = \sigma.$$

Proof. We will build a diagram of the following form

(6)
$$
\begin{tikzcd}
\sigma(Z) \arrow[rr, "\varrho"] \arrow[dd, "b"'] \arrow[dr, "a"] & & DX \arrow[dd] \arrow[dr, "Da"] & \\
& G_k(TZ) \arrow[rr, "\gamma"] \arrow[dl] & & G_k(TX) \arrow[dl] \\
Z \arrow[rr, "\sigma"] & & X &
\end{tikzcd}
$$

in stages. Here the left triangle of (6) is obtained by restricting the original triangle

(7)
$$
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "b"'] & G_k(TY) \arrow[dl, "p"] \\
Y &
\end{tikzcd}
$$

to the image subscheme $\sigma(Z)$, and the right triangle is the derived triangle as usual. We define γ to be the map induced by the derivative of σ.

To construct ϱ, we claim that the composite γa factors through DX_{nc}. To do so, pick $z \in Z$ and write $x = \sigma(z) \in X$, so $b(x) = z$. Set $w = \gamma a(x) \in G_k(TX)$. Our claim follows as soon as we can show that w is in the image of the embedding $Da: DX = G_k(\mathcal{F}) \to G_k(TX)$, and that w lies in DX_{nc}. To see this, write Π for the k-plane in $T_x X$ given by w. We need to show that $\partial b(\Pi) = \Sigma_x$, as k-planes in TY. Now $\Sigma_x = a(x)$ by definition, so the chain rule gives

$$\partial b(\Pi) = db(d\sigma(a(x))) = d(b \circ \sigma)(a(x)) = a(x),$$

because σ is a section of b. This proves our claim that γa factors through DX_{nc}, and constructs the map ϱ, such that we have $Da \circ \varrho = \gamma a$. But γ, by construction, is compatible with σ, hence so is ϱ. More precisely, the last compatibility gives

$$Db \circ \varrho = \sigma b,$$

which expresses the commutativity of the back square of (6). To prove the proposition, set $D\sigma = \varrho\sigma: Z \to DX_{\mathrm{nc}}$. Then

$$Db \circ D\sigma = Db \circ \varrho \circ \sigma = \sigma b \sigma = \sigma,$$

as was to be shown.

Set $D^0X=X$, and, for each integer $r>1$, define D^rX to be $D(D^{r-1}X)$, repeating the derived triangle construction. Hence D^rX fits into a derived triangle which we shall write as follows.

$$\begin{tikzcd} D^rX \arrow[r, "D^ra"] \arrow[d, "D^rb"] & G_k(TD^{r-1}X) \arrow[dl, "p"] \\ D^{r-1}X & \end{tikzcd}$$

For any integer $r\geq 0$, denote by $b_r\colon D^rX\to Y$ the composite projection $b\circ Db\circ\ldots\circ D^rb$. We define the *nonsingular part* D^rX_{ns} to be the complement, on D^rX_{nc}, of the union of the inverse images of the cuspidal divisors of DX through $D^{r-1}X$. In particular, D^rX_{ns} is a dense open subscheme of D^rX, and we have $D^1X_{\mathrm{ns}}=DX_{\mathrm{nc}}$.

Corollary 3.4. *For each $r\geq 1$, a section $\sigma\colon Y\to X$ induces a section $\sigma_r\colon Y\to D^rX_{\mathrm{ns}}$ of the projection b_r.*

Proof. We construct σ_r by induction on r. When $r=0$, we take $\sigma_0=\sigma$. When $r=1$, we can apply Proposition 3.3, with $Z=Y$, to the map b in (7). This gives σ_1 as the composite of $D\sigma\colon Z\to DX_{\mathrm{ns}}$ with σ_0. For $r>1$, we may assume that σ_{r-1} has been constructed already. To construct σ_r, let $Z=\sigma_{r-2}(Y)$ in $D^{r-2}X$. Then σ_{r-1} induces a section of $D^{r-1}b\colon D^rX\to D^{r-2}X$ with support on Z, so Proposition 3.3 gives an induced map $\alpha_r\colon Z\to D^rX$. We define σ_r to be the composite $\alpha_r\circ\sigma_{r-1}$. The compatibility condition of Proposition 3.3 shows immediately that σ_r is a section of b_r. This completes the proof.

4. Data

Fix a smooth connected S-scheme X, of dimension d, and choose an integer k between 0 and d. We define a sequence of schemes and maps

(1) $$\ldots \xrightarrow{b_k^{r+2}} D_k^{r+1}X \xrightarrow{b_k^{r+1}} D_k^rX \xrightarrow{b_k^r} D_k^{r-1}X \xrightarrow{b_k^{r-1}} \ldots \xrightarrow{b_k^2} D_k^1X \xrightarrow{b_k^1} X$$

as follows. Set $D_k^1X=G_k(TX)$, with b_k^1 the structure map $G_k(TX)\to X$. We define $b_k^2\colon D_k^2\to D_k^1X$ to be the vertical arrow in the derived triangle associated to the triangle

(2) $$\begin{tikzcd} G_k(TX) \arrow[r, "1"] \arrow[d, "b_k^1"] & G_k(TX) \arrow[dl, "p"] \\ X & \end{tikzcd}$$

For $r>2$, following the same process as at the end of the last section but with a more compact notation, we define $b_k^r\colon D_k^rX \to D_k^{r-1}X$ by induction on r, beginning with the last triangle. More precisely, suppose that we have already constructed $b_k^{r-1}\colon D_k^{r-1} \to D_k^{r-2}$, as part of the following triangle.

$$\begin{tikzcd} D_k^{r-1}X \arrow[r, "a_k^{r-1}"] \arrow[d, "b_k^{r-1}"'] & G_k(TD_k^{r-2}) \arrow[dl, "p"] \\ D_k^{r-2}X & \end{tikzcd}$$

Then the derived triangle provides both $b_k^r\colon D_k^rX \to D_k^{r-1}$ and the next step in the induction. In addition, the construction defines a natural $D_k^{r-1}X$-map

$$D_k^rX \xrightarrow{a_k^r} G_k(TD_k^{r-1}X)$$

at every stage. We call D_k^rX the scheme of *k-dimensional data of order r* on X.

Proposition 4.1. *Each map b_k^r is smooth, of relative dimension*

$$\dim(b_k^r) = k^r(d-k).$$

In particular, we have

$$\dim(D_k^r) = \begin{cases} d+r(d-1) & \text{if } k=1; \\ d+k(d-k)\dfrac{k^r-1}{k-1} & \text{if } k>1. \end{cases}$$

Proof. Both assertions follow directly from Corollary 2.2.

When $k=1$ we shall adopt a simpler notation, and write X_r for the data scheme D_1^rX.

Example 4.2. Consider $\mathbf{P}_r^n = D_k^r\mathbf{P}^n$. In [Se2], Semple investigates explicitly the first nontrivial data schemes, the varieties $\mathbf{P}_2^2$ and $\mathbf{P}_3^2$ of second- and third-order curvilinear data in the plane. The investigation of data on $\mathbf{P}^2$ continues in [CK1], [CK2]. By Proposition 4.1, we have

$$\dim(\mathbf{P}_r^2) = 2+r. \tag{3}$$

Second-order contacts of curves $\mathbf{P}^3$ have been studied recently by Rosselló [R1], [R2], but in a different spirit, based on Hilbert schemes. In the next section, advancing Semple's program, we shall examine $\mathbf{P}_2^3$ in detail, to obtain stronger results. We have

$$\dim(\mathbf{P}_r^3) = 3+2r, \tag{4}$$

again by Proposition 4.1.

Example 4.3. Now take $k=2$, and consider the data scheme $D_2^2\mathbf{P}^n$, which parametrizes the second-order information about moving planes in $\mathbf{P}^n$. We shall be especially interested in the case $n=3$, whose study was begun by Semple in [Se2]. By Proposition 4.1, we have

$$\dim(D_2^r\mathbf{P}^3) = 1+2^{r+1}. \tag{5}$$

In particular, the 9-fold $D_2^2\mathbf{P}^3$ projects onto the 5-fold $D_2^1\mathbf{P}^3 = G_2(T\mathbf{P}^3)$, which is a copy of the incidence correspondence of points and planes. We shall see later that the fiber of $D_2^2\mathbf{P}^3_{\mathrm{nc}}$ over a point p of the incidence correspondence can be identified with the 4-dimensional space of all bilinear forms on the plane in $T\mathbf{P}^3$ determined by p.

Functoriality and base change

Returning to the general theory, we suppose given an embedding

$$X \overset{i}{\hookrightarrow} Y$$

of smooth S-schemes. Then i induces an embedding $di\colon TX \to TY$.

Proposition 4.4. *The embedding $i\colon X \to Y$ induces a commutative ladder of the form*

$$\begin{tikzcd}
\cdots \arrow[r] & D_k^r X \arrow[r, "b_k^r"] \arrow[d] & D_k^{r-1} X \arrow[r] \arrow[d] & \cdots \arrow[r] & D_k^1 X \arrow[r, "b_k^1"] \arrow[d] & X \arrow[d] \\
\cdots \arrow[r] & D_k^r Y \arrow[r, "b_k^r"] & D_k^{r-1} Y \arrow[r] & \cdots \arrow[r] & D_k^1 Y \arrow[r, "b_k^1"] & Y
\end{tikzcd}$$

where all the vertical maps are embeddings.

Proof. This follows directly from Proposition 2.4 by induction on r, beginning with the map of triangles (2) given by i and di.

We denote by i_* the ladder of embeddings given by Proposition 4.4.

Corollary 4.5. *The commutative ladder i_* is functorial in the embedding i and commutes with any flat base change $T \to S$.*

Proof. This follows from Proposition 2.4 and Corollary 2.5.

Lifts

Suppose $k = \dim(X)$ and that $i: X \to Y$ is an embedding. Then one checks immediately that all the maps b^r_k reduce to isomorphisms, giving compatible S-embeddings

$$X \xhookrightarrow{i_r} D^r_k Y,$$

for all r. We shall call i_r and its image $i_r(X)$ the rth *lift* of i and X, respectively. When $k = \dim(X)$, we shall denote the rth lift of X by X_r. This direct global construction generalizes various curvilinear lifting constructions in [Se2], [RS1], [RS2], [RS3], [CK1] and [CK2], all defined first locally and then pieced together. The rth lift of X carries the rth order information given by the embedding i. In particular, the first lift is the Gauss map

$$X \longrightarrow G_k(TY).$$

Each lift is functorial in the embedding i, and is compatible with any flat base change $T \to S$.

Families

We now consider families of subvarieties. Let $\mathcal{X}$ denote the total space of a family of subvarieties of a variety Y, parametrized by an S-scheme T, which we shall assume is flat over S. More precisely, $\mathcal{X}$ is a T-scheme, and we have a T-inclusion

$$\mathcal{X} \subset Y \times T,$$

where $Y \times T$ is a T-scheme via the second projection. We assume that $\mathcal{X}$ is smooth over T, and write X_t for the fiber of $\mathcal{X}$ at closed point $t \in T$. By our assumptions, each X_t is a smooth variety. Write $T\mathcal{X}$ for the relative tangent bundle $T_{\mathcal{X}/T}$. Working over T instead of S, beginning with the triangle

$$\begin{tikzcd} G_k(T\mathcal{X}) \arrow[r, "1"] \arrow[d, "b^1_k"'] & G_k(T\mathcal{X}) \arrow[dl, "p"] \\ \mathcal{X} & \end{tikzcd}$$

of T-schemes, we obtain data schemes $D^r_k \mathcal{X}$, and embeddings

$$D^r_k(\mathcal{X}) \hookrightarrow D^r_k(Y \times T).$$

Proposition 4.6. *For $t \in T$, the derived schemes $D_k^r(X_t)$ form an algebraic family of subschemes of the derived scheme $D_k^r Y$, with total space $D_k^r(\mathcal{X})$.*

Proof. By invariance under the flat base change $T \to S$, we have $D_k^r(Y \times T) = (D_k^r Y) \times T$, and by invariance under the base change $\{t\} \to T$, we have $(D_k^r \mathcal{X})_t = D_k^r(X_t)$. The proposition now follows by functorality of direct images.

