Ark. Mat., **34** (1996), 179–198

An elementary approach to Carleman-type resolvent estimates

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Abstract. A new elementary approach to uniform resolvent estimates of the Carleman-type is developed. Schatten–von Neumann's $\mathfrak{S}_p$ perturbations of self-adjoint and unitary operators are considered. Examples of typical growth are provided.

1. Introduction. Scope of applications

We are interested in Carleman-type resolvent estimations for a linear operator on a Hilbert space H. Originally, T. Carleman [1] proved that for a *Volterra* operator A (i.e. a compact operator with zero spectrum $\sigma(A)=\{0\}$) belonging to the Hilbert–Schmidt class $\mathfrak{S}_2$ the resolvent $R_\lambda(A)=(\lambda I-A)^{-1}$ satisfies the following inequality

$$\|R_\lambda(A)\|\le c_1|\lambda|^{-1}\exp(c_2|\lambda|^{-2}),\quad \lambda\in\mathbf{C}\setminus\{0\}$$

where c_1, c_2 are constants depending on the norm $\|A\|_2=\|A\|_{\mathfrak{S}_2}$ only. Considerably later, L. Sakhnovich [12] obtained a similar estimation for operators with real spectrum and a Hilbert–Schmidt imaginary part $\operatorname{Im} A=(A-A^*)/2i$

$$\|R_\lambda(A)\|\le c_1|\operatorname{Im}\lambda|^{-1}\exp(c_2|\operatorname{Im}\lambda|^{-2}),\quad \lambda\in\mathbf{C}\setminus\mathbf{R}$$

with c_1 depending on $\|A\|$ and c_2 depending on $\|\operatorname{Im} A\|_{\mathfrak{S}_2}$. These results were then generalized to the case $\operatorname{Im} A\in\mathfrak{S}_p$, $1\le p<\infty$, (Schatten–von Neumann ideals of compact operators) replacing $|\operatorname{Im}\lambda|^{-2}$ by $|\operatorname{Im}\lambda|^{-p-1}$ (J. Schwartz, [13]) and finally to the case when $\operatorname{Im} A$ belongs to the Matsaev ideal $\mathfrak{S}_\omega$ (V. Matsaev, [7]). For a brief report on other generalizations until 1974 see [8, Sect. 1, 2.1.]

The main applications of resolvent estimates are, of course, within spectral theory, both pure and applied. First, they are indispensable for proving completeness theorems (it means completeness of eigenvectors and root vectors of an operator; T. Carleman, M. Keldysh, V. Lidsky, V. Matsaev, and others) and for existence of invariant subspaces and their use to decompose an operator into an integral with respect to a chain of its invariant subspaces (J. Wermer, L. Sakhnovich, Yu. Lyubich, V. Matsaev, J. Schwartz, I. Gohberg, M. Krein, J. Ringrose and others). Finally, resolvent estimates are important for semi-group theory (calculi, tauberian theorems, ergodic theorems, and so on), for the harmonic analysis-synthesis problem (recall the Carleman–Domar transform method), and for many other interesting fields. For some of such applications see [2], [4], [5], [6], [7], [8].

However, possible applications of resolvent estimates are not the subject of this paper. We present here a simple method to get the estimates themselves.

The paper is organized as follows. Section 2 explains the main ideas of the approach: the use of finite section approximation and a rank one majorant for triangular matrices. Sections 3–5 deal with technical realizations of this scheme and Section 6 contains the main results, Theorems 6.3 and 6.4. In Section 7 we discuss the sharpness of the estimates obtained. It should be stressed that the results of Theorems 6.3 and 6.4 are more or less well known; the novelty (and the subject of this communication) is our method of proof.

The contents of the paper was reported to the conference "Journées de théorie des opérateurs" (Luminy, France, 1993), [10].

The author is indebted to the referee for a careful reading and helpful comments on the manuscript.

2. Steps towards an estimate

2.1. Finite section approximation

Take a sequence of orthogonal projections $\{P_n\}_{n\geq 1}$ tending to the identity, $P_n x \to x$, $\forall x \in H$, and such that $P_n \leq P_{n+1}$ and $\operatorname{rank} P_n = n$, $n \geq 1$, and put

$$A_n = P_n A P_n, \quad n \geq 1.$$

2.2. Preparing the resolvent

Given a finite rank operator A_n we fix a *Schur basis* (that is an orthonormal basis of $P_n H$ making the matrix of A_n lower triangular: $A_n = \{a_{ij}^{(n)}\}$, $a_{ij}^{(n)} = 0$ for $j > i$) and put

$$B_n = A_n - \operatorname{diag}(i \operatorname{Im} A_n) = D_n + E_n$$

where $\operatorname{diag}(X)$ stands for the diagonal (operator) of X with respect to the chosen basis and

$$D_n = \operatorname{diag}(\operatorname{Re} A_n).$$

Then $R_\lambda(B_n)=(I-C_n)^{-1}R_\lambda(D_n)$ where $C_n=R_\lambda(D_n)E_n$. This implies that

$$\|R_\lambda(B_n)\| \le \frac{1}{\operatorname{Im}\lambda}\|(I-C_n)^{-1}\|, \quad \lambda\in\mathbf{C}\setminus\mathbf{R}.$$

2.3. Rank one majorant

For the lower triangular operator $C_n^2=\{c_{ij}^{(n)}\}$ there exist two vectors $p,q\in\mathbf{C}^n$ such that

$$|c_{ij}^{(n)}|\le p_i q_j, \quad j\le i$$

and $\|p\|=\|q\|=\|C_n\|$.

Passing to a continuous parameter we put

$$Qf(t)=p(t)\int_0^t q(y)f(y)\,dy$$

on the space $L^2(0,n)$ where $p=j\{p_i\}$, $q=j\{q_i\}$ and jx stands for the function

$$jx(t)= x_k \quad \text{if } k-1\le t<k,\ 1\le k\le n,\ x\in\mathbf{C}^n.$$

We get

$$\|(I-C_n^2)^{-1}x\|\le\|(I-Q)^{-1}jx\|.$$

2.4. Using an Euler formula

The standard Euler formula for first order linear differential equations implies

$$(I-Q)^{-1}f=f+\int_0^x k(x,s)f(s)\,ds$$

where $k(x,s)=p(x)q(s)\exp(\int_s^x p(t)q(t)\,dt)$.

2.5. Summing up

The previous steps lead directly to the desired estimate for $\|R_\lambda(A)\|$ in the case where $\operatorname{Im}A\in\mathfrak{S}_2$. The case $\operatorname{Im}A\in\mathfrak{S}_p$, $p\neq2$, requires a bit more attention.

And now, let us realize the sketched steps.

3. Finite rank approximation: spectral convergence

3.1. Remarks on spectral continuity

In order to apply the method sketched in Sect. 2 we need a kind of spectral continuity: having $\lim_n A_n = A$ (in some sense) we want $\lim_n \sigma(A_n) = \sigma(A)$ (in some sense) where, as above, $\sigma(A)$ stands for the spectrum of an operator A.

