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Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers

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Abstract. Some new characterizations of the class of positive measures γ on $\mathbf{R}^n$ such that $H_p^l \subset L_p(\gamma)$ are given, where H_p^l $(1<p<\infty, 0<l<\infty)$ is the space of Bessel potentials. This imbedding, as well as the corresponding trace inequality

$$\|J_l u\|_{L_p(\gamma)} \le C\, \|u\|_{L_p},$$

for Bessel potentials $J_l=(1-\Delta)^{-l/2}$, is shown to be equivalent to one of the following conditions.

(a) $J_l(J_l\gamma)^{p'} \le C J_l \gamma$ a.e.

(b) $M_l(M_l\gamma)^{p'} \le C M_l \gamma$ a.e.

(c) For all compact subsets E of $\mathbf{R}^n$

$$\int_E (J_l\gamma)^{p'}\, dx \le C \operatorname{cap}(E, H_p^l),$$

where $1/p+1/p'=1$, M_l is the fractional maximal operator, and $\operatorname{cap}(\,\cdot\,, H_p^l)$ is the Bessel capacity. In particular, it is shown that the trace inequality for a positive measure γ holds if and only if it holds for the measure $(J_l\gamma)^{p'}\,dx$. Similar results are proved for the Riesz potentials $I_l\gamma=|x|^{l-n}*\gamma$.

These results are used to get a complete characterization of the positive measures on $\mathbf{R}^n$ giving rise to bounded pointwise multipliers $M(H_p^m \to H_p^{-l})$. Some applications to elliptic partial differential equations are considered, including coercive estimates for solutions of the Poisson equation, and existence of positive solutions for certain linear and semi-linear equations.

1. Introduction

Let $M^+=M^+(\mathbf{R}^n)$ be the class of positive Borel measures on $\mathbf{R}^n$, finite on compact sets. For $l\in\mathbf{R}$ and $1\le p<\infty$, we define the space of Bessel potentials $H_p^l=H_p^l(\mathbf{R}^n)$ as the completion of all functions $u\in\mathcal{D}=C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|u\|_{H_p^l}=\|(1-\Delta)^{l/2}u\|_{L_p}$. For $l>0$, $u\in H_p^l$ if and only if $u=G_l*f$, where $f\in L_p$ and G_l is the Bessel kernel defined by $\widehat{G}_l(\,\cdot\,)=(1+|\cdot|^2)^{-l/2}$ (see [20]). (Note that

$G_l \geq 0$ and $G_l \in L_1(\mathbf{R}^n)$.) The operator $J_l f = G_l * f$ defined for functions $f \in L_p$ or measures $f \in M^+$ is called the Bessel potential of order $l\,(l>0)$. The Bessel capacity $\operatorname{cap} E = \operatorname{cap}(E, H_p^l)$ of a compact set $E \subset \mathbf{R}^n$ is defined by

$$\operatorname{cap} E = \inf\{\|u\|_{L_p}^p : J_l u \geq 1 \text{ on } E;\ u \geq 0,\ u \in L_p\}. \tag{1.1}$$

For $\gamma \in M^+$ and $E \subset \mathbf{R}^n$, we denote by γ_E the restriction of γ to E: $d\gamma_E = \chi_E\, d\gamma$, where χ_E is the characteristic function of E.

In this paper we consider the trace inequality for Bessel potentials

$$\|J_l u\|_{L_p(\gamma)} \leq \text{const}\, \|u\|_{L_p}, \tag{1.2}$$

where the L_p-norm of u on the right hand side is taken with respect to Lebesgue measure. It is well known that inequalities of this type are closely connected with spectral properties of the Schrödinger operator and lead to deep applications in partial differential equations, theory of Sobolev spaces, complex analysis, etc. (See [20], [22], [3], and Section 5 of this paper.)

The following result is due to Maz'ya [18], [19], Adams [1], and Dahlberg [8] (see also [20], [2], [10]).

Theorem 1.1. *Let $1<p<\infty$, $0<l<\infty$, and $\gamma \in M^+$. Then* (1.2) *holds if and only if, for all compact sets E in $\mathbf{R}^n$,*

$$\gamma(E) \leq C \cdot \operatorname{cap}(E, H_p^l). \tag{1.3}$$

(*Note that we may restrict ourselves to sets E such that* $\operatorname{diam} E \leq 1$ *in* (1.3). *See* [22].)

It is easily seen that (1.3) is equivalent to a "dual" condition [1]

$$\|J_l \gamma_E\|_{L_{p'}}^{p'} \leq C \gamma(E), \tag{1.4}$$

where $1/p + 1/p' = 1$. Kerman and Sawyer [14] showed that we may restrict ourselves to arbitrary cubes $E = Q$ ($\operatorname{diam} Q \leq 1$) in (1.4). One can also replace J_l by the corresponding fractional maximal function

$$M_l \gamma(x) = \sup\{|Q|^{1-l/n} \gamma(Q) : x \in Q, \operatorname{diam} Q \leq 1\}. \tag{1.5}$$

Thus, the non-capacitary condition (1.4) can be restated as [14]

$$\int_Q (M_l \gamma_Q)^{p'}\, dx \leq c\gamma(Q), \quad \operatorname{diam} Q \leq 1. \tag{1.6}$$

(See also [26] for a simplified proof of this result.) We observe that conditions (1.3), (1.4), and (1.6) are difficult to verify and sometimes not sufficient for applications. For instance, it is not straightforward that, if γ_1 and $\gamma_2 \in M^+$, and $J_l\gamma_2 \le J_l\gamma_1$ a.e., then $\sup \gamma_1(E)/\operatorname{cap} E<\infty$ implies $\sup \gamma_2(E)/\operatorname{cap} E<\infty$. In certain problems discussed below we need characterizations of the trace inequality in terms of potentials $J_l\gamma$, rather than the measure γ itself.

Our main result on the trace inequality (see Section 2) is as follows.

Theorem 1.2. *Let $\gamma\in M^+$, $1<p<\infty$, and $0<l<\infty$. Then* (1.2) *holds if and only if any one of the following conditions is valid.*

(a) *For all $u\in L_p$*

$$\int (J_l u)^p (J_l\gamma)^{p'}\,dx \le c\,\|u\|_{L_p}^p. \tag{1.7}$$

(b) *For all compact sets E*

$$\int_E (J_l\gamma)^{p'}\,dx \le c\cdot \operatorname{cap}(E, H_p^l). \tag{1.8}$$

(c) *For all compact sets E*

$$\int (J_l\gamma_E)^{p'}\,dx \le c\cdot \operatorname{cap}(E, H_p^l). \tag{1.9}$$

(d) *The potential $J_l\gamma(x)$ is finite a.e. and*

$$J_l(J_l\gamma)^{p'} \le c\,J_l\gamma \ \ a.e. \tag{1.10}$$

Note that in the simpler case $l>n/p$ it follows that (1.2) is equivalent to

$$\sup\{\gamma(Q) : \operatorname{diam} Q \le 1\} < \infty.$$

Analogous results are also given for Riesz potentials, $I_l u=(-\Delta)^{-l/2}u$, $0<l<n/p$ (Theorem 4).

In Section 2 we discuss some corollaries and examples. In particular, we show that the trace inequality holds if there exists $t>1$ such that, for all cubes Q, $(\operatorname{diam} Q\le 1)$

$$\left\{\frac{1}{|Q|}\int_Q (J_l\gamma)^{p't}\,dx\right\}^{1/pt} \le c\cdot |Q|^{-l/n}. \tag{1.11}$$

It should be noted that (1.11) is a strengthened version of the condition of C. Fefferman and D. Phong [9]

$$(1.11') \qquad \left\{ \frac{1}{|Q|} \int_Q \varrho^t dx \right\}^{1/pt} \le c \cdot |Q|^{-l/n},$$

where $d\gamma = \varrho(x)\, dx$. We show that (1.11) is less restrictive than (1.11′) and, obviously, applies to measures which are not necessarily absolutely continuous with respect to the Lebesque measure. An example demonstrating that one cannot set $t=1$ in (1.11) so that the trace inequality remains true is given.

We also prove that many operators of Harmonic Analysis (maximal functions, Hilbert transforms, g-functions etc.) are bounded in the space of measurable functions f such that

$$\int_E |f|^q dx \le c \cdot \operatorname{cap}(E, H_p^l)$$

for all compact set E. Here $1<p, q<\infty$, $0<l<\infty$.

Section 4 is devoted to the multiplier problem for a pair of potential spaces. We denote

$$M(H_p^m \to H_p^l) = \{g\colon u \in H_p^m \Rightarrow g \cdot u \in H_p^l\}.$$

For positive m and l, multipliers have been characterized by Maz'ya and Shaposhnikova [22]. In the case $m \cdot l < 0$, only some sufficient conditions were known. We characterize positive measures $\gamma \in M(H_p^m \to H_p^{-l})$ and show that, at least in this case, the sufficient conditions of Maz'ya and Shaposhnikova are also necessary.

Theorem 1.3. *Let* $\gamma \in M^+$, $1<p<\infty$, $l>0$ *and* $m>0$. *Then* $\gamma \in M(H_p^m \to H_p^{-l})$ *if and only if the following two conditions hold*:

$$(1.12) \qquad \int_E (J_l \gamma)^p dx \le c \cdot \operatorname{cap}(E, H_p^m),$$

$$(1.13) \qquad \int_E (J_m \gamma)^{p'} dx \le c \cdot \operatorname{cap}(E, H_{p'}^l),$$

for all compact sets $E \subset \mathbf{R}^n$.

Note that, in contrast to the assumption (b) of Theorem 1.2, the exponents on the left hand sides of (1.12) and (1.13) are the same as in the corresponding capacities on the right hand sides. In the simpler case $p=2$, $l=m$ this pair of conditions is equivalent to $\sup \gamma(E)/\operatorname{cap} E < \infty$ by Theorem 1.2. (Cf. [22, Theorem 1.5].)

In Section 5 we consider applications to some linear and non-linear problems for elliptic partial differential equations. We show, in particular, that solutions of the Poisson equation $-\Delta u=\gamma$, $\gamma\geq 0$ and $\gamma\in M(H_p^1\to H_p^{-1})$ satisfy the coercivity property: $D^l u\in M(H_p^1\to H_p^{-1})$ for all l, $|l|=2$.

