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Compression semigroups of open orbits in complex manifolds

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Introduction

Let $G_{\mathbf{C}}$ be a connected complex Lie group and $G\subseteq G_{\mathbf{C}}$ a real form, i.e., there exists an antiholomorphic involution σ of $G_{\mathbf{C}}$ such that $G=G_{\mathbf{C}}^{\sigma}=\{g\in G_{\mathbf{C}}:\sigma(g)=g\}$ is the group of fixed points. Now let $M=G_{\mathbf{C}}/P$ be a complex homogeneous space and suppose that the G-orbit $\mathcal{O}$ of the base point is open. We are interested in the semigroup

$$S(P):=\{g\in G_{\mathbf{C}}:gGP\subseteq GP\}=\{g\in G_{\mathbf{C}}:g.\mathcal{O}\subseteq \mathcal{O}\}$$

of *compressions* of the open G-orbit.

Such semigroups play a central role in the theory holomorphic extensions of unitary representations of the group G (cf. [HOØ], [FHO], [Ols], [N5], [N6], [N7], [St]).

More concretely we are dealing with the following two classes of homogeneous spaces, namely with complex flag manifolds and with certain embeddings of complex coadjoint orbits into complex homogeneous spaces.

The main results for complex flag manifolds are fairly easy to describe. Since everything decomposes nicely according to the decomposition of $G_{\mathbf{C}}$ into simple factors, we may assume that $G_{\mathbf{C}}$ is simple. In this case three mutually exclusive possibilities occur (cf. Proposition II.3, Theorem III.14):

(1) $S(P)=G_{\mathbf{C}}$.

(2) $\operatorname{int} S(P)=\emptyset$.

(3) $G_{\mathbf{C}}\neq\operatorname{int} S(P)\neq\emptyset$. Then G is a Hermitean simple Lie algebra and we have two possibilities. Let P_k be one of the two maximal parabolic subgroups containing the complexification $K_{\mathbf{C}}$ of the maximal compact subgroup K of G. Then $\operatorname{int} S(P)\neq\emptyset$ if and only if either $P\cap P_k$ or $P\cap\overline{P}_k$ contains a Borel subgroup. In the first case $S(P)=S(P_k)$ and $S(P)=S(P_k)^{-1}$ in the second case.

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According to these results one has a complete description of all non-trivial compression semigroups of open G-orbits on complex flag manifolds. One also has a characterization of the existence of non-trivial compression semigroups in terms of the projective realizations of the flag manifolds. Recall that each complex flag manifold $G_{\mathbf{C}}/P$ may be written as the G-orbit $G_{\mathbf{C}}.[v_\lambda]$ of the projective line through a highest weight vector v_λ for a holomorphic representation of $G_{\mathbf{C}}$ with highest weight λ. Then it turns out that the interior of $S(P)$ is non-empty if and only if the coadjoint orbit through $i\lambda$ is closed and its convex hull contains no lines (cf. Theorem III.16).

Compression semigroups can also be determined for more general, non-semi-simple, groups. All one has to assume a priori is the existence of a compactly embedded Cartan algebra in $\mathfrak{g}$. Then one can embed coadjoint orbits of G as open domains into a complex homogeneous space of $G_{\mathbf{C}}$ via a generalized version of the Borel embedding theorem (cf. Theorem I.3). For these domains the compression semigroups then split up nicely into the complex nilradical and a compression semi-group for a reductive subgroup (cf. Theorem II.8).

I. The Borel embedding and the compression semigroup

In this section $\mathfrak{g}$ denotes a finite dimensional real Lie algebra. A subalgebra $\mathfrak{a}\subseteq\mathfrak{g}$ is said to be *compactly embedded* if the closure of the subgroup generated by $e^{\operatorname{ad}\mathfrak{a}}$ in $\operatorname{Aut}(\mathfrak{g})$ is compact. We assume that $\mathfrak{g}$ contains a compactly embedded Cartan algebra $\mathfrak{t}$.

Associated to the Cartan subalgebra $\mathfrak{t}_{\mathbf{C}}$ in the complexification $\mathfrak{g}_{\mathbf{C}}$ is a root decomposition as follows. For a linear functional $\lambda\in\mathfrak{t}_{\mathbf{C}}^*$ we set

$$\mathfrak{g}_{\mathbf{C}}^{\lambda}:=\{X\in\mathfrak{g}_{\mathbf{C}}:(\forall Y\in\mathfrak{t}_{\mathbf{C}})[Y,X]=\lambda(Y)X\}$$

and

$$\Delta:=\Delta(\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}}):=\{\lambda\in\mathfrak{t}_{\mathbf{C}}^*\setminus\{0\}:\mathfrak{g}_{\mathbf{C}}^{\lambda}\neq\{0\}\}.$$

Then

$$\mathfrak{g}_{\mathbf{C}}=\mathfrak{t}_{\mathbf{C}}\oplus\bigoplus_{\lambda\in\Delta}\mathfrak{g}_{\mathbf{C}}^{\lambda},$$

$\lambda(\mathfrak{t})\subseteq i\mathbf{R}$ for all $\lambda\in\Delta$ and $\sigma(\mathfrak{g}_{\mathbf{C}}^{\lambda})=\mathfrak{g}_{\mathbf{C}}^{-\lambda}$, where σ denotes complex conjugation on $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{g}$. Let $\mathfrak{k}\supseteq\mathfrak{t}$ denote a maximal compactly embedded subalgebra. Then a root is said to be *compact* if $\mathfrak{g}_{\mathbf{C}}^{\lambda}\subseteq\mathfrak{k}_{\mathbf{C}}$. We write Δ_k for the set of compact roots and Δ_p for the set of non-compact roots. A subset $\Sigma\subseteq\Delta$ is called a *parabolic system of roots* if there exists an element $X\in i\mathfrak{t}$ such that $\Sigma=\{\alpha\in\Delta:\alpha(X)\geq 0\}$.

A *positive system* Δ^+ is a parabolic system with $\Delta^+\cap-\Delta^+=\emptyset$. The *Weyl group* associated to $\mathfrak{t}$ is the group

$$\mathcal{W}_{\mathfrak{k}}:=N_G(\mathfrak{t})/Z_G(\mathfrak{t})\cong N_K(\mathfrak{t})/Z_K(\mathfrak{t})$$

which coincides with the Weyl group of the compact Lie algbra $\mathfrak{k}$. A positive system Δ^+ is said to be $\mathfrak{k}$*-adapted* if Δ_p^+ is invariant under the Weyl group. The Lie algebra $\mathfrak{g}$ is said to have *cone potential* if for every non-compact root α and for every non-zero element $X_\alpha\in\mathfrak{g}_{\mathbf{C}}^\alpha$ we have that $[X_\alpha,\overline{X}_\alpha]\neq 0$.

We fix a simply connected complex Lie group $G_{\mathbf{C}}$ with $\mathbf{L}(G_{\mathbf{C}})=\mathfrak{g}_{\mathbf{C}}$. Then the complex conjugation $\sigma\colon\mathfrak{g}_{\mathbf{C}}\to\mathfrak{g}_{\mathbf{C}}$ integrates to an antiholomorphic automorphism of $G_{\mathbf{C}}$ which we also denote $\sigma\colon g\mapsto\overline{g}$. We write G for the subgroup $G_{\mathbf{C}}^\sigma$ of fixed points with respect to σ. We also define $K:=\exp\mathfrak{k}$ and $K_{\mathbf{C}}:=\langle\exp\mathfrak{k}_{\mathbf{C}}\rangle\subseteq G_{\mathbf{C}}$.

Definition I.1. Let $\omega\in\mathfrak{g}^*$ be a linear functional on $\mathfrak{g}$ which we also consider as a complex linear functional on $\mathfrak{g}_{\mathbf{C}}$. The *coadjoint action* of G on $\mathfrak{g}^*$ is given by

$$\mathrm{Ad}^*(g).\alpha=\alpha\circ\mathrm{Ad}(g)^{-1}$$

for $g\in G$ and $\alpha\in\mathfrak{g}^*$. Let $G^\omega:=\{g\in G:\mathrm{Ad}^*(g).\omega=\omega\}$ denote the stabilizer of ω. Then $\mathfrak{g}^\omega:=\{X\in\mathfrak{g}:\omega\circ\mathrm{ad}\,X=0\}$ is the Lie algebra of G^ω.

(a) A complex subalgebra $\mathfrak{b}\subseteq\mathfrak{g}_{\mathbf{C}}$ satisfying

$$\mathfrak{b}\cap\mathfrak{g}=\mathfrak{g}^\omega,\quad \omega([\mathfrak{b},\mathfrak{b}])=\{0\},\quad\text{and}\quad \mathfrak{b}+\overline{\mathfrak{b}}=\mathfrak{g}_{\mathbf{C}}$$

is called a *complex polarization in* ω. Note that for any complex polarization in ω we have $\mathfrak{b}\cap\overline{\mathfrak{b}}=(\mathfrak{g}^\omega)_{\mathbf{C}}$ and $\mathfrak{g}+\overline{\mathfrak{b}}=\mathfrak{g}_{\mathbf{C}}$ (cf. [N8]). Recall that the $\mathrm{Ad}(G^\omega)$-invariant complex polarizations are in one-to-one correspondence with pseudo-Kähler structures on the coadjoint orbit $\mathcal{O}_\omega:=\mathrm{Ad}^*(G).\omega$ which are compatible with the natural symplectic structure on $\mathcal{O}_\omega$ (cf. [N8]). This symplectic structure is defined by the 2-form Ω given by

$$\Omega(\beta)(\beta\circ\mathrm{ad}\,X,\beta\circ\mathrm{ad}\,Y)=\beta([X,Y])$$

(cf. [LM]).

A complex polarization is called *positive* if the corresponding pseudo-Kähler structure is positive, i.e., if the Hermitean form

$$(X,Y)\mapsto\omega([iX,\overline{Y}])$$

is positive semidefinite on $\mathfrak{b}$ which means that $\omega(i[X,\overline{X}])\geq 0$ for all $X\in\mathfrak{b}$. Note that always $\mathfrak{b}^\perp=(\mathfrak{g}^\omega)_{\mathbf{C}}$ holds with respect to this form.

(b) A coadjoint orbit $\mathcal{O}_\omega$ is called a *pseudo-Kähler orbit* if there exists a complex polarization in ω which is invariant under G^ω, and a *Kähler orbit* if there exists such a positive complex polarization in ω (cf. [OW] and [Woo, pp. 92, 103]).

In the following we identify the dual $\mathfrak{t}^*$ of the Cartan algebra $\mathfrak{t}$ always with the subspace $[\mathfrak{t},\mathfrak{g}]^\perp$ of $\mathfrak{g}^*$. This makes sense since $\mathfrak{g}=\mathfrak{t}\oplus[\mathfrak{t},\mathfrak{g}]$ is a direct sum of vector spaces. We will see that a pseudo-Kähler orbit of an element $\omega\in\mathfrak{t}^*$, where the polarization is defined by a subalgebra

$$\mathfrak{p}_\Sigma:=\mathfrak{t}_{\mathbb{C}}\oplus\bigoplus_{\alpha\in\Sigma}\mathfrak{g}_{\mathbb{C}}^\alpha$$

and $\Sigma=\{\alpha\in\Delta:\alpha(X)\geq 0\}$ is a parabolic system of roots, can be embedded as an open G-orbit into a complex homogeneous space of $G_{\mathbb{C}}$. The main achievement of this paper is the calculation of the compression semigroup of this open G-orbit. For the case of simple Hermitean Lie groups this was done for the orbit isomorphic to a symmetric space by Ol'shanskii ([Ols]) and independently by Stanton ([St]).

For the following lemma we recall from [N7, Lemma II.2] that for every parabolic system Σ the subalgebra $\mathfrak{b}:=\mathfrak{p}_\Sigma$ decomposes as a semidirect product $\mathfrak{b}\cong\mathfrak{b}_u\rtimes\mathfrak{b}_s$, where $\mathfrak{b}_s=\mathfrak{p}_{\Sigma\cap-\Sigma}$ and $\mathfrak{b}_u$ is a nilpotent ideal.

Lemma I.2. *Let $\Sigma\subseteq\Delta$ be a parabolic system of roots, $\mathfrak{b}:=\mathfrak{p}_\Sigma$, and $B:=\langle\exp_{G_{\mathbb{C}}}\mathfrak{b}\rangle$. Then the following assertions hold:*

(i) *The group B is closed, $B\cong B_u\rtimes B_s$, where $B_u=\exp\mathfrak{b}_u$ is simply connected, and $B_s=\langle\exp_{G_{\mathbb{C}}}\mathfrak{b}_s\rangle$.*

(ii) *The mapping $\overline{B}_u\times B_s\times B_u\to G_{\mathbb{C}}$, $(x,y,z)\mapsto xyz$ is a diffeomorphism onto an open subset of $G_{\mathbb{C}}$.*

(iii) *$B\cap\overline{B}=B_s$.*

Proof. (i) Since $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$ contained in $\mathfrak{b}$, the closedness of B follows from [B2, Ch. 7, §2, no. 1, Cor. 4]. The closedness of B_s follows with the same argument.

To see that B_u is closed, let $\widetilde{B}$ denote the simply connected covering group of B. Then $\widetilde{B}\cong\widetilde{B}_u\rtimes\widetilde{B}_s$ and it is clear that $\widetilde{B}_u$ is closed in $\widetilde{B}$.

Let $\mathfrak{a}:=\mathfrak{t}+iZ(\mathfrak{g})\subseteq\mathfrak{t}_{\mathbb{C}}\subseteq\mathfrak{b}_s$. We claim that $\mathfrak{a}$ is a maximal compactly embedded abelian subalgebra of $\mathfrak{b}$. To see this, suppose that $\mathfrak{a}'\supseteq\mathfrak{a}$ is compactly embedded and abelian. Then $\mathfrak{a}'\subseteq Z_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{t})=\mathfrak{t}_{\mathbb{C}}$ and the assertion follows immediately from the fact that every $\operatorname{ad}X$, $X\in\mathfrak{a}'$ has purely imaginary eigenvalues.

