Hankel operators between weighted Bergman spaces

Svante Janson

1. Introduction and main results

We will study Hankel operators on (weighted) Bergman spaces of analytic functions on the unit disc. These spaces and operators are defined as follows.

Let m denote the Lebesgue measure on the unit disc D and let, for $-1<\alpha<\infty$, μ_α be the measure $\frac{\alpha+1}{\pi}(1-|z|^2)^\alpha\, dm(z)$. Thus $L^2(d\mu_\alpha)$ is the space of all measurable functions on the unit disc for which the norm

$$\|f\|_\alpha^2 = \frac{\alpha+1}{\pi}\int_D |f(x)|^2(1-|z|^2)^\alpha\, dm(z) \tag{1.1}$$

is finite. (The normalization constant is chosen such that μ_α has unit mass.) The Bergman space A^α is the subspace of all analytic functions in $L^2(\mu_\alpha)$; this is a closed subspace so A^α is a Hilbert space. The orthogonal projection of $L^2(\mu_\alpha)$ onto A^α (known as the Bergman projection) will be denoted by P_α. We will also consider the space $\bar{A}^\alpha$ of anti-analytic functions in $L^2(\mu_\alpha)$ and the corresponding projection $\bar{P}_\alpha$.

There are (at least) two non-equivalent natural definitions of Hankel operators on the Bergman spaces. These have become known as the big and small Hankel operators.

Definition. Let f be a (measurable) function on D. The big and small Hankel operators with symbol f, in this paper denoted by H_f and $\tilde{H}_f$ respectively are defined by

$$H_f(g) = (I-P_\alpha)(\bar{f}g), \tag{1.2}$$

$$\tilde{H}_f(g) = \bar{P}_\alpha(\bar{f}g). \tag{1.3}$$

(g is an analytic function. We discuss the interpretation of the right hand sides when $\bar{f}g\notin L^2(\mu_\alpha)$ later.)

We will in this paper only consider analytic symbols.

Note that H_f maps into $A^{\alpha\perp}$, while $\tilde{H}_f$ maps into the much smaller space $\bar{A}^\alpha$. In fact, $\bar{P}_\alpha P_\alpha$ is the one-dimensional projection $f\to\int f\,d\mu_\alpha$ onto the constant functions, so $\tilde{H}_f-\bar{P}_\alpha H_f$ has rank (at most) one. Consequently, if H_f is bounded (compact, in S_p (see below)), then so is $\tilde{H}_f$. The converse is not obvious, but we will see that it holds in important cases.

We recall that an operator T in a Hilbert space, or from one Hilbert space into another, belongs to the Schatten—von Neumann class S_p if the sequence of singular numbers $\{s_n(T)\}_{n=0}^\infty=\{\inf\|T-K\|: \operatorname{rank}(K)\leqq n\}_{n=0}^\infty$ belongs to l^p.

Remark. S_∞ thus is the class of all bounded operators. Some authors prefer to let S_∞ be the class of compact operators, but we find that definition less natural. (The compact operators correspond to c_0, not to l^∞.)

The characterization of the symbols f such that the small Hankel operator $\tilde{H}_f$ belongs to S_p is due to Peller (1982a, 1982b) and Semmes (1984), see Section 4. The big Hankel operator H_f was first considered by Axler (1986) ($p=\infty$ and $\alpha=0$); the general case was studied by Arazy, Fisher and Peetre (1986). The result by Arazy, Fisher and Peetre shows a striking cut-off: If $1<p\leqq\infty$, then H_f belongs to S_p iff $\tilde{H}_f$ does (iff f belongs to a certain Besov space), but if $0<p\leqq 1$, then H_f never belongs to S_p (unless f is constant and thus $H_f=0$).

The purpose of the present paper is to investigate this phenomenon further by studying H_f as an operator of one Bergman space into another. More precisely, we use two parameters $\alpha,\beta\in(-1,\infty)$, and study H_f and $\tilde{H}_f$, defined using P_α as above, as operators of A^β into $L^2(\mu_\alpha)$. (There is some arbitrariness in this choice. One might use three parameters and study the operators from A^β into $L^2(\mu_\gamma)$, but we will consider only the case $\gamma=\alpha$.)

We let $S_p^{\beta\alpha}$ denote $S_p(A^\beta, L^2(\mu_\alpha))$, i.e. the class of S_p-operators of A^β into $L^2(\mu_\alpha)$. Furthermore for $0<p\leqq\infty$, and $-\infty<s<\infty$, B_p denotes the usual Besov space of analytic functions in the unit disc; if m is a non-negative integer and $m>s$, then

$$B_p^s=\{f\colon (1-|z|^2)^{m-s}D^m f(z)\in L^p((1-|z|^2)^{-1}\,dm)\}. \tag{1.4}$$

(In particular, $A^\alpha=B_2^{-(\alpha+1)/2}$.) The main results then can be stated as follows.

Theorem 1. *Let* $\alpha,\beta>-1$ *and* $0<p\leqq\infty$.

(i) *If* $1/p<1+\frac{1}{2}(\alpha-\beta)$, *then* $H_f\in S_p^{\beta\alpha}$ *iff* $f\in B_p^{1/p+(\beta-\alpha)/2}$.

(ii) *If* $1/p\geqq 1+\frac{1}{2}(\alpha-\beta)$, *except in the case* $p=\infty$ *and* $\beta=\alpha+2$, *then* $H_f\in S_p^{\beta\alpha}$ *only if f is constant (and thus* $H_f=0$*).*

For $p=\infty$, Theorem 1 gives necessary and sufficient conditions for H_f to be bounded, except when $\beta=\alpha+2$ (this exceptional case will not be treated here).

There are similar conditions for compactness (without exception). Define b^s_∞ to be the closure of polynomials in B^s_∞. If m is a non-negative integer greater than s then

$$b^s_\infty = \{f\colon (1-|z|^2)^{m-s}|D^m f(z)| \to 0 \quad \text{as} \quad |z| \to 1\}. \tag{1.5}$$

Theorem 2. *Let* $\alpha, \beta > -1$.