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2. Trace inequality for Riesz and Bessel potentials

For $l>0$ and $1<p<\infty$, we denote by h_p^l the completion of the space $\mathcal{D}=C_0^\infty$ with respect to the norm $\|u\|_{h_p^l}=\|(-\Delta)^{l/2}u\|_{L_p}$. If $0<l<n/p$ and $1<p<\infty$, then $u\in h_p^l$ if and only if $u=I_l v$, where $v\in L_p$, and the Riesz potential I_l is defined by $I_l v=|\cdot|^{l-n}*v$. In the same manner we define Riesz potentials of measures $\gamma\in M^+$:

$$I_l\gamma(x)=\int\frac{d\gamma(y)}{|x-y|^{n-l}},\quad x\in R^n.$$

Note that $I_l\gamma$ is finite a.e. (locally integrable) if and only if $\int_{|y|\geq 1}|y|^{l-n}d\gamma<\infty$ [17]. To any measurable set $E\subset\mathbf{R}^n$, we associate its Riesz capacity by [20]

$$\operatorname{cap}(E,h_p^l)=\inf\{\|u\|_{L_p}^p:I_l u\geq 1 \text{ on } E;\ u\geq 0,\ u\in L_p\}. \tag{2.1}$$

The (homogeneous) fractional maximal operator M_l, where $0<l<n$, is defined by

$$M_l\gamma(x)=\sup\left\{\frac{\gamma(Q)}{|Q|^{1-l/n}}:x\in Q\right\}. \tag{2.2}$$

It is easily seen that $M_l\gamma(x)\leq cI_l\gamma(x)$ for all $x\in\mathbf{R}^n$.

Now we are in a position to state our main result for Riesz potentials.

Theorem 2.1. *Let $\gamma\in M^+$, $1<p<n/p$. The following conditions are equivalent.*

(a) *The trace inequality*

$$\|I_l f\|_{L_p(\gamma)}\leq c\|f\|_{L_p} \tag{2.3}$$

holds for all $f \in L_p$.

(b) *For all compact sets* E

$$\gamma(E) \leq c \cdot \text{cap}(E, h_p^l). \tag{2.4}$$

(c) *For all compact sets* E

$$\int (I_l \gamma_E)^{p'} dx \leq c \cdot \gamma(E). \tag{2.5}$$

(d) *For all compact sets* E

$$\int (I_l \gamma_E)^{p'} dx \leq c \cdot \text{cap}(E, h_p^l). \tag{2.6}$$

(e) *For all compact sets* E

$$\int_E (I_l \gamma)^{p'} dx \leq c \cdot \text{cap}(E, h_p^l). \tag{2.7}$$

(f) *The potential* $I_l\gamma$ *is finite a.e. and*

$$I_l (I_l \gamma)^{p'} \leq c I_l \gamma \quad a.e. \tag{2.8}$$

Note that the equivalence of (a), (b) and (c) is known (see [20]). We can restrict ourselves to cubes $E=Q$ in (2.5) due to a result of Kerman and Sawyer [14]. It will be shown below that, for conditions (2.6) and (2.7), this is not true. The potential $I_l\gamma$ can be replaced by $M_l\gamma$ in (2.5)–(2.8).

Proof. It suffices to prove that (c) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (b).

Step 1. (c) $\Rightarrow$ (f). Suppose (2.5) holds. Let us show first that $I_l\gamma \in L_{p'}^{\text{loc}}$, and, in particular, $I_l\gamma < \infty$ a.e. Let $B = B_r(x)$ be the n-dimensional ball with radius $r>0$, centered at $x \in \mathbf{R}^n$. Then

$$\int_B (I_l \gamma_B)^{p'} dy \leq c\gamma(B). \tag{2.9}$$

For $x \in B$, we have

$$I_l \gamma_B(x) \geq c\gamma(B)(\text{diam}\, B)^{l-n}.$$

Hence, by (2.9)

$$\gamma(B_r(x)) \leq c \cdot r^{n-lp}. \tag{2.10}$$

We set $\gamma=\gamma_1+\gamma_2$, where $\gamma_1=\gamma_{2B}$ and $\gamma_2=\gamma_{(2B)^c}$. Here $2B=\{t:|x-t|<2r\}$. Then

$$\int_B (I_l\gamma)^{p'}dy \le c\left[\int_B (I_l\gamma_1)^{p'}dy+\int_B (I_l\gamma_2)^{p'}dy\right]. \tag{2.11}$$

By (2.9)

$$\int_B (I_l\gamma_1)^{p'}dy \le c\gamma(2B)<\infty. \tag{2.12}$$

To estimate the second integral on the right hand side of (2.11), note that, for all $y\in B_r(x)$,

(2.13)

$$I_l\gamma_2(y)=\int_{|x-t|\ge 2r}|y-t|^{l-n}d\gamma(t)\le 2^{n-l}\int_{|x-t|\ge r}|x-t|^{l-n}d\gamma(t)\le$$
$$\le c\int_r^\infty \frac{\gamma(B_\varrho(x))}{\varrho^{n-l+1}}d\varrho.$$

It follows from (2.10) that

$$\sup_{y\in B} I_l\gamma_2(y)\le c\int_r^\infty \varrho^{-l(p-1)-1}d\varrho<\infty.$$

Thus, we have proved that $I_l\gamma\in L_{p'}^{\mathrm{loc}}$. Now let us show that (2.9) implies (2.8). Note that

$$I_l(I_l\gamma)^{p'}(x)\le c\int_0^\infty \int_{B_r(x)}(I_l\gamma)^{p'}dy\frac{dr}{r^{n-l+1}}.$$

To estimate the right hand side of the preceding inequality, we use again the decomposition (2.11). By (2.12)

$$\int_0^\infty \int_{B_r(x)}(I_l\gamma_1)^{p'}dy\frac{dr}{r^{n-l+1}}\le c\int_0^\infty \gamma(2B_r(x))\frac{dr}{r^{n-l+1}}\le cI_l\gamma(x). \tag{2.14}$$

The estimate of the second term is more delicate. By (2.13)

$$\int_{B_r(x)}(I_l\gamma_2)^{p'}dy\le cr^n\left[\int_r^\infty \gamma(B_\varrho(x))\frac{d\varrho}{\varrho^{n-l+1}}\right]^{p'}.$$

For fixed $x\in\mathbf{R}^n$, let

$$\varphi(r)=\left[\int_r^\infty \gamma(B_\varrho(x))\frac{d\varrho}{\varrho^{n-l+1}}\right]^{p'}.$$

We claim that

$$\int_0^\infty \int_{B_r(x)} (I_l\gamma_2)^{p'}\,dy \frac{dr}{r^{n-l+1}} \le c\int_0^\infty r^{l-1}\varphi(r)\,dr \le c\,I_l\gamma(x).$$

To prove this, we note that

$$\sup_{r>0}\varphi(r) \le \left[\int_0^\infty \gamma(B_\varrho(x))\frac{d\varrho}{\varrho^{n-l+1}}\right]^{p'} \le c_1[I_l\gamma(x)]^{p'}. \tag{2.15}$$

Similarly, by (2.10)

$$\sup_{r>0} r^{lp}\varphi(r) \le c\sup_{r>0} r^{lp}\left[\int_r^\infty \varrho^{-l(p-1)-1}d\varrho\right]^{p'} \le c_2<\infty. \tag{2.16}$$

Clearly, for any fixed $R>0$,

$$\int_0^\infty r^{l-1}\varphi(r)\,dr \le c_1\int_0^R r^{l-1}dr\cdot\sup_{r>0}\varphi(r)+c_2\int_R^\infty r^{l(1-p)-1}dr\cdot\sup_{r>0} r^{lp}\varphi(r).$$

Applying (2.15) together with (2.16), we get

$$\int_0^\infty r^{l-1}\varphi(r)\,dr \le c_1[I_l\gamma(x)]^{p'}R^l+c_2R^{l(1-p)}.$$

Choosing $R=[I_l\gamma(x)]^{1/(l(p-1))}$, we have

$$\int_0^\infty r^{l-1}\varphi(r)\,dr \le c\cdot I_l\gamma(x).$$

We have proved that

$$\int_0^\infty \int_{B_r(x)} (I_l\gamma)^{p'}\,dy \frac{dr}{r^{n-l+1}} \le c\,I_l\gamma(x).$$

Thus, for all $x\in\mathbf{R}^n$,

$$I_l(I_l\gamma)^{p'}(x) \le c\,I_l\gamma(x).$$

The proof of Step 1 is complete.

Step 2. (f) $\Rightarrow$ (e). Suppose (2.8) holds and $I_l\gamma<\infty$ a.e. If, for some $x_0\in\mathbf{R}^n$, $I_l\gamma(x_0)<\infty$, then

$$M_l(I_l\gamma)^{p'}(x_0) \le c\,I_l(I_l\gamma)^{p'}(x_0) \le c\,I_l\gamma(x_0) < \infty,$$

where M_l is the fractional maximal operator defined by (2.2). Hence, for any cube Q, $x_0 \in Q$,

$$\int_Q (I_l\gamma)^{p'}\,dx \le c|Q|^{1-l/n} I_l\gamma(x_0) < \infty.$$

This implies that $I_l\gamma \in L_{p'}^{\mathrm{loc}}$. By (2.8)

$$[I_l(I_l\gamma)^{p'}]^{p'} \le c\,(I_l\gamma)^{p'} < \infty \quad \text{a.e.}$$

Setting $d\tilde{\gamma} = (I_l\gamma)^{p'}\,dx$ and integrating the preceding inequality over an arbitrary cube Q, we get

$$\int_Q (I_l\tilde{\gamma})^{p'}\,dx \le c\,\tilde{\gamma}(Q) < \infty.$$

This obviously implies

$$\int_Q (I_l\tilde{\gamma}_Q)^{p'}\,dx \le c\tilde{\gamma}(Q),$$

and, by a result of Kerman and Sawyer [14],

$$\tilde{\gamma}(E) \le c\cdot \operatorname{cap} E$$

for all compact sets E. The proof of Step 2 is complete.