Now we use [HN1, III.7.11] to see that $Z(\widetilde{B})\subseteq\exp_{\widetilde{B}}(\mathfrak{t}+iZ(\mathfrak{g}))$, which in turn yields $Z(\widetilde{B})\subseteq\widetilde{B}_s$. Let $\pi_1(B)\subseteq Z(\widetilde{B})$ denote the kernel of the covering $\widetilde{B}\to B$. Then $\pi_1(B)\widetilde{B}_u\cong\widetilde{B}_u\times\pi_1(B)$ is a closed subgroup and $\pi_1(B)\cap\widetilde{B}_u=\{\mathbf{1}\}$. We conclude that B_u is closed and simply connected because it is the injective image of $\widetilde{B}_u$.

To see that $B_u \cap B_s = \{\mathbf{1}\}$, we only note that $\widetilde{B}$ is a semidirect product and $\pi_1(B)$ is contained in the factor $\widetilde{B}_s$.

(ii) That the image is an open subset follows from the fact that $\mathfrak{b}_u + \mathfrak{b}_s + \overline{\mathfrak{b}}_u = \mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{g}_{\mathbb{C}}$. To see that the mapping is a diffeomorphism, in view of (i), it only remains to show that $B \cap \overline{B}_u = \{\mathbf{1}\}$.

Pick $E \in i\mathfrak{t}$ such that $\Sigma = \{\lambda \in \Delta : \lambda(E) \geq 0\}$. Set

$$\gamma_t(g) := \exp(-tE) g \exp(tE).$$

Then γ_t fixes B_s pointwise and, since $\lambda(E) > 0$ holds for all $\lambda \in \Sigma \setminus -\Sigma$, we conclude that $\lim_{t \to \infty} \gamma_t(b) = \mathbf{1}$ for all $b \in B_u$. On the other hand, for every $\overline{b} \in \overline{B}_u \setminus \{\mathbf{1}\}$, $\gamma_t(\overline{b})$ eventually leaves every compact subset of the closed subgroup $\overline{B}_u$. Hence we have $\overline{B}_u \cap B = \{\mathbf{1}\}$.

(iii) Since B_s is invariant under σ, the inclusion $B_s \subseteq B \cap \overline{B}$ is trivial. To see the converse, pick $b = b_s b_u \in B$ with $b_s \in B_s$ and $b_u \in B_u$. Then $b \in B \cap \overline{B}$ implies that $b_u \in B \cap \overline{B}$ and therefore $\overline{b}_u \in \overline{B}_u \cap B = \{\mathbf{1}\}$. Hence $b_u = \mathbf{1}$ and thus $B \cap \overline{B} \subseteq B_s$. □

The preceding lemma is a generalization of the well known result of Harish–Chandra for the case where G is simple Hermitean and $\Sigma = \Delta_k \cup \Delta_p^+$ (cf. [Hel, p. 388]). The following result is a generalization of the *Borel embedding theorem* for Hermitean symmetric spaces (cf. [Hel, Ch. VIII, §7]).

Theorem I.3. (The embedding theorem) *Let $\omega \in \mathfrak{t}^*$, $\Sigma \subseteq \Delta$ be a parabolic subset, and suppose that $\mathfrak{b} = \mathfrak{p}_\Sigma$ is a complex polarization in ω. We set $B := \langle \exp_{G_{\mathbb{C}}} \mathfrak{b} \rangle$, $M := G_{\mathbb{C}} / \overline{B}$, and write x_0 for the base point of M. Then the orbit mapping $G \to M, g \mapsto g.x_0$ induces an open embedding*

$$\mathcal{O}_\omega \cong G/G^\omega \to G_{\mathbb{C}} / \overline{B}$$

which is holomorphic with respect to the complex structure on $\mathcal{O}_\omega$ defined by the complex polarization $\mathfrak{b}$.

Proof. In view of Lemma I.2(i) and [N8, Prop. I.2], we only have to show that $B \cap G = G^\omega$. First we apply [N10, Thm. I.18] to see that $\mathcal{O}_\omega$ is simply connected. Let $p: \widetilde{G} \to G$ denote the universal covering group. Then $\widetilde{G}^\omega$ is connected and since $p(\widetilde{G}^\omega) = G^\omega$, it follows that G^ω is connected. Hence

$$G^\omega = \langle \exp \mathfrak{g}^\omega \rangle \subseteq B \cap G$$

because $\mathfrak{b} \cap \mathfrak{g} = \mathfrak{g}^\omega$.

On the other hand Lemma I.2(iii) tells us that

$$B \cap G = B_s^\sigma = \{b \in B_s : \sigma(b) = b\}.$$

Since $\mathfrak{b}$ is a complex polarization in ω, we have $\mathfrak{b}_s = \mathfrak{b} \cap \overline{\mathfrak{b}} = (\mathfrak{g}^\omega)_{\mathbb{C}}$ because $\mathfrak{g}_{\mathbb{C}} = \overline{\mathfrak{b}}_u \oplus \mathfrak{b}_s \oplus \mathfrak{b}_u$ is a direct vector space decomposition. Hence $\mathrm{Ad}(B_s).\omega = \omega$ holds in $\mathfrak{g}_{\mathbb{C}}^*$. We conclude that

$$B \cap G \subseteq G_{\mathbb{C}}^\omega \cap G = G^\omega. \quad \square$$

With the preceding theorem we have a realization of the coadjoint orbit $\mathcal{O}_\omega$ as an open subset $D = D_\omega$ of the complex homogeneous space $G_{\mathbb{C}}/\overline{B}$, namely as the open G-orbit of the base point. Next we study the semigroup

$$S(\overline{B}) = \{g \in G_{\mathbb{C}} : gG\overline{B} \subseteq G\overline{B}\} = \{g \in G_{\mathbb{C}} : g.D \subseteq D\}.$$

We call this semigroup the *compression semigroup of the domain D* or the *compression semigroup associated to $\overline{B}$*. Note that $S(\overline{B})$ only depends on B or, equivalently the parabolic set Σ, but not explicitly on ω. Consequently there are many elements ω and therefore many coadjoint orbits $\mathcal{O}_\omega$ which lead to the same semigroup.

Remark I.4. (a) Note that we have not assumed in Theorem I.3 that $\mathfrak{b}$ is a positive polarization in ω, so that this result also applies to pseudo-Kähler orbits.

(b) Proposition IV.16 in [N8] shows in particular that the embedding theorem applies to every coadjoint orbit in a reductive Lie algebra which meets the dual of a compactly embedded Cartan algebra. This can also be viewed as a result on adjoint orbits.

The following observation facilitates the determination of the semigroups $S(\overline{B})$.

Lemma I.5. *Let G be a topological group, D a subset of the locally compact G-space M, and $S = \{g \in G : g.D \subseteq D\}$. Then the following assertions hold:*

(i) *If D is open or closed then the semigroup S is closed in G.*

(ii) *If M is a homogeneous G-space and D is relatively compact then*

$$\operatorname{int} S = \{g \in G : g.\overline{D} \subseteq \operatorname{int} D\}.$$

Proof. (i) This is Lemma 8.34 in [HN2].

(ii) Let $s \in \operatorname{int} S$ and U be a neighborhood of $\mathbf{1}$ in G such that $Us \subseteq S$. Then

$$s.\overline{D} = \overline{s.D} \subseteq U(s.D) = (Us).D \subseteq \operatorname{int} D.$$

This shows $s.\overline{D} \subseteq \operatorname{int} D$.

Conversely, if $s.\overline{D} \subseteq \operatorname{int} D$, then there exists a neighborhood U of $\mathbf{1}$ in G such that $Us.\overline{D} \subseteq \operatorname{int} D$. In particular, it follows that $Us \subseteq S$, i.e., $s \in \operatorname{int} S$. $\square$

Hamiltonian functions and the compression semigroup

One first step towards the determination of the semigroups $S(\overline{B})$ will be a result showing that $S(\overline{B})$ is rather big whenever $\mathcal{O}_\omega$ is a Kähler orbit, i.e., $\mathfrak{b}=\mathfrak{p}_\Sigma$ is a positive polarization.

For $X\in\mathfrak{g}$ we write $H_X:\mathfrak{g}^*\to\mathbf{R}, \nu\mapsto\nu(X)$ for the associated function on $\mathfrak{g}^*$ and in particular on the orbit $\mathcal{O}_\omega$ which is a symplectic manifold. We define the cone B_ω consisting of all those $X\in\mathfrak{g}$ for which the function H_X is bounded from below on $\mathcal{O}_\omega$.

Let $\mathcal{X}$ denote the vector field on $\mathcal{O}_\omega$ defined by

$$\mathcal{X}(\beta)=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}^*(\exp(-tX)).\beta=-\,\mathrm{ad}^*(X)(\beta)=\beta\circ\mathrm{ad}\,X \quad \forall\beta\in\mathcal{O}_\omega.$$

Since the canonical symplectic form on $\mathcal{O}_\omega$ satisfies

$$\Omega(\beta)(\beta\circ\mathrm{ad}\,Y,\beta\circ\mathrm{ad}\,Z)=\beta([Y,Z]) \quad \forall Y,Z\in\mathfrak{g},\ \beta\in\mathcal{O}_\omega,$$

we find that

$$\Omega(\beta)(\mathcal{X}(\beta),\beta\circ\mathrm{ad}\,Y)=\beta([X,Y])=-\langle\beta\circ\mathrm{ad}\,Y,X\rangle=-dH_X(\beta)(\beta\circ\mathrm{ad}\,Y).$$

This means that $\mathcal{X}$ is the Hamiltonian vector field on $\mathcal{O}_\omega$ corresponding to the Hamiltonian function H_X.

Since the function H_X is constant on $\mathcal{O}_\omega$ if and only if the corresponding vector field vanishes, we see that the set of all $X\in\mathfrak{g}$ with constant Hamiltonian function H_X is the Lie algebra of the effectivity kernel of the action of G on $\mathcal{O}_\omega$. If $\mathcal{O}_\omega$ spans $\mathfrak{g}^*$, then the linearity of the vector field $\mathcal{X}$ yields that this Lie algebra consists of those elements for which $\mathrm{ad}\,X=0$, i.e.,

$$Z(\mathfrak{g})=\{X\in\mathfrak{g}:dH_X|_{\mathcal{O}_\omega}=0\}.$$

Definition I.6. (a) We say that an element $\omega\in\mathfrak{g}^*$ is *admissible* if the coadjoint orbit $\mathcal{O}_\omega$ is closed and its convex hull contains no lines. We call $\omega\in\mathfrak{g}^*$ *strictly admissible* if there exists a closed invariant convex set $C\subseteq\mathfrak{g}^*$ which contains no lines and which contains the coadjoint orbit $\mathcal{O}_\omega$ in its *algebraic interior*, i.e., the interior with respect to the affine subspace it generates. We say that $\mathcal{O}_\omega$ is (*strictly*) *admissible* if ω is (strictly) admissible. Note that strict admissibility implies admissibility ([HNP, Cor. 5.12]). It is clear that this property implies that the convex hull of $\mathcal{O}_\omega$ contains no lines.

An element $\omega\in\mathfrak{g}^*$ is said to be of *convex type* if the coadjoint orbit $\mathcal{O}_\omega$ lies in a closed pointed convex cone and of *strict convex type* if $\mathcal{O}_\omega$ lies in the algebraic interior of a pointed convex cone in $\mathfrak{g}^*$ which is invariant under the coadjoint action. We recall from [HNP, Lemma 5.9] that f is strictly admissible if and only if $(\omega,1)$ is of strict convexity type in $\mathfrak{g}^*\times\mathbf{R}$. ☐

Lemma I.7. *Let $\omega \in \mathfrak{t}^*$ be admissible and $X \in \operatorname{int} B_\omega$. Then the function H_X is proper on $\mathcal{O}_\omega$.*

Proof. This is a consequence of [HNP, Prop. 1.17]. □

In the following we say that $\omega \in \mathfrak{g}^*$ is *reduced* if $\mathfrak{z}(\mathfrak{g})$ is the largest ideal contained in the stabilizer algebra $\mathfrak{g}^\omega$. For a positive system Δ^+ of roots we define the cone

$$C_{\max} := C_{\max}(\Delta_p^+) := \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_p^+)\ i\alpha(X) \geq 0\}$$

and

$$C_{\min} := C_{\min}(\Delta_p^+) := \operatorname{cone}\{i[\overline{X}_\alpha, X_\alpha] : \alpha \in \Delta_p^+, X_\alpha \in \mathfrak{g}_{\mathbf{C}}^\alpha\},$$

where for a subset S of a vector space cone(S) denotes the smallest closed convex cone containing S. For a cone C in a vector space V we write $C^\star := \{\nu \in V^* : \nu(C) \subseteq \mathbf{R}^+\}$ for the dual cone. A closed convex cone is called a *wedge*.

Lemma I.8. *Let $\mathfrak{g}$ be a Lie algebra containing the compactly embedded Cartan algebra $\mathfrak{t}$. Then there exists a reductive subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ such that*

$$\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{l} \quad \text{and} \quad \mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus (\mathfrak{t} \cap \mathfrak{l}),$$

where $\mathfrak{n} = [\mathfrak{t}, \mathfrak{n}] + \mathfrak{z}(\mathfrak{g})$ is the nilradical.