As is well known the spectrum is not an upper semicontinuous function even for norm convergence, that is, in general, the inclusion $\sigma(A) \subset \overline{\lim}_n \sigma(A_n)$ fails to be true even for $\lim_n \|A_n - A\| = 0$. Probably the simplest example is given by the operators $A_n f = e^{it} f(t) - (1-1/n)\hat{f}(-1)$ on the space $L^2(-\pi, \pi)$; here $\sigma(A_n) = \mathbf{T}$ (the unit circle, $\mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$) for all $n = 1, 2, \ldots$ but the limit $\lim_n A_n = A$ is the orthogonal sum of the unilateral shift and its adjoint, and hence $\sigma(A) = \operatorname{clos} \mathbf{D}$, where $\mathbf{D}$ stands for the open unit disc, $\mathbf{D} = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$.

On the other hand, the spectrum is lower semicontinuous with respect to norm convergence (see [6]): if (on a Banach space) one has $\lim_n \|A_n - A\| = 0$ the inclusion $\overline{\lim}_n \sigma(A_n) \subset \sigma(A)$ follows (indeed, if $\lambda \notin \sigma(A)$ all operators $\mu I - A_n = (\mu I - A) \times (I + R_\mu(A)(A - A_n))$ are invertible for n large enough and μ close enough to λ because of $\|R_\mu(A)(A - A_n)\| \le \|R_\mu(A)\| \, \|A - A_n\| \le \text{const} \, \|A - A_n\| < 1$).

However, the same lower semicontinuity fails for weaker convergences, for instance for the strong (pointwise) operator convergence. (Example: $A_n = \lambda_n(\cdot, b_n) b_n$ where $\{b_n\}$ stands for an orthonormal sequence in a Hilbert space and $\lambda_n \in \mathbf{D}$ (an arbitrary sequence); then obviously, $\lim_n \|A_n x\| = 0$ for every x, and $\sigma(A_n) = \{0\} \cup \{\lambda_n\}$; hence, the set $\underline{\lim}_n \sigma(A_n)$, in general, is not contained in $\sigma(A) = \{0\}$.)

We need a kind of lower semicontinuity property in the spirit of perturbation theory: imposing additional requirements on the resolvents of the leading operators T_n and supposing $A_n = T_n + K_n$ to be "small" perturbations of T_n, one can guarantee the desired lower semicontinuity.

Now, we carry out a version of such a perturbed semicontinuity sufficient for our purposes (see Sect. 4).

Notation. For a Banach space X, $L(X)$ stands for the algebra of linear bounded operators on X; for $T \in L(X)$, T^* means the adjoint operator, $T^* \in L(X^*)$, and $\mathfrak{S}_\infty$ stands for the ideal of all compact operators on X.

3.2. Lemma. *Let X be a reflexive Banach space, $K \in \mathfrak{S}_\infty$, $T \in L(X)$ and $A = T + K$. Suppose also that $T_n \in L(X)$, $K_n \in \mathfrak{S}_\infty$ and $A_n = T_n + K_n$ $(n = 1, 2, \ldots)$ have the following properties*:

(1) $\lim_n \|K_n - K\| = 0$,

(2) $\lim_n T_n^* f = T^* f$, $\forall f \in X^*$ *(strong operator convergence),*

(3) *there exists a closed set* $\Omega\subset\mathbf{C}$ *such that* $\sigma(A)\subset\Omega$ *and* $\sup_n\|R_\lambda(T_n)\|<\infty$ *whenever* $\lambda\in\mathbf{C}\setminus\Omega$.
Then, the spectra $\sigma(A_n)$ *converge uniformly to* Ω, *i.e.* $\forall\varepsilon>0$ *one has* $\sigma(A_n)\subset\Omega+\varepsilon\mathbf{D}$ *for* $n>n(\varepsilon)$.

Proof. Let $\varepsilon>0$ and suppose there exist infinitely many n's and λ_n's such that

$$\lambda_n\in\sigma(A_n),\qquad \lambda_n\notin\Omega+\varepsilon\mathbf{D}.$$

Since $\sigma(T_n)\subset\Omega$ (see condition (3)) the Fredholm–Riesz theory implies that the difference $\sigma(A_n)\setminus\Omega$ consists of eigenvalues of A_n. Let $x_n\in X$, $\|x_n\|=1$ such that

$$(T_n+K_n-\lambda_nI)x_n=O.$$

If necessary, by passing to a subsequence, one can assume that the sequence $\{x_n\}$ converges weakly to a limit $x\in X$ and $\{\lambda_n\}$ converges to a $\lambda\in\mathbf{C}$. Hence, we get consecutively

$$\lim_n\|Kx_n-Kx\|=0,\quad \lim_n\|K_nx_n-Kx\|=0\quad\text{and}\quad \lim_n\|T_nx_n+Kx-\lambda x_n\|=0.$$

On the other hand, the sequence $\{T_nx_n\}$ converges weakly to Tx:

$$|(T_nx_n-Tx,f)|\le|((T_n-T)x_n,f)|+|(T(x_n-x),f)|\le\|T_n^*f-T^*f\|+|(x_n-x,T^*f)|$$

and the last two terms tend to zero. So,

$$Ax-\lambda x=Tx+Kx-\lambda x=\lim_n(T_nx_n+Kx-\lambda x_n)=O\quad\text{(weak limit)}.$$

Using (3) and the fact that $\lambda\notin\Omega$ we get $x=O$, and hence $\lim_n\|Kx_n\|=0$ and $\lim_n\|(T_n-\lambda I)x_n\|=0$.

Of course, this contradicts condition (3): $1=\|x_n\|\le\|R_\lambda(T_n)\|\,\|(T_n-\lambda I)x_n\|\le\text{const}\,\|(T_n-\lambda I)x_n\|\longrightarrow0$.

This finishes the proof. □

3.3. Corollary. *Let* T, T_n *be self-adjoint operators on a Hilbert space* ($T_n^*=T_n$) *and* $K,K_n\in\mathfrak{S}_\infty$ *such that* $\lim_n\|K_n-K\|=0$, $\lim_nT_nx=Tx$, $\forall x$ (*strong operator convergence*). *If* $\sigma(T+K)\subset\mathbf{R}$, *the spectra* $\sigma(T_n+K_n)$ *converge uniformly to the real line* $\mathbf{R}$.

Indeed, we put $\Omega=\mathbf{R}$ in the lemma and use the obvious observation that $\|R_\lambda(T_n)\|\le|\operatorname{Im}\lambda|^{-1}$ for $\lambda\in\mathbf{C}\setminus\mathbf{R}$. □

3.4. Corollary. *Let T be a self-adjoint operator on a Hilbert space H, $K\in\mathfrak{S}_\infty$ and P_n be orthogonal projections such that $P_n\le P_{n+1}$,* $\operatorname{rank} P_n=n$ *and* $\lim_n P_nx=x$, $\forall x$ *(norm convergence). Putting $T_n=P_nTP_n$, $K_n=P_nKP_n$, $A_n=T_n+K_n$ and $A=T+K$ and supposing $\sigma(A)\subset\mathbf{R}$ we conclude that the spectra $\sigma(A_n)$ converge uniformly to the real line* $\mathbf{R}$.