Step 3. (e) $\Rightarrow$ (d). Suppose that (2.7) holds. Let us prove first that $\int (I_l\gamma_E)^{p'}\,dx < \infty$ for any compact set E. Assuming $E \subset B = \{x : |x| \le R\}$, we have

$$\begin{aligned}\int (I_l\gamma_E)^{p'}\,dx &\le c\left\{\int_{2B} (I_l\gamma)^{p'}\,dx + \int_{(2B)^c} (I_l\gamma_B)^{p'}\,dx\right\}\\ &\le c\left\{\operatorname{cap} 2B + [\gamma(B)]^{p'} \int_{|x|\ge 2R} \frac{dx}{|x|^{(n-l)p'}}\right\} < \infty.\end{aligned}$$

To show that (2.7) implies (2.6), we need some facts from the non-linear potential theory. The non-linear potential of a measure $\gamma \in M^+$ introduced by Khavin and Maz'ya in [16] is defined by

$$V_{pl}\gamma = I_l(I_l\gamma)^{p'-1}.$$

Lemma 2.2. ([16], [22]) *For any compact set $E \subset \mathbf{R}^n$, there exists a measure $\nu = \nu^E$ such that*

(i) $\operatorname{supp} \nu \subset E$,

(ii) $\nu(E) = \operatorname{cap} E$,

(iii) $\|I_l\nu\|_{L_{p'}}^{p'} = \operatorname{cap} E$,

(iv) $V_{pl}\nu(x) \ge 1$ *quasi-everywhere on* $\mathbf{R}^n$,

(v) $V_{pl}\nu(x) \le K = K(p,l,n)$ *on* $\mathbf{R}^n$,

(vi) $\operatorname{cap}\{V_{pl}\nu \ge t\} \le At^{-\sigma} \operatorname{cap} E$ *for all* $t > 0$, *where* $\sigma = \min(1, p-1)$; $\operatorname{cap}(\,\cdot\,) = \operatorname{cap}(\,\cdot\,, h_p^l)$, *and the constant A is independent of E.*

The measure ν^E associated with E is called the capacitary (equilibrium) measure of E.

Remark 2.1. In what follows one can replace $V_{pl}\gamma$ by the potential

$$W_{pl}\gamma(x) = \int_0^\infty \left[\frac{\gamma(B_r(x))}{r^{n-lp}}\right]^{p'-1} \frac{dr}{r} \le c\cdot V_{pl}\gamma(x). \tag{2.17}$$

As was shown by Hedberg and Wolff (see [2]), W_{pl} is a good substitute for V_{pl} in many problems. In particular, the estimate (vi) holds for W_{pl} with $\sigma = p-1$.

We will need the following lemma.

Lemma 2.3. *Suppose* $0<l, m<n$. *Suppose* γ *and* $\nu \in M^+$. *Then*

$$I_l(I_m\nu\, d\gamma) \le c[I_l(I_m\gamma\, d\nu) + I_m\nu\cdot I_l\gamma]. \tag{2.18}$$

Proof of Lemma 2.3. By Fubini's theorem

$$I_l(I_m\nu\, d\gamma)(x) = \int \frac{d\gamma(y)}{|x-y|^{n-l}} \int \frac{d\nu(t)}{|y-t|^{n-m}} = \int d\nu(t) \int K(x,y,t)\, d\gamma(y),$$

where $K(x,y,t) = |x-y|^{l-n}|y-t|^{m-n}$. It is easily seen that

$$\begin{aligned}\int K(x,y,t)\, d\gamma(y) &\le \int_{|y-t|<|t-x|/2} K(x,y,t)\, d\gamma(y) + \int_{|y-t|\ge|t-x|/2} K(x,y,t)\, d\gamma(y) \\ &\le \frac{2^{n-l}}{|x-t|^{n-l}} I_m\gamma(t) + \frac{2^{n-m}}{|x-t|^{n-m}} I_l\gamma(x).\end{aligned}$$

Hence

$$I_l(I_m\nu\, d\gamma) \le 2^{n-l} I_l(I_m\gamma d\nu) + 2^{n-m} I_l\gamma\cdot I_m\nu.$$

The proof of Lemma 2.3 is complete.

Now we are in a position to complete the proof of Step 3. Let E be a compact set and let $\nu = \nu^E$ be its associated capacitary measure. Then

$$\int (I_l\gamma_E)^{p'}\, dx \le \int [I_l(I_l\varphi\, d\gamma_E)]^{p'}\, dx,$$

where $\varphi = (I_l\nu)^{p'-1}$. Applying Lemma 2.3 with $l=m, \gamma=\gamma_E$ and $d\nu = \varphi\, dx$, we get

$$\int (I_l\gamma_E)^{p'}\, dx \le c\left\{\int (I_l\varphi)^{p'}(I_l\gamma_E)^{p'}\, dx + \int [I_l(\varphi I_l\gamma_E)]^{p'}\, dx\right\} = c(A_1 + A_2). \tag{2.19}$$

To estimate A_1, we choose an arbitrary $r>1$ and apply Hölder's inequality

$$A_1 \le \|I_l\gamma_E\|_{L_{p'}}^{p'/r'} \left\{ \int (I_l\gamma_E)^{p'} (I_l\varphi)^{r'p'}\,dx \right\}^{1/r'}.$$

Recall that by assertion (v) of Lemma 2.2 one has $I_l\varphi(x) \le K = K(n,l,p)$ for all $x\in\mathbf{R}^n$. Then

$$\int (I_l\gamma_E)^{p'}(I_l\varphi)^{r'p'}\,dx \le \int_0^K \left\{ \int_{I_l\varphi\ge t} (I_l\gamma)^{p'}\,dx \right\} t^{r'p'-1}\,dt.$$

By (2.7) and assertion (vi) of Lemma 2.2

$$\int_{I_l\varphi\ge t} (I_l\gamma)^{p'}\,dx \le c\cdot\operatorname{cap}\{I_l f\ge t\} \le \frac{c}{t^\sigma}\operatorname{cap} E,$$

where $\sigma=\min(1,p-1)$. Hence

$$\int (I_l\gamma_E)^{p'}(I_l\varphi)^{r'p'}\,dx \le c\cdot\operatorname{cap} E \int_0^K t^{r'p'-\sigma-1}\,dt.$$

Choosing $r'>\sigma p'$, we obtain

$$A_1 \le c\,\|I_l\gamma_E\|_{L_{p'}}^{p'/r}(\operatorname{cap} E)^{1/r'}. \tag{2.20}$$

Let us get a similar estimate for the term A_2. By duality

$$A_2^{1/p'} = \sup_{\|g\|_{L_p}\le 1} \left| \int I_l(\varphi I_l\gamma_E) g\,dx \right| = \sup_{\|g\|_{L_p}\le 1} \left| \int I_l g\cdot I_l\gamma_E\cdot\varphi\,dx \right|.$$

Suppose first that $p\ge 2$. We set $s=p/(p-2)>1$. Then $1/p+1/p+1/s=1$. Applying Hölder's inequality for the three functions, φ, $\varphi_1=(I_l\gamma_E)^{(p-2)/(p-1)}$ and $\varphi_2=(I_l\gamma_E)^{1/(p-1)}I_l g$, we get

$$A_2^{1/p'} \le \|\varphi\|_{L_p} \|I_l\gamma_E\|_{L_{p'}}^{p'/s'} \sup_{\|g\|_{L_p}\le 1} \left\{ \int |I_l g|^p (I_l\gamma)^{p'}\,dx \right\}^{1/p}. \tag{2.21}$$

By Lemma 2.1

$$\|\varphi\|_{L_p} = \|I_l\nu\|_{L_{p'}}^{p'/p} = (\operatorname{cap} E)^{1/p}.$$

From (2.7) and the trace inequality for the measure $(I_l\gamma)^{p'}dx$ it follows that

$$\sup_{\|g\|_{L_p}\le 1} \left\{ \int |I_l g|^p (I_l\gamma)^{p'}\,dx \right\}^{1/p} \le c<\infty.$$

Thus

$$A_2 \le c(\operatorname{cap} E)^{p'/p} \|I_l \gamma_E\|_{L_{p'}}^{p'^2/s}. \tag{2.22}$$

Since $p \ge 2$, we can choose $r' = p/p' > \sigma/p'$ in (2.20). Then, combining (2.20) and (2.22), we have

$$A_1 + A_2 \le c(\operatorname{cap} E)^{p'/p} \|I_l \gamma_E\|_{L_{p'}}^{p'(1-p'/p)}.$$

We have shown above that $\|I_l \gamma_E\|_{L_{p'}} < \infty$ for all compact sets E. Thus (2.19), together with the preceding estimate, gives

$$\|I_l \gamma_E\|_{L_{p'}}^{p'} \le c \cdot \operatorname{cap} E.$$

In the case $1 < p < 2$ we estimate the right hand side of (2.21) in a different way. We set $s = p/(2-p)$ with $1/p' + 1/p' + 1/s = 1$. Using again Hölder's inequality for the three functions,

$$\varphi_1 = \varphi^{p-1}, \quad \varphi_2 = (\varphi |I_l g|)^{2-p} \quad \text{and} \quad \varphi_3 = |I_l g|^{p-1} I_l \gamma,$$

we obtain

$$A_2^{1/p'} \le \|\varphi\|_{L_p}^{p/p'} \sup_{\|g\|_{L_p} \le 1} \left\{ \int |I_l g|^p (I_l \gamma)^{p'} dx \right\}^{1/p'} \times \sup_{\|g\|_{L_p} \le 1} \left\{ \int |I_l g|^p \varphi^p dx \right\}^{1/s}. \tag{2.23}$$

As above $\|\varphi\|_{L_p} = (\operatorname{cap} E)^{1/p}$, and, by (2.7) and the trace inequality,

$$\sup_{\|g\|_{L_p} \le 1} \left\{ \int |I_l g|^p (I_l \gamma)^{p'} dx \right\}^{1/p'} \le c < \infty.$$

Let us show that

$$\sup_{\|g\|_{L_p} \le 1} \left\{ \int |I_l g|^p (I_l \nu)^{p'} dx \right\}^{1/s} \le c < \infty. \tag{2.24}$$

as well. Since ν is the capacitary measure, its non-linear potential is bounded: $V_{pl}\nu(x) \le K$ for all $x \in \mathbf{R}^n$. Then, for any compact set $e \subset \mathbf{R}^n$, we have

$$\int (I_l \nu_e)^{p'} dx = \int V_{pl} \nu_e \, d\nu_e \le K \cdot \nu(e).$$

By Steps 1 and 2 of the proof applied to the measure ν, the preceding estimate implies

$$\int_e (I_l\nu)^{p'}\,dx \le c\cdot \operatorname{cap} e$$

for all compact sets e. Hence, by the trace inequality for the measure $(I_l\nu)^{p'}dx$, we get (2.24). We have proved that, for $1<p<2$,

$$A_2 \le c\|\varphi\|^p_{L_p} = c\cdot \operatorname{cap} E.$$

Together with (2.20) it gives

$$\|I_l\gamma_E\|^{p'}_{L_{p'}} \le c\{\operatorname{cap} E + \|I_l\gamma_E\|^{p'/r}_{L_{p'}}(\operatorname{cap} E)^{1/r'}\}. \tag{2.25}$$

We have already shown that $\|I_l\gamma_E\|_{L_{p'}}<\infty$. Moreover, we may assume that $\|I_l\gamma_E\|^{p'}_{L_{p'}}\ge \operatorname{cap} E$. (Otherwise, the desired estimate (2.8) is obviously true.) Then it follows from (2.25) that

$$\|I_l\gamma_E\|^{p'}_{L_{p'}} \le c(\operatorname{cap} E)^{1/r'}\|I_l\gamma_E\|^{p'/r}_{L_{p'}}.$$

Since $r>1$, we have

$$\|I_l\gamma_E\|^{p'}_{L_{p'}} \le c\cdot \operatorname{cap} E.$$

The proof of Step 3 is complete.