Proof. (i) Using Lemma III.7.15 in [HN1], we find a Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ with $\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{r}) \oplus (\mathfrak{t} \cap \mathfrak{s})$ and $[\mathfrak{t} \cap \mathfrak{r}, \mathfrak{s}] = \{0\}$. Let $\mathfrak{t}_1$ be a vector space complement for the center in $\mathfrak{t} \cap \mathfrak{r}$. We set $\mathfrak{l} := \mathfrak{t}_1 \oplus \mathfrak{s}$. Then $\mathfrak{l}$ is reductive and $\mathfrak{l} \cap \mathfrak{n} \subseteq \mathfrak{t}_1 \cap \mathfrak{n} = \{0\}$ because $\mathfrak{t} \cap \mathfrak{n} = Z(\mathfrak{g})$. On the other hand $[\mathfrak{t}, \mathfrak{r}] \subseteq [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$ shows that $\mathfrak{r} \cong \mathfrak{n} \rtimes \mathfrak{t}_1$. Hence $\mathfrak{g} \cong \mathfrak{n} \rtimes \mathfrak{l}$. □

Proposition I.9. *Let $\omega \in \mathfrak{t}^*$ be strictly admissible and reduced. Then the following assertions hold:*

(i) *There exists a unique $\mathfrak{k}$-adapted positive system Δ_p^+ of non-compact roots such that $\omega \in \operatorname{int} C_{\min}^\star(\Delta_p^+)$ and a unique positive polarization $\mathfrak{b} = \mathfrak{p}_\Sigma$ with $\Delta_p^+ \subseteq \Sigma$ in ω.*

(ii) *$\overline{B}_\omega = W_{\max}$, where $W_{\max}$ denotes the unique invariant wedge in $\mathfrak{g}$ determined by $C_{\max} = W_{\max} \cap \mathfrak{t}$.*

(iii) *If an invariant wedge $W \subseteq \mathfrak{g}$ contains $C_{\max}$, then it contains $W_{\max}$.*

(iv) *$W_{\max}$ contains the nilradical $\mathfrak{n}$ of $\mathfrak{g}$.*

Proof. (i) First we use Theorem IV.23 in [N8] to find a $\mathfrak{k}$-adapted positive system such that $\omega \in \operatorname{int} C_{\min}^\star$. To see that the system Δ_p^+ is uniquely determined by this condition, let $\alpha \in \Delta_p^+$ and pick $X_\alpha \in \mathfrak{g}_{\mathbf{C}}^\alpha \setminus \{0\}$. Then, according to [N8, Thm. IV.23], the Lie algebra $\mathfrak{g}$ has cone potential, hence $[X_\alpha, \overline{X}_\alpha] \neq \{0\}$. Therefore $i[\overline{X}_\alpha, X_\alpha] \in C_{\min}$ and the choice of Δ^+ shows that $\omega(i[\overline{X}_\alpha, X_\alpha]) > 0$. Changing α to $-\alpha$ changes the sign of this expression. Therefore Δ_p^+ is uniquely determined. The second statement follows from [N8, Thm. IV.21, 23].

(ii) First we apply the convexity theorem for coadjoint orbits [HNP, Thm. 5.17] which yields that

$$p_{\mathfrak{t}^*}(\mathcal{O}_\omega) = \operatorname{conv}(\mathcal{W}_{\mathfrak{k}}.\omega) + \operatorname{cone}(i\Delta_p^+),$$

where $p_{\mathfrak{t}^*}: \mathfrak{g}^* \to \mathfrak{t}^*$ denotes the restriction mapping. Note that the assumptions of this theorem are met because $\omega \in \operatorname{int} C_{\min}^\star$. Therefore $X \in \mathfrak{t}$ is contained in B_ω if and only if H_X is bounded from below on the cone generated by $i\Delta_p^+$. This means that $X \in C_{\max}$. Therefore $\overline{B}_\omega \cap \mathfrak{t} = C_{\max}$. Since an invariant wedge $W \subseteq \mathfrak{g}$ is uniquely determined by its intersection with $\mathfrak{t}$ (cf. [N4, Prop. III.34]), it follows that $W_{\max} = \overline{B}_\omega$.

(iii) Suppose $C_{\max} \subseteq W$. Let W_1 denote the closed convex hull of $\operatorname{Ad}(G).C_{\max}$. Then W_1 is a generating invariant wedge contained in $W_{\max}$ such that $W_1 \cap \mathfrak{t} \supseteq C_{\max}$. On the other hand $W_1 \cap \mathfrak{t} \subseteq W_{\max} \cap \mathfrak{t} = C_{\max}$. Therefore $W_1 \cap \mathfrak{t} = C_{\max}$ and consequently $W_1 = W_{\max}$ by (ii). Now the assertion follows from the trivial observation that $W_1 \subseteq W$.

(iv) The nilradical $\mathfrak{n} \subseteq \mathfrak{g}$ can be written as $\mathfrak{n} = [\mathfrak{t}, \mathfrak{n}] + \mathfrak{z}(\mathfrak{g})$ (cf. Lemma I.8). Let $W := \overline{W_{\max} + \mathfrak{n}}$. Then W is an invariant wedge in $\mathfrak{g}$ and if $p_{\mathfrak{t}}: \mathfrak{g} \to \mathfrak{t}$ denotes the projection along $[\mathfrak{t}, \mathfrak{g}]$, then $p_{\mathfrak{t}}(W) = W \cap \mathfrak{t} = p_{\mathfrak{t}}(W_{\max}) = C_{\max}$. Therefore $W = W_{\max}$ follows from (ii) and therefore we have $\mathfrak{n} \subseteq W_{\max}$. □

Let S be a closed subsemigroup of the Lie group G. For the following proposition we recall the definition of the *tangent Lie wedge*

$$\mathbf{L}(S) := \{X \in \mathbf{L}(G) : \exp(\mathbf{R}^+ X) \subseteq S\}$$

of S. Recall that a wedge W in a Lie algebra $\mathfrak{g}$ is called a *Lie wedge* if $e^{\operatorname{ad} X} W = W$ holds for all $X \in W \cap (-W)$.

Proposition I.10. *Let $\omega \in \mathfrak{t}^*$ be strictly admissible and $\mathfrak{b}$ the positive complex polarization in ω. Then $i\overline{B}_\omega \subseteq \mathbf{L}(S(\overline{B}))$.*

Proof. We have already seen, in Lemma I.5, that the semigroup $S := S(\overline{B})$ is closed. Since $D \subseteq G_{\mathbf{C}}/\overline{B}$ is a G-orbit, it is clear that $G \subseteq S$, i.e., that $\mathfrak{g} \subseteq \mathbf{L}(S)$. Hence $\mathbf{L}(S) = \mathfrak{g} + iW$, where $W := (-i\mathbf{L}(S)) \cap \mathfrak{g}$ is an invariant wedge.

Next we note that we may without loss of generality assume that ω is strictly reduced because every ideal $\mathfrak{a}$ contained in $\mathcal{O}_\omega^\perp$ is contained in B_ω and on the other hand $\langle \exp(\mathfrak{a}_{\mathbf{C}}) \rangle \subseteq S(\overline{B})$ in this case.

It remains to show that $B_\omega \subseteq W$. In view of Proposition I.9(ii), (iii), it suffices to show that $\operatorname{int} C_{\max} = (\operatorname{int} B_\omega) \cap \mathfrak{t} \subseteq W$. Let $X \in \operatorname{int} C_{\max}$ and $\Phi: \mathcal{O}_\omega \to D \subseteq M$ denote the embedding of $\mathcal{O}_\omega$.

Since $\lim_{p\to\partial D} H_X(\Phi^{-1}(p))=\infty$ (Lemma I.7), it suffices to show that, for $t\geq 0$

$$(H_X\circ\Phi^{-1})(\exp itX.\Phi(\beta))\leq H_X(\beta)\quad \forall\beta\in\mathcal{O}_\omega.$$

Let $\mathcal{X}$ denote the vector field on $\mathfrak{g}^*$ defined by $\mathcal{X}(\beta)=\beta\circ\mathrm{ad}\,X$ for all $\beta\in\mathcal{O}_\omega$ and recall that this is the Hamiltonian vector field on $\mathcal{O}_\omega$ corresponding to the function H_X. Let J denote the tensor field defining the complex structure on $\mathcal{O}_\omega$. Then

$$\begin{aligned} dH_X(\beta)(d\Phi^{-1}(\Phi(\beta))(-id\Phi(\beta)\mathcal{X}(\beta))) &= -dH_X(\beta)(J(\beta)\mathcal{X}(\beta)) \\ &= -\Omega(\beta)(J(\beta)\mathcal{X}(\beta),\mathcal{X}(\beta)). \end{aligned}$$

This expression is always non-positive since $\mathfrak{b}$ is a positive complex polarization in ω and the corresponding Kähler structure is compatible with the symplectic form (cf. [N8]). Now the assertion follows from

$$\Phi(\mathrm{Ad}^*(\exp(itX)).\beta)=\exp(itX).\Phi(\beta).\quad\square$$

Let $\mathfrak{g}=\mathfrak{n}\rtimes\mathfrak{l}$ denote the decomposition described above. We set $N:=\exp\mathfrak{n}$, $L:=\langle\exp\mathfrak{l}\rangle$, $N_{\mathbb{C}}:=\exp\mathfrak{n}_{\mathbb{C}}$, $L_{\mathbb{C}}:=\langle\exp\mathfrak{l}_{\mathbb{C}}\rangle$, $\mathfrak{p}:=\mathfrak{b}\cap\mathfrak{l}_{\mathbb{C}}$, and $P:=\langle\exp\mathfrak{p}\rangle$. Note that

$$G\cong N\rtimes L\quad\text{and}\quad G_{\mathbb{C}}\cong N_{\mathbb{C}}\rtimes L_{\mathbb{C}},$$

so that $N_{\mathbb{C}}$ and $L_{\mathbb{C}}$ are simply connected.

Lemma I.11. *Let $N_{\mathbb{C}}$ be a complex nilpotent Lie group and N a connected subgroup such that $\mathbf{L}(N)$ is a real form of $\mathbf{L}(N_{\mathbb{C}})$. Let further $\mathfrak{b}$ be a complex subalgebra of $\mathfrak{n}_{\mathbb{C}}$ such that for every characteristic ideal $\mathfrak{a}\subseteq\mathfrak{n}$ we have that $\mathfrak{a}_{\mathbb{C}}=\mathfrak{a}+(\mathfrak{b}\cap\mathfrak{a}_{\mathbb{C}})$. Then $NB=N_{\mathbb{C}}$ holds for $B=\exp\mathfrak{b}$.*

Proof. Let

$$\mathfrak{n}^0:=\{0\}\subseteq\mathfrak{n}^1=\mathfrak{z}(\mathfrak{n})\subseteq\ldots\subseteq\mathfrak{n}^k=\mathfrak{n}$$

denote the ascending central series of $\mathfrak{n}$. Note that every ideal $\mathfrak{n}^j$ is characteristic. We write N^j for the associated analytic subgroups of N and $N^j_{\mathbb{C}}$ for the corresponding subgroups of $N_{\mathbb{C}}$. We show by induction over j that N^jB is a subgroup containing $N^j_{\mathbb{C}}$.

This is trivial for $j=0$. Suppose that $j<k$ and the claim holds for j. Set $B':=B\cap N^{j+1}_{\mathbb{C}}$. Then $(N^jB)\cap N^{j+1}_{\mathbb{C}}=N^jB'$ is a subgroup containing $N^j_{\mathbb{C}}$ and therefore the commutator group of $N^{j+1}_{\mathbb{C}}$. It follows in particular that this subgroup is normal. Therefore $N^{j+1}B'$ is a product of a subgroup and a normal subgroup, so that it

is again a subgroup. According to our assumption, the Lie algebra of this group contains $\mathfrak{n}^{j+1}+(\mathfrak{b}\cap\mathfrak{n}_{\mathbb{C}}^{j+1})=\mathfrak{n}_{\mathbb{C}}^{j+1}$. Hence $N_{\mathbb{C}}^{j+1}=N^{j+1}B'$. From this it follows that $N^{j+1}B\supseteq N^{j+1}B'=N_{\mathbb{C}}^{j+1}$. Thus it contains the subgroup $N_{\mathbb{C}}^{j+1}B$ because it is right B-invariant. This shows that $N^{j+1}B$ is equal to this subgroup and the assertion follows.

For $j=k$ we obtain the assertion of the lemma. □

Lemma I.12. *Let $\mathfrak{t}\subseteq\mathfrak{g}$ be a compactly embedded Cartan algebra, $\Sigma\subseteq\Delta$ a parabolic system and $\mathfrak{b}=\mathfrak{p}_{\Sigma}$. Then $\mathfrak{a}_{\mathbb{C}}\subseteq\mathfrak{a}+\mathfrak{b}$ holds for every characteristic ideal $\mathfrak{a}$ of $\mathfrak{g}$.*

Proof. Since $\mathfrak{a}$ is characteristic, its complexification is invariant under $\mathfrak{t}_{\mathbb{C}}$, hence adapted to the root decomposition. Let $\alpha\in\Delta$. If $\alpha\in\Sigma$, then $\mathfrak{a}_{\mathbb{C}}\cap\mathfrak{g}_{\mathbb{C}}^{\alpha}\subseteq\mathfrak{b}$ holds trivially. If $\alpha\in-\Sigma$, then

$$\mathfrak{a}_{\mathbb{C}}\cap\mathfrak{g}_{\mathbb{C}}^{\alpha}\subseteq\mathfrak{a}\cap(\mathfrak{g}_{\mathbb{C}}^{\alpha}+\mathfrak{g}_{\mathbb{C}}^{-\alpha})+\mathfrak{g}_{\mathbb{C}}^{-\alpha}\subseteq\mathfrak{a}+\mathfrak{b}. \quad \square$$

Proposition I.13. *Let $\mathfrak{t}\subseteq\mathfrak{g}$ be a compactly embedded Cartan algebra, $\omega\in\mathfrak{t}^*$, and $\Sigma\subseteq\Delta$ a parabolic system such that $\mathfrak{b}=\mathfrak{p}_{\Sigma}$ is a complex polarization in ω. Then the following assertions hold:*

(i) $N_{\mathbb{C}}\subseteq S(\overline{B})$.

(ii) $LN_{\mathbb{C}}=GN_{\mathbb{C}}$.

(iii) $L\overline{P}N_{\mathbb{C}}=G\overline{B}$.