Cuspidal subschemes

For X as above, denote by C^r the cuspidal divisor of $D_k^r X$, and by C_i^r the cuspidal subschemes of §3. (This generalizes the "divisors at infinity" of [CK1] and [CK2].) It is clear by the construction that these subschemes are functorial for direct images, compatible with flat base change, and therefore move naturally in families. More precisely, for a family $\{X_t\}_{t \in T}$ of subschemes of Y as above, with total space

$$\mathcal{X} \subset Y \times T,$$

denote by $\mathcal{C}_k^r$ the cuspidal subscheme of $\mathbf{X}/T$, and write $(C_k^r)_t$ for the cuspidal subscheme of $D_k^r X_t$. Then the family $\{(C_i^r)_t\}_{t \in T}$ is algebraic, with total space $\mathcal{C}_i^r/T$. The corresponding cycle classes, denoted

$$\mathbf{C}_i^r \in A_m(\mathcal{C}_i^r),$$

where m is the dimension over T, enjoy similar compatibilities.

Flat data on $\mathbf{P}^n$

To consider data schemes over the base space $X = \mathbf{P}^n$, fix an integer k with $0 < k < n$. Our goal is to construct a canonical, global section of the projection $D_k^2 \mathbf{P}^n \to D_k^1 \mathbf{P}^n$, assigning to each first-order datum a second-order datum corresponding to a flat or inflectional k-fold in $\mathbf{P}^n$. Our construction depends on special features of $\mathbf{P}^n$.

The first-order data of dimension k on $\mathbf{P}^n$ are parametrized by the first-order data space $D_k^1 \mathbf{P}^n = G_k(T\mathbf{P}^n)$, the Grassmann bundle of k-planes in the tangent bundle $T\mathbf{P}^n$. We have a canonical $\mathbf{P}^n$-isomorphism from $D_k^1 \mathbf{P}^n$ to the incidence correspondence

$$I = \{(p, \Pi) \in \mathbf{P}^n \times G_k(\mathbf{P}^n) \mid p \in \Pi\}$$

of points and k-planes in $\mathbf{P}^n$. Indeed, at a given point $p \in \mathbf{P}^n$, any k-plane $\Pi \subset \mathbf{P}^n$ through p determines the k-plane $T_p \Pi \subset T_p \mathbf{P}^n$. Conversely, any k-plane in $T_p \mathbf{P}^n$ is clearly represented by a unique k-plane Π. To obtain an isomorphism of schemes over $\mathbf{P}^n$, we only need to note that the correspondence just defined works equally well for the point functors. We therefore obtain a natural projection

$$D^1_k\mathbf{P}^n \xrightarrow{\pi} G_k(\mathbf{P}^n),$$

clearly a smooth map. Further, comparing the definitions, it is easy to see that the relative tangent bundle $T_\pi = T_{D^1_k\mathbf{P}^n/G_k(\mathbf{P}^n)}$ is the universal k-subbundle $\Sigma \subset p^*T\mathbf{P}^n$, up to a twist. Indeed, for each k-plane $\Pi \subset \mathbf{P}^n$, we have $\pi^{-1}(\Pi) \cong \{(x,\Pi) \in I \,|\, x \in \Pi\}$, and the relative tangent vectors over Π are, by definition, tangent to $\pi^{-1}(\Pi)$, and hence give Σ.

Write $\mathcal{F}$ for the Semple bundle which defines the second-order data scheme $D^2_k\mathbf{P}^n$ as $G_k(\mathcal{F})$. The inclusion of T_π in $TD^1_k\mathbf{P}^n$ induces an inclusion $\Sigma = T_\pi \hookrightarrow \mathcal{F}$, and this, in turn, induces a section

$$D^1_k\mathbf{P}^n = G_k(\Sigma) \xrightarrow{s} G_k(\mathcal{F}) = D^2_k\mathbf{P}^n, \tag{6}$$

of the structure map $b^2_k \colon D^2_k\mathbf{P}^n \to D^1_k\mathbf{P}^n$. We define the set of *flat data* in $D^2_k\mathbf{P}^n$ to be the image of the section s.

Heuristically, we can represent a second-order datum by a k-parameter first-order motion Δ of a point (x,Π) of I, as described in the introduction. Then Δ represents a flat datum exactly when the motion of the tangent k-plane Π is *stationary*. When $k=1$, this happens at flexes of embedded curves. When $k=2$, this happens at planar points of embedded surfaces.

5. Coordinate representations

Fix a smooth ambient space Y; our goal here will be to provide local coordinates on the data schemes D^r_kY which will allow us to give natural descriptions of the lifts of smooth k-folds in Y. In this way we will later recover the classical Gauss map and second fundamental form as local coordinate representations of global data objects. We begin at the triangle level, and later specialize to data schemes.

Coordinates in $G_k(TY)$

Fix a triangle

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & G_k(TY) \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

as before, but this time (as will be the case for data schemes) assume that the map a is an embedding. Fix a smooth k-fold $W \subset Y$; we shall assume also (as will be the case for lifts to data schemes) that the Gauss map $\gamma\colon W \to G_k(TY)$, defined by the assignment $w \mapsto T_w W$, factors through X. In other words, we have an embedding $i\colon W \to X$ such that $a \circ i = \gamma$ and $b \circ i$ is the inclusion $W \hookrightarrow Y$.

Fix a base point $w_0 \in W$; we begin by choosing a system of local parameters

$$t_1, \dots, t_k, u_1, \dots, u_l$$

in $\mathcal{O}_{Y,w_0}$ such that we can define W in Y, near w_0, by an equation of the form

$$u = f(t).$$

Here we write $t = (t_1, \dots, t_k)$ and $u = (u_1, \dots, u_l)$, where $f(t)$ is a power series in the t_i, and $k + l = \dim(Y)$. In characteristic zero, any smooth W admits such a representation; in characteristic $p > 0$ we shall restrict attention to those W which do.

To define local coordinates in $G_k(TY)$, we turn first to TY. Passing to a local trivialization in the usual way, a point of TY near the fiber over w_0 has the form

$$(t, u; \bar{t}, \bar{u}), \tag{1}$$

where $\bar{t} = (\bar{t}_1, \dots, \bar{t}_k)$ and $\bar{u} = (\bar{u}_1, \dots, \bar{u}_l)$ give coordinates in the fiber, and we use a semicolon to distinguish the fiber from the base. The local parameters $\bar{t}_1, \dots, \bar{t}_k$, $\bar{u}_1, \dots, \bar{u}_l$, by definition, trivialize TY in a Zariski neighborhood, denoted U, of the point w_0 in Y. Write P (resp. Q) for the subbundle of TY with coordinates $\bar{t}_1, \dots, \bar{t}_k$ (resp. $\bar{u}_1, \dots, \bar{u}_l$), so that we have a splitting $TY = P \oplus Q$, where P (resp. Q) has rank k (resp. l). In particular, the k-plane in $T_{w_0}Y$ with equation $\bar{u} = 0$ is the geometric fiber P_{w_0}.

Over U, the Y scheme $\mathrm{Hom}_Y(P, Q)$ embeds in $G_k(TY)$ as a standard open neighborhood of the k-plane P_{w_0}. In this embedding, each homomorphism $P \to Q$ maps to its graph, a k-plane in TY; in particular, the family of k-planes in TY given by P corresponds to the zero-map. In our coordinates, each element of $\mathrm{Hom}_Y(P, Q)$ can be represented in the form

$$\bar{u} = \alpha \bar{t}. \tag{2}$$

Here α denotes an $l \times k$ matrix $[\alpha_{i,j}] = [\alpha_{i,j}(t)]$ of regular functions of t. Because α determines its k-plane uniquely as a function of t, we can therefore write local coordinates on $G_k(TY)$ in the form

$$(t, u; \alpha), \tag{3}$$

with α as in (2). In these coordinates, we can describe the image $i(W)$ near w quite simply, by means of the local parametrization

$$t \longmapsto (t, f(t); Df(t)),$$

where $Df(t)$ denotes the derivative of f at $t=0$, here represented by the Jacobian matrix. In other words, the fiber coordinate α is given by the $l\times k$ matrix $[\partial u_i/\partial t_j]$.

The lift Di

Denote by $D\gamma\colon W\to G_k(TX)$ the Gauss map of the lifted embedding $i\colon W\to X$. Thus $D\gamma(w)$ is the k-plane in $T_{i(w)}X$ tangent to the image $i(W)$ at the point $i(w)$, for all $w\in W$.

Proposition 5.1. *The map $D\gamma$ factors through $DX\subset G_k(TX)$.*

Proof. The inclusion $W\hookrightarrow Y$ and the embeddings i and γ, the latter viewed as a map $G_k(TW)=W\to G_k(TY)$, define an embedding of triangles, of the following form.

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr, "1"] \arrow[dd, "1"'] & & X \arrow[dr, "a"] \arrow[dd] & \\
& G_k(TW) \arrow[rr, "\gamma"] \arrow[dl, "p"] & & G_k(TY) \arrow[dl, "p"] \\
W \arrow[rr] & & Y &
\end{tikzcd}$$

Hence the direct image construction gives an embedding $W=DW\to DX$. To obtain $D\gamma$, and complete the proof, we compose with the inclusion of DX in $G_k(TX)$.

Denote by Di the embedding $W\to DX$ given by Proposition 5.1. For the derived triangle

$$\begin{tikzcd}
DX \arrow[r, "Da"] \arrow[d, "Db"'] & G_k(TX) \arrow[dl, "p"] \\
X &
\end{tikzcd}$$

where Da is again an embedding, we now have the embedding $i\colon W\to X$ and a lifted embedding $Di\colon W\to DX$. These enjoy compatibilities analogous to those of the previous situation, namely $Da\circ Di=D\gamma$ and $Db\circ Di=i$. Hence we can iterate.

Coordinates for TX and $\mathcal{F}$

To write Di in coordinates, we first write the map $D\gamma$ in coordinates, and then describe the factorization through DX in terms of these coordinates. We have assumed that the map a embeds X in $G_k(TY)$, so, working as before, points of TX can be written in the form

$$(t, u, \alpha; \bar{t}, \bar{u}, \bar{\alpha}). \tag{4}$$

Here the coordinates t, u, α are understood to be constrained to X, and again the semicolon distinguished the new fiber coordinates, $\bar{t}, \bar{u}, \bar{\alpha}$. The key observation is that the Semple bundle $\mathcal{F} \subset TX$ is defined, even in the larger space $G_k(TY)$, by equation (2) above. Indeed,the map ∂b is given by the assignment

$$(t, u, \alpha; \bar{t}, \bar{u}, \bar{\alpha}) \longmapsto (t, u, \alpha; \bar{t}, \bar{u}),$$

and (2) defines the k-plane in b^*TY assigned by $a: X \to G_k(TY)$ to the point (t, u, α) of X, that is, (2) defines Σ on U.

It follows that $\bar{t}, \bar{\alpha}$ give local coordinates for the fibers of $\mathcal{F}$.

Coordinates in $DX = G_k(\mathcal{F})$

We follow the same procedure as we did for writing coordinates in X. This time, to describe k-planes in $\mathcal{F}$ as graphs of suitable linear maps, we begin by choosing suitable coordinates to obtain a local trivialization of $\mathcal{F}$. Denote by Π a k-plane in TX, over $(t, u, \alpha) \in X$, which is general in the sense that the projection $(\bar{t}, \bar{u}, \bar{\alpha}) \mapsto \bar{t}$ maps Π onto the k-dimensional space of all $\bar{t}$. If also $\Pi \subset \mathcal{F}$, the projection $\mathcal{F} \to \Sigma$ then carries Π isomorphically onto the fiber of Σ at (t, u, α), which happens exactly when Π represents a point of DX_{nc}. Such a Π can be given as the graph of a linear function of $\bar{t}$, of the form

$$\bar{t} \longmapsto L(\bar{t}) = (\bar{u}, \bar{\alpha}), \tag{5}$$

where $\bar{u}$ and $\bar{\alpha}$ are linear in $\bar{t}$, with coefficients regular in t. On the one hand, for the k-plane given by (5) to lie in $\mathcal{F}$, the linear function $\bar{u} = \bar{u}(t)$ must be given by (2). On the other hand, $\bar{\alpha} = \bar{\alpha}(\bar{t})$ can be written explicitly as

$$\bar{\alpha} = \beta \bar{t},$$

where β denotes a $kl \times k$ matrix, most naturally indexed in the form

$$\beta = [\beta_{i,(\eta,\zeta)}],$$

for $i=1,\dots,l$ and $\eta,\zeta=1,\dots,k$. Here, as the index i runs from 1 to l, we obtain β as a vertical stack of $k\times k$ matrices, each of the form $\beta_i=[\beta_{i,(\eta,\zeta)}]$.