3.5. *Remarks.* The resolvent convergence $\lim_n R_\lambda(A_n)=R_\lambda(A)$ (say, with respect to pointwise (or even weak operator) convergence) is a separate problem. For instance, under the hypotheses of Lemma 3.2, the formula

$$R_\lambda(A_n)=(I-R_\lambda(T_n)K_n)^{-1}R_\lambda(T_n)$$

reduces the question to a convergence of the resolvents $R_\lambda(T_n)$, $n=1,2,\dots$. The latter, in general, fails to be true: for example, if T_n are defined on the usual l^2 space by the equations $T_n^*x=T_n^*(x_0,x_1,x_2,\dots)=(x_n,x_0,x_1,\dots,x_{n-1},x_{n+1},\dots)$ (and so are unitary operators) we have $\lim_n \|T_n^*x-Sx\|=0$ where S is the unilateral shift $(Sx=(0,x_0,x_1,\dots))$; hence, $T=S^*$ and so the sequence $\{T_n^{-1}\}$ does not converge because T is not invertible.

However, *if we suppose not only strong operator convergence* $\lim_n T_n^*=T^*$ *but also* $\lim_n T_n=T$ (and, as in the lemma, the boundedness $\sup_n\|R_\lambda(T_n)\|<\infty$) *we can claim that $\lambda I-T$ is invertible and* $\lim_n R_\lambda(T_n)=R_\lambda(T)$ (and hence the same for $R_\lambda(A_n)$, see the above formula for $R_\lambda(A_n)$). Indeed, the sequence $\{R_\lambda(T_n)\}\subset L(X)$, being bounded on a reflexive Banach space, has a weak limit point, say L; and so the formula $(R_\lambda(T_n)x,(\lambda I-T_n)^*f)=(x,f)$ and the strong convergence $\lim_n T_n^*=T^*$ show that $(Lx,(\lambda I-T_n)^*f)=(x,f)$ for all $x\in X$, $f\in X^*$, that is $L(\lambda I-T)=I$ and $\lambda I-T$ is left invertible. In the same way we can see that T is right invertible. Now, the convergence is obvious: $\lim_n(R_\lambda(T_n)-R_\lambda(T))x=\lim_n R_\lambda(T_n)(T_n-T)R_\lambda(T)x=O$.

In particular, the resolvent convergence $\lim_n R_\lambda(A_n)=R_\lambda(A)$ takes place under the hypotheses of Corollaries 3.3 and 3.4.

4. Preparing the resolvent

4.1. Using a Schur basis

We are dealing with the situation described in Corollary 3.4. Let us consider the restrictions $T_n|_{H_n}$, $K_n|_{H_n}$ (where $H_n=P_nH$ stands for the range of the projection P_n) and keep the same notation T_n,K_n for these operators (they differ from the previous T_n,K_n by zero operators on the orthogonal complement $H_n^\perp$). Let $\mathcal{E}=$

$\{e_k\}_{k=1}^n$ be an orthonormal basis of H_n such that the matrix of $A_n = T_n + K_n$, $\{a_{ij}^{(n)}\}$ is lower triangular: $a_{ij}^{(n)} = 0$ for $j > i$.

Given an orthonormal basis $\mathcal{E}$ and an operator, say W, consider the diagonal part of W, i.e. an operator $\operatorname{diag} W$ with the matrix $\delta_{ij}(We_j, e_i)$, δ_{ij} stands for the Kronecker δ. The boundedness of $\operatorname{diag} W$ is obvious as well as the inequality $\|\operatorname{diag} W\| \leq \|W\|$. Moreover, if $\mathcal{E}$ is a Schur basis of W (on a finite dimensional space) the diagonal of W consists of eigenvalues of W and so we get the following equivalent form of Corollary 3.4.

4.2. Lemma. *In the notation and under the hypotheses of Corollary* 3.4 *we have*

$$\lim_n \|\operatorname{Im}(\operatorname{diag} A_n)\| = 0.$$

4.3. Corollary. *Let $B_n = A_n - i\operatorname{Im}(\operatorname{diag} A_n)$, $n \geq 1$. Then*

$$\lim_n B_n x = Ax, \quad x \in H,$$

and $B_n = D_n + E_n$, where $D_n = \operatorname{diag} B_n = \operatorname{Re}(\operatorname{diag} A_n)$ and E_n has a strictly lower triangular matrix (*with respect to the same Schur basis*). *Moreover,*

$$\|\operatorname{Im} B_n\|_{\mathfrak{S}_2} = \|\operatorname{Im} E_n\|_{\mathfrak{S}_2} = 2^{-1/2}\|E_n\|_{\mathfrak{S}_2}.$$

Indeed, $\operatorname{Im} B_n = \operatorname{Im}(D_n + E_n) = \operatorname{Im} E_n = (E_n - E_n^*)/2i$ and since E_n is strictly lower triangular we get

$$\|\operatorname{Im} E_n\|_{\mathfrak{S}_2}^2 = 2^{-2}\|E_n - E_n^*\|_{\mathfrak{S}_2}^2 = 2^{-1}\|E_n\|_{\mathfrak{S}_2}^2.$$

The limit formula is an obvious consequence of Lemma 4.2. □

4.4. Corollary. *For the above defined operators B_n, one has*

$$R_\lambda(B_n) = (I - R_\lambda(D_n)E_n)^{-1} R_\lambda(D_n), \quad \lambda \in \mathbf{C}\setminus\mathbf{R}$$

and so

$$\|R_\lambda(B_n)\| \leq \|(I - R_\lambda(D_n)E_n)^{-1}\|\, |\operatorname{Im}\lambda|^{-1}, \quad \lambda \in \mathbf{C}\setminus\mathbf{R}.$$

This is a standard manipulation with the resolvent of a perturbation:

$$\begin{aligned}(\lambda I - B_n)^{-1} &= (\lambda I - D_n - E_n)^{-1} = ((\lambda I - D_n)(I - R_\lambda(D_n)E_n))^{-1}\\ &= (I - R_\lambda(D_n)E_n)^{-1} R_\lambda(D_n),\end{aligned}$$

for $\lambda \in \mathbf{C}\setminus\mathbf{R}$. □

Resolvent estimates for $\mathfrak{S}_p$-perturbations require some more conditions on the diagonal part of the operator under consideration. The following is a well-known fact (for which we are at a loss to find an easy reference and shall provide it with a short proof). For a definition and properties of symmetrically normed operator ideals we refer to [4].