Step 4. (d) $\Rightarrow$ (b). This is easy. For an arbitrary $E\subset \mathbf{R}^n$ let $I_l u\ge 1$ on E; $u\ge 0$, $u\in L_p$. If (2.6) is valid, then by Hölder's inequality

$$\gamma(E)\le \int_E I_l u\,d\gamma = \int u I_l\gamma_E\,dx \le \|u\|_{L_p}\|I_l\gamma_E\|_{L_{p'}} \le c\|u\|_{L_p}(\operatorname{cap} E)^{1/p'}.$$

Now it follows from the definition of capacity (see (2.1)) that

$$\gamma(E)\le c\cdot \operatorname{cap} E,$$

which concludes the proof of Theorem 2.1.

Remark 2.2. Let $1<p<\infty$, $0<l<n/p$, and $q=1+1/p'$. Then assertion (e) of Theorem 2.1 can be rewritten as

$$V_{ql}\gamma(x)\le c I_l\gamma(x) \text{ a.e.} \tag{2.26}$$

where $V_{ql}=I_l(I_l\gamma)^{q'-1}$ is the non-linear potential of γ.

We observe that one cannot replace V_{ql} in (2.26) by the corresponding Hedberg–Wolff potential W_{ql} (see (2.17)). Note that $W_{ql}\gamma(x)\le cV_{ql}\gamma(x)$, but the converse is true only for $l\ge(2-q)/n$ [16]. Unfortunately, this is not the case when $q=1+1/p'$ and $0<l<n/p$.

In fact, the inequality $W_{ql}(x)\le cI_l\gamma(x)$ follows from the estimate

$$\gamma(B_r(x))\le r^{n-lp}, \quad (x\in \mathbf{R}^n,\ x>0),$$

which is weaker than the trace inequality.

Corollary 2.4. *Suppose* $1<p<\infty$ *and* $0<l<u/p$. *Suppose* $\gamma,\nu\in M^+$ *and* $I_l\gamma\leq I_l\nu$ *a.e. Then*

$$\sup\frac{\gamma(E)}{\operatorname{cap}E}\leq c\sup\frac{\nu(E)}{\operatorname{cap}E}$$

where the suprema are taken over all compact sets $E\subset\mathbf{R}^n$.

Corollary 2.4 follows from assertion (e) of Theorem 2.1.

The analogue of Theorem 2.1 for Bessel potentials (see Theorem 1.2 in the Introduction) can be proved in a similar fashion, and we do not go into details here. Note only that condition (1.10) can be replaced by

$$J_l(J_l\gamma)^{p'}\leq c\max(1,J_l\gamma),$$

since $J_l1=\int G_l(x)dx$. Hence, we can restrict ourselves to the set $\{x:J_l\gamma(x)\leq 1\}$ in (1.10), as well as we can consider only the sets E of $\operatorname{diam}E\leq 1$ in conditions (1.3), (1.8) and (1.9) (see [22]).

3. Some corollaries and examples

Let us show that we can put $M_l\gamma$ in place of $I_l\gamma$ in assertions (c)–(f) of Theorem 2.1. For (c) and (d) it is easy, since by a result of Muckenhoupt and Wheeden [23]

$$\int(I_l\gamma)^{p'}dx\leq c\int(M_l\gamma)^{p'}dx \tag{3.1}$$

with the constant c independent of γ; the reverse inequality is trivial.

We will need the following lemma, which shows that many operators of classical analysis are bounded in the space of functions f such that

$$\int_E|f|^q dx\leq c\cdot\operatorname{cap}E \tag{3.2}$$

for all compact sets E, $(1<q<\infty)$, if they are bounded in L_q-spaces with Muckenhoupt weights.

Recall that a weighted analogue of (3.1), namely

$$\int(I_l\gamma)^q w\,dx\leq c\int(M_l\gamma)^q w\,dx, \tag{3.3}$$

holds for $1<q<\infty$ and $w\in A_\infty$, where A_∞ is the union of the Muckenhoupt classes A_p, $1\leq p<\infty$, [23]. In particular, (3.3) is true for all A_1-weights w such that

$$Mw(x)\leq A\cdot w(x)\text{ a.e.}, \tag{3.4}$$

where $Mw=M_lw$ for $l=0$ is the Hardy–Littlewood maximal function. Moreover, the constant c in (3.3) depends only on l,q,n and the constant A from (3.4).

Lemma 3.1. *Let* $0<q<\infty$, $1<p<\infty$, *and* $0<l<n/p$. *Suppose that a function* $f\in L_q^{\text{loc}}$ *satisfies* (3.2) *with* $\operatorname{cap}(\cdot)=\operatorname{cap}(\cdot, h_p^l)$. *Suppose that, for all weights* $w\in A_1$,

$$\int_{\mathbf{R}^n} |g|^q w\,dx \le K\int_{\mathbf{R}^n} |f|^q w\,dx \tag{3.5}$$

with a constant K *depending only on* n, q, *and the constant* A *in the Muckenhoupt condition* (3.4). *Then*

$$\int_E |g|^q dx \le C\cdot\operatorname{cap} E \tag{3.6}$$

for all compact sets E, *with a constant* C *depending only on* l, p, n *and* K.

For $g=Mf$ and $q=p$, Lemma 3.1 is due to I. Verbitsky. (See [22], where it was used to derive an analogue of the Sobolev inequality for the spaces of functions defined by (3.2).) The idea of the proof is the same in the general case and we give here only a sketch of the proof.

Proof of Lemma 3.1. Suppose $\nu=\nu^E$ is the capacitary measure of $E\subset\mathbf{R}^n$ and $\varphi=V_{pl}\nu$ is its non-linear potential. Then, by Lemma 2.2,

(i) $\varphi(x)\ge 1$ quasi-everywhere on E;

(ii) $\varphi(x)\le B=B(n,p,l)$ for all $x\in\mathbf{R}^n$;

(iii) $\operatorname{cap}\{\varphi\ge t\}\le ct^{-\sigma}\operatorname{cap} E$, ($\sigma=\min(1,p-1)$, $t>0$), with the constant c independent of E. We need one more property of φ [22]:

(iv) $M\varphi^\delta(x)\le c\varphi^\delta(x)$ a.e., with a constant c independent of E, where $0<\delta<n/(n-l)$ for $1<p\le 2-l/n$, and $0<\delta<(p-1)n/(n-lp)$ for $2-l/n<p<\infty$. (Note that the bounds on δ are exact. If we use the Hedberg–Wolff potential $W_{pl}\nu$ instead of $V_{pl}\nu$, then one can show that (iv) holds for all $0<\delta<(p-1)n/(n-lp)$.)

Now, it follows from (iv) that $\varphi^\delta\in A_1$. Hence by (3.5)

$$\int |g|^q\varphi^\delta dx\le K\int |f|^q\varphi^\delta dx.$$

Applying this together with (i) and (ii), we get

$$\int_E |g|^q dx\le\int_{\mathbf{R}^n}|g|^q\varphi^\delta dx\le c\int_{\mathbf{R}^n}|f|^q\varphi^\delta dx=c\int_0^B\int_{\varphi\ge t}|f|^q dx\,t^{\delta-1}dt.$$

By (3.2) and (iii)

$$\int_{\varphi\ge t}|f|^q\,dx\le c\cdot\operatorname{cap}\{\varphi\ge t\}\le\frac{c}{t^\delta}\operatorname{cap} E.$$

Hence

$$\int_E |g|^q dx \le c \cdot \operatorname{cap} E \cdot \int_0^B t^{\delta-\sigma-1} dt.$$

Clearly, for all $0<l<n/p$, we can choose $\delta>\sigma=\min(1,p-1)$, so that $0<\delta<n/(n-l)$ if $1<p\le 2-l/n$, and $0<\delta<(p-1)n/(n-lp)$ if $2-l/n<p<\infty$. Then

$$\int_0^B t^{\delta-\sigma-1} dt < \infty,$$

which concludes the proof of Lemma 3.1.

We observe that Lemma 3.1 is also valid for Bessel capacities $\operatorname{cap}(\,\cdot\,, H_p^l)$, $0<l<\infty$ (see [22]).

In Section 5 we will need the boundedness of the Riesz transforms $R_j f = f * x_j/|x|^{n+1}$ $(j=1,2,\dots,n)$ in the spaces of functions defined by the capacitary condition (3.2).

Corollary 3.2. *Let* $1<p,q<\infty$ *and* $0<l<n/p$. *Then*

$$\sup \frac{\int_E |R_j f|^q dx}{\operatorname{cap} E} \le c \sup \frac{\int_E |f|^q dx}{\operatorname{cap} E}, \quad (j=1,2,\dots,n),$$

where the suprema are taken over all compact sets in $\mathbf{R}^n$ *and* $\operatorname{cap}(\,\cdot\,)=\operatorname{cap}(\,\cdot\,, h_p^l)$.

Proposition 3.3. *Suppose* $1<p<\infty$, $0<l<n/p$, *and* $\gamma \in M^+$. *Then the following three conditions are equivalent.*

(a) *For all sets* E

$$\int_E (I_l\gamma)^{p'} dx \le c \cdot \operatorname{cap} E. \tag{3.7}$$

(*b*) *For all sets* E

$$\int_E (M_l\gamma)^{p'} dx \le c \cdot \operatorname{cap} E. \tag{3.8}$$

(*c*) *The maximal function* $M_l\gamma$ *is finite a.e. and*

$$M_l(M_l\gamma)^{p'} \le c \cdot M_l\gamma. \tag{3.9}$$

Proof. Applying Lemma 3.1 with $g=I_l\gamma$, $f=M_l\gamma$, and $q=p'$, we see that (a) is equivalent to (b).