With this notation, each point of our neighborhood on $DX=G_k(\mathcal{F})\hookrightarrow G_k(TX)$ is given, over $(t,u,\alpha)\in X$, by a linear map

$$\bar{t}\mapsto \begin{bmatrix}\alpha\\ \beta\end{bmatrix}\bar{t}.$$

Here $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ denotes the $(l+kl)\times k$ matrix obtained by placing α over β, and we take a matrix product on the right. The first l rows give $\bar{u}$, while the remaining kl rows give $\bar{\alpha}$, repectively, as functions of $\bar{t}$. Combining our observations and omitting redundant terms, our coordinates on DX take the very simple form

$$(t,u,\alpha,\beta), \tag{6}$$

where α (resp. β) is an $l\times k$ (resp. $kl\times k$) matrix with entries regular in t.

Finally, it is easy to check that the lift Di is given parametrically, in these coordinates, by the assignment

$$t\longmapsto (t,u,Df,D^2f), \tag{7}$$

where D^2f denotes the second derivative of f, represented by the usual matrix

$$\beta=[\beta_{i,(\eta,\zeta)}]=\left[\frac{\partial}{\partial t_\eta}\alpha_{i,\zeta}\right]=\left[\frac{\partial^2 u_i}{\partial t_\eta \partial t_\zeta}\right].$$

In particular, β is *symmetric* in η and ζ.

Lifts to data schemes

We now specialize to data schemes. Consider the situation of §4, for a given smooth k-dimensional subscheme $X\subset Y$. To write down the rth lift $i_r\colon X\to D^r_kY$ in local coordinates, we represent X as the graph of an equation $u=f(t)$ for suitable local parameters on Y, and then iterate the construction of coordinates given above through the tower of data schemes. Coordinates on D^r_kY take the form

$$t\longmapsto (t,u,u',\dots,u^{(r)}), \tag{8}$$

where each $u^{(q)}$ is a matrix representation of a suitable q-linear map. To be precise, denote by A the k-dimensional vector space with coordinates $\{\bar{t}_i\}$, and write B for the l-dimensional vector space with coordinates $\{\bar{u}_j\}$. Then $u^{(r)}$ represents a linear map $A \to \mathrm{Hom}(A^{\otimes r-1}, B)$, that is, an element of $\mathrm{Hom}(A^{\otimes r}, B)$.

It follows that the rth lift $i_r(X)$ of the embedded k-fold X is given parametrically by the assignment

$$(9) \qquad t \longmapsto (t, f(t), Df(t), \dots, D^r f(t)),$$

so i_1 is the Gauss map $X \to D_k^1 Y$, hence i_2 is its first derivative, and so on. In particular, all the forms D^q which appear here are *symmetric*. In this way, in the special case $r=2$, we recover the local second fundamental form II of [GH] as the derivative of the classical Gauss map when $Y=\mathbf{P}^n$. Further, (9) shows that any point of $D_k^r X_{\mathrm{nc}}$ can be given by a suitable regular function $u=f(t)$, and we can even take $f(t)$ to be a polynomial of degree $\leq r$. This explains why we call such data *noncuspidal*.

Coordinates for cuspidal data

To obtain cuspidal data explicitly as limits of noncuspidal data, we must look directly at the k-planes in TX which define them. The coordinates we obtained for lifts above will serve this purpose well.

Example 5.2. (A space curve with an ordinary cusp.) Consider the singular rational curve W in $Y=\mathbf{P}^3$ parametrized, in the affine coordinates t, u_1, u_2, by the assignment

$$\lambda \longmapsto w(\lambda) = (\lambda^2, \lambda^3, \lambda^4).$$

This curve has as its only singularity an ordinary cusp at the origin, which we take as the base point w_0 in the discussion above. The tangent line to W at $w(\lambda)$, for a given λ, can be given by the equations $u_1=\frac{3}{2}\lambda t$, $u_2=2\lambda^2 t$. In particular, when $\lambda \to 0$, we obtain the t-axis as the cuspidal tangent. The lift of W to the first-order data space $\mathbf{P}_1^3 = PT\mathbf{P}^3$ is therefore given in local coordinates by

$$\lambda \longmapsto (t, u_1, u_2; \alpha_1, \alpha_2) = \left(\lambda^2, \lambda^3, \lambda^4; \tfrac{3}{2}\lambda, 2\lambda^2\right).$$

The derivative of this assignment is represented by the vector

$$(10) \qquad [\bar{t}, \bar{u}_1, \bar{u}_2, \bar{\alpha}_1, \bar{\alpha}_2] = \left[2\lambda, 3\lambda^2, 4\lambda^3, \tfrac{3}{2}, 4\lambda\right].$$

Now the relation (2), in this case

$$[\bar{u}_1, \bar{u}_2] = \left[\tfrac{3}{2}\lambda\bar{t}, 2\lambda^2\bar{t}\right],$$

shows that the second and third entries of (10) are redundant. Hence the line l in $\mathcal{F}$ which represents our second-order datum is given in local coordinates as the span of the vector

$$[\bar{t}, \bar{\alpha}_1, \bar{\alpha}_2] = \left[2\lambda, \tfrac{3}{2}, 4\lambda\right]. \tag{11}$$

Further, when $\lambda \to 0$, the vector (11) limits to $\left[0, \tfrac{3}{2}, 0\right]$. The first zero shows that the singular point w_0 determines a cuspidal second-order datum; indeed, the first coordinate represents the natural projection from l to the $\bar{t}$-line, and the latter, by definition, is the fiber of Σ at the first-order datum given by $w_0 \in W$. Because λ appears with exponent one in this coordinate, we also know that the closure of the lifted curve meets the cuspidal divisor of $\mathbf{P}^3_2$ transversally at w_0. In other words, the ordinary cusp at $w_0 \in W$ counts exactly once as a cuspical datum.

Example 5.3. (A quadric cone.) To illustrate what can happen at a surface singularity, we work out one second-order datum at the the vertex of a quadric cone in detail. Fix an origin $p \in \mathbf{P}^3$, take affine coordinates t_1, t_2, u at p, and consider the cone in $\mathbf{P}^3$, with vertex p, over the affine conic $\lambda \mapsto (1, \lambda, \lambda^2)$. This cone can be defined parametrically by the assignment

$$(\lambda_1, \lambda_2) \longmapsto (t_1, t_2, u) = (\lambda_1, \lambda_1\lambda_2, \lambda_1\lambda_2^2).$$

Since the tangent plane at a given (λ_1, λ_2) is $u = -\lambda_2^2 t_1 + 2\lambda_2 t_2$, the lift to $D_2^2\mathbf{P}^3$ is via

$$(\lambda_1, \lambda_2) \longmapsto (t_1, t_2, u; \alpha_1, \alpha_2) = (\lambda_1, \lambda_1\lambda_2, \lambda_1\lambda_2^2; -\lambda_2^2, 2\lambda_2).$$

This parametrization has as origin the first-order datum (p, Π), where Π denotes the plane $u=0$, which is certainly tangent to our cone at p, its vertex. The Jacobian matrix of the map above is

$$\begin{bmatrix} 1 & \lambda_2 & \lambda_2^2 & 0 & 0 \\ 0 & \lambda_1 & 2\lambda_1\lambda_2 & -2\lambda_2 & 2 \end{bmatrix}.$$

The middle column of this matrix is redundant by relation (2), so the relevant matrix is

$$\begin{bmatrix} 1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & -2\lambda_2 & 2 \end{bmatrix}.$$

Specializing to the origin, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Here the square matrix formed by the first two columns represents the projection from the plane in $\mathcal{F}$ which gives our second-order datum to the space of the coordinates $\bar{t}$ which parametrizes the fiber of Σ over the first-order datum (p, Π). Because the determinant of this 2×2 matrix is zero, we know that the limit second-order datum of our cone, corresponding to (p, Π), is well-defined, and lies in the cuspidal divisor; because the rank of the projection is precisely one, this datum lies in $C{=}C_1$, but not in the smaller cuspidal subscheme C_0. Further, the local representation of Σ as a direct summand of $\mathcal{F}$ produced by our coordinates agrees with the natural global splitting given at the end of §4, so the vanishing of the determinant given by the last two columns shows that the datum at hand also lies in the *parabolic* locus, to be discussed in §8 below.

6. Example: data for curves in $\mathbf{P}^3$

For concreteness, we shall work throughout this section over the base scheme $S{=}\mathrm{Spec}(k)$, where k is a field. Here we consider 1-dimensional data on $\mathbf{P}^3{=}\mathbf{P}^3_S$, that is, *curve elements.*

First-order data

The iteration starts with $\mathbf{P}^3_1$, the space of lines in the tangent bundle $T\mathbf{P}^3$. Denote by $G_1(\mathbf{P}^3)$ the classical Grassmann variety of lines in $\mathbf{P}^3$, and by $b\colon G_1(\mathbf{P}^3)\to \mathbf{P}^3$ its structure map. By the discussion at the end of §4, we identify $\mathbf{P}^3_1$ with the incidence correspondence

(1) $$I=\{(p,l)\in \mathbf{P}^3\times G_1(\mathbf{P}^3)\,|\,p\in l\}$$

of points and lines in $\mathbf{P}^3$.

The projections from $\mathbf{P}^3_1$ to $\mathbf{P}^3$ and $G_1(\mathbf{P}^3)$ give two key divisor classes. Denote by p the divisor class on $\mathbf{P}^3_1$ given by the pullback of a hyperplane in $\mathbf{P}^3$. Concretely, p represents the condition that a point-line pair (p,l) should have p on a given plane. Similarly, the Plücker class on $G_1(\mathbf{P}^3)$ pulls back to a divisor class, denoted g, which represents the condition that l should meet a given line. We have chosen our notation for these classes, by the way, to agree with Schubert's symbols for the corresponding conditions [Sch]. By Martinelli's computations ([M] or [F, Example 14.7.17, p. 277]) for example, we obtain

(2) $$A^{\boldsymbol{\cdot}}\mathbf{P}^3_1=\mathbf{Z}[p,g]/(p^4,g^3-2pg^2+2p^2g).$$

where the second relation is Schubert's incidence formula.

Notice that if we take I to be $G_2(T_{\mathbf{P}^3}(-2))$, then g becomes the tautological class $t=c_1(\mathcal{O}_I(1))$. Indeed, the composite inclusion

$$\Omega_{I/P3}(t-p)\subset b^*(\Omega_{\mathbf{P}^3}(1))\subset H^0(\mathbf{P}^3,\mathcal{O}_{\mathbf{P}^3}(1))\otimes\mathcal{O}_I$$

is the pullback to I of the inclusion of the universal subbundle on $G_1(\mathbf{P}^3)$ into the trivial bundle, so $g=-c_1(\Omega_{I/P3}(t-p))=t$ follows immediately.

This remark allows us to define three more divisor classes on I. Indeed, write α for the class on I corresponding, as above, to the condition that the line l of a pair (p,l) should pass through a given point. We obtain

(3) $$\alpha=c_2(\Omega_{I/P3}(t-p))=g^2-pg+p^2.$$

On the other hand, write β for the divisor class corrresponding to the condition that the pair (p,l) has the line l contained in a given plane. We find

(4) $$\beta=g^2-\alpha=pg-p^2.$$

Finally, write G for the divisor class wich represents the condition that the line l of a given pair (p,l) should coincide with a given line. Then we have

(5) $$G=\tfrac{1}{2}g^4=p^2g^2-2p^3g.$$

Second-order data

Now we determine the intersection ring of the first nontrivial derived scheme, the scheme $\mathbf{P}^3_2$ of second-order curvilinear data on $\mathbf{P}^3$. By Proposition 2.1, we have $\mathbf{P}^3_2=P(\mathcal{F})$, where $\mathcal{F}$ is the fiber product of $T\mathbf{P}^3_1$ and $\Sigma=b^*\mathcal{O}_{\mathbf{P}^3}(-1)$ over $b^*T\mathbf{P}^3$, where $b=b^1_1$ denotes the structure map $\mathbf{P}^3_1\to\mathbf{P}^3$. Twisting the diagram (1) of §3 so that the bottom row becomes the Euler sequence [F, B.5.8, p. 435], we obtain a commutative diagram with exact rows and columns, of the following form.