4.5. Lemma. *Let $\mathfrak{S}$ be a symmetrically normed operator ideal and $W\in\mathfrak{S}$. Then,* $\operatorname{diag} W\in\mathfrak{S}$ *and*

$$\|\operatorname{diag} W\|_{\mathfrak{S}} \le \|W\|_{\mathfrak{S}}.$$

Proof. Given an orthonormal basis $\mathcal{B}=\{b_n\}$ define a family of unitary operators as follows: $U_t b_n = e^{int} b_n$, $0\le t\le 2\pi$. Then the integral

$$J=\frac{1}{2\pi}\int_0^{2\pi} U_t^* W U_t\, dt$$

is weakly convergent (in $L(H)$, in the Riemann sense) and its value is precisely $\operatorname{diag} W$ since

$$\begin{aligned}(Jb_k,b_j) &= \left(\left\{\frac{1}{2\pi}\int_0^{2\pi} U_t^* W U_t\, dt\right\} b_k, b_j\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi} (U_t^* W U_t b_k, b_j)\, dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-j)t}(W b_k, b_j)\, dt\\ &= \delta_{kj}(Wb_k,b_j) = ((\operatorname{diag} W) b_k, b_j)\end{aligned}$$

for all k,j. Moreover, it is clear that the Riemann sums of J are uniformly bounded in $\mathfrak{S}$-norm with a bound $\|W\|_{\mathfrak{S}}$ (because of $\|U_t^* W U_t\|_{\mathfrak{S}}=\|W\|_{\mathfrak{S}}$ for all t). The lemma follows. □

5. Rank one majorant

5.1. The operator C_n

Defining an operator C_n by the following formula

$$C_n = R_\lambda(D_n) E_n$$

and using Corollary 4.4 we reduce the problem to an estimate of $\|(I-C_n)^{-1}\|$. As to the latter, it can be majorized using a pointwise matrix majorant for the square C_n^2.

5.2. Lemma. *Let C_n be a strictly lower triangular $n\times n$ matrix and $\{s_{kj}\}$ the matrix of the square C_n^2. There exist two vectors $p,q\in\mathbf{C}^n$ such that*

$$|s_{kj}|\le p_k q_j \text{ for } j<k \quad \text{and} \quad \|p\|=\|q\|=\|C_n\|_{\mathfrak{S}_2}.$$

Proof. Let $\{c_{ij}\}$ be the matrix of C_n, $c_{ij}=0$ for $j\ge i$. One has

$$s_{kj} = \sum_{i=j+1}^{k-1} c_{ki} c_{ij}, \quad 1\le j<k-1$$

and hence

$$|s_{kj}| \le \left(\sum_{i=1}^{k-1} |c_{ki}|^2\right)^{1/2} \left(\sum_{i=j+1}^{n} |c_{ij}|^2\right)^{1/2} =: p_k q_j$$

$$\|p\|^2 = \sum_{k=1}^{n} p_k^2 = \|C_n\|_{\mathfrak{S}_2}^2, \quad \|q\|^2 = \sum_{j=1}^{n} q_j^2 = \|C_n\|_{\mathfrak{S}_2}^2. \quad \square$$

In the next proposition we write $|x| = \{|x_k|\}_{k=1}^n$ for a vector $x = \{x_k\}_{k=1}^n \in \mathbf{C}^n$ and $|x| \le |y|$ if $|x_k| \le |y_k|$, $1 \le k \le n$, for $x, y \in \mathbf{C}^n$.

5.3. Corollary. *Let* $Q_n x = \left\{\sum_{j=1}^{k-1} p_k q_j x_j\right\}_{k=1}^n$, $x \in \mathbf{C}^n$. *Then*

$$|C_n^{2k} x| \le Q_n^k |x| \quad \textit{for all } x \in \mathbf{C}^n \textit{ and } k \ge 1,$$

and hence

$$\left\|\sum_{k\ge 0} C_n^{2k} x\right\| \le \left\|\sum_{k\ge 0} Q_n^k |x|\right\|, \quad x \in \mathbf{C}^n.$$

Indeed, the first inequality follows from the lemma and a straightforward induction, the second one is a consequence of the first. $\square$

5.4. Passing to a continuous parameter

To estimate the resolvent $(I - Q_n)^{-1}$ it is useful to pass to a continuous parameter and then apply an elementary formula from differential equations. To this end we use a step function isometric imbedding of $\mathbf{C}^n$ into $L^2(0, n)$ putting

$$jx(t) = x_i \quad \text{for } i-1 \le t < i \ (1 \le i \le n), \ x = \{x_i\} \in \mathbf{C}^n.$$

5.5. Lemma. *Let* $\{p_i\}$ *and* $\{q_i\}$ *be the vectors of* $\mathbf{C}^n$ *defined in Lemma* 5.2 *and let* $p = j\{p_i\}$, $q = j\{q_i\}$ *and*

$$Qf(t) = p(t) \int_0^t q(u) f(u)\, du, \quad t \in (0, n); \ f \in L^2(0, n).$$

Then

$$|(jQ_n^k x)(t)| \le (Q^k |jx|)(t), \quad t \in (0, n); \ x \in \mathbf{C}^n; \ k \ge 0.$$

Proof. Clearly, it is enough to check the inequality for $k=1$ and then use an induction on k. We have, for a vector $x \in \mathbf{C}^n$ and for $i-1 \le t < i$,

$$|jQ_n x(t)| = \left|\sum_{l=1}^{i-1} p_i q_l x_l\right| = \left|p(t) \int_0^{i-1} q(u)(jx)(u)\, du\right|$$

$$\le p(t) \int_0^t q(u)|jx|(u)\, du = (Q|jx|)(t). \quad \square$$

5.6. Corollary. *For a strictly lower triangular $n\times n$ matrix C_n we have*

$$\|(I-C_n^2)^{-1}\|\leq\|(I-Q)^{-1}\|$$

where Q stands for the operator defined in Lemma 5.5.

Indeed, by Corollary 5.3 and Lemma 5.5 we have

$$\|(I-C_n^2)^{-1}x\|\leq\|(I-Q_n)^{-1}x\|\leq\|(I-Q)^{-1}|jx|\|$$

for every $x\in\mathbf{C}^n$; moreover, $\|x\|=\|jx\|$. □

6. An Euler formula and completion of the estimates

6.1. Lemma. *Let Q be the operator defined in Lemma* 5.5. *Then, for every $g\in L^2(0,n)$, we have*

$$((I-Q)^{-1}g)(t)=g(t)+\int_0^t k(t,u)g(u)\,du,\quad t\in(0,n)$$

where

$$k(t,u)=p(t)q(u)\exp\left(\int_u^t q(s)p(s)\,ds\right)$$

for $0<u<t$ (and $k(t,u)=0$ for $u>t$).

Proof. The equation $(I-Q)^{-1}g=f$ is equivalent to $p^{-1}Qf=p^{-1}(f-g)$ (one can suppose without loss of generality that $p(t)$ is always greater than 0) and then (using $(p^{-1}Qf)(0)=0$ and $(p^{-1}Qf)'=qf$, at least for smooth functions g, which is enough) to a linear differential equation with respect to $G=p^{-1}(f-g)$, namely to

$$pqG+qg=G',\quad G(0)=0.$$

The well-known Euler formula for the solution,

$$G(t)=\int_0^t g(u)q(u)\exp\left(\int_u^t q(s)p(s)\,ds\right)du,$$

finishes the proof. □

Now, we are in a position to derive our main lemma.