Let us show that (a) implies (c). Note that the latter can be restated as

$$\int_Q (M_l\gamma)^{p'}\,dy \le c\cdot|Q|^{1-l/n} M_l\gamma(x) \tag{3.10}$$

for all $x\in Q$. As in the proof of Theorem 2.1, we set $\gamma=\gamma_1+\gamma_2$ ($\gamma_1=\gamma_{2Q}$ and $\gamma_2=\gamma_{(2Q)^c}$) and have

$$\int_Q (M_l\gamma)^{p'}\,dy \le c\left[\int_Q (M_l\gamma_1)^{p'}\,dy + \int_Q (M_l\gamma_2)^{p'}\,dy\right].$$

By Theorem 2.1, (a) implies that, for $x\in Q$,

$$\int_Q (M_l\gamma_1)^{p'}\,dy \le c\int_Q (I_l\gamma_1)^{p'}\,dy \le c\cdot\gamma(2Q) \le c\cdot|Q|^{1-l/n} M_l\gamma(x). \tag{3.11}$$

To estimate the second integral, note that

$$M_l\gamma_2(y) = \sup_{y\in Q'} \frac{\gamma(Q'\cap(2Q)^c)}{|Q'|^{1-l/n}}.$$

If $y\in Q'\cap Q$ and $Q'\cap(2Q)^c\ne\emptyset$, then clearly $Q\subset 5Q'$. Thus, for $y\in Q$,

$$M_l\gamma_2(y) \le \sup_{Q':Q\subset 5Q'} \frac{\gamma(Q')}{|Q'|^{1-l/n}} \le c\cdot \sup_{Q'\supset Q} \frac{\gamma(Q')}{|Q'|^{1-l/n}}.$$

Then

$$\begin{aligned}\int_Q (M_l\gamma_2)^{p'}\,dy &\le c|Q| \sup_{Q'\supset Q}[\gamma(Q')/|Q'|^{1-l/n}]^{p'} \\ &\le c|Q|^{1-l/n} M_l\gamma(x) \sup_{Q'} |Q'|^{l/n}[\gamma(Q')/|Q'|^{1-l/n}]^{p'-1}.\end{aligned}$$

It follows from (a) that $\gamma(Q)\le c|Q|^{1-lp/n}$, so that the last factor on the right hand side is finite. Combining this with (3.11) we get (3.10).

It remains to prove that (c) $\Rightarrow$ (b). It follows from (3.9) that

$$\int_Q [M_l(M_l\gamma)^{p'}]^{p'}\,dx \le c\int_Q (M_l\gamma)^{p'}\,dx$$

for all cubes Q. Letting $d\widetilde{\gamma}=(M_l\gamma)^{p'}\,dx$ we have

$$\int_Q (M_l\widetilde{\gamma})^{p'}\,dx \le c\widetilde{\gamma}(Q).$$

Applying again the result of Kerman and Sawyer we get

$$\widetilde{\gamma}(E)=\int_E (M_l\gamma)^{p'}\,dx \le c\cdot\operatorname{cap} E.$$

The proof of Proposition 3.3 is complete.

Let us consider some simpler conditions sufficient for the trace inequality to hold. It was shown by Fefferman and Phong [9] that (2.3) is true for the measure $d\gamma(x)=g(x)\,dx$ $(g\ge 0)$ if there exists $t>1$ such that

$$\frac{1}{|Q|}\int_Q g^t(x)\,dx \le c\cdot|Q|^{-lpt/n}. \tag{3.12}$$

We observe (see [14]) that this result is a consequence of two known estimates: Sawyer's inequality for the fractional maximal function [25]

$$\|M_l f\|_{L_p(\gamma)} \le c\|f\|_{L_p}[\sup |Q|^{lp/n-1}\gamma(Q)]^{1/p}, \tag{3.13}$$

and the Adams–Hedberg inequality [2], [12]

$$|I_l f| \le c(M_{lt}f)^{1/t}(Mf)^{1-1/t} \tag{3.14}$$

where $t>1$ and $0<l<n/t$. Actually, it follows from (3.12) and (3.13) that

$$\|M_{lt}f\|_{L_p(g^t dx)} \le c\|f\|_{L_p}. \tag{3.15}$$

Hence by (3.14) and the boundedness of the Hardy–Littlewood maximal operator, we have the Fefferman–Phong inequality

$$\|I_l f\|_{L_p}(g\,dx) \le c\|f\|_{L_p}\left(\frac{\int_Q g^t dx}{|Q|^{1-lpt/n}}\right)^{1/tp}. \tag{3.16}$$

Combining (3.16) with our Proposition 3.3, we obtain the following corollary.

Corollary 3.4. *Let $\gamma\in M^+$, $1<p<\infty$, and $0<l<n/p$. Then the trace inequality* (2.3) *holds if there exists $t>1$ such that*

$$\int_Q (M_l\gamma)^{p't}dx \le c|Q|^{1-lpt/n} \tag{3.17}$$

for all cubes Q.

It is of interest to note that condition (3.17) is stronger than the original Fefferman–Phong condition, and applies to measures not necessarily absolutely continuous with respect to the Lebesgue measure.

Proposition 3.5. *Let $d\gamma = g\,dx$, $g \geq 0$. Under the assumptions of Corollary* 3.4, (3.12) *implies* (3.17).

Proof. Suppose (3.12) holds. Then, by Hölder's inequality,

$$\gamma(Q) \leq c \cdot |Q|^{1-lp/n} \tag{3.18}$$

for all cubes Q. Using the preceding inequality, it is easy to see that (3.17) is equivalent to

$$\int_Q (M_l \gamma_Q)^{p't} dx \leq c \cdot |Q|^{1-lpt/n}. \tag{3.17$'$}$$

(See analogous statements in the proof of Proposition 3.3 or Theorem 2.1 based on the decomposition $d\gamma = \chi_{2Q}\,d\gamma + (1-\chi_{2Q})\,d\gamma$.) For $x \in Q$ we have

$$\begin{aligned}[M_l \gamma_Q(x)]^{p't} &= \sup_{x \in Q'} \left[|Q'|^{l/n-1} \int_{Q' \cap Q} g(y)\,dy \right]^{p't} \\ &\leq \sup_{x \in Q} \left[|Q'|^{-1} \int_{Q' \cap Q} g(y)\,dy \right]^t \left[|Q'|^{lp/n-1} \int_{Q'} g(y)\,dy \right]^{t(p'-1)}.\end{aligned}$$

Then, by (3.18),

$$[M_l \gamma_Q(x)]^{p't} \leq c \sup_{x \in Q'} \left[|Q'|^{-1} \int_{Q' \cap Q} g(y)\,dy \right]^t = c[M(\chi_Q g)]^t.$$

Since the maximal operator M is bounded in $L_t(\mathbf{R}^n)$, $t > 1$, we have

$$\int_Q (M_l \gamma_Q)^{p't} dx \leq c \int [M(\chi_Q g)]^t dx \leq c \int_Q g^t\,dx.$$

Now it is clear that (3.12) implies (3.17$'$). The proof of Proposition 3.5 is complete.

We observe that, for $t=1$, Corollary 3.4 is not true. In other words, we cannot restrict ourselves to cubes $E=Q$ in assertions (c) and (d) of Theorem 2.1.

Proposition 3.6. *There exists a measure γ with compact support such that*

$$\int_Q (I_l \gamma)^{p'} dx \leq c \cdot |Q|^{1-lp/n} \tag{3.19}$$

for all cubes Q, but the trace inequality (2.3) *does not hold.*

Note that a similar example for assertion (b) of Theorem 2.1 is well known. By a theorem of Frostman, there exists a measure ν with compact support e such that $\nu(Q)\le c|Q|^{1-lp/n}$, but $\text{cap}(e,h_p^l)=0$, which contradicts the condition $\nu(e)\le c\cdot\text{cap}(e,h_p^l)$. Unfortunately, the energy of the measure ν in this example is infinite; $\|I_l\nu_e\|_{L'_p}=\infty$. Hence it does not satisfy condition (3.19).

To construct a measure claimed in Proposition 3.6, we set

$$d\gamma(x)=\eta(x_n)\varphi(x')\,dx'\,dx_n$$

where

$$x=(x',x_n),\quad x'=(x_1,\dots,x_{n-1}),$$

$\eta(x_n)=1$ for $|x_n|\le 1$, $\eta(x_n)=0$ for $|x_n|>1$ and

$$\varphi(x')=\varphi_\beta(x')=\begin{cases}|x'|^{1-n}(\log(2/|x'|))^{-\beta}, & |x'|\le 1\\ 0, & |x'|>1.\end{cases}\tag{3.20}$$

Let $1<p<\infty$, $n\ge 2$, and $l=(n-1)/p$. We claim that, for $1+1/p'<\beta<p$ the estimate (3.19) is true, but the trace inequality is not valid.

For $0<r<1$, set $E_r=\{x:|x'|\le r,\ |x_n|\le 1\}$. It is known [20] that for $l=(n-1)/p$ the capacity of the cylinder E_r, $\text{cap}(E_r,h_p^l)=(\log 2/r)^{1-p}$.

Then

$$\frac{\gamma(E_r)}{\text{cap}\,E_r}\ge c\frac{\int_{|x'|\le r}|x'|^{1-n}\log(2/|x'|)^{-\beta}dx'}{(\log(2/r))^{1-p}}\ge\left(\log\frac{2}{r}\right)^{p-\beta}.$$

For $\beta<p$, $(\log(2/r))^{p-\beta}\to\infty$ as $r\to 0$. Thus the trace inequality is not valid. Now suppose $1+1/p'<\beta<p$. (Clearly, such β exists for any $1<p<\infty$.)

We show that

$$I_l\gamma(x)\le\begin{cases}c|x'|^{l+1-n}(\log(4/|x'|))^{-\beta+1} & \text{for } |x'|\le 2;\\ c|x|^{l-n} & \text{for } |x'|>2.\end{cases}\tag{3.21}$$

It is easily seen that

$$\begin{aligned}I_l\gamma(x)&=\int_{-1}^{1}\int_{|t'|\le 1}\frac{\varphi(t')\,dt'\,dt_n}{(|x'-t'|^2+|x_n-t_n|^2)^{(n-l)/2}}\\&\le\int_{|t'|\le 1}\varphi(t')\,dt'\int_{-\infty}^{\infty}\frac{dt_n}{(|x'-t'|^2+|x_n-t_n|^2)^{(n-l)/2}}\\&\le c\int_{|t'|\le 1}\frac{\varphi(t')\,dt'}{|x'-t'|^{n-l-1}}=cB.\end{aligned}$$

For $|x'|\le 2$, we have

$$\begin{aligned}
B &\le c|x'|^{l-n+1}\int_{|t'|\le|x'|/2}|t'|^{1-n}\left(\log\frac{2}{|t'|}\right)^{-\beta}dt'\\
&\quad+c|x'|^{1-n}\left(\log\frac{2}{|x'|}\right)^{-\beta}\int_{|x'|/2<|t'|<2|x'|}|x'-t'|^{l-n+1}dt'\\
&\quad+c\int_{2|x'|<|t'|<1}|t'|^{l-2n+2}\left(\log\frac{2}{|t'|}\right)^{-\beta}dt'\\
&=B_1+B_2+B_3.
\end{aligned}$$

By direct computation we get

$$B_1\le c|x'|^{l-n+1}\left(\log\frac{4}{|x'|}\right)^{1-\beta};$$

$$B_2\le c|x'|^{l-n+1}\left(\log\frac{4}{|x'|}\right)^{-\beta};$$

$$B_3\le c|x'|^{l-n+1}\left(\log\frac{4}{|x'|}\right)^{-\beta}.$$

Combining these estimates we see that

$$I_l\gamma(x)\le c|x'|^{l-n+1}\left(\log\frac{4}{|x'|}\right)^{1-\beta},\quad |x'|\le 2.$$

If $|x'|>2$, we have

$$I_l\gamma(x)\le c\int_{-1}^{1}\int_{|t'|\le 1}\frac{\varphi(t')\,dt'\,dt_n}{(|x'|^2+|x_n-t_n|^2)^{(n-l)/2}}\le c|x|^{l-n}\int_{|t'|\le 1}\varphi(t')\,dt'.$$

Since $\beta>1$, we have

$$\int_{|t'|\le 1}\varphi(t')\,dt'<\infty,$$

which gives (3.21) for $|x'|>2$.