(6)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T_{\mathbf{P}^3_1/\mathbf{P}^3}(g-2p) \arrow[r, equal] \arrow[d] & T_{\mathbf{P}^3_1/\mathbf{P}^3}(g-2p) \arrow[d] & & \\
0 \arrow[r] & \mathcal{F}(g-2p) \arrow[r] \arrow[d] & T\mathbf{P}^3_1(g-2p) \arrow[r] \arrow[d, "\partial b"] & T_{\mathbf{P}^3_1/\mathbf{P}^3} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{T\mathbf{P}^3} \arrow[r, "\iota"] \arrow[d] & b^*T\mathbf{P}^3(g-2p) \arrow[r] \arrow[d] & T_{\mathbf{P}^3_1/\mathbf{P}^3} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Due to the twist, the bottom row exhibits $\mathcal{O}_{T\mathbf{P}^3}$ as the universal 1-subbundle of $T\mathbf{P}^3(g-2p)$, with $T_{\mathbf{P}^3_1/\mathbf{P}^3}$ as the universal quotient. By definition, $PT_{\mathbf{P}^3_1/\mathbf{P}^3}$ is the cuspidal divisor C on the derived scheme $\mathbf{P}^3_2$; write k for its divisor class. Here is our result:

Proposition 6.1. *The intersection ring of* $\mathbf{P}^3_2$ *is*

$$A^{\cdot}\mathbf{P}^3_2 = \mathbf{Z}[p,g,k]/R,$$

where R denotes the ideal generated by the previous relations p^4 and $g^3-2pg^2+2p^2g$, together with

$$k^3+(5g-6p)k^2+(7g^2-16pg+10p^2)k.$$

Proof. Twisting $\mathcal{F}$ does not change the underlying scheme of $\mathbf{P}^3_2=P(\mathcal{F})$, but it does change the canonical line bundle: on $P(\mathcal{F}(g))$, we have $\mathcal{O}(1)=k$, either by formula (4) of §3 and [F, B.5.5, p. 434], or by the evident generalization of [H, 2.6, p. 371], applied to the left column of the diagram (6) above.

The key observation here is that the first column in the above diagram (6) splits. Indeed, by the construction at the end of §4, we have a section $i\colon \mathbf{P}^3_1 \to \mathbf{P}^3_2$ which associates to any pair (p,l) the tangent direction defined by the motion of the pair (q,l), as the point q moves in l. Because the image of i does not meet the cuspidal divisor C, it follows that $T_{\mathbf{P}^3_1/\mathbf{P}^3}(g-2p)$ is a direct summand of $\mathcal{F}(g-2p)$, with the complementary summand $\mathcal{O}_{T\mathbf{P}^3_1}$ given by the section i. Denote by F the image of this section, and write k' for the class of F, which represents the locus of flex data. We have

$$k' = k^2+(5g-6p)k+7g^2-16pg+10p^2,$$

because k' represents the zero-locus of the section

$$\mathcal{O}_{\mathbf{P}^3_2} \longrightarrow b_1^{2*}(\mathcal{F}(g))(k) \longrightarrow b_1^{2*}T_{\mathbf{P}^3_1/\mathbf{P}^3}(-2p+g+k).$$

This splitting obviously generalizes to second-order data in any projective space; for curvilinear data in $\mathbf{P}^2$, see [CK1].

The second-order data scheme $\mathbf{P}^3_2$ suffices for enumerating second-order contacts among space curves moving in fairly general families, as long as the general members have no flexes. For families of curves whose general members do have flexes, however, all we need to do is blow up the flex locus above, in order to turn it into a divisor. Geometrically, the flex locus is precisely the closed subscheme of second-order data for which the osculating plane is indeterminate, and the blowup associates to each flex datum a given choice of this plane.

Denote by B the blowup of $\mathbf{P}^3_2$ along F. It is easy to see, by the way, that B is the second-order data scheme corresponding to the projection $I' \to \mathbf{P}^3$, where I' is the point-plane incidence correspondence. We now obtain

Proposition 6.2. *The intersection ring of B is*

$$A^{\cdot}B = \mathbf{Z}[p,g,k,e]/R',$$

where R' is the ideal generated by the relations of R, together with the extra relations

$$e^2+(6p-5g)e+k^2+(5g-6p)k+7g^2-16pg+10p^2$$

and

$$ke.$$

Here, by abuse of notation, the residues of p, g, k represent the pullbacks to X of the classes in $\mathbf{P}_2^3$ denoted by the same letters, and the residue of e represents the exceptional divisor.

Proof. This follows directly from [F, Proposition 6.7 and Example 8.3.9]. In detail, consider the natural commutative diagram

$$\begin{tikzcd} \widetilde{F} \arrow[r, "j"] \arrow[d, "g"] & B \arrow[d, "f"] \\ F \arrow[r, "i"] & \mathbf{P}_2^3 \end{tikzcd}$$

where $\widetilde{F}$ is the exceptional divisor; our notation for the maps follows [F].

First we compute the intersection ring of $\widetilde{F}=P(N)$, where N denotes the normal bundle of F in $\mathbf{P}_2^3$. Observe that the restriction of the projection b_1 gives an isomorphism $F \to I$. Since F is the zero-locus of a section of $(b_1^2)^* T_{\mathbf{P}_1^3/\mathbf{P}^3}(-2p+g+k)$, we have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow (b_1^2)^* T_{\mathbf{P}_1^3/\mathbf{P}^3}(-2p+g+k) \longrightarrow \mathcal{I}_F(-6p+5g+2k) \longrightarrow 0$$

on $\mathbf{P}_2^3$. Indeed, the twist on $\mathcal{I}_F$ is precisely the first Chern class of the middle bundle, because we have $c_1(T_{\mathbf{P}_1^3/\mathbf{P}^3})=-2p+3g$. Therefore, $\check{N}$ is the restriction to F of $(b_1^2)^* T_{\mathbf{P}_1^3/\mathbf{P}^3}(4p-4g-k)$, which, under the above isomorphism, is just

$$\check{N} = T_{\mathbf{P}_1^3/\mathbf{P}^3}(4p-4g).$$

Hence we obtain the intersection ring

$$A^{\cdot}(\widetilde{F}) = \mathbf{Z}[p,g,e]/(p^4, g^3-2pg+2p^2g, t^2+(6p-5g)t+(10p^2-16pg+7g^2)).$$

By [F, Proposition 6.7], for each k, we have an exact sequence

$$0 \longrightarrow A_k(F) \xrightarrow{\alpha} A_k(\widetilde{F}) \oplus A_k(\mathbf{P}^3_2) \xrightarrow{\beta} A_k(B) \longrightarrow 0, \tag{7}$$

defined by the maps

$$\alpha(x) = (c_1(E) \cap g^* x, -i_* x),$$
$$\beta(\tilde{x}, y) = j_* \tilde{x} + f^* y,$$

where E denotes the universal quotient bundle in $\widetilde{F} = P(N)$, so we have $c_1(E) = t - 6p + 5g$.

In codimension one, we obtain the classes p, q, k as pullbacks from $\mathbf{P}^3_2$, as well as the class $e = j_*[\widetilde{F}]$. For the multiplicative structure, we apply [F, Example 8.3.9]. As a sample, to compute e^2, we have

$$e^2 = j_*([\widetilde{F}]) \cdot j_*([\widetilde{F}]) = j_*(c_1(j\mathcal{O}_B(\widetilde{F})) \cdot [F] \cdot [F]) = -j_*(t).$$

But now, the exact sequence (7) for $k=5$ gives $\alpha[F] = (t - 6p + 5g, -k')$, hence

$$j_*(t) = j_*(6p - 5g) + fk' = 6pe + 5ge + k^2 + (5g - 6p)k + 7g^2 - 16pg + 10p^2.$$

Incidentally, the last relation of R follows from the two new relations. It is also interesting to note that we now have $e^7 = 0$.

We shall show in a moment that every point of B has a well-defined osculating plane, assigned by a morphism $B \to \check{\mathbf{P}}^3$. Hence we can impose the condition that this plane should pass through a given point, by pulling back a suitable hyperplane in $\check{\mathbf{P}}^3$, and we shall denote by μ the resulting divisor class on B. More precisely, we shall first construct a *rational map*, denoted $\varrho\colon \mathbf{P}^3_2 \to \check{\mathbf{P}}^3$ which assigns the osculating plane on the complement of F. We define ϱ as a composite

$$P(\mathcal{F}(3p-2g)) \longrightarrow P(T_{I/\mathbf{P}^3}(p-2g)) = P(\Omega_{I/\mathbf{P}^3}(p-g)) \hookrightarrow P(b^*(\Omega_{\mathbf{P}^3}(1))) \longrightarrow \check{\mathbf{P}}^3,$$

which we now explain. Up to a twist, the first arrow in ϱ is induced by the projection $\mathcal{F} \to T_{I/\mathbf{P}^3}$ given by the splitting of the first column of (6). Hence this arrow represents a rational map $\mathbf{P}^3_2 \to P(T_{I/\mathbf{P}^3})$, whose domain of definition is the complement $\mathbf{P}^3_2 \setminus F$, and which becomes a morphism when we blow up $\mathbf{P}^3_2$ along F to obtain B. For the identification $P(T_{I/\mathbf{P}^3}(p-2g)) = P(\Omega_{I/\mathbf{P}^3}(p-g))$ which comes next, we use the fact that the relative tangent bundle $T_{I/\mathbf{P}^3}$ has rank 2, so that the perfect pairing

$$T_{I/\mathbf{P}^3} \otimes T_{I/\mathbf{P}^3} \longrightarrow \bigwedge^2 T_{I/\mathbf{P}^3}$$

identifies $T_{I/\mathbf{P}^3}$ with a twist of the dual bundle $\Omega_{I/\mathbf{P}^3}$, and hence, after a twist, gives the required identification. The inclusion which comes next is induced, after a twist, by the bottom row of (6). The final arrow in the composite for β, the projection from the point-plane incidence correspondence to $\check{\mathbf{P}}^3$, is induced by the Euler sequence for $\Omega_{\mathbf{P}^3/S}$. Given the global maps, the reader can now check directly, in local coordinates, that β does indeed assign the osculating plane to each curve (or curve germ) representing a given noninflectional datum.

Returning to the main discussion, the linear system which gives the rational map ϱ corresponds to the tautological class in

$$P(\mathcal{F}(g)(3p-3g)),$$

namely $k-(3p-3g)$. Since ϱ becomes a morphism when we blow up F, the pullback to B of a hyperplane of $\check{\mathbf{P}}^3$ gives the class

$$\mu = k-3p+3g-e. \tag{8}$$

We obtain dual bases for the intersection pairing by straightforward computation, based on the multiplicative structure of $A^{\cdot}B$. In particular, the bases

$$\{p, g, k, e\} \subset A^1(B)$$

and

$$\{e\mu G-3p^3g^2e, k\mu^3p^2+3\mu Gp+3p^3g^2e, \mu Gp+p^3g^2e, -p^3g^2e\} \subset A^6(B)$$

are dual. Therefore, if C is a curve in $\mathbf{P}^3$, with class $[C]$ in B, set

$$\begin{aligned}
m &= \int_B p\cap[C], &&\text{the degree of } C,\\
r &= \int_B g\cap[C], &&\text{the degree of its tangential developable surface,}\\
\beta &= \int_B k\cap[C], &&\text{the number of cusps of } C,\\
f &= \int_B e\cap[C], &&\text{the number of flexes of } C.
\end{aligned}$$

By duality, we obtain

$$[C] = m(e\mu G-3p^3g^2e)+r(k\mu^3p^2+3\mu Gp+3p^3g^2e)+\beta(\mu Gp+p^3g^2e)-fp^3g^2e.$$

To each surface S in $\mathbf{P}^3$, we can associate a cycle on B in the following way: at each smooth point of S we take, for each line contained in the tangent plane, all planes containing it, and, inside each of them, the second-order datum at the point of the hyperplane section. The closure of the set of all these elements, as the given smooth point moves on S, defines a four-dimensional subvariety of B. If now Σ is a general two-dimensional family of surfaces in $\mathbf{P}^3$, write $[\Sigma]$ for the divisor class it represents in B. Then set

$$T=\int_B \mu GP\cap[\Sigma]=\int_B \mu p^3 g^2\cap[\Sigma],$$

the number of times that a surface passes through a given point in a given direction; set

$$K'=\int_B e\mu G\cap[\Sigma],$$

the number of times a given line is a flex for a surface; and set

$$K=\int_B k\mu^3 p^2\cap[\Sigma],$$

the number of times that a given point has a cuspidal direction contained in a given plane. Also observe that we have $p^3g^2e\cap[\Sigma]=0$, because, counting dimensions, it follows immediately that no surface moving in a general two-dimensional family can have a flex at a given general point in a given general direction. Hence we can generalize formula (121a) of [Sch], to find the number of second-order contacts between a general curve C, which we, in contrast to Schubert and Rosselló ([R1], [R2]), allow to have flexes, and the members of a general two-dimensional family Σ of surfaces with the invariants above. We obtain

$$\int_B [C]\cap[\Sigma]=mK'+r(K+3T)+\beta T$$

contacts.