6.2. Lemma. *Let C_n be a strictly lower triangular $n\times n$ matrix and m an integer, $m\geq 1$. Then,*

$$\|(I-C_n)^{-1}\|\leq\left(\sum_{k=0}^{2m-1}\|C_n^k\|\right)(1+\|C_n^m\|_2^2\cdot e^{\|C_n^m\|_2^2}).$$

Proof. Starting from the formula

$$(I-C_n)^{-1}=\left(\sum_{k=0}^{2m-1}C_n^k\right)(I-C_n^{2m})^{-1}$$

we will consecutively apply Lemma 5.2, Corollary 5.3 and Lemma 5.5 for the matrix C_n^m instead of C_n. Then, we get an operator Q (constructed in Lemma 5.5, but for C_n^m instead of C_n), and Corollary 5.6 and Lemma 6.1 imply

$$\|(I-C_n^{2m})^{-1}\|\leq\|(I-Q)^{-1}\|\leq 1+\left(\iint_S|k(t,u)|^2\,dt\,du\right)^{1/2}$$

where S stands for the square $S=(0,n)\times(0,n)$. Since

$$\begin{aligned}|k(t,u)|^2&\leq|p(t)|^2|q(u)|^2\exp\left(2\int_0^n p(s)q(s)\,ds\right)\\&\leq|p(t)|^2|q(u)|^2\exp\left(\int_0^n(p^2+q^2)\,ds\right)\end{aligned}$$

(see Lemma 5.2)

$$\leq|p(t)|^2|q(u)|^2e^{2\|C_n^m\|_2^2}$$

we have

$$\|(I-Q)^{-1}\|\leq 1+\|p\|\,\|q\|e^{\|C_n^m\|_2^2}\leq 1+\|C_n^m\|_2^2e^{\|C_n^m\|_2^2}.$$

The lemma follows. □

Deriving the main inequality we distinguish the case of an operator A with $\operatorname{Im}A\in\mathfrak{S}_2$; the general case $\operatorname{Im}A\in\mathfrak{S}_p$ will be derived from this one.

6.3. Theorem. *Let A be a Hilbert space operator satisfying* $\operatorname{Im}A\in\mathfrak{S}_2$ *and* $\sigma(A)\subset\mathbf{R}$. *Then,*

$$\|R_\lambda(A)\|\leq\frac{1}{|\operatorname{Im}\lambda|}\left(1+\frac{2^{3/2}\|\operatorname{Im}A\|_2}{|\operatorname{Im}\lambda|}\right)\left(1+\frac{8\|\operatorname{Im}A\|_2^2}{|\operatorname{Im}\lambda|^2}\right)e^{8\|\operatorname{Im}A\|_2^2/|\operatorname{Im}\lambda|^2}.$$

Proof. The matrices A_n, B_n, C_n, D_n, E_n are the same as in Section 4.1, Corollary 4.3 and Section 5.1. We have

$$\|C_n\|_2 \le \|R_\lambda(D_n)\|\,\|E_n\|_2 \le |\operatorname{Im}\lambda|^{-1}\|E_n\|_2$$

and (see Corollary 4.3)

$$\|E_n\|_2^2 = 2\|\operatorname{Im} E_n\|_2^2 = 2\|\operatorname{Im} A_n - \operatorname{Im}(\operatorname{diag} A_n)\|_2^2 = 2\|\operatorname{Im} A_n - \operatorname{diag}(\operatorname{Im} A_n)\|_2^2$$

(see Lemma 4.5)

$$\le 8\|\operatorname{Im} A_n\|_2^2 \le 8\|\operatorname{Im} A\|_2^2.$$

Let us combine these estimates with Lemma 6.2 for $m=1$. Since

$$1+\|C_n\|_2^2 e^{\|C_n\|_2^2} \le 1+8(\operatorname{Im}\lambda)^{-2}\|\operatorname{Im} A_n\|_2^2 e^{8(\operatorname{Im}\lambda)^{-2}\|\operatorname{Im} A\|_2^2}$$

we get using Corollaries 5.6 and 4.4 with Lemma 6.2 (for $m=1$)

$$\begin{aligned}
\|R_\lambda(B_n)\| &\le |\operatorname{Im}\lambda|^{-1}\|(I-C_n)^{-1}\| \\
&\le |\operatorname{Im}\lambda|^{-1}(1+\|C_n\|)(1+\|C_n\|_2^2 e^{\|C_n\|_2^2}) \\
&\le |\operatorname{Im}\lambda|^{-1}(1+|\operatorname{Im}\lambda|^{-1}2^{3/2}\|\operatorname{Im} A\|_2)(1+\|C_n\|_2^2 e^{\|C_n\|_2^2}) \\
&\le |\operatorname{Im}\lambda|^{-1}(1+|\operatorname{Im}\lambda|^{-1}2^{3/2}\|\operatorname{Im} A\|_2)(1+8(\operatorname{Im}\lambda)^{-2}\|\operatorname{Im} A\|_2^2) \\
&\quad \times e^{8(\operatorname{Im}\lambda)^{-2}\|\operatorname{Im} A\|_2^2}.
\end{aligned}$$

The desired inequality follows from passing to the limit and using Corollary 4.3 and Remark 3.5. $\square$

Now, the general case $\operatorname{Im} A\in\mathfrak{S}_p$ follows in a more or less standard way (for instance, see [2], [4]).

6.4. Theorem. *Let A be a Hilbert space operator satisfying $\operatorname{Im} A\in\mathfrak{S}_p$, $0<p<\infty$ and $\sigma(A)\subset\mathbf{R}$. Then,*

$$\|R_\lambda(A)\| \le |\operatorname{Im}\lambda|^{-1}(1+2k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im} A\|_p)^{4m-1}\cdot e^{M_p(\operatorname{Im}\lambda)^{-2m}\|\operatorname{Im} A\|_p^{2m}}$$

for $\lambda\in\mathbf{C}\setminus\mathbf{R}$, where m is an integer such that $2m-2<p\le 2m$ and k_p stands for a constant satisfying $k_p=k_{p/(p-1)}$ and $k_p\le pe^{-2/3}(\log 2)^{-1}$ for $p\ge 2$, $M_p=(2k_p)^{2m}$. For $|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im} A\|_p>1$ one has

$$\|R_\lambda(A)\| \le |\operatorname{Im}\lambda|^{-1}(1+2k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im} A\|_p)^{2p+3}\cdot e^{M_p|\operatorname{Im}\lambda|^{-p-2}\|\operatorname{Im} A\|_p^{p+2}}.$$

Proof. We use the same notation as in the previous theorem.