(iv) $S(\overline{B})=N_{\mathbb{C}}S(\overline{P})$*, where* $S(\overline{P})$ *is the compression semigroup of the open L-orbit of the base point in* $L_{\mathbb{C}}/\overline{P}$.

Proof. (i) First we use Lemma I.12 to see that the subalgebra $\mathfrak{n}_{\mathbb{C}}\cap\overline{\mathfrak{b}}$ satisfies the assumptions of Lemma I.11. Thus $N\overline{B}=N_{\mathbb{C}}\overline{B}$ and therefore

$$N_{\mathbb{C}}G\overline{B}=GN_{\mathbb{C}}\overline{B}=GN\overline{B}=G\overline{B}.$$

(ii) Since $G=NL$, we have that $GN_{\mathbb{C}}=LNN_{\mathbb{C}}=LN_{\mathbb{C}}$.

(iii) First we note that (i) implies that $N_{\mathbb{C}}\subseteq S(\overline{B})$. Hence

$$G.\overline{B}=N_{\mathbb{C}}(G.\overline{B}).$$

Since both subalgebras $\mathfrak{l}$ and $\mathfrak{n}$ are invariant under $\mathfrak{t}$, it follows that their complexifications are adapted to the root space decomposition. Now it follows from $\mathfrak{g}=\mathfrak{l}+\mathfrak{n}$ that $\mathfrak{g}_{\mathbb{C}}^{\alpha}=(\mathfrak{g}_{\mathbb{C}}^{\alpha}\cap\mathfrak{n}_{\mathbb{C}})\oplus(\mathfrak{g}_{\mathbb{C}}^{\alpha}\cap\mathfrak{l}_{\mathbb{C}})$ holds for all roots $\alpha\in\Delta$. It follows that $\mathfrak{b}=(\mathfrak{b}\cap\mathfrak{l}_{\mathbb{C}})+(\mathfrak{b}\cap\mathfrak{n}_{\mathbb{C}})$ which in turn yields that

$$\mathfrak{n}_{\mathbb{C}}+\overline{\mathfrak{p}}=\mathfrak{n}_{\mathbb{C}}+(\overline{\mathfrak{b}}\cap\mathfrak{l}_{\mathbb{C}})=\mathfrak{n}_{\mathbb{C}}+\overline{\mathfrak{b}},$$

and therefore $\overline{P}N_{\mathbb{C}}=\overline{B}N_{\mathbb{C}}$. Using (i), this leads to

$$L\overline{P}N_{\mathbb{C}}=L\overline{B}N_{\mathbb{C}}=LN_{\mathbb{C}}\overline{B}=GN_{\mathbb{C}}\overline{B}=N_{\mathbb{C}}(G\overline{B})=G\overline{B}.$$

(iv) Since $G_{\mathbb{C}}\cong N_{\mathbb{C}}\rtimes L_{\mathbb{C}}$, it is clear that, for $g\in L_{\mathbb{C}}$,

$$g(L\overline{P})N_{\mathbb{C}}\subseteq(L\overline{P})N_{\mathbb{C}} \quad\Longleftrightarrow\quad g(L\overline{P})\subseteq L\overline{P}.$$

Therefore $S(\overline{B})\cap L_{\mathbb{C}}=S(\overline{P})$ and the assertion follows from $N_{\mathbb{C}}\subseteq S(\overline{B})$. □

In view of the preceding proposition, the investigation of the semigroup $S(\overline{B})$ boils down to the semigroup $S(\overline{P})$ associated to the reductive group $L_{\mathbb{C}}$.

Lemma I.14. *Let $\nu:=\omega|_{\mathfrak{l}}$. Then $\mathfrak{p}=\mathfrak{p}_{\Sigma'}$ is a complex polarization in ν. If $\mathfrak{b}$ is positive, then the same holds for $\mathfrak{p}$.*

Proof. Since $\mathfrak{b}=\mathfrak{p}_{\Sigma}$ holds for a parabolic system Σ of roots, it follows that the set $\Sigma':=\{\lambda\in\Sigma:\mathfrak{g}_{\mathbb{C}}^{\lambda}\cap\mathfrak{l}_{\mathbb{C}}\neq\{0\}\}$ is also parabolic. Therefore $\mathfrak{p}=\mathfrak{b}\cap\mathfrak{l}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{l}_{\mathbb{C}}$. Since $[\mathfrak{l},\mathfrak{n}]\subseteq[\mathfrak{t},\mathfrak{n}]$ ([HN1, III.7.15]), we find that

$$\begin{aligned}\mathfrak{l}^{\nu}&=\{X\in\mathfrak{l}:\nu([X,\mathfrak{l}])=\{0\}\}=\{X\in\mathfrak{l}:\nu([X,\mathfrak{g}])=\{0\}\}\\&=\{X\in\mathfrak{l}:\omega([X,\mathfrak{g}])=\{0\}\}=\mathfrak{g}^{\omega}\cap\mathfrak{l}=\mathfrak{p}\cap\mathfrak{l}.\end{aligned}$$

Hence $(\mathfrak{l}^{\nu})_{\mathbb{C}}=\mathfrak{p}\cap\overline{\mathfrak{p}}$ and consequently $\mathfrak{p}$ is a complex polarization in ν.

If, in addition, $\mathfrak{b}$ is a positive polarization, then

$$(X,Y)\mapsto\nu(i[X,\overline{Y}])=\omega(i[X,\overline{Y}])$$

defines a positive semidefinite Hermitean form on $\mathfrak{p}$ and therefore $\mathfrak{p}$ is also positive. □

This result entails that we find exactly the same situation as in $\mathfrak{g}$ in the reductive Lie algebra $\mathfrak{l}$. As we have seen in [N8], the subalgebra $\mathfrak{p}$ is well adapted to the decomposition of $\mathfrak{l}_{\mathbb{C}}$ into simple ideals. It follows immediately that the space $G_{\mathbb{C}}/\overline{P}$ decomposes accordingly, and the same holds for the G-orbit of the base point. Hence $S(\overline{P})$ is a direct product of semigroups of the same type corresponding to the simple factors. So it remains to study the simple case.

In the next section we conclude the determination of the semigroup $S(\overline{B})$ corresponding to a strictly admissible coadjoint orbit $\mathcal{O}_{\omega}$. The determination of this semigroup for the case where $\mathcal{O}_{\omega}$ corresponds to a Hermitean symmetric domain is due to Ol'shanskii (cf. [Ols]).

II. The case of simple Hermitean groups

In the next two sections we consider the following problem. Let G be a linear simple Lie group, $G_{\mathbf{C}}$ its complexification, $M=G_{\mathbf{C}}/P$ a complex flag manifold and $\mathcal{O}\subseteq M$ an open G-orbit. We assume that P is the stabilizer of a point in $\mathcal{O}$. Then

$$S(P)=\{g\in G_{\mathbf{C}}:g.\mathcal{O}\subseteq\mathcal{O}\}.$$

We will show in Section III that these semigroups have non-empty interior different from $G_{\mathbf{C}}$ if and only if G is Hermitean and the orbit $\mathcal{O}$ is a Borel embedding of an admissible coadjoint orbit.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}_0$ be a Cartan decomposition, $\mathfrak{t}\subseteq\mathfrak{k}$ a Cartan subalgebra, and $\mathfrak{h}=\mathfrak{t}+\mathfrak{a}=Z_{\mathfrak{g}}(\mathfrak{t})$ the corresponding Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbf{C}}=(\mathfrak{k}+i\mathfrak{p}_0)+(i\mathfrak{k}+\mathfrak{p}_0)$ is a Cartan decomposition of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{h}':=i\mathfrak{t}+\mathfrak{a}$ is a maximal abelian subalgebra of $i\mathfrak{k}+\mathfrak{p}_0$.

To study our semigroup $S(P)$ we first need some knowledge on the G-double coset decomposition of $G_{\mathbf{C}}$.

Proposition II.1. *Let G be semisimple and $\mathfrak{t}_1,\dots,\mathfrak{t}_n$ representatives for the conjugacy classes of Cartan subalgebras in $\mathfrak{g}$. Then the set*

$$\bigcup_{i=1}^{n} GN_{G_{\mathbf{C}}}(\mathfrak{t}_i)G$$

contains an open dense subset of $G_{\mathbf{C}}$.

Proof. For $g\in G_{\mathbf{C}}$ we set $g^*:=\sigma(g)^{-1}$, where σ denotes complex conjugation on $G_{\mathbf{C}}$. Let $G_{\mathbf{C}}^{\mathrm{reg}}$ denote the set of regular elements in $G_{\mathbf{C}}$,

$$G'_{\mathbf{C}}:=\{g\in G_{\mathbf{C}}:gg^*\in G_{\mathbf{C}}^{\mathrm{reg}}\},$$

and $T_i:=Z_{G_{\mathbf{C}}}(\mathfrak{t}_i)$ the Cartan subgroups corresponding to the Cartan subalgebras $\mathfrak{t}_{i\mathbf{C}}\subseteq\mathfrak{g}_{\mathbf{C}}$. Then it follows from [OM, p. 400] that the open set $G'_{\mathbf{C}}$ is a union of the finitely many open sets $G^i_{\mathbf{C}}:=GH'_iG$, where

$$H_i:=\{g\in G_{\mathbf{C}}:gg^*\in T_i\}\quad\text{and}\quad H'_i:=H_i\cap G'_{\mathbf{C}}.$$

To obtain a better description of the representatives of the double cosets, we need to shrink the sets H_i. So let $g=hh^*\in T_i\cap G_{\mathbf{C}}^{\mathrm{reg}}$. Then $\mathrm{Ad}(g)$ fixes $\mathfrak{t}_{i\mathbf{C}}$ pointwise. Since

$$\mathrm{Ad}(h^*)=\sigma\circ\mathrm{Ad}(h)^{-1}\circ\sigma,$$

it follows that $\mathrm{Ad}(h)^{-1}$, and therefore $\mathrm{Ad}(h)$, commutes with σ on $\mathfrak{t}_{i\mathbf{C}}$. Hence $\mathrm{Ad}(h)(\mathfrak{t}_i)$ is a Cartan subalgebra of $\mathfrak{g}$ and we find $h'\in G$ and $j\in\{1,\dots,n\}$ with $\mathrm{Ad}(h')\,\mathrm{Ad}(h)\mathfrak{t}_i=\mathfrak{t}_j$. Then $i=j$ by Corollary 2.4 in [Ro72]. Hence

$$(h'h)(h'h)^* = h'(hh^*)h'^{-1} = h'gh'^{-1}.$$

We conclude that $GH_i'G\subseteq GN_{G_{\mathbf{C}}}(\mathfrak{t}_i)G$ and the assertion follows. □

Corollary II.2. *A subsemigroup $S\subseteq G_{\mathbf{C}}$ containing G with dense interior is completely determined by its intersections with the groups $N_{G_{\mathbf{C}}}(\mathfrak{t}_i)$. More presicely*

$$S\cap G_{\mathbf{C}}' = \bigcup_{i=1}^{n} G(S\cap N_{G_{\mathbf{C}}}(\mathfrak{t}_i)')G.$$

Proof. Since $G_{\mathbf{C}}'$ is open and dense in $G_{\mathbf{C}}$ and $\operatorname{int} S$ is open and dense in S, it follows that $S\cap G_{\mathbf{C}}'$ is dense in S. On the other hand $G\subseteq S$ shows that

$$S\cap(GN_{G_{\mathbf{C}}}(\mathfrak{t}_i)G) = G(S\cap N_{G_{\mathbf{C}}}(\mathfrak{t}_i))G. \quad \square$$

Proposition II.3. *Let G be a linear simple Lie group, $G_{\mathbf{C}}$ its complexification, and $M=G_{\mathbf{C}}/P$ a complex flag manifold. Assume that $S(P)^0:=\operatorname{int} S(P)\neq\emptyset$. Then either*

(1) *$S(P)=G_{\mathbf{C}}$, G acts transitively on $G_{\mathbf{C}}/P$, or*

(2) *G is Hermitean and $S(P)^0\cap\exp(iZ(\mathfrak{k}))\neq\emptyset$.*

Proof. Let $S:=S(P)$ and suppose that this semigroup is non-empty. Then Proposition II.1 shows that there exists a Cartan subalgebra $\mathfrak{t}_j\subseteq\mathfrak{g}$ such that $S^0\cap N_{G_{\mathbf{C}}}(\mathfrak{t}_j)\neq\emptyset$. Let $\mathfrak{a}_j\subseteq i\mathfrak{t}_j$ denote the vector part of $i\mathfrak{t}_j$. Then an application of [HN2, Cor. 1.20] entails that $S^0\cap\exp\mathfrak{a}_j\neq\emptyset$ because $\exp(\mathfrak{t}_j)\subseteq S$ and therefore $S^0\exp(\mathfrak{t}_j)=S^0$. The subspace $i\mathfrak{a}_j\subseteq\mathfrak{g}$ is abelian and compactly embedded. Hence there exists $g\in G$ such that $\mathrm{Ad}(g)i\mathfrak{a}_j\subseteq\mathfrak{t}$ (cf. [HN2, Prop. 7.3]). Then $\mathrm{Ad}(g)\mathfrak{a}_j\subseteq i\mathfrak{t}$ and consequently $S^0\cap\exp(i\mathfrak{t})\neq\emptyset$.

Let $C:=\exp|_{i\mathfrak{t}}^{-1}(S^0)$. Then C is an open subsemigroup of $i\mathfrak{t}$ which is invariant under the Weyl group $\mathcal{W}_{\mathfrak{k}}$. Let $c\in C$. Then

$$c_0 := \sum_{\gamma\in\mathcal{W}_{\mathfrak{k}}} \gamma(c)\in C$$

is a fixed point for $\mathcal{W}_{\mathfrak{k}}$. There are two possible cases:

Case (1): $c_0=0$. Then $0\in C$ and $\mathbf{1}\in\operatorname{int} S$. This means that $S=G_{\mathbf{C}}$ because $G_{\mathbf{C}}$ is connected.