(i) *If* $\beta < \alpha + 2$, *then* H_f *is compact from* A^β *into* $L^2(\mu_\alpha)$ *iff* $f \in b_\infty^{(\beta-\alpha)/2}$.

(ii) *If* $\beta \geqq \alpha + 2$, *then* H_f *is never compact from* A^β *into* $L^2(\mu_\alpha)$ *unless* f *is constant.*

For the small Hankel operators there is no cut-off; otherwise the conditions are the same.

Theorem 3. *Let* $\alpha, \beta > -1$ *and* $0 < p \leqq \infty$. *Then* $\tilde{H}_f \in S_p^{\beta\alpha}$ *iff* $f \in B_p^{1/p+(\beta-\alpha)/2}$.

Theorem 4. *Let* $\alpha, \beta > -1$. *Then* $\tilde{H}_f$ *is compact from* A^β *into* $\bar{A}^\alpha$ *iff* $f \in b_\infty^{(\beta-\alpha)/2}$.

We include Theorems 3 and 4 although they are not new; as is well-known (cf. e.g. Rochberg (1985)), they are easily reduced to results by Peller and Semmes. (We do this reduction in Section 4.) See also Burbea (1987). (When $\beta = \alpha$ and $p \geqq 1$, Theorems 3 and 4 are given in Janson, Peetre and Rochberg (1987) and Arazy, Fisher and Peetre (1986).)

Taking $\beta = \alpha$ in Theorems 1 and 2 we recover the result by Arazy, Fisher and Peetre. If $\beta \neq \alpha$ we have a similar result, but with a different cut-off. In particular, if $\alpha > \beta$ then there is a positive result also for $p = 1$. This enables us to prove the theorem (in one direction) by interpolating between $p = 1$ and $p = \infty$ (see Section 7). Thus our approach gives a new proof of the theorem by Arazy, Fisher and Peetre, which in some parts is simpler than the original proof (where it was impossible to use $p = 1$ directly), although it is fair to add that other parts are more complicated because the loss of (isometric) Möbius invariance when $\alpha \neq \beta$.

Our approach is inspired by Rochberg (1982), who introduced weights (corresponding to Bergman spaces) in order to overcome a similar problem at the other endpoint $p = \infty$ for Hankel operators on H^2.

This approach can also be applied to big Hankel operators in several complex variables. See Wallstén (1988) for details.

The remainder of the paper contains the proof of the theorems. This is organized as follows.

Various preliminaries are taken care of in Section 2.

Theorems 1 (ii) and 2 (ii) are proved in Section 3.

In Section 4 we study the small Hankel operators and prove Theorems 3 and 4. Since $H_f \in S_p^{\beta\alpha} \Rightarrow \tilde{H}_f \in S_p^{\beta\alpha}$, as we remarked earlier, Theorem 3 shows that $H_f \in S_p^{\beta\alpha} \Rightarrow f \in B_p^{1/p+(\beta-\alpha)/2}$, which proves one implication in Theorem 1 (i). Similarly, Theorem 4 implies one half of Theorem 2 (i). It remains to show that

If

$$(1.6)\qquad 1/p < 1+\tfrac{1}{2}(\alpha-\beta),\quad \text{then}\quad f\in B_p^{1/p+(\beta-\alpha)/2} \Rightarrow H_f\in S_p^{\beta\alpha}$$

and

If

$$(1.7)\qquad \beta-\alpha < 2,\quad \text{then}\quad f\in b_\infty^{(\beta-\alpha)/2} \Rightarrow H_f\quad \text{is compact.}$$

This is done in several steps. In Section 5 we prove (1.6) for $p\leqq 1$. The case $p=\infty$ is proved in Section 6. The remaining case $1<p<\infty$ is proved in Section 7, where we also prove (1.7).

We remark that the case $p=2$ (Hilbert—Schmidt) of Theorem 1 may be proved directly by computing $\|H_z m(z^n)\|_\alpha$ (cf. (3.2) below) and by then deducing

$$\|H_f\|_{S_2^{\beta\alpha}}^2 \asymp \sum_1^\infty n^{\beta-\alpha+1}|f(n)|^2 \asymp \|f\|_{B_2^{(\beta-\alpha+1)/2}}^2,\quad \beta-\alpha < 1,$$

but we will not use this. Observe, however, that when $\alpha=\beta$, there is an interesting *isometry* $\|H_f\|_{S_2}^2=\sum_1^\infty n|\hat{f}(n)|^2$, see Arazy, Fisher and Peetre (1986).

2. Preliminaries

We begin by recalling some facts about $S_p=S_p(H_1, H_2)$, the class of S_p-operators from a Hilbert space H_1 to another Hilbert space H_2, equipped with the "norm" $\|T\|_{S_p}=\|\{s_n(T)\}_0^\infty\|_{l_p}$. For further details we refer to e.g. Simon (1979) and McCarthy (1967).

(i) If $p\geqq 1$ then $\|\ \|_{S_p}$ is a norm and S_p is a Banach space.

(ii) If $p<1$ then $\|\ \|_{S_p}^p$ is subadditive and S_p is a Fréchet space. Thus

$$(2.1)\qquad \|S+T\|_{S_p}^p \leqq \|S\|_{S_p}^p+\|T\|_{S_p}^p,\quad 0<p\leqq 1.$$

(iii) There is a Hölder-type inequality

$$(2.2)\qquad \|ST\|_{S_r} \leqq \|S\|_{S_p}\|T\|_{S_q},$$

provided $1/r=1/p+1/q$, $S\in S_p$, $T\in S_q$.