Let m be an integer such that $m-1<\frac{1}{2}p\leq m$. Using a Hölder-type inequality for $\mathfrak{S}_p$-norms (see [4], [2]) we get $\|C_n^m\|_2\leq\|C_n^m\|_{p/m}\leq\|C_n\|_p^m$, and by Lemma 6.2

$$\|(I-C_n)^{-1}\|\leq\left(\sum_{k=0}^{2m-1}\|C_n^k\|\right)(1+\|C_n\|_p^{2m}\cdot e^{\|C_n\|_p^{2m}}).$$

Moreover, $\|C_n\|_p\leq|\operatorname{Im}\lambda|^{-1}\|E_n\|_p$ (here we use the famous Matsaev inequality for quasi-nilpotent operators, with $k_p=k_{p/(p-1)}$ and $k_p\leq pe^{-2/3}(\log 2)^{-1}$ for $p\geq 2$; see [4], [5])

$$\begin{aligned}&\leq k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}E_n\|_p=k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}A_n-\operatorname{Im}(\operatorname{diag}A_n)\|_p\\&=k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}A_n-\operatorname{diag}(\operatorname{Im}A_n)\|_p\end{aligned}$$

(see Lemma 4.5)

$$\leq 2k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}A_n\|_p\leq 2k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}A\|_p.$$

Finally, we get

$$\begin{aligned}\|R_\lambda(B_n)\|&\leq|\operatorname{Im}\lambda|^{-1}\|(I-C_n)^{-1}\|\\&\leq|\operatorname{Im}\lambda|^{-1}\left(\sum_{k=0}^{2m-1}\|C_n\|^k\right)(1+\|C_n\|_p^{2m}\cdot e^{\|C_n\|_p^{2m}})\\&\leq|\operatorname{Im}\lambda|^{-1}(1+\|C_n\|)^{2m-1}(1+\|C_n\|_p^{2m})\cdot e^{\|C_n\|_p^{2m}}\\&\leq|\operatorname{Im}\lambda|^{-1}(1+\|C_n\|_p)^{4m-1}\cdot e^{M_p(\operatorname{Im}\lambda)^{-2m}\|\operatorname{Im}A\|_p^{2m}}\\&\leq|\operatorname{Im}\lambda|^{-1}(1+2k_p|\operatorname{Im}\lambda|^{-1}\|\operatorname{Im}A\|_p)^{4m-1}\cdot e^{M_p(\operatorname{Im}\lambda)^{-2m}\|\operatorname{Im}A\|_p^{2m}}.\end{aligned}$$

To finish the proof we refer again (as in the proof of Theorem 6.3) to Corollary 4.3 and Remark 3.5. □

6.5. *Remarks.* The inequalities obtained (Theorems 6.3 and 6.4) depend continuously on $\operatorname{Im}A$; in particular, if $\operatorname{Im}A=O$ we get the standard estimate for the resolvent of a self-adjoint operator $\|R_\lambda(A)\|\leq|\operatorname{Im}\lambda|^{-1}$. As is well known, the latter *formally* (i.e. without referring to the spectral theory) implies a stronger inequality $\|R_\lambda(A)\|\leq\operatorname{const}(\operatorname{dist}(\lambda,\sigma(A)))^{-1}$ with an absolute constant const (see, for instance, [11]; in fact, for self-adjoint operators const$=1$); a similar transformation (i.e. deriving an estimate depending on $\operatorname{dist}(\lambda,\sigma(A))$ only) is possible for the estimates of Theorems 6.3 and 6.4.

On the other hand, if we do not need an estimate depending on $\|\operatorname{Im} A\|_p$ only, we can simplify the expression inside brackets of the right hand side of Theorems 6.3 and 6.4 by replacing $2k_p\|\operatorname{Im} A\|_p$ by $2\|A\|$ (because of $\|C_n\|\leq 2|\operatorname{Im}\lambda|^{-1}\|A\|$).

The last remark is about perturbations of unitary operators. The above approach does not work directly for operators $B=U+C$ where U is unitary and C is a "smooth" operator. However, in a partial case when $\mathbf{T}\backslash\sigma(B)\neq\emptyset$ one can use the Cayley transform. Namely, let, say, $1\notin\sigma(B)$; then the operator $A=i(I+B)(I-B)^{-1}$ is well defined and bounded and we have

$$\begin{aligned} A-A^* &= i(I-B^*)^{-1}((I-B^*)(I+B)+(I+B^*)(I-B))(I-B)^{-1} \\ &= 2i(I-B^*)^{-1}(I-B^*B)(I-B)^{-1}. \end{aligned}$$

Therefore, as is well known, the operator A is a perturbation of a self-adjoint operator of the same class as B is of a unitary operator: $\operatorname{Im} A\in\mathfrak{S} \Leftrightarrow (I-B^*B)\in\mathfrak{S}$ where $\mathfrak{S}$ stands for any symmetrically normed operator ideal; in particular, $\operatorname{Im} A\in\mathfrak{S}$ if $C\in\mathfrak{S}$. Moreover, for $z\neq 1$ we can write $z=(\lambda-i)(\lambda+i)^{-1}$, $\lambda\neq -i$ and get

$$\begin{aligned} R_z(B) &= ((\lambda-i)(\lambda+i)^{-1}-B)^{-1} = ((\lambda-i)(\lambda+i)^{-1}-(A-iI)(A+iI)^{-1})^{-1} \\ &= (A+iI)(\lambda+i)((\lambda-i)(A+iI)-(A-iI)(\lambda+i))^{-1} \\ &= (A+iI)((\lambda+i)/2i)R_\lambda(A). \end{aligned}$$

And so, we have for $|z-1|\geq\delta>0$

$$\|R_z(B)\|\leq \text{const}\,\|R_\lambda(A)\|\leq \text{const}\cdot\varphi(|\operatorname{Im}\lambda|)$$

where φ stands for the right hand side of the inequalities of Theorems 6.3 or 6.4, and hence

$$\|R_z(B)\|\leq \text{const}\cdot\varphi(|1-|z|^2|\,|1-z|^{-2})\leq \text{const}\cdot\varphi(\text{const}\,|1-|z||). \quad \square$$

7. Examples of resolvent growth

7.1. One point spectrum

Much more is known for the special case when $\sigma(A)=\{0\}$. Namely, if A is a Volterra operator (i.e. A is compact and $\sigma(A)=\{0\}$) with the singular numbers $s_n(A)$ satisfying $s_n(A)=O(n^{-1/p})$, $p>0$, for $n\to\infty$ the following Keldysh–Matsaev estimate holds (see [4] for details)

$$\|R_\lambda(A)\|\leq \text{const}\cdot\exp(\text{const}/\lambda^p), \quad \lambda\in\mathbf{C}\backslash\{0\}.$$

Of course, this is true for $A\in\mathfrak{S}_p$, $\sigma(A)=\{0\}$.

On the other hand, if one is interested in an estimate depending on a smoothness condition for $\operatorname{Im} A$ (or $\operatorname{Re} A$, or $I-B^*B$) only, one has to restrict oneself to the case $p\in(1,\infty)$ because even the conditions $\operatorname{rank}\operatorname{Im} A=1, \sigma(A)=\{0\}$ does not imply anything better than $s_n(A)=O(n^{-1})$ (and $A\in\mathfrak{S}_p$, $p>1$; see the operator J_1 below); and therefore, in this case the Keldysh–Matsaev estimate is available with $p=1$ only (see again the operator J_1 below). This is a bit stronger than that which we had with our approach. Moreover, the following example shows that this is the best possible estimate.

Example 1. Let J_α, $\alpha>0$ be the fractional integration operator on the space $L^2(0,1)$ defined by the formula

$$J_\alpha f(x)=\Gamma(\alpha)^{-1}\int_0^x (x-y)^{\alpha-1} f(y)\,dy, \quad 0<x<1$$

where Γ stands for the Euler gamma-function.