Case (2): $c_0\neq 0$. Then c_0 is a non-zero $\mathcal{W}_{\mathfrak{k}}$-invariant element in $i\mathfrak{t}$. It follows that $i\mathbf{R}c_0\subseteq Z(\mathfrak{k})$, and in particular that $Z(\mathfrak{k})\neq\{0\}$. Hence $\mathfrak{g}$ is a Hermitean simple Lie algebra (cf. [Hel, p. 382]) and $S^0\cap\exp(iZ(\mathfrak{k}))\neq\emptyset$. □

Proposition II.4. *Let $\mathfrak{g}$ be simple Hermitean, $\mathfrak{t}\subseteq\mathfrak{g}$ be a compactly embedded Cartan algebra, Δ^+ a positive $\mathfrak{k}$-adapted system, and $C_{\max}=(i\Delta_{\mathfrak{p}}^+)^\star$ the corresponding maximal cone. Suppose that $S\subseteq G_{\mathbf{C}}$ is a subsemigroup with dense interior containing G. If*

$$S\cap\exp(i\mathfrak{t})\subseteq\exp(iC_{\max}\cup -iC_{\max}),$$

then

$$S\subseteq\overline{GN_{G_{\mathbf{C}}}(\mathfrak{t})G}.$$

Proof. In view of Corollary II.2, we have to show that

$$\operatorname{int}(S)\cap N_{G_{\mathbf{C}}}(\mathfrak{t}')=\emptyset$$

for every Cartan subalgebra $\mathfrak{t}'\subseteq\mathfrak{g}$ which is not conjugate to $\mathfrak{t}$.

As before, let $\mathfrak{k}\subseteq\mathfrak{g}$ be the unique maximal compactly embedded subalgebra containing $\mathfrak{t}$ and pick a Cartan subalgebra $\mathfrak{t}'\subseteq\mathfrak{g}$ not conjugate to $\mathfrak{t}$. Using [PR, 1.3], we may assume that $\mathfrak{t}'$ is invariant under the Cartan involution determined by $\mathfrak{k}$, i.e.,

$$\mathfrak{t}'=(\mathfrak{t}'\cap\mathfrak{k})+(\mathfrak{t}'\cap\mathfrak{p}),$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Cartan–Killing form. Moreover, since all compactly embedded Cartan algebras are conjugate by [PR, 1.4], and $\mathfrak{t}'\cap\mathfrak{k}$ may be extended to a Cartan subalgebra of $\mathfrak{k}$, we even may assume that $\mathfrak{t}'\cap\mathfrak{k}\subseteq\mathfrak{t}$.

Now we consider the group $N':=N_{G_{\mathbf{C}}}(\mathfrak{t}')$. Its Lie algebra coincides with $\mathfrak{t}'$ and $A':=\exp(i(\mathfrak{t}'\cap\mathfrak{k})+(\mathfrak{t}'\cap\mathfrak{p}))$ is a normal subgroup such that N'/A' is compact. Let $S':=\operatorname{int}(S)\cap N$ and suppose that $S'\neq\emptyset$. Then $S'A'/A'$ is an open subsemigroup of the compact group N'/A' and therefore it is a subgroup ([HN2, 1.21]). It follows in particular that it contains the unit element. This means that $A'\cap S'\neq\emptyset$. Since $\exp(\mathfrak{t}'\cap\mathfrak{p})\subseteq G\subseteq S$, it even follows that

$$S'\cap\exp(i(\mathfrak{t}'\cap\mathfrak{k}))=\operatorname{int}(S)\cap\exp(i(\mathfrak{t}'\cap\mathfrak{k}))\neq\emptyset.$$

Let $X\in i(\mathfrak{t}'\cap\mathfrak{k})\subseteq i\mathfrak{t}$. If there exists no non-compact root vanishing on X, then $Z_{\mathfrak{g}}(X)\subseteq\mathfrak{k}$ and therefore $\mathfrak{t}'\cap\mathfrak{p}\subseteq Z_{\mathfrak{g}}(X)\subseteq\mathfrak{k}$ yields a contradiction. Thus we find an element $X\in i\mathfrak{t}\cap i\mathfrak{t}'$ and a non-compact root α such that $\alpha(X)=0$ and $\exp(X)\in\operatorname{int}(S)$. This is impossible since $S\cap\exp(i\mathfrak{t})\subseteq\exp(iC_{\max}\cup -iC_{\max})$. $\square$

We want to apply Proposition II.4 to study the semigroup $S(\overline{B})$. The following lemma can be found in [N1, IV.6]. Its proof is basically obtained by reduction to the case of $\mathrm{sl}(2,\mathbf{R})$.

Lemma II.5. *Let G be simple Hermitean, Δ^+ a positive $\mathfrak{k}$-adapted system, and $B\subseteq G_{\mathbf{C}}$ the corresponding Borel subgroup. Then*

$$S(\overline{B})\cap\exp(it)\subseteq\exp(-iC_{\max}).$$

Before we can compute the compression semigroup for the Kähler orbits, we need a result on the open G-orbits in flag manifolds.

Lemma II.6. *Let G be simple Hermitean, Δ^+ a positive $\mathfrak{k}$-adapted system, $B\subseteq G_{\mathbf{C}}$ the corresponding Borel subgroup,*

$$\mathcal{W}:=N_{G_{\mathbf{C}}}(\mathfrak{t})/Z_{G_{\mathbf{C}}}(\mathfrak{t}) \quad \text{and} \quad \mathcal{W}_{\mathfrak{k}}:=N_G(\mathfrak{t})/Z_G(\mathfrak{t})$$

the Weyl groups. Let $\pi: G_{\mathbf{C}}\to G_{\mathbf{C}}/B$ denote the orbit mapping and $x_0=\pi(\mathbf{1})$ the base point. Then the action of $N_{G_{\mathbf{C}}}(\mathfrak{t})$ on the orbit through x_0 factors to an action of $\mathcal{W}$ on $G_{\mathbf{C}}/B$ and the open G-orbits are the orbits through the points in the orbit of $\mathcal{W}(G_{\mathbf{C}})$. If $\gamma,\gamma'\in\mathcal{W}$, then

$$G.(\gamma.x_0)=G.(\gamma'.x_0) \iff \gamma'\gamma^{-1}\in\mathcal{W}_{\mathfrak{k}}.$$

Proof. This is Corollary 4.7 in [Wol]. □

Proposition II.7. *Let G be a simple Lie group which is Hermitean or compact, and $P\subseteq G_{\mathbf{C}}$ a parabolic subgroup such that the set $\Sigma_{\mathfrak{p}}:=\{\alpha\in\Delta:\mathfrak{g}_{\mathbf{C}}^{\alpha}\subseteq\mathfrak{p}\}$ satisfies $\Sigma_{\mathfrak{p}}\cap\Delta_p=\Delta_p^+$, where Δ^+ is a $\mathfrak{k}$-adapted positive system. Then the following assertions hold:*

(i) *If G is compact, then $S(\overline{P})=G_{\mathbf{C}}$.*

(ii) *If G is Hermitean, then $S(\overline{P})=G\exp(-iW_{\max})$, where $W_{\max}$ is the generating invariant cone in $\mathfrak{g}$ such that*

$$W_{\max}\cap\mathfrak{t}=C_{\max}=\operatorname{cone}(i\Delta_p^+)^{\star}.$$

Proof. (i) Since an open G-orbit is also compact in the manifold $G_{\mathbf{C}}/\overline{P}$, it follows that G acts transitively on $G_{\mathbf{C}}/\overline{P}$. Hence $S(\overline{P})=G_{\mathbf{C}}$ (cf. Proposition II.3).

(ii) (cf. [Ols], [OH]) Let $B\subseteq G_{\mathbf{C}}$ denote the Borel subgroup defined by the positive system $\Delta^+\subseteq\Delta$ and $P_{\max}$ the unique maximal parabolic subgroup such that $\Sigma_{\mathfrak{p}_{\max}}=\Delta_k\cup\Delta_p^+$. We also write N for the commutator group of B and $P^+:=\exp(\bigoplus_{\lambda\in\Delta_p^+}\mathfrak{g}_{\mathbf{C}}^{\lambda})$. Then $B\subseteq P\subseteq P_{\max}$. We claim that $GB=GP_{\max}$. To see this, we consider the Levi decomposition $P_{\max}=P^+\rtimes K_{\mathbf{C}}$ and the Iwasawa decomposition $K_{\mathbf{C}}=K\exp(it)(N\cap K_{\mathbf{C}})$. Then

$$GP_{\max}=GK_{\mathbf{C}}P^+=GK\exp(it)(N\cap K_{\mathbf{C}})P^+=G\exp(it)N=GT\exp(it)N=GB.$$

It follows in particular that $GP=GB$. Applying the complex conjugation on $G_{\mathbb{C}}$, we also see that $G\overline{P}=G\overline{B}$. Therefore

$$S(\overline{P})=\{g\in G_{\mathbb{C}}:gG\overline{P}\subseteq G\overline{P}\}=\{g\in G_{\mathbb{C}}:gG\overline{B}\subseteq G\overline{B}\}.$$

So we may assume that $P=B$. To see that the G-orbit in $G_{\mathbb{C}}/\overline{B}$ comes from a coadjoint orbit, we pick $\omega\in-\operatorname{int}C_{\min}(\Delta_p^+)^\star$ such that $\mathcal{O}_\omega$ has maximal dimension, i.e., $\mathfrak{g}^\omega=\mathfrak{t}$ (cf. [N8, Lemma II.4]). Then $\mathfrak{p}=\mathfrak{b}$ is a positive complex polarization in ω (cf. [N8]).

It follows that $\mathcal{O}_\omega$ corresponds to the open domain $G.x_0$ in $G_{\mathbb{C}}/\overline{P}$. Hence Proposition I.10 implies that $i\overline{B}_\omega=-iW_{\max}\subseteq\mathbf{L}(S(\overline{P}))$ and therefore

$$\mathbf{L}(S(\overline{P}))\supseteq\mathfrak{g}-iW_{\max}.$$

We conclude that $G\exp(-iW_{\max})\subseteq S(\overline{P})$.

It remains to show that $S(\overline{P})\subseteq G\exp(-iW_{\max})$. From Lemma II.5 we infer that

$$S(\overline{B})\cap\exp(i\mathfrak{t})=S(\overline{P})\cap\exp(i\mathfrak{t})\subseteq\exp(-iC_{\max}).$$

Then Proposition II.4 applies and since $S(\overline{P})$ has dense interior, we find that

$$S(\overline{P})\subseteq\overline{GN_{G_{\mathbb{C}}}(\mathfrak{t})G}$$

and that $G(S(\overline{P})\cap N_{G_{\mathbb{C}}}(\mathfrak{t}))G$ is dense in $S(\overline{P})$. Let $s\in N_{G_{\mathbb{C}}}(\mathfrak{t})\cap S(\overline{P})$. Then $s.x_0\in G.x_0$, $G.(s.x_0)=G.x_0$, and Lemma II.6 yields that $s\in N_G(\mathfrak{t})Z_{G_{\mathbb{C}}}(\mathfrak{t})$. Using $Z_{G_{\mathbb{C}}}(\mathfrak{t})=\exp(\mathfrak{t}_{\mathbb{C}})$, we conclude that

$$\begin{aligned}G(S(\overline{P})\cap N_{G_{\mathbb{C}}}(\mathfrak{t}))G&=G(S(\overline{P})\cap Z_{G_{\mathbb{C}}}(\mathfrak{t}))G=G(S(\overline{P})\cap\exp(\mathfrak{t}_{\mathbb{C}}))G\\&=G(S(\overline{P})\cap\exp(i\mathfrak{t}))G\subseteq G(\exp(-iC_{\max}))G\subseteq G\exp(-iW_{\max}).\end{aligned}$$

Now Corollary II.3 shows that $S(\overline{P})=G\exp(-iW_{\max})$. $\square$

We subsume the results obtained in the first two sections in the following general theorem.

Theorem II.8. (Theorem on compression semigroups of Kähler orbits) *Let $\mathfrak{g}$ be a Lie algebra with the compactly embedded Cartan algebra $\mathfrak{t}$, $\omega\in\mathfrak{t}^*$ strictly admissible and reduced, Δ^+ a $\mathfrak{k}$-adapted positive system, and Σ a parabolic set of roots such that $\mathfrak{b}:=\mathfrak{p}_\Sigma$ is the unique positive complex polarization in ω. Write $G_{\mathbb{C}}$ for a simply connected Lie group with $\mathbf{L}(G_{\mathbb{C}})=\mathfrak{g}_{\mathbb{C}}$. Then $B:=\langle\exp\overline{\mathfrak{b}}\rangle\subseteq G_{\mathbb{C}}$ is closed,* *the G-orbit of the base point in $G_{\mathbf{C}}/\overline{B}$ is open, and the compression semigroup $S(\overline{B})$ of this orbit is given by*

$$S(\overline{B}) = N_{\mathbf{C}} L \exp(-iW_{\max}),$$

where $G = N \rtimes L$ is a semidirect decomposition into the nilradical and a reductive Lie group L, and $W_{\max}$ is the maximal invariant cone in $\mathfrak{l}$ corresponding to the $\mathfrak{k}$-adapted positive system Δ^+.

Note that if ω is not reduced, then one obtains a reduction to the reduced case by factoring the ideal $\mathfrak{a} := \mathcal{O}_\omega^\perp$ and looking at ω as a functional in $(\mathfrak{g}/\mathfrak{a})^* \cong \mathfrak{a}^\perp \subseteq \mathfrak{g}^*$.

III. Compression semigroups and highest weight orbits

In the first two sections we have determined the compression semigroups of all open orbits in $G_{\mathbf{C}}/\overline{B}$ arising via the Borel embedding of strictly admissible coadjoint orbits. If G is simple, this includes all the semigroups $S(P)$, where P is a parabolic contained in one of the two K-invariant maximal parabolics $P_{\max} = K_{\mathbf{C}} P^+$ and $\overline{P}_{\max} = K_{\mathbf{C}} P^-$.