(iv) If $\{e_n\}$ is any ON-basis in H_1, then

$$(2.3)\qquad \text{if}\quad 0<p\leqq 2,\quad \|T\|_{S_p}^p \leqq \sum \|Te_n\|^p.$$

$$(2.4)\qquad \text{if}\quad 2\leqq p\leqq\infty,\quad \|T\|_{S_p}^p \geqq \sum \|Te_n\|^p.$$

The Hilbert space A^α has a reproducing kernel given by

$$K_\alpha(z, w) = (1-z\bar{w})^{-\alpha-2} \tag{2.5}$$

I.e.

$$f(z) = \int f(w) K_\alpha(z, w)\, d\mu_\alpha(w) \tag{2.6}$$

for $f \in A^\alpha$, and more generally

$$P_\alpha f(z) = \int f(w) K_\alpha(z, w)\, d\mu_\alpha(w), \quad f \in L^2(\mu_\alpha). \tag{2.7}$$

As a special case of (2.6)

$$\|K_\alpha(\cdot, w)\|_\alpha^2 = K_\alpha(w, w) = (1-|w|^2)^{-\alpha-2}. \tag{2.8}$$

We will also write $K_{\alpha,w}(z) = K_\alpha(z, w)$.

We have defined H_f by (1.2), although obviously $P_\alpha(\bar{f}g)$ is not always defined. To begin with, we note that $P_\alpha \bar{f} = \bar{f}(0)$ for any analytic function f and any reasonable extension of P_α. Hence $H_f(1) = \bar{f} - \bar{f}(0)$ for any reasonable definition of H_f, so if H_f maps A^β into $L^2(\mu_\alpha)$ then necessarily $\bar{f} - \bar{f}(0) \in L^2(\mu_\alpha)$ and thus $f \in L^2(\mu_\alpha)$. Consequently it suffices to consider symbols in A^α, and for such symbols (1.2) makes sense at least for all bounded g. If (1.2) then defines an operator which is continuous in the norms $\| \|_\beta$ and $\| \|_\alpha$, we extend H_f to A^β by continuity. In fact, we will see in Section 6 that when H_f defines a bounded operator, and thus $f \in B_\infty^{(\beta-\alpha)/2}$ by Theorem 1 (or 3), then H_f may be written as a well-defined integral operator on A^β. We will now derive its kernel. First we note that by (2.7) and (2.6), assuming $f \in A^\alpha$,

$$P_\alpha(\bar{f}K_{\alpha,w})(z) = \langle \bar{f}K_{\alpha,w}, K_{\alpha,z}\rangle = \overline{\langle fK_{\alpha,z}, K_{\alpha,w}\rangle} = \overline{f(w)K_\alpha(w, z)} = \overline{f(w)}K_\alpha(z, w) \tag{2.9}$$

and thus

$$H_f(K_{\alpha,w})(z) = \overline{(f(z) - f(w))}K_\alpha(z, w). \tag{2.10}$$

Since $g = \int g(w) K_{\alpha,w}\, d\mu_\alpha(w)$ by (2.6), we have, at least formally,

$$\begin{aligned} H_f g &= \int g(w) H_f(K_{\alpha,w})(z)\, d\mu_\alpha(w) = \int (\bar{f}(z) - \bar{f}(w)) K_\alpha(z, w) g(w)\, d\mu_\alpha(w) \\ &= \frac{\alpha+1}{\beta+1} \int (\bar{f}(z) - \bar{f}(w))(1-z\bar{w})^{-\alpha-2}(1-|w|^2)^{\alpha-\beta} g(w)\, d\mu_\beta(w). \end{aligned} \tag{2.11}$$

In fact, it is easily seen that, for each fixed z, and with P_α in the definition (1.2) extended by (2.7) to $L^1(\mu_\alpha)$, both the left and right hand sides of (2.11) are continuous functionals of $g \in A^\alpha$ and that they coincide when $g = K_{\alpha,\zeta}$ for some $\zeta \in D$. Consequently (2.11) holds for all $g \in A^\alpha$, and in particular for $g \in H^\infty$ (in which case the original definition of P_α suffices).

We have thus represented H_f as an integral operator. There are, however, many other representations that give the same result for analytic g; the difference is seen if we extend them to non-analytic g. In fact, the integrals in (2.11) are well defined for any $g\in L^2(\mu_\alpha)$, and they vanish for $g\in A^{\alpha\perp}$; consequently they define the operator $H_f P_\alpha$ on $L^2(\mu_\alpha)$. For us, it would be more natural to use the integral representation corresponding to the operator $H_f P_\beta$ (since it is equivalent to study H_f on A^β and $H_f P_\beta$ on $L^2(\mu_\beta)$. Unfortunately, we do not know any simple expression for the kernel of this operator, so we will instead use (2.11) or (2.14) below (the latter corresponds to $H_f P_{\alpha+1}$). We will see in Section 6 that, when α, β and f are appropriate, (2.11) defines an integral operator mapping $L^2(\mu_\beta)$ into $L^2(\mu_\alpha)$ for most values of α and β, but for certain (small) values we need the alternative (2.14) which always defines a bounded operator on $L^2(\mu_\beta)$.

We derive this alternative expression as follows. We observe that

$$(2.12)\quad \bar{w}\frac{\partial}{\partial\bar{w}}K_\alpha(z,w) = \bar{w}(\alpha+2)z(1-z\bar{w})^{-\alpha-3} = (\alpha+2)\big(K_{\alpha+1}(z,w)-K_\alpha(z,w)\big).$$

Hence, differentiating (2.10) with respect to $\bar{w}$,

$$(2.13)\quad \begin{aligned} H_f(K_{\alpha+1,w})(z) &= H_f(K_{\alpha,w})(z)+\frac{1}{\alpha+2}\bar{w}\frac{\partial}{\partial\bar{w}}H_f(K_{\alpha,w})(z) \\ &= \big(\bar{f}(z)-\bar{f}(w)\big)K_{\alpha+1}(z,w)-\frac{1}{\alpha+2}\bar{w}\overline{f'(w)}K_\alpha(z,w). \end{aligned}$$