7. Higher-order data, global second fundamental forms

We begin with a derived triangle of the form

$$\begin{tikzcd} DX \arrow[r, "Da"] \arrow[d, "Db"'] & G_k(TX) \arrow[dl, "p"] \\ X & \end{tikzcd}$$

as in §2.

Structure of DX_{nc}

Recall that $DX = G_k(\mathcal{F})$, for the Semple bundle $\mathcal{F}$. By the fiber square which defines $\mathcal{F}$, the universal k-subbundle $\mathcal{U} \subset \mathcal{F}_{DX}$ of Proposition 2.3 fits naturally into a commutative square

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & TX_{DX} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\partial b} \\ \Sigma_{DX} & \xrightarrow{\iota} & b^*TY_{DX} \end{array}$$

of bundles on DX, with α as in §3. By Proposition 3.2, the left vertical α is an isomorphism precisely on the open set DX_{nc}, the complement of the cuspidal divisor.

Proposition 7.1. *The X-scheme DX_{nc} is a principal homogeneous space for the X-group scheme* $\mathrm{Hom}(\Sigma, T_{X/Y})$. *Furthermore, a section*

$$X \xrightarrow{\sigma} DX_{\mathrm{nc}}$$

of the structure map $DX_{\mathrm{nc}} \to X$ determines a unique X-isomorphism

$$\mathrm{Hom}(\Sigma, T_{X/Y}) \xrightarrow{\sim} DX_{\mathrm{nc}}$$

such that the zero-section of $\mathrm{Hom}(\Sigma, T_{X/Y})$ *maps to σ.*

Proof. To show that DX_{nc} is a principal homogeneous space for $\mathrm{Hom}(\Sigma, T_{X/Y})$, fix a T-point $f: T \to X$. On the one hand, a T-point of $\mathrm{Hom}(\Sigma, T_{X/Y})$ over f is a homomorphism $h: \Sigma_T \to (T_{X/Y})_T$. On the other hand, by the discussion above, a T-point of DX_{nc} over f is, in effect, a lift $\phi: \Sigma_T \to TX_T$ of the inclusion $\Sigma_T \hookrightarrow b^*TY_T$. (Indeed, each lift splits $\mathcal{F}_T \to \Sigma_T$, hence gives a sub*bundle* of TX_T.) Because $T_{X/Y} = \ker(\partial b) = \ker(\partial b_{\mathcal{U}})$, the assignment

$$(h, \phi) \longmapsto h + \phi$$

defines an action of the discrete group $\mathrm{Hom}(\Sigma, T_{X/Y})_T$ on the set $(DX_{\mathrm{nc}})_T$. Because any two lifts g will differ by a unique h, the action is transitive and free. Further, the composition law is clearly functorial in X/T. It follows that our assignment defines a transitive, free, group-scheme action of $\mathrm{Hom}(\Sigma, T_{X/Y})$ on DX_{nc}, hence DX_{nc} is indeed a principal homogeneous space for $\mathrm{Hom}(\Sigma, T_{X/Y})$. Now take for T the identity point of DX_{nc}. Then the universal k-subbundle $\mathcal{U}$ gives a canonical lift

$$\Sigma_{DX_{\mathrm{nc}}} \xrightarrow{\phi} \mathcal{F}_{DX_{\mathrm{nc}}}.$$

To obtain a distinguished map $s: \Sigma \to \mathcal{F}$ over X, we pull ϕ back via σ. Given s, a section of $\mathcal{F} \to \Sigma$, the last assertion is clear, by the correspondence (5) of §3.

The second fundamental form

Now we place ourselves under the hypotheses of §4, and apply Proposition 7.1 to triangle (2) of §4, which defines the second-order data scheme D_k^2X. In this case, Σ is the universal k-subbundle of $TX_{G_k(TX)}$, and we have an exact sequence

$$0 \longrightarrow \Sigma \longrightarrow TX_{G_k(TX)} \longrightarrow \Lambda \longrightarrow 0,$$

where Λ denotes the universal quotient.

Proposition 7.2. *The D_k^1X-scheme $D_k^2X_{\mathrm{nc}}$ is a principal homogeneous space for the D_k^1X-group scheme* $\mathrm{Hom}(\Sigma\otimes\Sigma, \Lambda)$. *Further, a section*

$$D_k^1X \xrightarrow{\sigma} D_k^2X_{\mathrm{nc}}$$

of the structure map $D_k^2X_{\mathrm{nc}} \to D_k^1X$ determines a unique X-isomorphism

$$\mathrm{Hom}(\Sigma\otimes\Sigma, \Lambda) \xrightarrow{\sim} D_k^2X_{\mathrm{nc}}$$

such that the zero-section of $\mathrm{Hom}(\Sigma\otimes\Sigma, \Lambda)$ *maps to σ.*

Proof. The relative tangent bundle $T_{G_k(TX)/X}$ is $\mathrm{Hom}(\Sigma, \Lambda)$, so the first assertion of Proposition 7.1 tells us that $D_k^2X_{\mathrm{nc}}$ is a principal homogeneous space for $\mathrm{Hom}(\Sigma, \mathrm{Hom}(\Sigma, \Lambda))$. The target parametrizes bilinear maps $\Sigma\times\Sigma \to \Lambda$, so the first assertion of Proposition 7.2 follows. The second assertion of Proposition 7.2 follows directly from this.

Now suppose that we have an embedding $X \hookrightarrow Y$ of smooth S-schemes, where $\dim(X)=k$. Here we shall write Σ and Λ for the universal bundles as above, but defined on Y. Suppose further that we have a section $\sigma\colon D_k^1Y \to D_k^2Y_{\mathrm{nc}}$, giving an isomorphism as in Proposition 7.2. Then the second lift X_2 maps into $D_k^2Y_{\mathrm{nc}}$, by smoothness. Hence the isomorphism of Proposition 7.2 defines an embedding, denoted II,

$$X \xrightarrow{\mathrm{II}} \mathrm{Hom}(\Sigma_X\otimes\Sigma_X, \Lambda_X),$$

where the subscripts indicate restriction to $X_2=X$.

Corollary 7.3. *Under the assumptions just above, for any $x\in X$, the bilinear form* $\mathrm{II}(x)$ *identifies canonically with the Hessian at x. In particular, the image of* II *consists entirely of symmetric forms.*

Proof. At $x\in X$, the fiber Σ_x identifies with the tangent space T_xX, while Λ_x may be regarded as the normal space at x of X in Y. It follows easily, by differentiating the Gauss map, i.e. the first lift $i_1\colon X \to G_k(TY) = D^1_k Y$, that $\mathrm{II}(x)$ defines the Hessian, because the derivative of i_1 gives the inclusion $i_2\colon X_2 \to D^2_k Y$.

We call the map II the *second fundamental form* on the embedded subscheme X, and write II_x for $\mathrm{II}(x)$. When X moves in a flat family of embedded subschemes $\{X_t\}_{t\in T}$, so does II, by Proposition 4.6. For each $t\in T$, we denote by II_t the second fundamental form on X_t.

Higher-oder data

Now we apply Proposition 7.1 to the triangle which defines the general data scheme $D^r_k X$, again under the hypotheses of §4, for any $r\geq 2$. We continue with the exact sequence

$$0 \longrightarrow \Sigma \longrightarrow TX_{D^1_k X} \longrightarrow \Lambda \longrightarrow 0,$$

on $D^1_k X = G_k(TX)$, which we hold fixed as we climb higher in the tower of data schemes. As before, we write $D^r_k X_{\mathrm{ns}}$ for the dense open subscheme obtained by removing from $D^r_k X$ the pullbacks of the cusp divisors of the data schemes below D^r_k. Here is our main result.

Theorem 7.4. *The $D^{r-1}_k X_{\mathrm{ns}}$-scheme $D^r_k X_{\mathrm{ns}}$ is a principal homogeneous space for the $D^{r-1}_k X$-group scheme* $\mathrm{Hom}(\Sigma^{\otimes r}, \Lambda)_{D^{r-1}_k X_{\mathrm{ns}}}$.

Proof. We shall write $\mathcal{F}_r$ for the pullback to $D^r_k X$ of the Semple bundle on $D^r_k X$ which constructs D^{r+1}_k. First, we have a natural isomorphism

$$(A_r) \qquad \mathcal{U}_r|_{D^r_k X_{\mathrm{ns}}} \cong \Sigma_{D^r_k X_{\mathrm{ns}}}.$$

Indeed, this follows immediately from the first part of Proposition 3.2, by induction on r. Next, we have a natural isomorphism

$$(B_r) \qquad T_{D^r_k X/TD^{r-1}_k X}|_{D^r_k X_{\mathrm{ns}}} \cong \mathrm{Hom}(\Sigma^{\otimes r}, \Lambda)_{D^r_k X_{\mathrm{ns}}}.$$

This time, the construction follows from the second part of Proposition 3.2, again by induction on r. By (A_{r-1}) and (B_{r-1}), we have

$$\begin{aligned}\mathrm{Hom}(\mathcal{U}_{r-1}, T_{D^{r-1}_k X/TD^{r-2}_k X})_{D^{r-1}_k X_{\mathrm{ns}}} &= \mathrm{Hom}(\Sigma, \mathrm{Hom}(\Sigma^{\otimes r-1}, \Lambda))_{D^{r-1}_k X_{\mathrm{ns}}} \\ &= \mathrm{Hom}(\Sigma^{\otimes r}, \Lambda)_{D^{r-1}_k X_{\mathrm{ns}}}.\end{aligned}$$

Here, by Proposition 3.1, the universal bundle $\mathcal{U}_{r-1}$ plays the part of Σ in Proposition 7.1, and the theorem follows.

The multilinear forms in the structure group

$$\mathrm{Hom}(\Sigma^{\otimes r},\Lambda)_{D_k^{r-1}Y_{\mathrm{ns}}}$$

of $D_k^r Y_{\mathrm{ns}}$ allow us to *compare* different nonsingular rth order data over a given point $\xi \in D_k^{r-1}Y$. For example, ξ could be shared by the lifts $i_{r-1}(X)$ and $i_{r-1}(X')$ of two different subvarieties X and X' whose rth lifts differ over ξ, by a canonically determined r-multilinear map from $\Sigma(\xi)$ to $\Lambda(\xi)$.

For second-order data, a section $\sigma\colon D_k^1 X \to D_k^2 X_{\mathrm{nc}}$ gives a canonical trivialization of $D_k^2 X$ over $D_k^1 X$. Such sections appear in practice, as we have seen in §4, formula (6), for $X=\mathbf{P}^n$. However, such sections do not, in general, induce trivializations of the higher-order data bundles, as we observed in §3. Hence, for third- and higher-order data, a description of *individual* data points in terms of multilinear maps must necessarily be local.

Example 7.5. Even the case $Y=\mathbf{P}^2$ makes clear that a multilinear description, as above, of the third-order data must, at best, be local. Indeed, look at the curve given in affine coordinates by $y=x^2+x^4$, at the point $(0,0)$. It is easy to check that the vanishing of the coefficient of x^3 in the local series expansion is not projectively invariant, even under the very simple substitution $x \mapsto x+y$.

Symmetric data

We define $S_k^r X$ to be the closure, in $D_k^r X$, of the set of rth order data given by all possible smooth k-dimensional subschemes W of X. In other words, $S_k^r X$ is the closure of the union, denoted U_k^r, of all possible rth lifts $i_r(W)$.

We now give $S_k^r X$ a natural scheme sutructure. As a set of geometric points, it follows directly from the local description of §5 that $U_k^r X$ is stable for the action of $\mathrm{Hom}(S^r\Sigma,\Lambda)_{D_k^{r-1}X_{\mathrm{ns}}}$ obtained by restricting the free action of the $D_k^r X_{\mathrm{ns}}$-group scheme $\mathrm{Hom}(\Sigma^{\otimes r},\Lambda)_{D_k^{r-1}X_{\mathrm{ns}}}$. But, by the same token, more is true: $\mathrm{Hom}(S^r\Sigma,\Lambda)_{D_k^{r-1}X_{\mathrm{ns}}}$ acts *transitively* on the fibers of $U_k^r X$ over $D_k^{r-1}X$. It follows that there is a unique scheme structure on $U_k^r X$ such that $U_k^r X$ agrees with $\mathrm{Hom}(S^r\Sigma,\Lambda)_{D_k^{r-1}X_{\mathrm{ns}}}$ when restricted to small enough open subsets of their common base space $D_k^{r-1}X_{\mathrm{ns}}$. We shall write $S_k^r X_{\mathrm{ns}}$ for $U_k^r X$ equipped with this scheme structure, and we shall denote by $S_k^r X$ its closure as a subscheme of $D_k^r X$. Because of its construction, we shall call $S_k^r X$ the space of *symmetric* rth order data of dimension k on X, and refer to its subscheme $S_k^r X_{\mathrm{ns}}$ as the space of *nonsingular* symmetric data. Clearly the second is an open subscheme of the first.