In this section we will extend these results to general parabolics and therefore complete the whole puzzle for simple groups. Recall that we have already reduced matters to the case where G is simple Hermitean in Proposition II.3.

We start with a simple lemma from linear algebra.

Lemma III.1. *Let V be a real vector space and $g \in \mathrm{Gl}(V)$ be diagonalizable with positive real eigenvalues, and $[v] \in \mathbf{P}(V)$. Then $[v_0] := \lim_{n\to\infty} g^n.[v]$ exists and $g.v_0 = \lambda_{\max} v_0$, where $\lambda_{\max}$ is the largest eigenvalue of g on the smallest g-invariant subspace of V containing v.*

Proof. We may assume that the smallest g-invariant subspace of V containing v coincides with V. Let $\lambda_0 < \lambda_1 < \ldots < \lambda_k = \lambda_{\max}$ denote the different eigenvalues of g on V. Then $v = \sum_{i=0}^k v_i$ with $g.v_i = \lambda_i v_i$ and therefore

$$g^n.[v] = [g^n.v] = \left[\sum_{i=0}^k g^n.v_i\right] = \left[\sum_{i=0}^k \lambda_i^n.v_i\right] = \left[\sum_{i=0}^k \left(\frac{\lambda_i}{\lambda_{\max}}\right)^n.v_i\right] \to [v_k].$$

This proves the lemma. $\square$

It will be useful to review a few facts about invariant control sets for semigroups acting on spaces: An *invariant control set* for a subsemigroup S of G acting on a set $\mathbf{X}$ is a set C_S which satisfies

$$\overline{S.c} = \overline{C}_S \quad \forall c \in C_S$$

and is maximal with respect to this property.

Proposition III.2. *Let* $\mathbf{X}$ *be a* G*-space and* $S \subseteq G$ *a subsemigroup. Then the following assertions hold:*

(i) *Each closed invariant control set for* S *is* S*-invariant.*

(ii) *If* S *has non-empty interior* S^0*, and* G *acts transitively, then each invariant control set is closed and the set* $C_S^0 := S^0.C_S$ *is open dense in* C_S*. It satisfies* $C_S^0 = S^0.c$ *for all* $c \in C_S^0$*.*

(iii) *If* G *is semisimple,* $\mathbf{X} = G/P$ *is a flag manifold, and* S *has non-empty interior, then there is a unique invariant control set given by*

$$C_S = \bigcap_{x \in \mathbf{X}} \overline{S.x}.$$

Proof. (i), (ii) [HN2, Prop. 8.1].

(iii) [HN2, Prop. 8.2]. □

Proposition III.3. *Let* G *be a semisimple Lie group,* P *a parabolic subgroup, and* L *a closed subgroup such that* LP *is open in* G*. Suppose that*

$$S(L,P) := \{g \in G : gLP \subseteq LP\}$$

has non-empty interior. If $\Omega = LP/P$ *is the open* L*-orbit of the base point, then the closure* $\overline{\Omega}$ *of* Ω *in* G/P *is the invariant control set for* $S(L,P)$*.*

Proof. This is a special case of [HN2, Prop. 8.8]. □

Let us return to the setting where G is simple Hermitean and

$$S(P)^0 \cap \exp(iZ(\mathfrak{k})) \neq \emptyset$$

(cf. Proposition II.3). Then $S(P)$ is a subsemigroup of $G_{\mathbb{C}}$ with non-empty interior containing G and $C_{S(P)} = \overline{\mathcal{O}}$ is the corresponding unique invariant control set on M.

Highest weight modules

In this subsection $\mathfrak{g}_{\mathbb{C}}$ denotes a simple complex Lie algebra, $\mathfrak{g}$ a real form, and $\mathfrak{t} \subseteq \mathfrak{g}$ a Cartan algebra. Let V be a $\mathfrak{g}_{\mathbb{C}}$-module. For a linear functional λ on $\mathfrak{t}_{\mathbb{C}}$ we write

$$V^\lambda := \{v \in V : (\forall X \in \mathfrak{h}_{\mathbb{C}}) X.v = \lambda(X)v\}$$

for the *weight space* of weight λ. We set $\mathcal{P}_V := \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* : V^\lambda \neq \{0\}\}$ and call this set the set of *weights* of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. An element $\omega \in \mathcal{P}_V$ is called a *highest weight* with respect to the positive system Δ^+ if

$$(\omega + \Delta^+) \cap \mathcal{P}_V = \emptyset.$$

Next let $\varkappa$ denote the Cartan–Killing form of $\mathfrak{g}_{\mathbf{C}}$ and $(\cdot,\cdot)$ the bilinear form on the dual $\mathfrak{g}_{\mathbf{C}}^*$ of $\mathfrak{g}_{\mathbf{C}}$ induced by $\varkappa$. Then $(\cdot,\cdot)$ is positive definite and real on $\operatorname{span}_{\mathbf{R}} \Delta$ and for every root $\lambda \in \Delta$ there exists an element $\check{\lambda} \in \mathfrak{t}_{\mathbf{C}}$ such that

$$\mu(\check{\lambda}) = \frac{2(\lambda,\mu)}{(\lambda,\lambda)} \quad \forall \mu \in \mathfrak{t}_{\mathbf{C}}^*.$$

We write $\check{\mathcal{R}}$ for the abelian subgroup of $\mathfrak{t}_{\mathbf{C}}$ generated by $\check{\Delta}$, define the *weight lattice*

$$\mathcal{P} := \{\mu \in \mathfrak{t}_{\mathbf{C}}^* : \mu(\check{\mathcal{R}}) \subseteq \mathbf{Z}\},$$

and define the set of *dominant weights* by

$$\mathcal{P}^+ := \mathcal{P}^+(\Delta^+) := \{\mu \in \mathcal{P} : (\forall \alpha \in \Delta^+) \langle \mu, \check{\alpha} \rangle \in \mathbf{N}_0\}.$$

Note that if Υ is a basis of the root system, then a basis of $\mathcal{P}$ is given by

$$\{\omega_\alpha : \alpha \in \Upsilon\}, \quad \text{where } \omega_\alpha(\check{\beta}) = \begin{cases} 0 & \text{if } \beta \neq \alpha, \\ 1 & \text{if } \beta = \alpha. \end{cases}$$

Then $\mathcal{P}^+ = \sum_{\alpha \in \Upsilon} \mathbf{N}_0 \omega_\alpha$.

Proposition III.4. *Let V be a finite dimensional $\mathfrak{g}_{\mathbf{C}}$-module and $\Delta^+ \subseteq \Delta$ a positive system. Then the following assertions hold:*

(i) $\mathcal{P}_V \subseteq \mathcal{P}$.

(ii) $V = \bigoplus_{\mu \in \mathcal{P}_V} V^\mu$.

(iii) *If V is irreducible, then $\mathcal{P}_V \cap \mathcal{P}^+$ contains a highest weight with respect to Δ^+.*

(iv) *For every $\lambda \in \mathcal{P}^+$ there exists, up to isomorphy, a unique irreducible $\mathfrak{g}_{\mathbf{C}}$-module called V_λ such that λ is a highest weight with respect to Δ^+ in $\mathcal{P}_{V_\lambda}$.*

Proof. (i), (ii) ([B2, Ch. 8, §7, no. 1, Prop. 1]).
(iii), (iv) ([B2, Ch. 8, §7, no. 2]). □

Let V be an irreducible $\mathfrak{g}_{\mathbf{C}}$-module, $G_{\mathbf{C}}$ a connected group with $\mathbf{L}(G_{\mathbf{C}}) = \mathfrak{g}_{\mathbf{C}}$, and suppose that the representation of $\mathfrak{g}_{\mathbf{C}}$ integrates to a representation of $G_{\mathbf{C}}$ (this is always the case if $G_{\mathbf{C}}$ is simply connected). We write $\mathbf{P}(V)$ for the projective space of V. Then the representation of $G_{\mathbf{C}}$ on V induces an action of $G_{\mathbf{C}}$ on $\mathbf{P}(V)$ defined by

$$g.[v] = [g.v] \quad \forall g \in G_{\mathbf{C}},\ v \in V \setminus \{0\},$$

where $V \setminus \{0\} \to \mathbf{P}(V), v \mapsto [v]$ is the quotient mapping.

Proposition III.5. *Let $\lambda\in\mathcal{P}_V$ be a highest weight with respect to the positive system Δ^+ and $v_\lambda\in V^\lambda$ a highest weight vector. Then the following assertions hold:*

(i) *The stabilizer of $[v_\lambda]\in\mathbf{P}(V)$ is a parabolic subgroup P_λ associated with Δ^+.*

(ii) *Let $\lambda=\sum_{\alpha\in\Upsilon} n_\alpha\omega_\alpha$. Then $\mathfrak{p}_\lambda:=\mathbf{L}(P_\lambda)=\mathfrak{p}_\Sigma$ with*

$$\Sigma=\Delta^+\cup(-\Delta^+\cap\operatorname{span}_{\mathbf{R}}\{\alpha\in\Upsilon:n_\alpha=0\}).$$

(iii) *If $\beta,\beta'\in\mathcal{P}_V$ with $G_{\mathbf{C}}.[v_\beta]=G_{\mathbf{C}}.[v_{\beta'}]$, then $\beta\in\mathcal{W}.\lambda$. In particular if β is the highest weight, then β' is an extremal weight.*

Proof. (i) Let $\alpha\in\Delta^+$. For $X\in\mathfrak{g}_{\mathbf{C}}^\alpha$ we have that $X.v_\lambda\in V^{\lambda+\alpha}=\{0\}$. Hence v_λ is a common eigenvector for the Borel subgroup $B=B(\Delta^+)$. Thus B fixes the point $[v_\lambda]$ in the projective space. This means that the stabilizer of $[v_\lambda]$ is a subgroup which contains B, hence parabolic.

(ii) It follows from [B2, Ch. 8, §7, no. 2, Prop. 3] that $\mathfrak{g}_{\mathbf{C}}^{-\alpha}\subseteq\mathfrak{p}_\lambda$ holds for $\lambda\in\Delta^+$ if and only if $\lambda(\check{\alpha})=0$. Let $\lambda=\sum_{\alpha\in\Upsilon} n_\alpha\omega_\alpha$ and $\alpha\in\Upsilon$. Then $\lambda(\check{\alpha})=n_\alpha=0$.

(iii) According to our assumption, there exists $g\in G_{\mathbf{C}}$ such that $[v_{\beta'}]=g.[v_\beta]$. Hence the stabilizer $P_{\beta'}$ of $[v_{\beta'}]$ satisfies $P_{\beta'}=gP_\beta g^{-1}$. Since $\operatorname{Ad}(g)\mathfrak{t}_{\mathbf{C}}\subseteq\mathfrak{p}_{\beta'}:=\mathbf{L}(P_{\beta'})$ is a Cartan algebra, there exists $p\in P_{\beta'}$ such that $\operatorname{Ad}(p)\operatorname{Ad}(g)\mathfrak{t}_{\mathbf{C}}=\mathfrak{t}_{\mathbf{C}}$. Now $pg\in N_{G_{\mathbf{C}}}(\mathfrak{t}_{\mathbf{C}})$. Hence $\gamma:=\operatorname{Ad}(pg)|_{\mathfrak{t}_{\mathbf{C}}}\in\mathcal{W}$ satisfies $\gamma.\beta=\beta'$ since $g.v_\beta\subseteq\mathbf{C}v_{\beta'}$. □

Since every parabolic subalgebra $\mathfrak{p}\subseteq\mathfrak{g}_{\mathbf{C}}$ associated with $\mathfrak{b}(\Delta^+)$ occurs as some $\mathfrak{p}_\Sigma$ for a parabolic subset $\Sigma\supseteq\Delta^+$, let $\Sigma':=\Upsilon\cap\Sigma$. We consider the weight

$$\omega_\Sigma:=\sum_{\alpha\notin\Sigma'}\omega_\alpha$$

and the corresponding highest weight module V. Then the preceding proposition shows that $P=P_\Sigma=P_{\omega_\Sigma}$ arises as the stabilizer of a highest weight vector $[v_{\omega_\Sigma}]$ in $\mathbf{P}(V)$. Thus we have obtained a realization of the flag manifold $G_{\mathbf{C}}/P$ as a compact submanifold of the projective space $\mathbf{P}(V)$. This realization will turn out to be crucial for the investigation of the compression semigroups.

For the following we recall that for a finite dimensional irreducible highest weight module V the weight spaces corresponding to the extremal weights are one-dimensional (cf. [B2, Chap. 7, §7, Prop. 5]).

Definition III.6. (a) Let Y be a diagonalizable endomorphism of the complex vector space V with real eigenvalues. An element $v\in V$ is said to be *generic* with respect to Y if the smallest Y-invariant subspace containing v contains eigenvectors for the maximal and minimal eigenvalues of Y.

(b) If V is a finite dimensional module of the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$ and $\mathcal{P}_V$ the corresponding set of weights, then we say that an element $X\in i\mathfrak{t}$ is *weight separating* if the values $\alpha(X),\alpha\in\mathcal{P}_V$ are pairwise distinct.