An argument similar to the one after (2.11) shows that, e.g. for $g\in A^\alpha$,

$$(2.14)\quad \begin{aligned} H_f g(z) &= \int g(w)H_f(K_{\alpha+1,w})(z)\,d\mu_{\alpha+1}(w) \\ &= \int \big(\overline{f(z)}-\overline{f(w)}-(\alpha+2)^{-1}(1-z\bar{w})\bar{w}\overline{f'(w)}\big)(1-z\bar{w})^{-\alpha-3}g(w)\,d\mu_{\alpha+1}(w) \\ &= (\beta+1)^{-1}\int \big((\alpha+2)(\overline{f(z)}-\overline{f(w)})-(1-zw)\bar{w}\overline{f'(w)}\big)(1-z\bar{w})^{-\alpha-3}(1-|w|^2)^{\alpha+1-\beta} \\ &\qquad \times g(w)\,d\mu_\beta(w). \end{aligned}$$

Next we consider the action of Möbius transformations on the Hankel operators. Let φ be a Möbius function and define the operator V_φ^α by

$$(2.15)\quad V_\varphi^\alpha g(z) = g\circ\varphi(z)\big(\varphi'(z)\big)^{\alpha/2+1}.$$

Then V_φ^α is an isometry of $L^2(\mu_\alpha)$ onto itself which maps A^α onto itself. Consequently $P_\alpha V_\varphi^\alpha = V_\varphi^\alpha P_\alpha$. This yields, if $M_{\varphi'}^s$ denotes the multiplication operator $g\to(\varphi')^s g$,

$$(2.16)\quad \begin{aligned} V_\varphi^\alpha H_f(g) &= (I-P_\alpha)V_\varphi^\alpha(\bar{f}g) = (I-P_\alpha)\big(\bar{f}\circ\varphi g\circ\varphi(\varphi)^{\alpha/2+1}\big) \\ &= (I-P_\alpha)\big(\overline{f\circ\varphi}(\varphi')^{(\alpha-\beta)/2}V_\varphi^\beta(g)\big) = H_{f\circ\varphi}M_{\varphi'}^{(\alpha-\beta)/2}V_\varphi^\beta(g). \end{aligned}$$

Hence, as operators from A^β to $L^2(\mu_\alpha)$,

(2.17) $$H_f \text{ is unitarily equivalent to } H_{f\circ\varphi}M_{\varphi'}^{(\alpha-\beta)/2}.$$

Since φ' and $(\varphi')^{-1}$ are bounded, $M_{\varphi'}^{(\alpha-\beta)/2}$ is invertible and we conclude that

(2.18) $$H_f\in S_p \quad \text{iff} \quad H_{f\circ\varphi}\in S_p,$$

although the norms in general are different. (It is obvious that the situation is simpler when $\alpha=\beta$, see Arazy, Fisher and Peetre (1986), where the Möbius invariance is effectively exploited.)

We will also perform some computations with the monomials $\{z^n\}$. We set $\gamma_{n,\alpha}=\|z^n\|_\alpha$, and find explicitly

(2.19) $$\gamma_{n,\alpha}^2=\|z^n\|_\alpha^2=\frac{\alpha+1}{\pi}\int|z|^{2n}(1-|z|^2)^\alpha dm(z)$$

$$=(\alpha+1)\int x^n(1-x)^\alpha dx=\frac{\Gamma(n+1)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}\asymp(n+1)^{-\alpha-1}.$$

Thus

(2.20) $$\gamma_{n,\alpha}\asymp n^{-(\alpha+1)/2},\quad n\geqq 1.$$

We observe that $\{z^n/\gamma_{n,\alpha}\}$ is an ON-basis in A^α.

3. The cut-off

In order to understand the cut-off $1/p\geqq 1+1/2(\alpha-\beta)$, let us first consider the simplest non-trivial symbol, viz. z. It is easily seen that $\{H_z(z^n)\}_{n=0}^\infty$ are orthogonal, and since $z^n/\gamma_{n,\beta}$ is an ON-basis in A^β, it follows that the singular numbers of H_z are given by $\{\|H_z(z^n/\gamma_{n,\beta})\|_\alpha\}_0^\infty$, rearranged in decreasing order. Furthermore, for $n\leqq 1$,

(3.1) $$P_\alpha(\bar z z^n)=\frac{\langle\bar z z^n, z^{n-1}\rangle_\alpha}{\|z^{n-1}\|_\alpha^2}z^{n-1}=\frac{\gamma_{n,\alpha}^2}{\gamma_{n-1,\alpha}^2}z^{n-1}$$

and, using (2.15),

(3.2) $$\|H_z(z)\|_\alpha^2=\|\bar z z^n\|_\alpha^2-\|P_\alpha(\bar z z^n)\|_\alpha^2=\gamma_{n+1,\alpha}^2-\frac{\gamma_{n,\alpha}^4}{\gamma_{n-1,\alpha}^2}\asymp n^{-\alpha-3}.$$

Hence

(3.3) $$\|H_z(z^n/\gamma_{n,\beta})\|_\alpha\asymp n^{-\alpha/2-1+\beta/2}.$$

Consequently, $H_z\in S_p^{\beta\alpha}$ iff $p((\alpha-\beta)/2+1)>1$ $(0<p<\infty)$, $H_z\in S_\infty^{\beta\alpha}$ iff $(\alpha-\beta)/2+1\geqq 0$ and H_z is compact iff $(\alpha-\beta)/2+1>0$.