The next result follows, by induction on r, directly from Theorem 7.4, the local description of §5, and the scheme structure we have just defined.

Theorem 7.6. *For each $r>1$, we have a natural projection $S_k^r X_{\mathrm{ns}} \to S_k^{r-1} X_{\mathrm{ns}}$, which is a principal homogeneous space for the $S_k^{r-1} X_{\mathrm{ns}}$-group scheme* $\mathrm{Hom}(S^r\Sigma, \Lambda)_{D_k^{r-1} X_{\mathrm{ns}}}$.

It is clear that the symmetric data schemes $S_k^r X$ enjoy the same functorial properties and compatibilities as do the $D_k^r X$, from which they inherit their structure.

8. Example: data for surfaces in $\mathbf{P}^3$

As in §6, we will work over the base scheme $S=\mathrm{Spec}(k)$, where k is a field. Here we consider two-dimensional data on $\mathbf{P}^3$, that is, *surface elements.*

First-order data

The first-order data scheme $D_2^1\mathbf{P}^3 = G_2(T\mathbf{P}^3)$ identifies naturally with the usual point-plane incidence correspondence

$$J = \{(p, P) \in \mathbf{P}^3 \times \check{\mathbf{P}}^3 \mid p \in P\},$$

a smooth 5-fold. Denote by p the pullback to J of the hyperplane class in $\mathbf{P}^3$. Thus p corresponds to the condition that a pair (p, P) should have p on a given plane. Dually, we write P for the pullback to J of the hyperplane class in $\check{\mathbf{P}}^3$. This class represents the condition that the plane P of a pair (p, P) should pass through a given point. Here the intersection ring is

$$A^{\cdot} D_2^1\mathbf{P}^3 = \mathbf{Z}[p, P]/(p^4, P^3 - pP^2 + p^2 P - p^3),$$

where the second relation is, as in the previous case, a classical incidence formula.

Second-order data

Here it will be convenient to work with a slightly modified version of diagram (1) of §3. Twisting the tangent bundle, we write J in the equivalent form

$$J = G_2(T\mathbf{P}^3(-1)).$$

This particular twist gives $b^*\mathcal{O}_{\mathbf{P}^3}(1) = \mathcal{O}(p)$. By abuse of notation, we shall denote by Σ the universal subbundle of $b^*T\mathbf{P}^3(-1)$, and by Λ the universal quotient. Then our diagram, of bundles on J, takes the following form.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & T_{J/\mathbf{P}^3}(-p) \arrow[r, equal] \arrow[d] & T_{J/\mathbf{P}^3}(-p) \arrow[d] & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & TJ(-p) \arrow[r] \arrow[d, "\partial b"] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Sigma \arrow[r, "\iota"] \arrow[d] & b^*(T\mathbf{P}^3(-1)) \arrow[r] \arrow[d] & \Lambda \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
\tag{1}
$$

Here, again by abuse of notation, we have written $\mathcal{F}$ for the pullback, a bundle of rank 4. By the construction at the end of §5, the first column splits, via the map $\Sigma \to \mathcal{F}$ given by differentiating the assignment, to each pair (p, P) of the two-parameter motion of a pair (p', P), where the point p' moves in a line l with $p \in l \subset P$, and, independently, the line l turns through the plane P around the axis p. Further, by our choice of twist, we find $\Lambda = \mathcal{O}(P)$, while for the kernels at the top we have $T_{J/\mathbf{P}^3} = \mathrm{Hom}(\Sigma, \Lambda) = \check{\Sigma} \otimes \Lambda$.

We know that $D_2^2\mathbf{P}^3$ is a smooth 8-fold. Write

$$0 \longrightarrow S \longrightarrow (b_2^2)^*\mathcal{F} \longrightarrow Q \longrightarrow 0$$

for the universal exact sequence on $G_2(\mathcal{F})$, write $c_1, \ldots, c_4$ for the Segre classes of the universal sub-bundle S, and write d_1, d_2 for the Chern classes of the universal quotient bundle. As usual, we shall use the symbols p and P for the pullbacks to $G_2(\mathcal{F})$ of the classes they denote.

Proposition 8.1. *We have*

$$A^{\cdot} D_2^2\mathbf{P}^3 = \mathbf{Z}[p, P, c_1, c_2]/R$$

where the ideal R of relations is generated by the known relations p^4 and $p^3 - p^2P + pP^2 - P^3$, together with the additional relations

$$2p^2P - 2pP^2 - p^2c_1 - P^2c_1 + 2pc_2 - 2Pc_2 + c_1^3 - 2c_1c_2$$

and

$$7p^2P^2-p^2c_1^2-P^2c_1^2-3p^2c_2+8pPc_2-3P^2c_2+2pc_1c_2-2Pc_1c_2-c_1^2c_2+c_2^2.$$

Proof. We have the following Chern polynomials:

$$\begin{aligned} c(\Sigma)&=1+(p-P)t+(p^2-pP+P^2)t^2,\\ c(\check{\Sigma}(P-p))&=1+(3P-3p)t+(3p^2-5pP+3P^2),\\ c(F)&=1+(2p-2P)t+(p^2+P^2)t^2+(-2p^2P+2pP^2)t^3.\end{aligned}$$

These yield

$$\begin{aligned} c_1&=d_1+2p-2P,\\ c_2&=d_2+2pd_1-2Pd_1+3p^2-8pP+3P^2,\\ c_3&=2pd_2-2Pd_2+3p^2d_1-8pPd_1+3P^2d_1-14p^2P+14pP^2,\\ c_4&=3p^2d_2-8pPd_2+3P^2d_2-14p^2Pd_1+14pP^2d_1+23p^2P^2.\end{aligned}$$

Now, eliminating d_1 and d_2, and using the Schubert relations [F, §14.7], we find

$$\begin{aligned} c_3&=2c_1c_2-c_1^3,\\ c_4&=c_1c_3+c_2^2-c_1^2c_2,\end{aligned}$$

and the proposition follows.

Symmetric second-order data

Assume that we have, at a point p in $\mathbf{P}^3$, a smooth analytic branch of a surface S. Choose affine coordinates t_1, t_2, u at p, such that S is given as the graph of a regular function $u=f(t_1,t_2)$ such that $u=0$ defines the tangent plane of S at p. Recall from §5 that we have local coordinates

$$(t_1,t_2,u,\alpha_1,\alpha_2;\beta_{11},\beta_{12},\beta_{21},\beta_{22})$$

for $D_2^2\mathbf{P}^3$. The coordinates before the semicolon represent a point (t_1,t_2,u), together with a plane in the tangent space at that point, defined by the linear equation $\bar{u}=\alpha_1\bar{t}_1+\alpha_2\bar{t}_2$. This tangent plane corresponds to the affine plane $U-u=\alpha_1(T_1-t_1)+\alpha_2(T_2-t_2)$ through p in $\mathbf{P}^3$ whose closure, denoted Π, gives the corresponding point (p,Π) of J. The coordinates after the semicolon give the plane in the subbundle $\mathcal{F}$ of TJ spanned by the rows of the matrix

$$\begin{bmatrix}1 & 0 & \beta_{11} & \beta_{21}\\ 0 & 1 & \beta_{12} & \beta_{22}\end{bmatrix}.$$

Here the first two columns represent the derivative $\bar{t}\mapsto\bar{t}$ of the first coordinate $t\mapsto t$ of our parametization of S, and, as in §5, we only need to consider the four coordinates $\bar{t}_1, \bar{t}_2, \bar{\alpha}_1, \bar{\alpha}_2$, because $\mathcal{F}$ is defined by the equation $\bar{u}=\alpha_1\bar{t}_1+\alpha_2\bar{t}_2$.

Denote by p_{ij} the Plücker coordinate given by the 2×2 minor corresponding to the columns $i+1$ and $j+1$ in the matrix above. Then the symmetry condition $\beta_{12}=\beta_{21}$ can be written in the form

$$p_{02}+p_{13}=0.$$

This final equation, however, makes sense globally, and defines a smooth hyperplane section of the Grassmannian. To interpret the equation $p_{02}+p_{13}=0$ more concretely, note that the global splitting of $\mathcal{F}=\check{\Sigma}(P-p)\oplus\Sigma$ is locally represented by $\{(\bar{t}_1,\bar{t}_2,\bar{\alpha}_1,\bar{\alpha}_2)\}=\{(\bar{t}_1,\bar{t}_2)\}\oplus\{(\bar{\alpha}_1,\bar{\alpha}_2)\}$. Consider now a matrix of the form

$$\begin{bmatrix}\bar{t}_{11} & \bar{t}_{21} & \bar{\alpha}_{11} & \bar{\alpha}_{21}\\ \bar{t}_{12} & \bar{t}_{22} & \bar{\alpha}_{12} & \bar{\alpha}_{22}\end{bmatrix},$$

view the rows of its left 2×2 submatrix as vectors of $\check{\Sigma}(P-p)$, and view the rows of its right 2×2 submatrix as vectors of Σ. Skipping the twist, since we are working locally, the vector $(\bar{t}_{1i},\bar{t}_{2i})$ corresponds to the linear form $(\bar{\alpha}_1,\bar{\alpha}_2)\mapsto\bar{t}_{1i}\bar{\alpha}_1+\bar{t}_{2i}\bar{\alpha}_2$. We now examine the corresponding element of $\mathrm{Hom}(\Sigma,\Sigma)=\check{\Sigma}(P-p)\otimes\Sigma\subset\bigwedge^2\mathcal{F}$, where again we can ignore the twist. As an antisymmetric tensor on $\mathcal{F}$, we have

$$(\bar{t}_{11},\bar{t}_{21})\otimes(\bar{\alpha}_{12},\bar{\alpha}_{22})-(\bar{t}_{12},\bar{t}_{22})\otimes(\bar{\alpha}_{11},\bar{\alpha}_{21});$$

this represents the endomorphism of Σ, denoted ε, defined by the assignment of the matrix product

$$(\bar{\alpha}_1,\bar{\alpha}_2)\longmapsto[\bar{\alpha}_1,\bar{\alpha}_2]\cdot\begin{bmatrix}\bar{t}_{11}\bar{\alpha}_{12}-\bar{t}_{12}\bar{\alpha}_{11} & \bar{t}_{11}\bar{\alpha}_{22}-\bar{t}_{12}\bar{\alpha}_{21}\\ \bar{t}_{21}\bar{\alpha}_{12}-\bar{t}_{22}\bar{\alpha}_{11} & \bar{t}_{21}\bar{\alpha}_{22}-\bar{t}_{22}\bar{\alpha}_{21}\end{bmatrix}.$$

In particular, the matrix representing ε is

$$\begin{bmatrix}p_{02} & p_{03}\\ p_{12} & p_{13}\end{bmatrix},$$

so the condition $p_{02}+p_{13}=0$ boils down to $\mathrm{Tr}(\varepsilon)=0$.

Remark. In Example 5.3, the quadric cone, the sum of minors $p_{02}+p_{13}$ is again zero at the vertex of the cone, so the corresponding cuspidal datum is symmetric as a limit of symmetric data. For this symmetric cuspidal datum, strange to say, the 2×2 submatrix on the right is not symmetric.

Now we interpret, in our new language, several classical terms for special points on an embedded surface such as S. A point is *parabolic* if the corresponding pencil of directions in $\mathcal{F}$ meets the subspace defined by the lift of Σ, which gives the flat data. In particular, even a cuspidal datum could be parabolic, since a pencil of directions could meet both Σ and $\check{\Sigma}(P-p)$; this is a *pinch-point.* A point is *planar* if the pencil of directions in $\mathcal{F}$ is entirely contained in Σ. Finally, we shall say that a point of a surface is *completely cuspidal* if the pencil of directions is contained in $\check{\Sigma}(P-p)$.

Denote $S_2^2\mathbf{P}^3$ the subscheme of $D_2^2\mathbf{P}^3=G_2(\mathcal{F})$ consisting of those second-order data on $\mathbf{P}^3$ having symmetric second fundamental form.