Lemma III.7. *Let V be an irreducible finite dimensional $G_{\mathbf{C}}$-module, $v_\lambda \in V$ a highest weight vector, $\mathcal{O} \subseteq G_{\mathbf{C}}.[v_\lambda]$ an open G-orbit, and $Y \in \exp(it)$. Then there exists a vector $v \in V$ which is generic with respect to Y such that $[v] \in \mathcal{O}$.*

Proof. The fact that V is a simple $G_{\mathbf{C}}$-module entails that V is spanned by the set $\{v \in V : [v] \in G_{\mathbf{C}}.[v_\lambda]\}$, and, by the analyticity of the orbit mapping, it is even spanned by $\{v \in V : [v] \in U\}$ for every open subset $U \subseteq G_{\mathbf{C}}.[v_\lambda]$. This applies in particular to the G-orbit $\mathcal{O}$. Let $v = \sum v_\alpha$ denote the decomposition of a vector $v \in V$ into Y-eigenvectors, where v_α is an eigenvector with eigenvalue α. Write $\lambda_{\min}$ and $\lambda_{\max}$ for the minimal and maximal eigenvalue. Then, since $\mathcal{O}$ spans V, we first find $[v'] \in \mathcal{O}$ with $v'_{\lambda_{\max}} \neq 0$. We note that the complement of this set is an analytic set, hence nowhere dense. So we even find $[v] \in \mathcal{O}$ with $v_{\lambda_{\max}} \neq 0$ and $v_{\lambda_{\min}} \neq 0$. Now the smallest Y-invariant subspace containing v also contains $v_{\lambda_{\max}}$ and $v_{\lambda_{\min}}$. □

Proposition III.8. *Suppose that $\mathfrak{g}$ contains a compactly embedded Cartan algebra and that the complex flag manifold M is realized as a $G_{\mathbf{C}}$-orbit of a highest weight vector in $\mathbf{P}(V)$. Then the following assertions hold:*

(i) *Every open G-orbit in M contains an element $[v_\lambda]$, where $\lambda \in \mathcal{P}_V$ is an extremal weight.*

(ii) *Every G-orbit of an extremal weight ray is open in M.*

Proof. [HN2, Prop. 8.25]. □

Lemma III.9. *Let $S \subseteq G_{\mathbf{C}}$ be a subsemigroup with non-empty interior containing G, M a complex flag manifold realized as a highest weight orbit $G_{\mathbf{C}}.[v_\lambda]$, $C_S \subseteq M$ the invariant control set for S, and $\mathcal{P}_{C_S}$ the set of all extremal weights α with $[v_\alpha] \in C_S$. Then $\mathcal{P}_{C_S}$ has the following properties:*

(i) $\mathcal{W}_{\mathfrak{k}}.\mathcal{P}_{C_S} = \mathcal{P}_{C_S}$.

(ii) *If $X \in it$ is weight separating with $\exp X \in S$, then*

$$\alpha(X) = \max\{\beta(X) : \beta \in \mathcal{P}_V\}$$

implies that $\alpha \in \mathcal{P}_{C_S}$.

(iii) *If $X \in it$ is weight separating with $\exp X \in S$, $\mu \in \mathcal{P}_{C_S}$, $\alpha \in \Delta$, and $s_\alpha \in \mathcal{W}$ is the corresponding reflection, then $s_\alpha(\mu)(X) > \mu(X)$ implies that $s_\alpha(\mu) \in \mathcal{P}_{C_S}$.*

(iv) $\mathcal{P}_{C_S} = \{\alpha \in \mathcal{P}_V : [v_\alpha] \in \operatorname{int} C_S\}$.

Proof. (i) Let $\gamma \in \mathcal{W}_{\mathfrak{k}} \cong N_K(\mathfrak{t})/Z_K(\mathfrak{t})$. Then there exists $k \in K$ with $\mathrm{Ad}(k)|_{\mathfrak{t}_{\mathbf{C}}} = \gamma$. It follows that $k.[v_\alpha] = [k.v_\alpha] = [v_{\alpha \circ \gamma^{-1}}] = [v_{\gamma.\alpha}]$.

(ii) Since the invariant control set C_S is G-invariant with non-empty interior, it contains at least one open G-orbit $\mathcal{O}$ ([HN2, Prop. 8.10(ii)]). Using Lemma III.7, we find $[v] \in \mathcal{O}$ such that v is generic for X. Then, according to Lemma III.1,

$$[v'] := \lim_{n \to \infty} \exp(X)^n.[v] \in C_S$$

exists in $\mathbf{P}(V)$ and v' is an eigenvector of $\exp(X)$ for the maximal eigenvalue $e^{\alpha(X)}$, hence a weight vector of weight α for $\mathfrak{t}_{\mathbf{C}}$ because X is weight separating. Finally

$$\alpha(X) = \max\{\beta(X) : \beta \in \mathcal{P}_V\}$$

and the weight separating property of X show that α is extremal.

(iii) Let $G_{\mathbf{C}}(\alpha)$ denote the analytic subgroup of $G_{\mathbf{C}}$ with

$$\mathfrak{g}_{\mathbf{C}}(\alpha) := \mathbf{L}(G_{\mathbf{C}}(\alpha)) = \mathfrak{g}_{\mathbf{C}}^{\alpha} + \mathfrak{g}_{\mathbf{C}}^{-\alpha} + [\mathfrak{g}_{\mathbf{C}}^{\alpha}, \mathfrak{g}_{\mathbf{C}}^{-\alpha}] \cong \mathrm{sl}(2, \mathbf{C}).$$

Further let W denote the smallest $\mathfrak{g}_{\mathbf{C}}(\alpha)$-submodule containing v_μ. This module is irreducible with highest weight vector v_μ and lowest weight vector $v_{\mu'}$, where $\mu' := s_\alpha(\mu)$ ([B2, Ch. 8, §7, no. 2, Prop. 3]).

It follows that the $G_{\mathbf{C}}(\alpha)$-orbit M_α of $[v_\mu]$ contains exactly two weight rays, namely $[v_\mu]$ and $[v_{\mu'}]$ (cf. [B2, Ch. 8, §7, no. 2, Prop. 5]). The orbits of these elements under the group $G_\alpha := \langle \exp(\mathfrak{g}_{\mathbf{C}}(\alpha) \cap \mathfrak{g})\rangle$ are relatively open in M_α (Proposition III.8).

Since $\mathfrak{g}_{\mathbf{C}}(\alpha)$ is invariant under $\mathrm{Ad}(\exp i\mathfrak{t})$, it follows that

$$\exp(X).M_\alpha = \exp(X).(G_{\mathbf{C}}(\alpha).[v_\mu]) = G_{\mathbf{C}}(\alpha).((\exp X).[v_\mu]) = G_{\mathbf{C}}(\alpha).[v_\mu] = M_\alpha.$$

On the other hand, the orbit $G_\alpha.v_\mu$ spans W, so it contains a generic vector v for X on W (Lemma III.7). Note that $G_\alpha.[v_\mu] \subseteq G.[v_\mu] \subseteq C_S$ since $\mu \in \mathcal{P}_{C_S}$. Now our assumption $\mu'(X) > \mu(X)$ shows that the maximal eigenvalue of X on W is $\mu'(X)$. Hence

$$[v_{\mu'}] = \lim_{n \to \infty} \exp(X)^n.[v] \in \overline{G.[v]} \subseteq C_S,$$

so that $\mu' \in \mathcal{P}_{C_S}$.

(iv) Since every G-orbit of an extremal weight ray is open by Proposition III.8, and C_S is the closure of a union of open G-orbits ([HN2, Prop. 8.10(ii)]), the condition $[v_\alpha] \in C_S$ even implies that $[v_\alpha] \subseteq \operatorname{int} C_S$. ☐

We apply these results in the special case where G is simple Hermitean and the interior of $S(P)$ intersects $\exp(iZ(\mathfrak{k}))$ non-trivially. We fix an element $Z_k \in S(P)^0 \cap \exp(iZ(\mathfrak{k}))$ and consider a realization of the flag manifold $M = G_{\mathbf{C}}/P$ as a $G_{\mathbf{C}}$-orbit of a highest weight ray $[v_\lambda]$ in a highest weight module V of $G_{\mathbf{C}}$. Let $\mathcal{P}_V$ denote the corresponding set of weights. Then the extreme points of the convex hull of $\mathcal{P}_V$ consist precisely of the Weyl group orbit $\mathcal{W}.\lambda$ of the highest weight λ ([B2, Ch. 8, §7, no. 2, Prop. 5]).

We choose a weight $\alpha \in \mathcal{P}_V$ such that $\alpha(Z_k)$ is maximal. Then there exists a weight separating element $Z_\alpha \in i\mathfrak{t}$ arbitrarily close to Z_k such that

$$\alpha(Z_\alpha) = \max\{\beta(Z_\alpha) : \beta \in \mathcal{P}_V\},$$

and $\exp(Z_\alpha) \in S(P)^0$. Now Lemma III.9(iii), (iv) yield $[v_\alpha] \in \operatorname{int} C_{S(P)}$. So we have shown that $\mathcal{P}_{C_{S(P)}}$ contains every weight α, where $\alpha(Z_k)$ is maximal.

To evaluate this condition, we need the following lemma.

Lemma III.10. *Let $\Delta^+ \subseteq \Delta$ be a $\mathfrak{k}$-adapted positive system. Then the following assertions hold:*

(i) *Let $\mu \in C^\star_{\min}$. Then $\mathcal{W}.\mu \cap C^\star_{\min} = \mathcal{W}_{\mathfrak{k}}.\mu$.*

(ii) *$\mathcal{W}_{\mathfrak{k}} = \{\gamma \in \mathcal{W} : \gamma.C^\star_{\min} = C^\star_{\min}\}$.*

(iii) *If $Z \in iZ(\mathfrak{k})$ such that $\alpha(Z) > 0$ holds for the positive non-compact roots α and $\mu \in it^*$, then*

$$i\mu \in C^\star_{\min} \iff \mu(Z) = \max\{(\gamma.\mu)(Z) : \gamma \in \mathcal{W}\}.$$

Proof. (i) That the right hand side is contained in the left hand side follows from the invariance of $C_{\min}$ under the small Weyl group $\mathcal{W}_{\mathfrak{k}}$. Suppose that $\gamma \in \mathcal{W}$ with $\gamma.\mu \in C^\star_{\min}$. Then $(\gamma.\mu)(i\check{\alpha}) \leq 0$ for all $\alpha \in \Delta^+_p$. Thus there exists $\gamma' \in \mathcal{W}_{\mathfrak{k}}$ such that

$$((\gamma'\gamma).\mu)(i\check{\alpha}) \leq 0 \quad \forall \alpha \in \Delta^+.$$

On the other hand there exists $\gamma'' \in \mathcal{W}_{\mathfrak{k}}$ with $(\gamma''.\mu)(i\check{\alpha}) \leq 0$ for all $\alpha \in \Delta^+$. Thus $(\gamma'\gamma).\mu = \gamma''.\mu$ ([B1, Ch. 5, §3, no. 3.3, Thm. 2]) and therefore

$$\gamma.\mu = (\gamma')^{-1}\gamma''.\mu \in \mathcal{W}_{\mathfrak{k}}.\mu.$$

(ii) That $\mathcal{W}_{\mathfrak{k}}$ leaves $C^\star_{\min}$ invariant is clear. To prove (ii), pick $\mu \in C^\star_{\min}$ such that the stabilizer of μ in $\mathcal{W}$ is trivial. Suppose that $\gamma \in \mathcal{W}$ leaves the cone $C^\star_{\min}$ invariant. Then $\gamma.\mu \in C^\star_{\min} \subseteq \mathcal{W}_{\mathfrak{k}}.\mu$ shows that $\gamma \in \mathcal{W}_{\mathfrak{k}}$.

(iii) Let $\alpha \in it^* \subseteq \mathfrak{t}^*_{\mathbf{C}}$ such that $\alpha(Z) = \max\{(\gamma.\alpha)(Z) : \gamma \in \mathcal{W}\}$. Pick a positive non-compact root β and consider the reflection s_β at the hyperplane $\ker \beta$. Then

$$s_\beta(\alpha) = \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta = \alpha - \alpha(\check{\beta})\beta.$$

Since $s_\beta(\alpha)(Z) \leq \alpha(Z)$, this means that $\alpha(\check{\beta})\beta(Z) \geq 0$. This shows that α is non-negative on all the elements $[X, \overline{X}]$ for $X \in \mathfrak{g}^\beta_{\mathbf{C}}$, $\beta \in \Delta^+_p$. In view of [HN2, Lemma 7.7], this means that $i\alpha \in C^\star_{\min}$, since $\check{\beta} \in \mathbf{R}^+[X, \overline{X}]$.

If, conversely, $i\alpha \in C^\star_{\min}$, then $(\gamma.\alpha)(Z) = \alpha(Z)$ holds for all $\gamma \in \mathcal{W}_{\mathfrak{k}}$. So, as in (i), we may assume that $\alpha(\check{\beta}) \geq 0$ holds for all $\beta \in \Delta^+$. This means that α is contained in the positive Weyl chamber corresponding to Δ^+. It follows that

$$\mathcal{W}.\alpha \subseteq \alpha - \sum_{\beta \in \Delta^+} \mathbf{R}^+\beta$$

(cf. [He2, p. 459]), so that the assertion follows from $\beta(Z) = 0$ if β is a compact root and $\beta(Z) > 0$ if β is non-compact. $\square$

Lemma III.11. *If $G_{\mathbf{C}} \neq S(P)^0 \neq \emptyset$ and $M = G_{\mathbf{C}}.[v_\lambda]$, then there exists a $\mathfrak{k}$-adapted positive system Δ^+ such that $i\lambda \in C_{\min}(\Delta_p^+)^\star$ is a highest weight, and $\mathcal{O} = G.[v_\lambda]$ holds for a highest weight vector v_λ in V with respect to Δ^+.*

Proof. If $\alpha \in \mathcal{P}_V$ is such that $\alpha(Z_k)$ is maximal among the Weyl group translates of α, then $\alpha(Z_k) \geq 0$ and Lemma III.10 yields that $i\alpha \in C_{\min}^\star$.