We may now prove parts (ii) of Theorems 1.1 and 1.2. Now assume that $1/p \geqq 1+(\alpha-\beta)/2$, with strict inequality if $p=\infty$, and that $H_f \in S_p^{\beta\alpha}$. Define for $|\zeta| \leqq 1$,

$$f_\zeta(z) = \begin{cases} (f(\zeta z)-f(0))/\zeta & \zeta \neq 0 \\ f'(0)z & \zeta = 0. \end{cases}$$

Then $\zeta \to f_\zeta$ is analytic and $\zeta \to H_{f_\zeta}$ is anti-analytic in the unit disc, and if $|\zeta|=1$ then $\|H_{f_\zeta}\|_{S_p} = \|H_f\|_{S_p} < \infty$. Since the maximum modulus principle holds in S_p, see Kalton (1986), it follows that $H_{f_\zeta} \in S_p^{\beta\alpha}$ for every $\zeta \in D$. In particular, we may take $\zeta=0$, but since we just have shown that that $H_z \notin S_p^{\beta\alpha}$, it follows that $f'(0)=0$.

We may apply the same argument to $f\circ\varphi$ for any Möbius function φ, because $H_{f\circ\varphi} \in S_p^{\beta\alpha}$ by (2.18). Hence $(f\circ\varphi)'(0)=0$ and $f'(\varphi(0))=0$. Since $\varphi(0)$ can be any point in D, f has to be constant. This proves Theorem 1.1 (ii), and Theorem 1.2 (ii) follows by the same argument.

4. The small Hankel operator

We compute the matrix elements of $\tilde{H}_f$ relative to the bases $\{z^n/\gamma_{n,\beta}\}$ in A^β and $\{\bar{z}^m/\gamma_{m,\alpha}\}$ in $\bar{A}^\alpha$:

$$\langle \tilde{H}_f z^n/\gamma_{n,\beta}, \bar{z}^m/\gamma_{m,\alpha}\rangle_\alpha = \gamma_{m,\alpha}^{-1}\gamma_{n,\beta}^{-1}\langle \bar{f}(z)z^n, \bar{z}^m\rangle_\alpha = \gamma_{m,\alpha}^{-1}\gamma_{n,\beta}^{-1}\gamma_{m+n,\alpha}^2 \overline{\hat{f}(m+n)}. \tag{4.1}$$

Hence, using (2.20), $\tilde{H}_f \in S_p^{\beta\alpha}$ iff the matrix

$$\left((m+1)^{(\alpha+1)/2}(n+1)^{(\beta+1)/2}\gamma_{m+n,\alpha}^2 \hat{f}(m+n)\right)_{m,n=0}^\infty$$

defines an S_p-operator on l^2. By Peller (1982a) and (1982b) (the cases $1 \leqq p \leqq \infty$ and $0<p<1$ respectively), or Semmes (1984) $(0<p<1)$, this holds iff $\{\gamma_{k,\alpha}^2 \hat{f}(k)\}$ is the Fourier transform of a function in $B_p^{1/p+(\alpha+1)/2+(\beta+1)/2}$, i.e. iff

$$f \in B_p^{1/p+(\alpha+1)/2+(\beta+1)/2-\alpha-1}.$$

This proves Theorem 3. Theorem 4 follows by the same argument from the remarks after Proposition 1 (and after Lemma 2) in Peller (1982a).

We remark that the matrix above defines $\tilde{H}_f g$ for polynomials g for any f. $\tilde{H}_f g$ may then be defined by continuity (when $\tilde{H}_f$ is continuous), also when (1.3) is not directly applicable.

5. The case $p \leqq 1$

We will use the following atomic decomposition of B_p^s, due to Coifman and Rochberg. The case $s<0$ is a special case of Coifman and Rochberg (1980), Theorem 2 (with different notation); the general case follows by integrations (which increase s by 1 each time). Cf. also Rochberg (1985).

Lemma 0. *If $0<p\leqq 1$, $-\infty<s<\infty$ and N is large enough ($N>1/p-s$ will do), then there exists a sequence $\{\zeta_i\}\subset D$ such that every function f in B_p^s can be decomposed as a (countable) sum*

$$f(z)=\sum_i \lambda_i(1-|\zeta_i|^2)^{N+s-1/p}(1-\zeta_i z)^{-N}$$

with

$$\sum_i |\lambda_i|^p \leqq C\|f\|_{B_p^s}^p.$$

(We may take N to be a positive integer, but, in fact N may be any real number, except a non-positive integer.)

In order to prove the direct part of Theorem 1 for $p\leqq 1$, it is therefore sufficient to obtain good S_p-estimates for symbols of the form $(1-\zeta z)^{-N}$, with N large but fixed. We will do this in several steps.

Convention. In this section ζ is an arbitrary point in D. C denotes constants that can be chosen independently of ζ (although they may depend on α, β, p, N, etc.).

Let M_ζ^s $(-\infty<s<\infty)$ denote the multiplication operator

$$M_\zeta^s g(z)=(1-\zeta z)^{-s}g(z). \tag{5.1}$$

Lemma 1. *If $0<p\leqq 2$, $\alpha-\beta>2/p$, $s<\frac{1}{2}[\alpha-\beta]$ and $\alpha-\beta$ is not an integer, then*

$$\|M_\zeta^s\|_{S_p(A^\beta,A^\alpha)}\leqq C. \tag{5.2}$$

Proof. Suppose first that $s<1/2$. Then

$$\|M^s(z_\zeta^n)\|_\alpha^2=\int|1-\zeta z|^{-2s}|z|^{2n}d\mu_\alpha(z)\leqq C\int_0^1 r^{2n}(1-r)^\alpha dr\leqq C(n+1)^{-\alpha-1}.$$

Consequently, using (2.20), $\|M_\zeta^s(z/\gamma_{n,\beta})\|_\alpha\leqq C(n+1)^{-(\alpha-\beta)/2}$. Since $p(\alpha-\beta)/2>1$, (5.2) now follows by (2.3).