Using (3.21), we see that, for any cube Q

$$\begin{aligned}
&\int_Q(I_l\gamma)^{p'}dx\\
&\quad\le c|Q|^{1/n}\left\{\int_{|x'|\le 2}|x'|^{(l-n+1)p'}\left(\log\frac{4}{|x'|}\right)^{(1-\beta)}p'dx'+\int_{|x'|>2}|x'|^{(l-n)p'}dx'\right\}.
\end{aligned}$$

Recall that $l=(n-1)/p$. Thus $p'(n-l-1)=n-1$ and $1/n=1-lp/n$. Since $\beta > 1+1/p'$ and $(n-l)p'>n-1$, both integrals on the right hand side of the preceding inequality are finite. We obtain that

$$\int_Q (I_l\gamma)^{p'}\,dx \le c\,|Q|^{1-lp/n}$$

which concludes the proof of Proposition 3.6.

Remark 3.1. It can be shown that, for the measure

$$d\gamma = \eta(x_n)\varphi_\beta(x')\,dx'\,dx_n$$

constructed in the proof of Proposition 3.6, the trace inequality holds if and only if $\beta \ge p$.

The estimates of $I_l\gamma$ given by (3.21) are easily seen to be sharp. In fact, on the support of γ, $B_0=\{(x',x_n):|x'|\le 1, |x_n|\le 1\}$, we have $I_l\gamma(x)\asymp c|x'|^{l-n+1}\big(\log\frac{4}{|x'|}\big)^{1-\beta}$.

For $x\notin 2B_0$ we clearly have $I_l\gamma(x)\asymp c|x|^{l-n}$. Using these estimates and taking into account that $l=(n-1)/p$, one can show that, for $\beta\ge p$, condition (f) of Theorem 2.1 is valid.

4. Positive measures as multipliers

Recall that h_p^l and H_p^l are the spaces of Riesz and Bessel potentials, respectively. We define the class of multipliers for a pair of potential spaces as

$$M(h_p^m \to h_p^l) = \left\{ \gamma \in \mathcal{D}' : \sup_{u\in\mathcal{D}} \frac{\|\gamma u\|_{h_p^l}}{\|u\|_{h_p^m}} < \infty \right\}.$$

A similar definition is valid for Bessel potentials.

A complete characterization of the classes

$$M(h_p^m \to h_p^l) \quad \text{and} \quad M(H_p^m \to H_p^l)$$

(as well as multipliers of some other spaces of differentiable functions) is due to Maz'ya and Shaposhnikova [22], mostly in the case when $l\cdot m\ge 0$. For $l\cdot m<0$, some sufficient conditions were given.

In this section, we characterize positive measures which are multipliers for a pair of potential spaces when $l\cdot m<0$. (Since by duality $M(h_p^m\to h_p^{-l})=M(h_{p'}^l\to h_{p'}^{-m})$, we can assume $m>0$ and $l<0$.) As in Sections 2 and 3, we give full proofs only for

Riesz potentials. The case of Bessel potentials requires minor modifications (mainly in the case when $m\geq n/p$ or $l\geq n/p'$), but we do not give the details here.

Let

$$1<p<\infty, \quad 0<m<n/p, \quad 0<l<n/p', \quad \text{and} \quad \gamma\in M^+.$$

Then, clearly, $\gamma\in M(h_p^m\to h_p^{-l})$ if and only if

$$\left|\int u\cdot v\, d\gamma\right|\leq c\|u\|_{h_{p'}^l}\|v\|_{h_p^m}. \tag{4.1}$$

(Note that, by duality, $h_{p'}^l=(h_p^{-l})^*$, where $1/p+1/p'=1$, $0<l<n/p'$.)

Letting $u=I_l f$ and $v=I_m g$ in (4.1), we restate it as

$$\left|\int I_l f\cdot I_m g\, d\gamma\right|\leq c\|f\|_{L_{p'}}\|g\|_{L_p}, \tag{4.1$'$}$$

where the functions f and g may be assumed to be positive.

By Hölder's inequality

$$\left|\int I_l f\cdot I_m g\, d\gamma\right|\leq c\|I_l f\|_{L_{p'}(\gamma)}\|I_m g\|_{L_p(\gamma)}.$$

Suppose that

$$\gamma(E)\leq c\cdot\operatorname{cap}(E,h_{p'}^l); \quad \gamma(E)\leq c\cdot\operatorname{cap}(E,h_p^m) \tag{4.2}$$

for all compact sets E. Then it follows from the trace inequality for the spaces $h_{p'}^l$ and h_m^l that (4.1$'$) holds, and hence $\gamma\in M(h_p^m\to h_p^{-l})$.

For $p=2$ and $m=l$, (4.2) is also necessary in order that $\gamma\in M(h_2^l\to h_2^{-l})$. (See [22].) Indeed, letting $f=g$ in (4.1$'$) we see that it implies the trace inequality

$$\|I_l f\|_{L_2(\gamma)}\leq c\|f\|_{L_2}.$$

Thus, $\gamma(E)\leq c\cdot\operatorname{cap} E$ for all compact sets E.

Unfortunately, conditions (4.2) are not necessary when $p\neq 2$ or $l\neq m$. (See an example at the end of this section.) However, it follows from Theorem 2.1 that, for $p=2$ and $l=m$, (4.2) is equivalent to

$$\int_E (I_l\gamma)^2\,dx\leq c\cdot\operatorname{cap}(E,h_2^l).$$

It is this condition, rather than (4.2) that can be extended to characterize positive measures $\gamma\in M(h_p^m\to h_p^{-l})$ in the general case.

Theorem 4.1. *Let*

$$1<p<\infty, \quad 0<m<n/p, \quad 0<l<n/p', \quad \textit{and} \quad \gamma\in M^+.$$

Then $\gamma\in M(h_p^m \to h_p^{-l})$ *if and only if the following two conditions hold*:

$$\int_E (I_l\gamma)^p dx \le c\cdot \operatorname{cap}(E, h_p^m); \tag{4.3}$$

$$\int_E (I_m\gamma)^{p'} dx \le c\cdot \operatorname{cap}(E, h_{p'}^l). \tag{4.4}$$

Proof. It follows from (4.1′) that $\gamma\in M(h_p^m \to h_p^{-l})$ if and only if

$$\left|\int f I_l(I_m g\, d\gamma)\, dx\right| \le c\|f\|_{L_{p'}}\|g\|_{L_p}.$$

By duality, this is equivalent to

$$\|I_l(I_m g\, d\gamma)\|_{L_p} \le c\|g\|_{L_p} \tag{4.5}$$

for all $g\in L_p$, $g\ge 0$. Changing the roles of $I_l f$ and $I_m g$, we get in a similar fashion that (4.5) is also equivalent to

$$\|I_m(I_l f\, d\gamma)\|_{L_{p'}} \le c\|f\|_{L_{p'}} \tag{4.6}$$

for all $f\in L_{p'}$, $f\ge 0$.

We recall that by Lemma 2.3

$$I_l(I_m g\, d\gamma) \le c[I_l(g I_m\gamma) + I_m g\cdot I_l\gamma]. \tag{4.7}$$

Thus

$$\|I_l(I_m g\, d\gamma)\|_{L_p} \le c\{\|I_l(g I_m\gamma)\|_{L_p} + \|I_m g\|_{L_p(\nu)}\},$$

where $d\nu=(I_l\gamma)^p dx$. Suppose that assumptions (4.3) and (4.4) hold. Then it follows from (4.3) that

$$\|I_m g\|_{L_p(\nu)} \le c\|g\|_{L_p}. \tag{4.8}$$

Similarly, it follows from (4.4) that

$$\|I_l f\|_{L_{p'}(\sigma)} \le c\|f\|_{L_{p'}},$$

where $d\sigma=(I_m\gamma)^{p'}$. The dual form of the preceding inequality is

$$\|I_l(f\,d\sigma)\|_{L_p}\le c\|f\|_{L_p(\sigma)}.$$

Letting $f=(I_m\gamma)^{1-p'}g$, we get

$$I_l(f\,d\sigma)=I_l(gI_m\gamma)\quad\text{and}\quad\|f\|_{L_p(\sigma)}=\|g\|_{L_p}.$$

Thus

$$\|I_l(gI_m\gamma)\|_{L_p}\le c\|g\|_{L_p}.$$

Combining this with (4.8) we obtain (4.5). We have proved that (4.3) and (4.4) imply $\gamma\in M(h_p^m\to h_p^{-l})$.