Thus there exists $\gamma \in \mathcal{W}_\mathfrak{k}$ such that $(\gamma.\alpha)(\check{\beta}) \geq 0$ for all $\beta \in \Delta_k^+$ since $i\alpha \in C_{\min}^\star$ and the $\mathcal{W}_\mathfrak{k}$-translates of the positive Weyl chamber cover $C_{\min}^\star$ ([B1, Ch. 5, §4, no. 4, Cor. 1]). Thus $\lambda = \gamma.\alpha$ is a highest weight for V with respect to the positive system Δ^+. Pick $g \in G$ such that $\gamma = \mathrm{Ad}(g)|_{\mathfrak{t}_{\mathbf{C}}}$. Then $g.[v_\alpha] = [v_{\gamma.\alpha}]$ entails that $\mathcal{O} = G.[v_\lambda]$. This proves the lemma. □

Proposition III.12. *Let $\mathcal{O} = G.x$, P_x be the stabilizer of x in $G_{\mathbf{C}}$ and $B \subseteq P_x$ be a Borel subgroup. Then $S(B) \subseteq S(P_x)$, and equality holds if $\mathfrak{b} = \mathfrak{p}_{\Delta^+}$ for a $\mathfrak{k}$-adapted positive system and $P_x \cap G$ is compact. In this case $S(B) = S(P_x) = G\exp(iW_{\max})$.*

Proof. Let $g \in S(B)$. Then $gGB \subseteq GB$ and therefore $gG \subseteq GB \subseteq GP_x$ entails

$$gGP_x \subseteq (GP_x)P_x = GP_x,$$

i.e., $g \in S(P_x)$.

Now suppose that $\mathfrak{b} = \mathfrak{p}_{\Delta^+}$ for a $\mathfrak{k}$-adapted positive system, and that $P_x \cap G$ is compact. Let Υ_k denote the set of all compact base roots in Υ. Then $\Upsilon = \Upsilon_k \cup \{\delta\}$ and the parabolic $P := P_x$ is a parabolic containing B. We also write N for the commutator group of B. We claim that $GB = GP$. To see this, write $\mathfrak{p} := \mathbf{L}(P)$ as $\mathfrak{p}_\Sigma$ for a parabolic subset $\Sigma \subseteq \Delta$. Then the compactness of $P \cap G$ is equivalent to $\Sigma \cap -\Sigma \subseteq \Delta_k$. Now Proposition II.7 shows that $S(P) = S(B) = G\exp(iW_{\max})$. □

Proposition III.13. *Let Δ^+ be a $\mathfrak{k}$-adapted positive system, $\lambda \in \mathcal{P}^+$ such that $i\lambda \in C_{\min}(\Delta_p^+)^\star$, V_λ a corresponding highest weight module, and $\mathcal{O} := G.[v_\lambda] \subseteq G_{\mathbf{C}}.[v_\lambda]$ the corresponding open G-orbit. Then $\beta \in \mathcal{P}_V$ and $[v_\beta] \in \mathcal{O}$ implies that $\beta \in \mathcal{W}_\mathfrak{k}.\lambda \subseteq -iC_{\min}(\Delta_p^+)^\star$.*

Proof. Since $[v_\beta] = G.[v_\lambda]$, there exists $g \in G$ with $g.[v_\lambda] = [v_\beta]$. We first use [HN2, Thm. 8.30] to find a G-invariant pseudo-Hermitean structure J on V such that the corresponding moment mapping is given by

$$\Phi : \Omega \to \mathfrak{g}^*, \quad [v] \mapsto \left(X \mapsto i\frac{J(X.v, v)}{J(v,v)} \right),$$

where $\Omega = \{[v] \in \mathbf{P}(V) : J(v,v) \neq 0\}$. Then $\Phi([v_\lambda]) = i\lambda$ and

$$i\beta = \Phi([v_\beta]) = \mathrm{Ad}^*(g)\Phi([v_\lambda]) = \mathrm{Ad}^*(g).i\lambda$$

because Φ is equivariant. Let $p: \mathfrak{g}^* \to \mathfrak{t}^*$ denote the restriction mapping. Then, according to the convexity theorem for coadjoint orbits ([HNP, Thm. 5.17]),

$$i\beta = p(i\beta) \in p(\mathrm{Ad}^*(G).i\lambda) \subseteq \mathrm{conv}(\mathcal{W}_{\mathfrak{k}}.i\lambda) + C^\star_{\max} \subseteq C^\star_{\min}.$$

Since, according to Proposition III.5, $\beta \in \mathcal{W}.\lambda$, we conclude with Lemma III.10 that

$$\beta \in \mathcal{W}.\lambda \cap -iC^\star_{\min} = \mathcal{W}_{\mathfrak{k}}.\lambda. \quad \square$$

Theorem III.14. *Let P be a parabolic in the complexification $G_{\mathbb{C}}$ of the Hermitean simple Lie group G. Then $\emptyset \neq \mathrm{int}\, S(P)$ if and only if P contains a Borel subgroup associated to a positive $\mathfrak{k}$-adapted system Δ^+. In this case $S(P)$ is the maximal Ol'shanskii semigroup $S_{\max} := G\exp(iW_{\max})$ associated to Δ^+_p.*

Proof. It remains to show that $S_{\max} = S(P)$ holds in all these cases, where $S := S(P)$ has non-empty interior.

We want to apply Proposition II.4. So we have to show that

$$S \cap \exp(i\mathfrak{t}) \subseteq \exp(iC_{\max}).$$

To do this, we return to the realization of the complex flag manifold $G_{\mathbb{C}}/P$ as an orbit of a highest weight ray $[v_\lambda]$ in the projective space $\mathbf{P}(V)$ of a highest weight module. Let $X \in i\mathfrak{t} \setminus iC_{\max}$ be weight separating and suppose that $\exp(X) \in \mathrm{int}(S)$. Then there exists a non-compact root α with $\alpha(X) < 0$. Since every short non-compact root is the average of two long non-compact roots ([Pa, p. 219]), we even find a long non-compact root α such that $\alpha(X) < 0$. We also know from [Pa, p. 220] that the long non-compact roots are conjugate under $\mathcal{W}_{\mathfrak{k}}$. Hence $\mathcal{W}_{\mathfrak{k}}.\alpha$ generates the same cone as Δ^+_p.

Since $\lambda(\check{\beta}) \geq 0$ for all $\beta \in \Delta^+_p$, it follows that $(\lambda, \beta) \geq 0$ for all $\beta \in \Delta^+_p$. We find in particular that $(\lambda, \mathcal{W}_{\mathfrak{k}}.\alpha) \subseteq \mathbf{R}^+$. Since, on the other hand, $\mathcal{W}_{\mathfrak{k}}.\alpha$ generates $i\mathfrak{t}^*$, there exists $\gamma \in \mathcal{W}_{\mathfrak{k}}$ such that $(\lambda, \gamma.\alpha) > 0$. The semigroup S is invariant under conjugation by $N_G(\mathfrak{t})$, so that we can replace X by $\gamma.X$ and therefore assume that $X \notin iC_{\max}$, $\alpha(X) < 0$, and $\lambda(\check{\alpha}) > 0$. Hence

$$s_\alpha(\lambda)(X) = \lambda(X) - \langle \lambda, \check{\alpha} \rangle \alpha(X) > \lambda(X).$$

Thus, using Lemma III.9(iii) and (iv), we see that $[v_\lambda] \in C_S$ implies that $[v_{s_\alpha.\lambda}] \in C_S$. Let $\beta := s_\alpha.\lambda$. Then $\beta \in \mathcal{W}_{\mathfrak{k}}.\lambda$ by Proposition III.13. As before, let $Z \in \mathfrak{z}(\mathfrak{k})$ such that $\Delta^+_p = \{\beta \in \Delta : \beta(Z) > 0\}$. Then $\beta(Z) = \lambda(Z)$, so that $\lambda(\check{\alpha})\alpha(Z) = 0$, contradicting the fact that $\lambda(\check{\alpha}) > 0$ and $\alpha(Z) > 0$.

So we have proved that $S\cap\exp(i\mathfrak{t})\subseteq\exp iC_{\max}$. Now Proposition II.4 shows that $S\subseteq\overline{GN_{G_{\mathbf{C}}}(\mathfrak{t})G}$.

Let $s\in S\cap N_{G_{\mathbf{C}}}(\mathfrak{t})$. Then $s.[v_\lambda]=[v_{\gamma.\lambda}]$, where $\gamma\in\mathcal{W}$ is the element of the big Weyl group represented by s, i.e., $\gamma=\mathrm{Ad}(s)|_{\mathfrak{t}_{\mathbf{C}}}$. Now, again using Proposition III.13, we find that $\gamma.\lambda\in\mathcal{W}_{\mathfrak{k}}.\lambda$. This means that $s.[v_\lambda]\in N_G(\mathfrak{t}).[v_\lambda]$. The same argument applies to every other weight vector in $\mathcal{W}_{\mathfrak{k}}.[v_\lambda]$. Thus $\gamma.(\mathcal{W}_{\mathfrak{k}}.\lambda)\subseteq\mathcal{W}_{\mathfrak{k}}.\lambda$. Let $\beta:=\sum_{w\in\mathcal{W}_{\mathfrak{k}}}w.\lambda\in-i\,\mathrm{int}\,C^\star_{\min}$. Then $\beta\neq0$ and $\gamma.\beta=\beta$. It follows that γ preserves the set of Weyl chambers containing $\mathbf{R}^+\beta$. Since the small Weyl group $\mathcal{W}_{\mathfrak{k}}$ acts simply transitively on this set of Weyl chambers and $\mathcal{W}$ acts simply transitively on the set of all Weyl chambers, it follows that $\gamma\in\mathcal{W}_{\mathfrak{k}}$. Hence $s\in S\cap N_{G_{\mathbf{C}}}(\mathfrak{t})$ is represented by an element in $\mathcal{W}_{\mathfrak{k}}$, so that

$$S\cap N_{G_{\mathbf{C}}}(\mathfrak{t})\subseteq N_G(\mathfrak{t})Z_{G_{\mathbf{C}}}(\mathfrak{t})=N_G(\mathfrak{t})\exp(\mathfrak{t}_{\mathbf{C}})\subseteq G\exp(i\mathfrak{t}).$$

For $s\in S\cap N_{G_{\mathbf{C}}}(\mathfrak{t})$ this implies the existence of $g\in G$ such that $gs\in\exp(i\mathfrak{t})\cap S\subseteq\exp(iC_{\max})$. So $S\cap N_{G_{\mathbf{C}}}(\mathfrak{t})\subseteq S_{\max}$ and since $G(S\cap N_{G_{\mathbf{C}}}(\mathfrak{t}))G$ is dense in S, we conclude that $S\subseteq S_{\max}$. $\square$

We collect the information obtained in Sections I–III in the following theorem.

Theorem III.15. *Let $\mathfrak{t}\subseteq\mathfrak{g}$ be a compactly embedded Cartan algebra, $\omega\in\mathfrak{t}^*$, Σ a parabolic system of roots, $\mathfrak{b}=\mathfrak{p}_\Sigma$ a complex polarization in ω, and $\mathfrak{p}=\mathfrak{b}\cap\mathfrak{l}_{\mathbf{C}}$. Then*

$$S(\overline{B})=N_{\mathbf{C}}S(\overline{P})$$

and if $\mathfrak{l}=\mathfrak{z}(\mathfrak{l}_0)\oplus\bigoplus_{j=1}^k\mathfrak{l}_j$ denotes the decomposition of $\mathfrak{l}$ into simple ideals, then

$$S(\overline{P})=Z(L_{\mathbf{C}})_0\times\prod_{j=1}^k S(\overline{P}_j),$$

where $\mathfrak{p}_j:=\mathfrak{p}\cap(\mathfrak{l}_j)_{\mathbf{C}}$. More precisely, $S(\overline{P}_j)=(L_j)_{\mathbf{C}}$ holds if and only if $\mathfrak{l}_j$ is not simple Hermitean or if $\mathfrak{p}_j=(\mathfrak{l}_j)_{\mathbf{C}}$. The interior of $S(\overline{P}_j)$ is non-empty and different from $(L_j)_{\mathbf{C}}$ if and only if $\mathfrak{l}_j$ is Hermitean and $\mathfrak{p}_j$ contains a Borel algebra associated to a $\mathfrak{k}$-adapted positive system of roots. In this case $S(P_j)=L_j\exp(-iW_{\max})$, where $W_{\max}\subseteq\mathfrak{l}_j$ denotes the corresponding maximal invariant cone.

Proof. This is a collection of the results from Proposition I.13, Proposition II.3, and Theorem III.14. $\square$

Compression semigroups and admissible orbits

In this last subsection we want to relate the non-triviality of the semigroups $S(P)$ to a convexity property of the coadjoint orbit $\mathcal{O}_{i\lambda}$ associated to P by realizing $G_{\mathbf{C}}/P$ as a highest weight orbit $G_{\mathbf{C}}.[v_\lambda]$.

Theorem III.16. *Let G be a simple Lie group contained in a complexification $G_{\mathbf{C}}$ and P be a parabolic in $G_{\mathbf{C}}$. We realize the flag manifold $G_{\mathbf{C}}/P$ as a highest weight orbit $G_{\mathbf{C}}.[v_\lambda]$. Then the following are equivalent:*

(i) $\emptyset \neq \operatorname{int} S(P) \neq G_{\mathbf{C}}$.

(ii) $\mathcal{O}_{i\lambda}$ *is of convex type and not zero.*

If these conditions are satisfied, then G is Hermitean.

Proof. (i) $\Rightarrow$ (ii): Proposition II.3 shows that G is Hermitean. Then we can apply Lemma III.11 to see that $i\lambda$ is contained in an invariant cone. Moreover $i\lambda$ cannot be zero since that would imply $P = G_{\mathbf{C}}$ and hence $\operatorname{int} S(P) = G_{\mathbf{C}}$.

(ii) $\Rightarrow$ (i): If $\mathcal{O}_{i\lambda}$ is of convex type and non-zero, then $\mathfrak{g}$ contains a non-trivial invariant cone and hence is Hermitean. Moreover [HNP, Thm. 5.20] shows that there exist a $\mathfrak{k}$-adapted positive system such that $\lambda \in -iC^\star_{\min}$. Then $i\lambda$ permits a complex polarization $\mathfrak{p}_\Sigma$ such that $\Sigma \supseteq \Delta^+$. This implies $P = P_\Sigma$ and therefore Proposition III.12 shows that $\operatorname{int} S(P) \neq \emptyset$. Finally the proof of [HN2, Thm. 8.49] shows that $\operatorname{int} S(P) \neq G_{\mathbf{C}}$. $\square$

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