Proposition 8.2. *The intersection ring of $S_2^2\mathbf{P}^3$ is generated by the restriction of the classes p, P, c_1, c_2 (which we shall denote by the same symbols), modulo the known relations p^4 and $p^3-p^2P+pP^2-P^3$, together with the new relations*

$$c_1^2-2c_2+c_1p-c_1P-2pP$$

and

$$c_1^4-12p^2P^2-4p^2Pc_1+4pP^2c_1+5p^2c_1^2-6pPc_1^2+5P^2c_1^2-4pc_1^3+4Pc_1^3.$$

Proof. The Plücker embedding of $G_2(\mathcal{F})$ gives an immersion in $P(\bigwedge^2\mathcal{F})$, and the relative hyperplane section corresponds to the class $c_1(\check{S})=c_1$. Also, we have

$$\bigwedge^2\mathcal{F}=\bigwedge^2(\check{\Sigma}(P-p)\oplus\Sigma)=\mathcal{O}(3P-3p)\oplus\mathcal{O}(p-P)\oplus(\Sigma\otimes\check{\Sigma})(P-p). \tag{2}$$

The symmetry of the second fundamental form is equivalent, as we observed above, to the vanishing of the component $\mathcal{O}(P-p)$ of $(\Sigma\otimes\check{\Sigma})(P-p)$. In other words, the locus $S_2^2\mathbf{P}^3$ is the restriction to $G_2\mathcal{F}$ of the zero locus of the section

$$\mathcal{O}\longrightarrow b^*\mathcal{F}\otimes\mathcal{O}_{P(\bigwedge^2\mathcal{F})}(1)\longrightarrow\mathcal{O}(P-p)\otimes\mathcal{O}_{P(\bigwedge^2\mathcal{F})}(1).$$

In other words, $S_2^2\mathbf{P}^3$ is the zero-locus of a section of $\mathcal{O}(P-p+c_1)$.

Our local computations show that the restriction of this section to any fiber $G_2(\mathbf{P}^3)$ of b is a smooth linear complex, whose intersection ring is generated exactly by the image, under restriction, of the intersection ring of the Grassmannian. Hence we conclude that the intersection ring of $S_2^2\mathbf{P}^3$ is also generated by the restriction of the intersection ring computed in Proposition 8.1. To complete the proof, we simply check that the first relation in Proposition 8.1 is exactly

$$(c_1-p+P)(c_1^2-2c_2+c_1p-c_1P-2pP),$$

which yields our first relation. The second relation follows from the second relation in Proposition 8.1 in a similar way, once we eliminate c_2 from the latter, and the proposition follows.

For explicit computations, note that we can forget the class c_2, which depends on c_1 once we tensor with $\mathbf{Q}$.

Now we interpret some of our classes geometrically. First, the divisor of flexes corresponds, via the decomposition (2), to the vanishing of the coordinate in the component $\bigwedge^2 \Sigma = \mathcal{O}(p-P)$. So, arguing as above, we find that the cuspidal divisor has class

$$k = c_1 + p - P. \tag{3}$$

In the same way, it follows that the divisor of parabolic points has class

$$f = c_1 + 3P - 3p. \tag{4}$$

Combining (3) and (4), we obtain the key relation

$$f + 4p = k + 4P. \tag{5}$$

For a surface S in $\mathbf{P}^3$ of degree d, denote by e the number of points in S in a given plane whose tangent plane passes through a given point, and denote by k and f the degrees of the curves of cuspidal and parabolic points, respectively.

Proposition 8.3. *For a surface S as above, we have*

$$f + 4d = k + 4e.$$

Proof. We simply multiply relation (5) above by the product of p with the class of the lifting of S to $S_2^2\mathbf{P}^3$.

Corollary 8.4. *If S is a general (and thus smooth) surface of degree $d>1$ in $\mathbf{P}^3$, then the degree of the curve of parabolic points is $4d(d-2)$.*

Proof. We just apply Proposition 8.3. Since S is smooth, we have $k=0$. To find e, we fix a point p and a plane P. The condition of the tangent plane to pass through p is equivalent to the vanishing of the polar of S at p, which has degree $d-1$. Hence we obtain e by intersecting S, of degree d, with the above polar, of degree $d-1$, and with the plane P. We obtain $e=d(d-1)$, and hence $f=4e-4d=4d(d-2)$, as was to be shown.

For another derivation of the formula $f=4e-4d=4d(d-2)$, observe that we can obtain the parabolic curve by intersecting S with the locus where the Hessian determinant vanishes.

Planar and completely cuspidal points

A second-order datum represents a completely cuspidal point if all the directions of the approaching point-plane pair limit to cuspidal directions. More precisely, this happens when $S \to b^*\mathcal{F} = b^*\check{\Sigma}(P-p) \oplus b^*\Sigma$ maps into the first factor. Hence the completely cuspidal-point locus of $G_2(\mathcal{F})$ is the zero-locus of the map $S \to b^*\Sigma$. Since we have

$$\begin{aligned} c(b^*\Sigma - S) = & 1 + (c_1 + p - P)t + (c_2 + pc_1 - Pc_1 + p^2 - pP + P^2)t^2 \\ & + (3pc_2 - 3Pc_2 - pPc_1 + 2p^2P - 2pP^2)t^3 + \ldots, \end{aligned}$$

it follows from the Porteous formula [F, Theorem 14.4, p. 254] that the class of this locus is given by

$$\begin{aligned} K = & c_2^2 - pc_1c_2 + Pc_1c_2 - p^2c_2 + 4pPc_2 - P^2c_2 \\ & + p^2c_1^2 - pPc_1^2 + P^2c_1^2 - 3p^2Pc_1 + 3pP^2c_1 + 3p^2P^2. \end{aligned}$$

The pinch-point locus lies in $S_2^2\mathbf{P}^3$, and in fact one can write

$$K = (c_1 - p + P)(c_1c_2 - 2pc_2 + 2Pc_2 + 2p^2c_1 - pPc_1 + 2P^2c_1 - 4p^2P + 4pP^2),$$

so that the class of pinch-points in $S_2^2\mathbf{P}^3$ is

$$K = c_1c_2 - 2pc_2 + 2Pc_2 + 2p^2c_1 - pPc_1 + 2P^2c_1 - 4p^2P + 4pP^2. \tag{6}$$

In the same way, the locus of planar points in $G_2(\mathcal{F})$ is the zero-locus of the map $S \to b^*\check{\Sigma}(P-p)$, and since

$$\begin{aligned} c(b^*\check{\Sigma}(P-p) - S) = & 1 + (c_1 - 3p + 3P)t + (c_2 - 3pc_1 + 3Pc_1 + 3p^2 - 5pP + 3P^2)t^2 \\ & + (-pc_2 + Pc_2 + 2p^2c_1 - 5pPc_1 + 2P^2c_1 + 2p^2P - 2pP^2)t^3 + \ldots \end{aligned}$$

this class is

$$F = (c_1 - p + P)(c_1c_2 - 6pc_2 + 6Pc_2 + 8p^2c_1 - 13pPc_1 + 8P^2c_1).$$

Hence, its class in $S_2^2\mathbf{P}^3$ is given by

$$F = c_1c_2 - 6pc_2 + 6Pc_2 + 8p^2c_1 - 13pPc_1 + 8P^2c_1. \tag{7}$$

Of course, we also have $fK = Fk = 0$.

Suppose now, for example, that we have a one-parameter family of surfaces of degree d, and that we want to compute the number of planar points in the family.

First of all, there is a direct way to compute this number, without the use of data schemes. Write $\mathbf{P}^N$ for the projective space parametrizing surfaces of degree d in $\mathbf{P}^3$. Let h denote the hyperplane class in $\mathbf{P}^3$ and write H for the hyperplane class in $\mathbf{P}^N$, which represents the condition that a surface in our family should pass through a given general point. In the product $X=\mathbf{P}^3\times\mathbf{P}^N$, the incidence relation has class $dh+H$ and a one-parameter family pulls back to aH^{N-1}, where a counts the members of the family passing through a general point. Now a smooth point of a surface $f=0$ is planar when the rank of the Hessian of f is two. This condition, expressed on X, means that the natural symmetric map

$$\mathcal{O}_X^4 \longrightarrow \mathcal{O}_X((d-2)h+H)^4$$

given by the Hessian matrix has rank two. Using [HT], this locus has class $10((d-2)h+H)^3$, so that the class we want is $10aH^{N-1}\cdot(dh+H)\cdot((d-2)h+H)^3$, which is the class of $20(d-2)^2(2d-1)$ points.

To obtain this number from the intersection ring of our variety $S_2^2\mathbf{P}^3$, we consider the relation

$$F=K-2pfk+2Pfk-20(P^3-p^3),$$

which follows readily from the relations above. If our family is general, we have no completely cuspidal points, and there will only be a finite number of singular surfaces, precisely $4a(d-1)^3$, as we can see, for example, with the same method as above. At each singular point there is a tangent cone, hence a one-dimensional family of tangent planes, and only a finite number of these can be parabolic. Hence only a finite number of pinch data should appear, therefore none after we impose the further condition that the singular point lies in a plane, or that the tangent plane should pass through a point. This seems to indicate that the threefold swept out by our family does not meet any of the classes K, pfk, Pfk. Thus the above formula would give us the number $-20(b-a)$, where b is the number of surfaces tangent to a general given plane. To compute the number b, observe that a surface is tangent to a plane P exactly when it intersects P in a singular curve. Hence b is the number of singular curves in the family obtained by intersecting our family of surfaces with the plane P. As before, we obtain that $b=3a(d-1)^2$, and hence $-20(b-a)=-20a(3d^2-6d+2)$, which is negative.

What can we learn from this surprise? Observe first that the difference between the actual number and the one just obtained is $40a(d-1)^4$, which is a multiple of the number of singular points. At each singular point p, however, any plane through p is obtained as a limit of tangent planes to nearby smooth surfaces in the family, so any plane through such a p appears as a *completely cuspidal* datum for the family. It follows that the intersection of our family with the cycle K must be improper, since it has dimension two, not zero. Hence, in order to use our machinery to count contacts between moving surfaces, we would need to modify our parameter space so that the locus of completely cuspidal data will properly intersect the data given by our family. We conjecture that a single blowup will suffice.

9. Connections

Throughout this section, E will denote a vector bundle of rank n on a scheme X, with structure map $p: E \to X$. To define frames in E, we fix an n-dimensional vector space V over the ground field k, and denote by G the linear group $GL(V)$. The tangent bundle TE of E has *two* vector bundle structures: via its structure map $q: TE \to E$, and via the derivative $dp: TE \to TX$. We shall be concerned especially with the second structure.

The frame bundle

Write V_X for the pullback of V under the constant map $X \to \mathrm{Spec}(k)$. The *frame bundle* of E is

$$R = \mathrm{Isom}(V_X, E),$$

the open subscheme of $\mathrm{Hom}(V_X, E)$ whose sections are $\mathcal{O}_X$-isomorphisms $V_X \to E$. In particular, the natural projection $p: R \to X$ is a locally trivial fibration, with fibers isomorphic to G. The group scheme G acts on R by composition on the right. Clearly we have $X = R/G$ as geometric quotient, and R is easily seen to be a principal homogeneous space over X under this action.

We can recover E from R as the associated fiber space with fiber-type V. By definition, in the spirit of [D3, 16.14.7, p. 94, and 20.1.4, p. 237], which treats the C^∞ case, the associated fiber space is

$$R \times^G V = (R \times V)/G,$$

where G acts on the right on $R \times V$, via

$$(r, v) \cdot g = (rg, g^{-1}v)$$

for $g \in G$, $r \in R$, and $v \in V$. Following [D3, 16.14], for $r \in R$ and $g \in G$, we shall often write $r \cdot v$ for the image of (r, v) in $R \times^G V$. Under the identification $E = R \times^G V$, we have $r \cdot v = r(v)$ in E; conversely, mapping $R \times V$ to E by the assignment $(r, v) \mapsto r(v)$, it is easy to check that E is the quotient as claimed.

Connections

From here on, we shall restrict attention to the case $k=\mathbf{C}$. In this setting, V, X and G have natural complex structures, hence, in particular, they are real C^∞ manifolds. Similarly, the algebraic, hence complex vector bundle E and the frame bundle R are C^∞ manifolds as well.