Conversely, suppose that $\gamma\in M(h_p^m\to h_p^{-l})$. Then (4.1′) is valid which implies (4.5) and (4.6). Letting $f=g=\chi_Q$ in (4.1′), we get

$$\gamma(Q)\cdot|Q|^{-l/n}|Q|^{-m/n}\le c\int_Q I_lf\cdot I_mg\,d\gamma\le c\|f\|_{L_{p'}}\|g\|_{L_p}=c|Q|.$$

Thus

$$\gamma(Q)\le c|Q|^{1-(l+m)/n} \tag{4.9}$$

for all cubes Q. Similarly, substituting $f=\chi_Q$ and $g=\chi_Q$ in (4.5) and (4.6) gives

$$\int_Q(I_l\gamma_Q)^p dx\le c\cdot|Q|^{1-mp/n};\quad \int_Q(I_m\gamma_Q)^{p'}dx\le c\cdot|Q|^{1-lp'/n}. \tag{4.10}$$

(In fact, it is easily seen that any one of the preceding estimates implies (4.9).) As in the proof of Theorem 2.1, (4.10) together with (4.9), implies

$$\int_Q(I_l\gamma)^p dx<\infty;\quad \int_Q(I_m\gamma)^{p'}dx<\infty$$

for all cubes Q. Setting $g=\chi_Q(I_m\gamma)^{p'-1}$ we see that

$$\|g\|_{L_p}^p=\int_Q(I_m\gamma)^{p'}dx<\infty.$$

By (4.5) we get

$$\int_Q[I_l(I_mg\,d\gamma)]^p dx\le c\int_Q(I_m\gamma)^{p'}dx. \tag{4.11}$$

Clearly, for $x\in Q$, we have

$$\sup_{x\in Q'} |Q'|^{l/n-1} \int_{Q'} I_m g\, d\gamma = M_l(I_m g\, d\gamma)(x) \le c I_l(I_m g\, d\gamma)(x).$$

By our choice of g,

$$\int_{Q'} I_m g\, d\gamma = \int g I_m \gamma_{Q'}\, dy \ge \int_Q (I_m \gamma_{Q'})^{p'}\, dy.$$

Thus

$$\sup_{x\in Q'} |Q'|^{l/n-1} \int_Q (I_m \gamma_{Q'})^{p'}\, dy \le I_l(I_m g\, d\gamma)(x) \tag{4.12}$$

for $x\in Q$. Now from (4.11) and (4.12) it follows that

$$\int_Q \sup_{x\in Q'} \left\{ |Q'|^{l/n-1} \int_{Q\cap Q'} (I_m \gamma_{Q'})^{p'}\, dy \right\}^p dx \le c \int_Q (I_m \gamma)^{p'}\, dx. \tag{4.13}$$

Let us show that we can replace $I_m\gamma_{Q'}$ by $I_m\gamma$ in the preceding inequality. We have

$$\int_{Q\cap Q'} (I_m\gamma)^{p'}\, dy \le c \int_{Q\cap Q'} (I_m\gamma_1)^{p'}\, dy + c \int_{Q'} (I_m\gamma_2)^{p'}\, dy,$$

where $d\gamma_1 = \chi_{2Q} d\gamma$ and $d\gamma_2 = (1-\chi_{2Q}) d\gamma$. Then by (4.13)

$$\int_Q \sup_{x\in Q'} \left[|Q'|^{l/n-1} \int_{Q\cap Q'} (I_m\gamma_1)^{p'}\, dy \right]^p dx \le c \int_Q (I_m\gamma)^{p'}\, dx. \tag{4.14}$$

To estimate the second term note that, for $x, y\in Q'$ and $t\in (2Q')^c$, we have $|t-y| \asymp |t-x|$. Hence

$$I_m\gamma_2(y) = \int \frac{d\gamma_2(t)}{|t-y|^{n-m}} \le c \int \frac{d\gamma_2(t)}{|t-x|^{n-m}} = c I_m \gamma_2(x).$$

Consequently,

$$\begin{aligned} \sup_{x\in Q'} |Q'|^{l/n-1} \int_{Q'} (I_m\gamma_2)^{p'}\, dy &\le c \sup_{x\in Q'} |Q'|^{l/n} [I_m\gamma_2(x)]^{p'} \\ &\le c [I_m\gamma(x)]^{p'-1} \cdot \sup_{x\in Q'} |Q'|^{l/n} I_m \gamma_2(x). \end{aligned} \tag{4.15}$$

Let $\operatorname{diam} Q'=r$. Making use of estimate (4.9) we get

$$\sup_{x\in Q'} |Q'|^{l/n} I_m\gamma_2(x) \le c \sup_{r>0} r^l \int_r^\infty \frac{\gamma(B_\varrho(x))}{\varrho^{n-m+1}}\,d\varrho \le c<\infty,$$

with the constant C independent of $x\in Q$. This, together with (4.15), implies

$$\int_Q \sup_{x\in Q'} \left[|Q'|^{l/n-1} \int_{Q'\cap Q} (I_m\gamma_2)^{p'}\,dy\right]^p dx \le c\int_Q (I_m\gamma)^{p'}\,dx. \tag{4.16}$$

Combining (4.14) and (4.16), we obtain

$$\int_Q [M_l(\chi_Q\nu)]^p\,dx \le c\nu(Q),$$

where $d\nu=(I_m\gamma)^{p'}\,dx$. Applying again the result of Kerman and Sawyer [14], we conclude that the trace inequality

$$\|I_l f\|_{L_{p'}(\nu)} \le c\|f\|_{L_{p'}}$$

holds. Thus assertion (b) of Theorem 4.1 is valid.

Substituting $f=\chi_Q(I_l\gamma)^{p-1}$ into (4.6), we derive in a similar way that, for $d\sigma=(I_l\gamma)^{p'}\,dx$

$$\int_Q [M_m(\chi_Q\sigma)]^{p'}\,dx \le c\sigma(Q)$$

for all cubes Q, which implies assertion (b). The proof of Theorem 4.1 is complete.

Corollary 4.2. *Under the conditions of Theorem* 4.1 *it is true that* $\gamma\in M(h_p^m\to h_p^{-l})$ *if the following two relations hold*

$$I_l(I_m\gamma)^{p'}(x)\le c(I_m\gamma)^{p'-1}(x)<\infty, \quad a.e., \tag{4.17}$$

$$I_m(I_l\gamma)^{p}(x)\le c(I_l\gamma)^{p-1}(x)<\infty, \quad a.e. \tag{4.18}$$

Proof. It follows from (4.17) and (4.18) that

$$I_l[I_l(I_m\gamma)^{p'}]^p \le cI_l(I_m\gamma)^{p'}<\infty, \quad \text{a.e.},$$
$$I_m[I_m(I_l\gamma)^{p}]^{p'} \le cI_m(I_l\gamma)^{p}<\infty, \quad \text{a.e.}$$

By Theorem 2.1, this gives

$$\int_E (I_m\gamma)^{p'}\,dx \le c\cdot\operatorname{cap}(E,h_{p'}^l),$$
$$\int_E (I_l\gamma)^{p}\,dx \le c\cdot\operatorname{cap}(E,h_p^m)$$

for all compact sets E. Applying Theorem 4.1 we conclude that $\gamma \in M(h_p^m \to h_p^{-l})$. For $p=2$ and $l=m$, the assumptions of Corollary 4.1 coincide with the estimate

$$I_l(I_l\gamma)^2 \le c I_l \gamma < \infty, \quad \text{a.e.}$$

By Theorem 2.1 the preceding condition is valid if and only if

$$\gamma(E) \le c \cdot \operatorname{cap}(E, h_2^l).$$

As mentioned above, this is equivalent to $\gamma \in M(h_2^l \to h_2^{-l})$.

Remark 4.1. If $p \ne 2$ and $l \ne m$, conditions (4.2) are not necessary for $\gamma \in M(h_p^m \to h_p^{-l})$.

To show this one can use the same idea as in Proposition 3.6. For $p \ne 2$ we set $l=(n-1)/p'$ and $m=(n-1)/p$. Let $d\gamma = \eta(x_n)\varphi(x')\,dx'dx_n$ where $\varphi(x')$ is defined by (3.21) with $\beta=2$; $\eta(x_n)=1$ for $|x_n| \le 1$ and $\eta(x_n)=0$ for $|x_n|>1$. For

$$E_r = \{x=(x',x_n) : |x'| \le r,\ |x_n| \le 1\}, \quad 0<r<1,$$

we have $\operatorname{cap}(E_r, h_{p'}^l) \asymp (\log(2/r))^{1-p}$ and $\operatorname{cap}(E_r, h_m^p) \asymp (\log(2/r))^{1-p}$ [20]. Then

$$\frac{\gamma(E_r)}{\operatorname{cap}(E_r, h_p^m)} \ge c\, \frac{\int_{|x'| \le r} |x'|^{1-n} (\log(2/|x'|))^{-2} dx'}{(\log(2/r))^{1-p}} \ge c \left(\log \frac{2}{r} \right)^{p-2}.$$

Similarly

$$\frac{\gamma(E_r)}{\operatorname{cap}(E_r, h_{p'}^l)} \ge c \left(\log \frac{2}{r} \right)^{p'-2}.$$

Letting $r \to 0$, we see that for $p \ne 2$, one of the conditions (4.2) is violated. In the opposite direction, one can use estimates (3.21) (see also Remark 3.1) to show that, for $l=(n-1)/p'$ and $m=(n-1)/p$, conditions (4.17) and (4.18) are valid. Hence, $\gamma \in M(h_p^m \to h_p^{-l})$, but (4.2) is not true.

Setting $p=2$, $l=(n-2)/2$, $m=(n-1)/2$, $(n \ge 4)$, one can construct an analogous example showing that (4.2) is not necessary even in the case $p=2$, $l \ne m$.

There is another generalization of the fact that $\gamma \in M^+ \cap M(h_2^l \to h_2^{-l})$ if and only if the L_2-trace inequality holds.

Proposition 4.3. *Let $1<p<\infty$, $0<l<n/p$, and $\gamma \in M^+$. Then $\gamma \in M(h_p^l \to h_{p'}^{-l})$ if and only if*

(4.19) $$\|I_l f\|_{L_2(\gamma)} \le c \|f\|_{L_p}.$$

The proof is the same as for $p=2$. By duality, $\gamma\in M(h_p^l\to h_{p'}^{-l})$ if and only if

$$\left|\int u\cdot v\,d\gamma\right|\le \|u\|_{h_p^l}\|v\|_{h_p^l}.$$

Substituting $u=v=I_l f$ in the preceding inequality, we see that (4.19) holds. Conversely, it follows from the Schwartz inequality that

$$\left|\int u\cdot v\,d\gamma\right|\le c\|u\|_{L_2(\gamma)}\|v\|_{L_2(\gamma)}.$$

Applying (4.19), which is obviously equivalent to

$$\|u\|_{L_2(\gamma)}\le c\|u\|_{h_p^l},$$

we get that $\gamma\in M(h_p^l\to h_{p'}^{-l})$.

Note that, for $p<2$, by a result of D. Adams (see [3]) (4.19) holds if and only if

$$\gamma(Q)\le c|Q|^{2(1/p-l/n)}$$

for all cubes Q. For $p>2$, we arrive at the "upper triangle case" of the trace inequality considered in [21]. According to the Maz'ya–Netrusov result (4.19) is equivalent to

$$\int_0^\infty\left[\frac{t}{\nu(t)}\right]^{2/(p-2)}dt<\infty, \tag{4.20}$$

where $p>2$ and $\nu(t)=\inf\{\operatorname{cap}(E,h_p^l):\nu(E)\ge t\}$, $t>0$. A non-capacitary characterization of the trace inequality in the "upper triangle case" based on different ideas was given by Verbitsky [26].