In general, let $m=[\alpha-\beta]\geqq 1$. We may assume that $p\leqq 2/m$ (otherwise we replace p by $2/m$). If $\alpha(i)=\beta+\dfrac{i}{m}(\alpha-\beta)$, the case just proved shows that

$$\|M_\zeta^{s/m}\|_{S_{mp}(A^{\alpha(i)},A^{\alpha(i+1)})}\leqq C,\quad i=0,1,\ldots,\quad m-1,$$

and (5.2) follows by the Schatten—Hölder inequality (2.2). □

Remark. The conditions in the lemma are not sharp. It is easily seen that $s=\frac{1}{2}[\alpha-\beta]$ will do as well; we guess that $s<\frac{1}{2}(\alpha-\beta)$ is enough.

Lemma 2. *If* $\alpha-\beta>-1$, $2s<2\alpha-\beta+3$ *and* $N\geqq\alpha+3$, *then*

$$\|H_{(1-\zeta z)^N} M_\zeta^s\|_{S_2^{\beta\alpha}} \leqq C. \tag{5.3}$$

Proof. Let $f(z)=(1-\zeta z)^N$. If ξ is any point on the line between z and w, then $|f'(\xi)| = N|\zeta||1-\zeta\xi|^{N-1} \leqq N(|1-\zeta w|+|\xi-w|)^{N-1} \leqq C|1-\zeta w|^{N-1}+C|z-w|^{N-1}$. Consequently,

$$\begin{aligned}|f(z)-f(w)| &\leqq C|z-w||1-\zeta w|^{N-1}+C|z-w|^N \\ &\leqq C|1-z\bar{w}|(|1-\zeta w|^{N-1}+|1-z\bar{w}|^{N-1}).\end{aligned} \tag{5.4}$$

We use the kernel in (2.14) (the kernel in (2.11) can often be used, but not always), and recall that the integrals in (2.14) define an extension $H_f P_{\alpha+1}$ of H_f to $L^2(\mu_\beta)$. Consequently we can estimate the Hilbert—Schmidt norm by (observing that $s<\alpha+2\leqq N-1$, and using (2.8) in the last inequality)

$$\|H_{(1-\zeta z)^N} M_\zeta^s\|_{S_2^{\beta\alpha}}^2 \leqq \|H_f P_{\alpha+1} M_\zeta^s\|^2_{S^2(L^2(\mu_\beta),L^2(\mu_\alpha))} = (\beta+1)^{-2}$$

$$\times\iint\left|\frac{(\alpha+2)\overline{(f(z)}-\overline{f(w))}-(1-z\bar{w})\bar{w}\overline{f'(w)}}{(1-z\bar{w})^{\alpha+3}}(1-|w|^2)^{\alpha+1-\beta}(1-\zeta w)^{-s}\right|^2 d\mu_\beta(w)d\mu_\alpha(z)$$

$$\leqq C\iint\frac{|1-z\bar{w}|^2(|1-\zeta w|^{2N-2}+|1-z\bar{w}|^{2N-2})}{|1-z\bar{w}|^{2\alpha+6}}|1-\zeta w|^{-2s}(1-|zw|^2)^{2\alpha+2-\beta}dm(w)d\mu_\alpha(z)$$

$$\leqq C\iint |1-z\bar{w}|^{-2\alpha-4}(1-|w|^2)^{2\alpha+2-\beta}d\mu_\alpha(z)\,dm(w)$$

$$+C\iint|1-\zeta w|^{-2s}(1-|w|^2)^{2\alpha+2-\beta}dm(w)d\mu_\alpha(z)$$

$$\leqq C\int(1-|w|^2)^{\alpha-\beta}dm(w)+C\int(1-|w|)^{2\alpha+2-\beta-2s}dm(w) = C. \quad \square$$

Lemma 3. *If* $0<p\leqq 1$, $\alpha-\beta>2/p-2$ *and* $N\geqq\alpha+3$, *then*

$$\|H_{(1-z)^N} M_\zeta^{\alpha-\beta}\|_{S_p^{\beta\alpha}} \leqq C.$$

Proof. Define q by $1/q=1/p-1/2\geqq 1/2$. Choose γ such that $\alpha+1>\gamma>\beta+2/q$, $\gamma>\alpha-\beta$ and $\gamma-\beta$ is not an integer. Let $s=\frac{1}{2}(\gamma-\beta)-1<\frac{1}{2}[\gamma-\beta]$. Then, by Lemma 1,

$$\|M_\zeta^s\|_{S_q(A^\beta,A^\gamma)} \leqq C.$$

Since $\alpha-\gamma>-1$ and $2(\alpha-\beta-s)=2\alpha-2\beta-\gamma+\beta+2<2\alpha-\gamma+3$, we have by Lemma 2 also

$$\|H_{(1-\zeta z)^N} M_\zeta^{\alpha-\beta-s}\|_{S_2(A^\gamma,L^2(\mu_\alpha))} \leqq C.$$

Consequently the lemma follows by (2.2). $\square$

Lemma 4. *If* $0<p\leqq 1$, $\alpha-\beta>2/p-2$ *and* $N\geqq\alpha+3$, *then*

$$\|H_{(1-\bar\zeta z)^{-N}}\|_{S_p^{\beta\alpha}} \leqq C(1-|\zeta|^2)^{(\alpha-\beta)/2-N}.$$

Proof. Let $\varphi(z)=\dfrac{\zeta-z}{1-\bar\zeta z}$. Then

$$1-\bar\zeta\varphi(z)=\frac{1-|\zeta|^2}{1-\bar\zeta z}, \quad\text{and}\quad \varphi'(z)=-\frac{1-|\zeta|^2}{(1-\bar\zeta z)^2}.$$

By (2.17), $H_{(1-\bar\zeta z)^{-N}}$ is unitarily equivalent to

$$H_{((1-|\zeta|^2)/(1-\bar\zeta z))^{-N}}(1-|\zeta|^2)^{(\alpha-\beta)/2}M_\zeta^{\alpha-\beta}=(1-|\zeta|^2)^{(\alpha-\beta)/2-N}H_{(1-\bar\zeta z)^N}M_\zeta^{\alpha-\beta},$$

and the lemma follows by Lemma 3. □

Lemmas 0 (with $s=1/p+(\beta-\alpha)/2$) and 4 together with (2.1) now yield (1.6), and thus Theorem 1, for $p\leqq 1$.