It is well known that $\{J_\alpha,\ \alpha>0\}$ forms a semi-group, $J_{\alpha+\beta}=J_\alpha J_\beta$, and $\sigma(J_\alpha)=\{0\}$, $\alpha>0$. Moreover, the singular numbers of J_α satisfy power-like asymptotics $s_n(J_\alpha)\sim(\pi n)^{-\alpha}$, $n\to\infty$ (see, for example, [5, Appendix, Sect. 6]). As for the resolvent, we can compute it explicitly (for any $\lambda\neq0$):

$$R_\lambda(J_\alpha)=\lambda^{-1}(I-\lambda^{-1}J_\alpha)^{-1}=\sum_{n\geq0}\lambda^{-n-1}J_\alpha^n=\sum_{n\geq0}\lambda^{-n-1}J_{n\alpha}=\lambda^{-1}I+V_K$$

where V_K stands for a Volterra operator $V_K f(x)=\int_0^x K(x,y)f(y)\,dy$ with the kernel K,

$$K(x,y)=\sum_{n\geq1}\lambda^{-n-1}\Gamma(n\alpha)^{-1}(x-y)^{n\alpha-1}, \quad 0<y<x.$$

Hence, we have

$$\begin{aligned}\|R_\lambda(J_\alpha)\|\geq\|R_\lambda(J_\alpha)1\|_2\geq\|R_\lambda(J_\alpha)1\|_1&=\lambda^{-1}+\int_0^1\int_0^x K(x,y)\,dy\,dx\\ &=\lambda^{-1}+\sum_{n\geq1}\lambda^{-n-1}\Gamma(n\alpha)^{-1}\int_0^1 (n\alpha)^{-1}x^{n\alpha}\,dx\\ &=\lambda^{-1}+\sum_{n\geq1}\lambda^{-n-1}\Gamma(n\alpha)^{-1}(n\alpha)^{-1}(n\alpha+1)^{-1}\\ &=\lambda^{-1}+\sum_{n\geq1}\lambda^{-n-1}\Gamma(n\alpha+2)^{-1}=S(\lambda).\end{aligned}$$

The latter expression is essentially the classical Mittag-Leffler functions $E_{\alpha,\beta}(z)=\sum_{n\geq 0} z^n\Gamma(n\alpha+\beta)^{-1}$, having well-known asymptotics at infinity (for instance, see [3]). In particular,

$$\|R_\lambda(J_\alpha)\| \geq \text{const}\cdot\lambda^{3/\alpha+\alpha}e^{(1/\lambda)^{1/\alpha}}$$

for $\lambda\to+0$. In fact, to check the last inequality we do not need any theory: for $0<\alpha\leq 1$ it is sufficient to choose an integer $n\geq 1$ such that $k-1< n\alpha+2\leq k$ (for every $k\geq 3$ there exists an n) and then use $\Gamma(k)\geq\Gamma(n\alpha+2)$ and $\lambda^{-n-1}\geq\lambda^{-(k-3)/\alpha-1}$ $(0<\lambda<1)$ which implies $S(\lambda)\geq\text{const}\cdot\lambda^{3/\alpha}e^{(1/\lambda)^{1/\alpha}}$ with a suitable constant; for $\alpha>1$ one uses the fact that between $n\alpha+2$ and $(n+1)\alpha+2$ there are no more than $[\alpha]+1$ integers and $\Gamma(n\alpha+2)\leq\Gamma(k)$ for all of them; hence, one can majorize $\sum_{k\geq 0}\lambda^{-k/\alpha}\Gamma(k)^{-1}$ by a product (*a polynomial in* λ^{-1} *of degree* $\leq[\alpha]+1)\cdot S(\lambda)$ which implies the desired estimate.

7.2. Contractions or dissipative operators

This section is about contractions or dissipative operators (not necessarily with one point spectrum) which are trace class perturbations of unitary, respectively selfadjoint operators. Then much more is known (see [5], [14], [9]); for instance, if B is a completely non-unitary contraction, $I-B^*B\in\mathfrak{S}_1$ and $\mathbf{D}\backslash\sigma(B)\neq\emptyset$ one has

$$(1-|z|)\|R_z(B)\| \leq \text{const}\,|m(z)|^{-1}, \quad z\in\mathbf{D}\backslash\sigma(B)$$

where m stands for a non-zero bounded analytic function in the disc $\mathbf{D}$ (i.e. for an H^∞ function). In particular, $\|R_z(B)\|\leq\text{const}\cdot\exp(1-|z|)^{-1}$ and even $\|R_z(B)\|\leq\text{const}\cdot\exp(\operatorname{dist}(z,\sigma(B)))^{-1}$ if $\sigma(B)\subset\mathbf{T}$.

These bounds are attainable: for a given H^∞ function m with $|m(z)|\leq 1$ there exists a contraction B with $\operatorname{rank}(I-B^*B)=\operatorname{rank}(I-BB^*)=1$ such that

$$(1-|z|)\|R_z(B)\| \leq |m(z)|^{-1} \leq 1+2(1-|z|)\|R_z(B)\|, \quad z\in\mathbf{D}.$$

Example 2. Taking $m(z)=\exp(a(z+1)(z-1)^{-1})$, $a>0$ we get an exponential rate of growth (for $0<z<1$, $z\to 1$) for a rank one perturbation of a unitary operator.

Similar facts hold for dissipative operators (i.e. $\operatorname{Im}A\geq O$) instead of contractions (with a substitution of $\operatorname{Im}\lambda(1+|\lambda|^2)^{-1}$ for $1-|z|$, $z\in\mathbf{D}$).

7.3. Uniformly large resolvents

All previous examples are of "anisotropic" growth of the resolvent: that is when approached from one side of the spectrum the resolvent is large, but if approached along other paths, the growth is moderate (for instance, $\|R_\lambda(A)\|\le|\operatorname{Im}\lambda|^{-1}$ for a dissipative operator A and $\operatorname{Im}\lambda<0$). Examples of a uniform growth near the spectrum are not so sharp as anisotropic ones (or probably, uniform majorants obtained in Sect. 6 are not so sharp). We start with smooth perturbations of unitary operators.

Example 3. Let e_n, $n\in\mathbf{Z}$ be the standard orthonormal basis of the space $l^2(\mathbf{Z})$ and B an operator on $l^2(\mathbf{Z})$ defined by the equations

$$Be_n=\mu_n e_{n+1},\quad n\in\mathbf{Z}$$

where $\mu_n=1+(n+1)^{-\alpha}$ for $n\ge 0$, $0<\alpha<1$, and $\mu_n=1/\mu_{-n-1}$ for $n<0$.

The operator B differs from the unitary shift S, $Se_n=e_{n+1}$, $B=S(I+C)$ by an operator SC satisfying $s_n(SC)=s_n(C)\sim(n+1)^{-\alpha}$, $n\to\infty$ and hence $SC\in\mathfrak{S}_p$, $p>\alpha^{-1}$. It is also clear that $\sigma(B)=\mathbf{T}$.