Following the approach of [D3, 17.16.3.1] for smooth manifolds over $\mathbf{R}$, we define a *connection* on E to be a C^∞ morphism

$$TX \times_X E \xrightarrow{\mathsf{C}} TE,$$

over X, such that, for $x \in X$, $\mathsf{k} \in T_x X$ and $\mathsf{u} \in E_x$, we have

(a) $$\mathsf{C}(\mathsf{k},\mathsf{u}) \overset{dp}{\longmapsto} \mathsf{k} \quad \text{and} \quad \mathsf{C}(\mathsf{k},\mathsf{u}) \overset{q}{\longmapsto} \mathsf{u},$$

(b) $$\mathsf{k} \longmapsto \mathsf{C}(\mathsf{k},\mathsf{u}) \text{ gives a } \mathbf{C}\text{-linear map } T_x X \longrightarrow T_{\mathsf{u}} E,$$

and

(c) $$\mathsf{u} \longmapsto \mathsf{C}(\mathsf{k},\mathsf{u}) \text{ gives a } \mathbf{C}\text{-linear map } E_x \longrightarrow TE_{\mathsf{k}},$$

where TE_{k} denotes the fiber $(dp)^{-1}(\mathsf{k})$. We call $\mathsf{C}(\mathsf{k},\mathsf{u})$ the *horizontal lift* of k at u.

For the record, we obtain the usual covariant derivative ∇ from the connection C as follows. Given a global section s of E, and a vector $\mathsf{k} \in T_x X$, write $\mathsf{u}=s(x)$. We have $ds(\mathsf{k}) \in T_{\mathsf{u}} E$, with $dp(ds(\mathsf{k}))=\mathsf{k}$. Given a connection C on E, by (1) we also have $\mathsf{C}(\mathsf{k},\mathsf{u}) \in T_{\mathsf{u}} E$, with $dp(\mathsf{C}(\mathsf{k},\mathsf{u}))=\mathsf{k}$, so the vector $ds(\mathsf{k})-\mathsf{C}(\mathsf{k},\mathsf{u}) \in T_{\mathsf{u}} E$ is *vertical*, that is, tangent to the fiber E_x. Denote by τ_{u} the natural isomorphism $T_{\mathsf{u}}(E_x) \xrightarrow{\sim} E_x$ and set

$$\nabla_{\mathsf{k}} s = \tau_{\mathsf{u}}(ds(\mathsf{k}) - \mathsf{C}(\mathsf{k},\mathsf{u}));$$

this is the covariant derivative.

Remark. This simple observation has important consequences when E is the tangent bundle TX. Then a metric on E is Kähler exactly when the covariant derivative of the associated Riemannian connection is complex-linear [KN, discussion following Corollary 4.4, p. 445]. This happens exactly when the metric induces a connection in our sense, as the last discussion shows. This holds for the tangent bundle on $\mathbf{P}^n_{\mathbf{C}}$, because the standard metric on $\mathbf{P}^n_{\mathbf{C}}$ is Kähler. We shall return to this point shortly.

Principal connections

Given an algebraic principal bundle R over X with projection $p\colon R\to X$ and structure group G, we shall write q for the structure map $TR\to R$. We have a natural right G-action on TR; to be precise, for each $g\in G$, the corresponding automorphism of TR is given by the derivative of the automorphism $r\mapsto rg$ of R. In the spirit of [D4, 20.2.2], we define a *principal connection* on R to be an X-map

$$TX\times_X R\xrightarrow{\mathsf{P}} TR$$

such that, for $x\in X$, $\mathsf{k}\in T_xX$ and $r\in R_x=p^{-1}(x)$, we have

$$\text{(a)}\qquad \mathsf{P}(\mathsf{k},r)\overset{dp}{\longmapsto}\mathsf{k} \quad\text{and}\quad \mathsf{P}(\mathsf{k},r)\overset{q}{\longmapsto} r,$$

$$\text{(b)}\qquad \mathsf{k}\longmapsto\mathsf{P}(\mathsf{k},r) \text{ gives a } \mathbf{C}\text{-linear map } T_xX\longrightarrow T_rR,$$

and

$$\text{(c)}\qquad \forall g\in G, \text{ we have } \mathsf{P}(\mathsf{k},rg)=\mathsf{P}(\mathsf{k},r)g,$$

where, in (c), the group G acts on TR as defined above. We call $\mathsf{P}(\mathsf{k},r)$ the *horizontal lift* of k at r.

Given a principal G-bundle R and a finite-dimensional vector space V on which G acts on the right, denote by E the associated vector bundle $R\times^G V$, just as above. (Here we assume that the quotient $R\times^G V$ exists, as it will in our applications.) For a fixed $v\in V$, consider the map $R\to E$ given by the assignment $r\mapsto r\cdot v$. Its derivative at $r\in R$ gives a map $T_rR\to T_{r\cdot v}E$, whose action, following [D3, 16.14.7.3, p. 95], we shall write as $\mathsf{h}\mapsto\mathsf{h}\cdot v$. Given a principal connection $\mathsf{P}\colon TX\times_X R\to TR$, set

$$\mathsf{C}(\mathsf{k},\mathsf{u})=\mathsf{P}(\mathsf{k},r)\cdot v,$$

for $\mathsf{k}\in T_xX$ and $\mathsf{u}\in E_x$, with $r\in R$ and $v\in V$ such that $\mathsf{u}=r\cdot v$, via the surjection $R\times V\to E$. This is well-defined, by the same argument as in the C^∞ case [D4, 20.5.1.1, p. 253], and clearly C is a connection on E.

Conversely, using moving frames, each connection C on a vector bundle E is induced by a suitable principal connection P on the bundle $R=\mathrm{Isom}(V_X,E)$ of frames in E. Here the C^∞ analogue is [D4, 20.5.2, p. 255], whose proof provides a template for the case at hand.

Grassmannian bundles

We now return to E, our given vector bundle of rank n on the scheme X over $\mathbf{C}$, to our fixed n-dimensional complex vector space V, and to the associated complex frame bundle $R = \mathrm{Isom}(V_X, E)$, a principal bundle over X with structure group $G = GL(V)$. Again we identify E with $R \times^G V$.

Now fix an integer k, $0 \le k \le n$, and consider the Grassmannian bundle $G_k(E)$. To obtain patching data for $G_k(E)$, we simply apply the Grassmannian functor G_k to the patching data for E, so it is clear that we have a canonical identification $G_k(E) = R \times^G G_k(V)$, where the right-hand side denotes the associated fiber space with fiber $G_k(V)$. Indeed, the quotient $(R \times G_k(V))/G$ is $G_k(E)$, and in particular the quotient exists, by the same reasoning which shows that E is the analogous quotient $(R \times V)/G$.

Given a connection C on E, induced by a principal connection P on the frame bundle R, set

$$\mathsf{G}(\mathsf{k}, \sigma) = \mathsf{P}(\mathsf{k}, r) \cdot w,$$

where $\mathsf{k} \in T_x X$, $\sigma \in G_k(E)_x$ for $x \in X$, and where $r \in R_x$ and $w \in G_k(V)$ are such that $s = r \cdot w$. That $\mathsf{G}(\mathsf{k}, \sigma)$ is well-defined follows by the same argument as for the analogous step in the passage from P to C.

Proposition 9.1. *The assignment* $(\mathsf{k}, \sigma) \mapsto \mathsf{G}(\mathsf{k}, \sigma)$ *defines a* C^∞*-map*

$$TX \times_X G_k(E) \xrightarrow{\mathsf{G}} TG_k(E)$$

over X*, with properties analogous to* (a)–(c) *for* P.

Proof. That G is an X-map is clear; that it enjoys the required properties follows immediately from the construction.

Data on a scheme with a connection

Here we suppose that the smooth S-scheme X comes equipped with a connection, as above, on the tangent bundle $TX = T_{X/S}$. Here $D_k^1 X = G_k TX$; write p for the structure map $G_k(TX) \to X$, write Σ for the universal k-subbundle of $p^* TX$, and write $\mathcal{F}$ for the pullback of Σ to $T(G_k TX)$ under ∂p, as in §2; then $D_k^2 X = G_k(\mathcal{F})$.

For a point $s \in G_k(TX)$, let $x = p(s)$. We identify the fiber Σ_s with the k-plane in $T_x X$ to which it corresponds. Then a point of $\mathcal{F}_s$ is a tangent vector $\mathsf{v} \in T_s(G_k TX)$ with $dp(\mathsf{v}) \in \Sigma_s$. By Proposition 9.1, the connection C induces a C^∞ morphism

$$TX \times_X G_k TX \xrightarrow{\mathsf{G}} T(G_k TX)$$

with properties (a)–(c) as above.

We now define a C^∞ bundle map $\alpha: \Sigma \to \mathcal{F}$. For a point of Σ given by a vector $\mathsf{k} \in \Sigma_s \subset T_x X$ as above, we set

$$\alpha(\mathsf{k}) = \mathsf{G}(\mathsf{k}, s).$$

Fix $s \in G_k TX$; then, as k varies, the induced map on the fibers $\Sigma_s \to T_s(G_k TX)$ is linear by property (b), and it follows easily that α is a bundle map $\Sigma \to T(G_k TX)$. By property (a), the diagram

(1)
$$\begin{tikzcd} & T(G_k TX) \arrow[d, "\partial p"] \\ \Sigma \arrow[ur, "\alpha"] \arrow[r, hook] & p{*}TX \end{tikzcd}$$

commutes, so α does map into $\mathcal{F}$. Write s_k for the induced map $D_k^1 X = G_k \Sigma \to G_k \mathcal{F} = D_k^2 X$.

Proposition 9.2. *The map s_k is a C^∞ section of the natural map $b_k^2: D_k^2 X \to D_K^1 X$, and the image of s_k lies in $D_k^2 X_{\mathrm{nc}}$.*

Proof. Because diagram (1) commutes, it follows that α splits the projection $\mathcal{F} \to \Sigma$, so the proposition follows from Proposition 3.3.

The image in $D_k^2 X_{\mathrm{ns}}$ of the section s_k is called the *flat subspace*; we denote it by F. In the C^∞ splitting

(2)
$$\mathcal{F} = \Sigma \oplus T_{D_k^1 X/X}$$

given by α, the first summand corresponds to F. As in §7, denote by Σ the universal k-subbundle on $G_k(TX)$; then we have an exact sequence

$$0 \longrightarrow \Sigma \longrightarrow TX_{G_k(TX)} \longrightarrow \Lambda \longrightarrow 0,$$

where Λ denotes the universal quotient.

Corollary 9.3. *Given the connection C, suppose further that the section s_k is a morphism of schemes. Then we obtain a natural identification*

$$D_k^2 X_{\mathrm{ns}} = \mathrm{Hom}(\Sigma \otimes \Sigma, \Lambda)_{D_k^1 X_{\mathrm{ns}}}$$

of $D_k^1 X_{\mathrm{ns}}$-schemes.

Proof. This follows from Proposition 7.2.

Under the isomorphism of Corollary 9.3, the zero-section on the right corresponds to the flat subspace F, which is now a sub*scheme.*

Classically, a C^∞ arc T on X is a *geodesic* if its tangent vector moves horizontally, relative to the connection C. In our language, we have two maps $T \to X_2 = D_1^2 X$: the second lift i_2 and the composite $\mathsf{s}_1 \circ i_1$, where i_1, the first lift, assigns the tangent direction. Then an arc T is a geodesic exactly when $i_2(T) = \mathsf{s}_1 \circ i_i(T)$ on the data scheme $D_1^2 X$. For algebraic subvarieties, whatever the dimension, we have a natural analogue: relative to C, we say that a smooth algebraic k-fold on X is *flat* if we have $i_2(T) = \mathsf{s}_1 \circ i_1(T)$ on $D_k^2 X$, where i_1 denotes the first lift $T \to D_k^1 X$. Evidently flat algebraic subvarieties will be rare, but, under the assumptions of Corollary 9.3, we can always ask about the pullback of F under i_2, whose support consists of the *flat locus* on T, relative to C.

When $X = \mathbf{P}^n_{\mathbf{C}}$, as we have remarked above, the standard connection is given by a Kähler metric, so the preceding discussion applies. The resulting section s_k coincides with the section we defined directly in §4, whose image in $D_k^2 X$ is the space of flat data we defined there for $\mathbf{P}^n$. In particular, while the connection C in this case is definitely C^∞, the resulting section is algebraic, so Corollary 9.3 applies.

Example 9.4. For $X = \mathbf{P}^2$, equipped with its standard connection, lines are flat. For more general curves, take $k=1$ and $r=2$. Then F gives the inflectional second-order data of [RS1], [RS2], [RS3], [CK1] and [CK2]. In particular, the flexes of a general curve T on X are the points at which the second lift of T meets F on $\mathbf{P}^2_2$.

Example 9.5. For $X = \mathbf{P}^3$ equipped with its standard connection, take $k=2$ and $r=2$. Then F parametrizes the second-order data given by flat points on surfaces, as in §7, and the set of flat points of an embedded surface T is precisely the support of i_2^F.

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