6. The case $p=\infty$

It is convenient to prove a more general result on boundedness of H_f on the L^q-analogues of the spaces $L^2(\mu_\alpha)$ defined in Section 1. It will, however, be convenient to use a different parametrization, whence we define, for $-\infty<s<\infty$ and $1\leqq p\leqq\infty$,

$$L_p^s=\{f\colon\ (1-|z|^2)^{-s}f(z)\in L^p((1-|z|^2)^{-1}dm(z))\} \tag{6.1}$$

Thus $L^2(\mu_\alpha)=L_2^{-(\alpha+1)/2}$. We have chosen the notation such that, if $s<0$, then B_p^s is the subspace of analytic functions in L_p^s. We begin by studying the integral operator (2.11).

Lemma 5. *Suppose that* $\alpha>-1$, $s<1$, $-1<\gamma<\alpha$ *and* $-1<\gamma+s<\alpha$. *Then, for any analytic f,*

$$\int_D\frac{|f(z)-f(w)|}{|1-z\bar w|^{\alpha-2}}(1-|z|^2)^\gamma dm(z)\leqq C(1-|w|^2)^{\gamma-\alpha+s}\|f\|_{B_\infty^s}. \tag{6.2}$$

Proof. We may assume that $\|f\|_{B_\infty^s}=1$. Furthermore, by symmetry, it suffices to prove (6.2) for $w=\varrho\geqq 0$. We study three cases separately. We will freely use the fact that $1-r^2=(1+r)(1-r)\asymp 1-r$, $0\leqq r<1$.

(i) $0<s<1$. By the definition of B_∞^s,

$$|f'(z)|\leqq(1-|z|^2)^{s-1}. \tag{6.3}$$

We begin by estimating $|f(z)-f(\varrho)|$, and claim that if $z=re^{i\theta}$, $-\pi<\theta\leqq\pi$ then

$$|f(z)-f(\varrho)| \leqq C((1-r)+(1-\varrho)+|\theta|)^s. \tag{6.4}$$

To see this, suppose that $r\leqq\varrho$ (the case $r\geqq\varrho$ then follows by symmetry). If $|\theta|\leqq 1-r$, then by (6.3),

$$|f(z)-f(\varrho)| \leqq |f(re^{i\theta})-f(r)|+|f(r)-f(\varrho)|$$
$$\leqq |\theta|(1-r)^{s-1}+\int_r^\varrho (1-x)^{s-1}dx \leqq (1+1/s)(1-r)^s,$$

and if $|\theta|>1-r$, then, with $a=\max(1-|\theta|, 0)$,

$$|f(z)-f(\varrho)| \leqq |f(re^{i\theta})-f(ae^{i\theta})|+|f(ae^{i\theta})-f(a)|+|f(a)-f(\varrho)|$$
$$\leqq \frac{2}{s}(1-a)^s+|\theta|(1-a)^{s-1} \leqq C|\theta|^s.$$

This proves (6.4). Furthermore, it is easy to see that

$$|1-z\varrho| \asymp (1-r)+(1-\varrho)+|\theta|. \tag{6.5}$$

Consequently, since $s-\alpha-1<-\gamma-1<0$,

$$\int_{-\pi}^{\pi} \frac{|f(re^{i\theta})-f(\varrho)|}{|1-re^{i\theta}\varrho|^{\alpha+2}}\,d\theta \leqq C\int_0^\pi ((1-r)+(1-\varrho)+\theta)^{s-\alpha-2}d\theta \tag{6.6}$$
$$\leqq C((1-r)+(1-\varrho))^{s-\alpha-1}$$

whence we achieve, substituting $r=1-x$,

$$\int_D \frac{|f(z)-f(\varrho)|}{|1-z\varrho|^{\alpha+2}}(1-|z|^2)^\gamma dm(z) \leqq C\int_0^1 (1-\varrho+x)^{s-\alpha-1}x^\gamma dx \leqq C(1-\varrho)^{s-\alpha+\gamma},$$

proving (6.2).

(ii) $s<0$. In this case $|f(z)|\leqq C(1-|z|^2)^s$. Since

$$\int_{-\pi}^{\pi} |1-re^{i\theta}\bar{w}|^{-\alpha-2}d\theta \asymp (1-r|w|)^{-\alpha-1},$$

we obtain (with $x=1-r$),

$$\int \frac{|f(z)|}{|1-z\bar{w}|^{\alpha+2}}(1-|z|^2)^\gamma dm(z) \leqq C\int_0^1 (1-|r|^2)^{s+\gamma}(1-r|w|)^{-\alpha-1}dr \leqq \tag{6.7}$$
$$\leqq C\int_0^1 x^{s+\gamma}(1-|w|+x)^{-\alpha-1}dx \leqq C(1-|w|)^{s+\gamma-\alpha}.$$

Similarly,

$$\int \frac{|f(w)|}{|1-z\bar{w}|^{\alpha+2}}(1-|z|^2)^\gamma dm(z) \leqq Cf(w)|(1-|w|)^{\gamma-\alpha} \leqq C(1-|w|)^{s+\gamma-\alpha}. \tag{6.8}$$

(iii) $s=0$. This case is similar to (i), with some logarithms entering in the computations. (Alternatively, one may interpolate between small positive and negative values of s.) □

Lemma 6. *Suppose that* $-1<\alpha<\infty$, $-\infty<s<1$. *Let* $f\in B^s_\infty$ *and define*

$$K(z,w)=\frac{\overline{f(z)}-\overline{f(w)}}{(1-z\bar{w})^{\alpha+2}}. \tag{6.9}$$

If $0<t<\alpha+1$, $0<s+t<\alpha+1$ *and* $1\leqq q\leqq\infty$, *then the mappings* $u(z)\to\int|K(z,w)|\,u(w)\,d\mu_\alpha(w)$ *and* $u(z)\to\int K(z,w)u(w)\,d\mu_\alpha(w)$ *map* L^{-s-t}_q *into* L^{-t}_q. *In particular,* H_f *then maps* B^{-s-t}_q *into* L^{-t}_q.