It is well known that B is unitarily equivalent to the same shift $S=S_\alpha$ but on the weighted space $l^2(\mathbf{Z},w_n)=\{x=\{x_n\}:\sum_{n\in\mathbf{Z}}|x_n|^2w_n^2<\infty\}$ where $w_0=1$, $w_n=\prod_{k=0}^{n-1}\mu_k$ for $n>0$ and $w_n=w_{-n}$ for $n<0$ (the equivalence is realized by a diagonal operator D: $\{x_n\}\to\{x_n/w_n\}$: $DBD^{-1}=S_\alpha$). Therefore, for $|z|<1$,

$$\begin{aligned}\|R_z(B)\|&=\|R_z(S_\alpha)\|\ge\|R_z(S_\alpha)e_0\|=\Big\|-\sum_{n\ge0}z^nS_\alpha^{-n-1}e_0\Big\|\\&=\Big\|-\sum_{n\ge0}z^ne_{-n-1}\Big\|=\Big(\sum_{n\ge0}|z|^{2n}w_{n+1}^2\Big)^{1/2}\\&\ge\sup_{n\ge0}|z|^nw_{n+1}=\sup_{n\ge0}|z|^n\exp\Big((1+o(1))\sum_{k=0}^{n}(k+1)^{-\alpha}\Big)\\&=\sup_{n\ge0}|z|^n\exp((1-\alpha)^{-1}(1+o(1))n^{1-\alpha}).\end{aligned}$$

And hence we get from an elementary extremum computation

$$\|R_z(B)\|=\exp((1+o(1))\alpha(1-\alpha)^{-1}(\log(1/|z|))^{-1/\alpha+1})\ge\exp(c(\alpha)(1-|z|)^{-1/\alpha+1})$$

where $c(\alpha)=\frac12\alpha(1-\alpha)^{-1}$ and $r(\alpha)<|z|<1$.

It is quite easy to see that the obtained growth rate is sharp:

$$\begin{aligned}\|R_z(S_\alpha)\|&=\Big\|-\sum_{n\ge0}z^nS_\alpha^{-n-1}\Big\|\le\sum_{n\ge0}|z|^n\|S_\alpha^{-n-1}\|\\&=\sum_{n\ge0}|z|^nw_{n+1}=\sum_{n\le2x(\alpha)}|z|^nw_{n+1}+\sum_{n>2x(\alpha)}|z|^nw_{n+1}\end{aligned}$$

(where $x(\alpha)=(1/\log|z|^{-1})^{1/\alpha}$ stands for the extremum point from the previous estimate: $\varphi'(x(\alpha))=0$ where $\varphi(x)=(1-\alpha)^{-1}x^{1-\alpha}-x\log|z|^{-1}$; below, we also use that $\varphi'(x)\leq -k(\alpha)\log|z|^{-1}$ for $x\geq 2x(\alpha)$ with a constant $k(\alpha)>0$)

$$\leq 2x(\alpha)\sup_{n\geq 0}|z|^n w_{n+1}+\Big(\sup_{n\geq 0}|z|^n w_{n+1}\Big)\sum_{m\geq 0}e^{-mk(\alpha)\log|z|^{-1}}$$
$$\leq \mathrm{const}\cdot(1-|z|)^{1/\alpha+1}\exp(3c(\alpha)(1-|z|)^{-1/\alpha+1}).$$

The case when $|z|>1$ can be considered similarly (with the same result).

Example 4. In order to exhibit a smooth perturbation of a self-adjoint operator with a uniformly large resolvent, we cannot use the standard Cayley transform of the constructed operator B because in this case the transform produces an unbounded operator. Instead, we use the Zhukovsky transform

$$A=\tfrac{1}{2}(B+B^{-1})$$

of the operator B of Example 3. Since $B=S(I+C)$ one has

$$A-A^*=\tfrac{1}{2}\{S(I+C-(I+C^*)^{-1})+((I+C)^{-1}-(I+C^*))S^{-1}\}$$

(using $(I+C)^{-1}=I-C+C^2(I+C)^{-1}$)

$$=\tfrac{1}{2}\{S(C+C^*-C^{*2}(I+C^*)^{-1})+(-C-C^*+C^2(I+C)^{-1})S^{-1}\}.$$

Therefore, the operator $\operatorname{Im}A$ has the same smoothness class as C, and, for instance, $C\in\mathfrak{S}_p\Rightarrow\operatorname{Im}A\in\mathfrak{S}_p$.

Now, let $\lambda=\frac{1}{2}(z+z^{-1})$; then,

$$\lambda I-A=\tfrac{1}{2}(zI-B+z^{-1}I-B^{-1})=\tfrac{1}{2}B^{-1}(zI-B)(B-z^{-1}I)$$

and hence

$$R_\lambda(A)=-2R_z(B)R_{1/z}(B)B=2(z^{-1}-z)^{-1}(R_{1/z}(B)-R_z(B))B.$$

Supposing $|z|<1$, replacing B by the unitarily equivalent operator S_α (see Example 3) and using that $R_z(S_\alpha)=-\sum_{n\geq 0}z^nS_\alpha^{-n-1}$, $R_{1/z}(S_\alpha)=\sum_{n\geq 0}z^{n+1}S_\alpha^n$ we get

$$\|R_\lambda(A)\|=\left\|2(z^{-1}-z)^{-1}\Big(\sum_{n\geq 0}z^{n+1}S_\alpha^n+\sum_{n\geq 0}z^nS_\alpha^{-n-1}\Big)S_\alpha\right\|$$
$$=2|z^{-1}-z|^{-1}\Big\|\sum_{n\in\mathbf{Z}}z^{|n|}S_\alpha^n\Big\|\geq 2|z^{-1}-z|^{-1}\Big\|\sum_{n\in\mathbf{Z}}z^{|n|}S_\alpha^ne_0\Big\|_{l^2(\mathbf{Z},w_n)}$$
$$=2|z^{-1}-z|^{-1}\Big(\sum_{n\in\mathbf{Z}}|z|^{2|n|}w_n^2\Big)^{1/2}\geq 2|z^{-1}-z|^{-1}\Big(\sup_{n\in\mathbf{Z}}|z|^{|n|}w_n\Big)$$
$$\geq 2|z^{-1}-z|^{-1}\exp(c(\alpha)(1-|z|)^{-1/\alpha+1})$$

by the previous example.

On the other hand, the spectral mapping theorem shows that $\sigma(A)=[-1,1]$, and the formula $\operatorname{Im}\lambda=\operatorname{Im}\frac{1}{2}z(1-|z|^{-2})$ with the geometric meaning of the Zhukovsky function $\frac{1}{2}(z+z^{-1})$ (conformally mapping the disc $\mathbf{D}$ on $\mathbf{C}\setminus[-1,1]$) imply that for $|\operatorname{Re}\lambda|<1-\delta$, $\delta>0$ and $|\operatorname{Im}\lambda|<\text{const}$ we have $|\operatorname{Im}\lambda|>\varepsilon(1-|z|)$ (and $|z^{-1}-z|^{-1}\geq c$) with an ε depending on δ and const. Hence, we obtain the following lower estimate for $\|R_\lambda(A)\|$ (uniform in $|\operatorname{Re}\lambda|<1-\delta$):

$$\|R_\lambda(A)\|\geq\exp(\text{const}\cdot|\operatorname{Im}\lambda|^{-1/\alpha+1}).$$

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Received September 20, 1994

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