5. Applications to partial differential equations

In this section we outline possible applications of the trace inequality and capacitary estimates found above to some elliptic partial differential equations. We mention here only simple cases of several model problems without any attempts of generalization. However, we treat both linear and non-linear equations, sometimes in the non-Hilbert case $p\neq 2$, so that the elements of non-linear potential theory used in the proofs above are essential.

Some of the applications are known (see [20], [3], [14]), and we discuss them briefly, emphasizing interesting connections with other parts of Analysis. Note that even in this case known results are stated in a new analytical form; all criterions are close to being necessary and sufficient, and many particular cases can be derived easily from them.

We start with a few problems for the Schrödinger equation

$$Lu = -\Delta u - \gamma u = 0, \tag{5.1}$$

with $\gamma \in M^+$, related to the trace inequality

$$\|u\|_{L_2(\gamma)} \leq \text{const} \cdot \|\nabla u\|_{L_2}, \tag{5.2}$$

or, equivalently,

$$\|I_1 u\|_{L_2(\gamma)} \leq \text{const} \cdot \|u\|_{L_2}, \tag{5.2$'$}$$

where I_1 is the Riesz potential of order $l=1$. Note that (5.1) and (5.2) are obviously connected through the equation

$$\langle Lu, u\rangle = \|\nabla u\|^2_{L_2} - \int |u|^2 d\gamma. \tag{5.3}$$

We would like to mention the following problems for the Schrödinger operator:

(1) Spectral properties of L.
(2) Positivity of solutions.
(3) Unique continuation property.

Problem 1 has been studied in great detail from the point of view of imbedding theorems since the work of Friedrichs (see [18], [20], [9], [11], [14]). It follows from (5.3) and our Theorem 2.1 that if $L>0$, then

$$I_1(I_1\gamma)^2(x) \leq c \cdot I_1\gamma(x) < \infty \quad \text{a.e.} \tag{5.4}$$

Moreover, there exists a constant $c_n>0$ such that if (5.4) holds for $c<c_n$, then the Schrödinger operator is positive. A sufficient condition for $L>0$ is given by

$$\left\{ \frac{1}{|Q|} \int_Q (I_1\gamma)^{2p} dx \right\}^{1/2p} \leq c \cdot |Q|^{-1/n} \tag{5.5}$$

for some $p>1$ and all cubes Q, if $c<c_n$. As was mentioned above, (5.5) is a refined version of the Feffermann–Phong condition applicable to measures γ not necessarily absolutely continuous with respect to the Lebesgue measure. Many other applications to distribution of eigenvalues, semiboundedness, discreteness and finiteness of the negative part of spectra, etc., can be found in the cited literature.

The second problem has attracted attention of specialists in partial differential equations as well as in stochastic processes. A necessary and sufficient condition for existence of positive solutions to the Schrödinger equation (5.1) for positive potentials γ was given by R. Khas'minsky [15] in terms of the Brownian motion (see also [7] and the papers cited there). It was shown later that Problem 2 reduces to Problem 1 under minor restrictions on the potential γ, not necessarily positive. (See Agmon [4] where the case of general second order elliptic operators on Riemannian manifolds is considered; γ is assumed to be in L_p^{loc}, $p>n$.)

We note that a standard substitute $u=e^v$ yields that Problem 2 is equivalent to the existence of solutions of the n-dimensional Riccati's equation

$$-\Delta v = |\nabla v|^2 + \gamma. \tag{5.6}$$

As was pointed out by K. Hansson (see Proposition 5.2 below), one can obtain directly a criterion for existence of solutions of (5.6) in the following form. There exists a constant $C_n>0$ such that, if

$$\gamma(E) \le C\cdot \operatorname{cap}(E, h_2^1) \tag{5.7}$$

for $C<C_n$ and all compact sets E, then (5.6) has a solution (in a weak sense) in $\mathbf{R}^n$. Conversely, if a solution exists, then (5.7) is valid.

Problem 3, first considered for the Schrödinger equation by T. Carleman (see [6], [13], [25]), is related to the inequality

$$\|u\|_{L_2(\varrho)} \le c\|\Delta u\|_{L_2(\varrho^{-1})} \tag{5.8}$$

where ϱ is an arbitrary non-negative weight. It is easy to see that (5.8) is equivalent to (5.2) with $d\gamma=\varrho\,dx$, for any weight ϱ. Hence again the solution can be given in terms of condition (5.7).

Next, we obtain coercive estimates for solutions of the equation

$$-\Delta u = \gamma, \tag{5.9}$$

where γ is a measure from $M(h_p^1 \to h_p^{-1})$, $1<p<\infty$. (Similar results are also valid for the equation $-\Delta u + u = \gamma$ if we replace h_p^l by H_p^l and use the corresponding Bessel capacity.) The proof is again based on Theorem 2.1 and Lemma 2.3.

Proposition 5.1. *Let* $\gamma\in M^+$ *and let* u *be a solution of* (5.9) *such that*

$$\int_{r<|x|<2r} |u|\,dx = o(r^{n+1}) \quad \text{as } r\to\infty. \tag{5.10}$$

Then the following properties are equivalent

(a) $\gamma\in M(h_p^1 \to h_p^{-1})$.

(b) $\nabla u \in M(h_p^1 \to L_p)\cap M(h_{p'}^1 \to L_{p'})$.

(c) $D^l u \in M(h_p^1 \to h_p^{-1})$ *for all* l, $|l|=2$.

Moreover, the following estimates hold

$$\begin{aligned}\sum_{|l|=2} \|D_l u\|_{M(h_p^1\to h_p^{-1})} &\le c_1(\|\nabla u\|_{M(h_p^1\to L_p)}+\|\nabla u\|_{M(h_{p'}^1\to L_{p'})})\\ &\le c_2\|\Delta u\|_{M(h_p^1\to h_p^{-1})}\\ &\le c_3\sum_{|l|=2}\|D_l u\|_{M(h_p^1\to h_p^{-1})}.\end{aligned} \tag{5.11}$$

Proof. Suppose $\gamma\in M(h_p^1\to h_p^{-1})$. Then by Theorem 2.1

$$\int_E (I_1\gamma)^p dx \le c\cdot \operatorname{cap}(E,h_p^1), \quad \int_E (I_1\gamma)^{p'} dx\le c\cdot\operatorname{cap}(E,h_{p'}^1). \tag{5.12}$$

Let $\eta\in C^\infty$, $\eta(x)=1$ for $|x|<1$ and $\eta(x)=0$ for $|x|>2$. Put $\eta_r(x)=\eta(x/r)$. From (5.9) it follows

$$-\Delta(\eta_r u)=\eta_r\gamma-2\nabla\eta_r\nabla u-u\Delta\eta_r,$$

which yields

$$\eta_r u=I_2(\eta_r\gamma-2\nabla\eta_r\nabla u-u\Delta\eta_r).$$

After integrating by parts this is rewritten as

$$\eta_r u=I_2(\eta_r\gamma)+I_2(u\Delta\eta_r)-2\operatorname{div} I_2(u\nabla\eta_r).$$

By differentiating we obtain that on the ball $|x|<r/2$ there holds the estimate

$$|\nabla u|\le c(n)\left(I_1\gamma+r^{-n-1}\int_{r<|y|<2r}|u(y)|\,dy\right)$$

where the constant $c(n)$ depends only on n. By using (5.10) and taking the limit as $r\to\infty$ we obtain the estimate

$$|\nabla u|\le c(n)I_1\gamma.$$

Now (5.12) implies

$$\int_E|\nabla u|^p dx\le c\cdot\operatorname{cap}(E,h_p^1),\quad \int_E|\nabla u|^{p'}dx\le c\cdot\operatorname{cap}(E,h_{p'}^1). \tag{5.13}$$

We have proved that $\nabla u\in M(h_p^1\to L_p)\cap M(h_{p'}^1\to L_{p'})$. Thus, (a) $\Rightarrow$ (b).

Now suppose $\nabla u \in M(h_p^1 \to L_p) \cap M(h_{p'}^1 \to L_{p'})$. Then by a theorem of Maz'ya and Shaposhnikova ([22], Section 1.5), $D^l u \in M(h_p^1 \to h_p^{-1})$ for all $|l|=2$ and

$$\|D^l u\|_{M(h_p^1 \to h_p^{-1})} \le c(\|\nabla u\|_{M(h_p^1 \to L_p)} + \|\nabla u\|_{M(h_{p'}^1 \to L_{p'})}),$$

from which we conclude that (b) $\Rightarrow$ (c).

The implication (c) $\Rightarrow$ (a) is trivial, because if $D^l u \in M(h_p^1 \to h_p^{-1})$, then $-\Delta u = \gamma \in M(h_p^1 \to h_p^{-1})$. Obviously, estimate (5.11) follows from the above argument. The proof of Proposition 5.1 is complete.

Now let us consider two non-linear problems

$$-\Delta u = u^q + \lambda\gamma \quad \text{on } \Omega,\ u \ge 0; \tag{5.14}$$

$$-\Delta u = \alpha|\nabla u|^q + \gamma \quad \text{on } \Omega; \tag{5.15}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{5.16}$$

where Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary and γ is a positive measure with compact support on Ω. Moreover, $1<q<\infty$, λ and α are positive constants.

The semi-linear problem (5.14), (5.16) was treated by Baras and Pierre [5]. A necessary and sufficient condition for existence of solutions (in a weak sense) was given in terms of a certain non-linear functional. Later Adams and Pierre [3] showed that (5.14) has a solution, for sufficiently small $\lambda>0$, if and only if, for all compact sets $E \subset \Omega$,

$$\gamma(E) \le c \cdot \operatorname{cap}(E, h_p^2), \tag{5.17}$$

where $p=q'$. The proof is based on capacitary estimates and certain weighted L_p-estimates, as in our Lemma 2.3.

The generalized Riccati's equation (5.15) was considered by K. Hansson. The proof of the following result is to appear.

Proposition 5.2. (K. Hansson) *If the problem* (5.15)–(5.16) *has a solution (in a weak sense), then for all compact sets* $E \subset \Omega$

$$\gamma(e) \le c \cdot \operatorname{cap}(E, h_p^1). \tag{5.18}$$

Conversely, (5.18) *implies that* (5.15)–(5.16) *has a solution for sufficiently small* $\alpha>0$.

Hansson's proof of the second assertion is based on our Theorem 2.1 and an iteration procedure. Clearly, both (5.17) and (5.18) can be given in a different form by using results of the present paper.

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