Proof. By interpolation, it suffices to consider the cases $q=1$ and $q=\infty$. These two cases follow easily from Lemma 5 with $\gamma=t-1$ and $\gamma=\alpha-s-t$, respectively. We omit the details. □

Taking $q=2$, $t=(\alpha+1)/2$ and $s=(\beta-\alpha)/2$, we obtain (1.6) (for $p=\infty$), provided $s<1$ and $s+t<\alpha+1$, i.e. $\beta-\alpha<2$ and $\beta-\alpha<\alpha+1$. In order to avoid the restriction $\beta-\alpha<\alpha+1$ (which is restrictive only when $\alpha<1$), we consider instead the integral operator (2.14). The following analogue of Lemmas 5 and 6 holds, and the choice $q=2$, $t=(\alpha+1)/2$ and $s=(\beta-\alpha)/2$ gives a complete proof of (1.6) with $p=\infty$.

Lemma 7. *Suppose that* $\alpha>-1$ *and* $s<1$. *Let* $f\in B^s_\infty$ *and define*

$$K(z,w)=\frac{\bar{f}(z)-\bar{f}(w)}{(1-z\bar{w})^{\alpha+3}}-(\alpha+2)^{-1}\frac{\bar{w}\overline{f'(w)}}{(1-z\bar{w})^{\alpha+2}}. \tag{6.10}$$

If $-1<\gamma<\alpha$ *and* $-1<\gamma+s<\alpha+1$, *then*

$$\int|K(z,w)|(1-|z|^2)^\gamma dm(z)\leqq C(1-|w|^2)^{\gamma-\alpha-1+s}\|f\|_{B^s_\infty}. \tag{6.11}$$

If $-1<\gamma<\alpha+1$ *and* $0<\gamma+s<\alpha+1$, *then*

$$\int|K(z,w)|(1-|w|^2)^\gamma\,dm(w)\leqq C(1-|z|^2)^{\gamma-\alpha-1+s}\|f\|_{B^s_\infty}. \tag{6.12}$$

Consequently, if $0<t<\alpha+1$, $0<s+t$ *and* $1\leqq q\leqq\infty$, *then the mappings* $u(z)\to\int|K(z,w)|\,u(w)d\mu_{\alpha+1}(w)$ *and* $u(z)\to\int K(z,w)u(w)d\mu_{\alpha+1}(w)$ *map* L^{-s-t}_q *into* L^{-t}_q. *In particular,* H_f *then maps* B^{-s-t}_q *into* L^{-t}_q.

Proof. $K(z,w)$ is defined in (6.10) as a difference of two terms; we estimate the two terms separately in (6.11) and (6.12). The estimates for the first term follows by substituting $\alpha+1$ for α in Lemma 5; the estimates for the second term follows by substituting f' for f and $s-1$ for s in (6.8) and (6.7).

The final assertions are proved as Lemma 6, using (6.11) with $\gamma=t-1$ for $q=1$, and (6.12) with $\gamma=\alpha+1-s-t$ for $q=\infty$. □

7. The case $1<p<\infty$ and compactness

We have proved (1.6) and thus Theorem 1 for $p\leqq 1$ and for $p=\infty$. The intermediate case $1<p<\infty$ follows by interpolation, but because of the cut-off we have to be a little careful.

Suppose that $\alpha, \beta>-1$, $1<p<\infty$ and $1/p<1+(\alpha-\beta)/2$. The case $\beta<\alpha$ is no problem; in general we choose γ and δ with $\beta=\gamma-2\delta/p$, $-1<\gamma<\alpha+2$ and $-1<\gamma-2\delta<\alpha$. (The reader may verify that this is possible.) Define the fractional integration I^s, for any complex s, by

$$I^s g(z)=\sum_0^\infty \hat{g}(n)(1+n)^{-s}z^n,$$

and define $T_z(f)$ to be the operator $H_f I^{\delta z}$.

Then, by (2.19), I^s is an isomorphism of A^γ onto $A^{\gamma-2\operatorname{Re} s}$ (provided $\gamma-2\operatorname{Re} s>-1$). It follows that the family $\{T_z\}$ of anti-linear mappings map $B_\infty^{(\gamma-\alpha)/2}$ into $S_\infty^{\gamma\alpha}$ when $\operatorname{Re} z=0$, and $B_1^{1+(\gamma-2\delta-\alpha)/2}$ into $S_1^{\gamma\alpha}$ when $\operatorname{Re} z=1$. By the abstract Stein interpolation theorem (see e.g. Cwikel and Janson (1984)),

$$T_{1/p} \text{ maps } B_p^{1/p+(\gamma-2\delta/p-\alpha)/2} \text{ into } S_p^{\gamma\alpha}.$$

Thus, if $f\in B_p^{1/p+(\beta-\alpha)/2}$, $H_f=T_{1/p}(f)I^{-\delta/p}\in S_p^{\beta\alpha}$.

This completes the proof of Theorem 1.

It is now easy to prove (1.7) and thus Theorem 2. Suppose that $\beta-\alpha<2$. Then, by Theorem 1, $f\to H_f$ maps $B_\infty^{(\beta-\alpha)/2}$ continuously into $S_\infty^{\beta\alpha}$, and if f is a polynomial, then H_f belongs to the closed subspace of compact operators (in fact, $H_f\in S_p$ for sufficiently large p). Consequently, $f\to H_f$ maps $b_\infty^{(\beta-\alpha)/2}$ into the space of compact operators.

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S. Janson
Matematiska institutionen
Thunbengsvägen 3
S-752 38 Uppsala
Sweden