On the regularity of difference schemes
Part II. Regularity estimates for linear and nonlinear problems

Wolfgang Hackbusch

1. Preliminaries

1.1. Discrete regularity estimate

Let L be an elliptic differential operator of second order. Usually, the differentiability of the solution u of

$$(1.1) \qquad Lu = f \quad (\Omega), \quad u|_\Gamma = 0,$$

is two orders larger than the order of differentiability of f. This property can be expressed in terms of Sobolev spaces,

$$(1.2a) \qquad \|L^{-1}\|_{H^s(\Omega)\to H^{2+s}(\Omega)} \leqq C$$

or in terms of Hölder spaces,

$$(1.2b) \qquad \|L^{-1}\|_{C^s(\bar\Omega)\to C^{2+s}(\bar\Omega)} \leqq C \qquad (s > 0, \quad s \neq \text{integer}).$$

For the notation of the various spaces and of the norm, see Section 1.3.

The discretization of the boundary value problem is written as

$$(1.3) \qquad L_h u_h = f_h,$$

where h denotes the discretization parameter (usually: grid size). Let $H_h^s(\Omega_h)$ be the discrete analogue of $H^s(\Omega)$ (derivatives replaced by differences). Then we want to prove the counterpart of (1.2a):

$$(1.4) \qquad \|L_h^{-1}\|_{H_h^s(\Omega_h)\to H_h^{2+s}(\Omega_h)} \leqq C \quad \text{uniformly in } h.$$

This inequality is called the *discrete regularity estimate*. It differs from usual stability conditions. For example, the l_2-stability of L_h is expressed by

$$\|L_h^{-1}\|_{H_h^0(\Omega_h)\to H_h^0(\Omega_h)} \leqq C \quad \text{uniformly in } h, \tag{1.5}$$

since $l_2=H_h^0(\Omega_h)$. Note that (1.4) implies stability with respect to $H_h^s(\Omega_h)$.

1.2. Results of this paper

In the recent paper [6] we proved (1.4) for $s\in(-3/2, -1/2)$. Section 2 contains quite a different technique for proving the regularity estimate (1.4) also for larger orders s. While [6] makes no use of (1.2a), the new approach does. The following general statement is proved: If the discrete regularity (1.4) holds for some s_0, if the continuous regularity estimate (1.2) is satisfied for $s\in[s_0, t]$ and if an additional consistency condition is fulfilled, then the discrete regularity (1.4) holds for $s\in[s_0, t]$, too. This theorem is not restricted to Sobolev spaces.

In Section 2.1 we consider the special case of a square $\Omega=(0, 1)\times(0,1)$. The square (or rectangle) is easier to treat since the boundary condition $u|_\Gamma=0$ requires no irregular discretization. There are some papers proving (1.4) with $s=0$ for a square (cf. Guilinger [5]) or for a similar situation (cf. Dryja [4]). Here we show H_h^4-regularity:

$$\|L_h^{-1}\|_{\hat{H}_h^2(\Omega_h)\to H_h^4(\Omega_h)} \leqq C, \tag{1.6}$$

where $\hat{H}_h^2$ differs from H_h^2 only slightly.

There are several papers on *interior* regularity, i.e. estimates of u_h in an interior region (cf. Thomée [16], Thomée and Westergren [17], Shreve [14]). [16] contains an interior Schauder estimate. But there is no paper known to the author considering the (global) discrete Hölder regularity for a square. For this reason we show $C_h^{2+\alpha}(\Omega_h)$-regularity $(0<\alpha<2,\ \alpha\neq 1)$:

$$\|L_h^{-1}\|_{\hat{C}_h^\alpha(\Omega_h)\to C_h^{2+\alpha}(\Omega_h)} \leqq C, \tag{1.7}$$

where $\hat{C}_h^\alpha$ is a modification of $C_h^\alpha(\Omega_h)$.

An arbitrary region Ω requires irregular discretizations of the boundary condition. In Section 2.4 we analyse the Shortley—Weller scheme and the difference method with composed meshes.

Section 3 contains some results for the nonlinear problem $\mathscr{L}(u)=0$. Let $\mathscr{L}_h(u_h)=0$ be its discretization. We show that $u\in H^t(\Omega)$ [or $u\in C^t(\bar{\Omega})$] implies that u_h is bounded in $H_h^t(\Omega_h)$ [or $C_h^t(\bar{\Omega})$, respectively] uniformly with respect to h, provided certain discrete regularity estimates hold for the linearized scheme. Our approach is different from D'jakonov's method [3], but similar to the technique of Lapin [9]. Two examples are discussed. The first one contains a Schauder estimate of the discrete solution. The second one is Lapin's problem. We show the same results under weaker assumptions.

1.3. Notation

$W^{m,p}(\Omega)$ ($m \geqq 0$ integer, $1 \leqq p \leqq \infty$, $\Omega \subset \mathbf{R}^d$) denotes the space of functions on Ω with all derivatives of order $\leqq m$ in $L^p(\Omega)$. Its norm is $\sum_{|\alpha| \leqq m} \|D^\alpha u\|_{L^p(\Omega)}$, where α is a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \geqq 0$, and

$$|\alpha| = \alpha_1 + \ldots + \alpha_d, \quad D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}).$$

For $p=2$ we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. $H^s(\Omega)$ for real $s \geqq 0$ is introduced, e.g., in [10]. $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm of $H^s(\Omega)$.

$C^\lambda(\bar{\Omega})$ $(0 < \lambda < 1)$ is the space of functions that are Hölder continuous with exponent λ. Its norm is $\|u\|_0 + |u|_\lambda$, where

$$\|u\|_0 = \sup \{|u(x)| : x \in \Omega\},$$

$$|u|_\lambda = \sup \{|u(x) - u(x')| / \|x - x'\|^\lambda : x, x' \in \Omega, x \neq x'\}.$$

$C^{m+\lambda}(\bar{\Omega})$ ($m = 0, 1, 2, \ldots$, $0 < \lambda < 1$) contains Hölder continuously differentiable functions with finite norm $\sum_{|\alpha| \leqq m} \|D^\alpha u\|_0 + \sum_{|\alpha| = m} |D^\alpha u|_\lambda$.

The norm of a Banach space X is always denoted by $\|\cdot\|_X$ (e.g. $\|\cdot\|_{H^m(\Omega)}$). If X and Y are two Banach spaces, the canonical norm of operators $A: X \to Y$ is

$$\|A\|_{X \to Y} = \sup \{\|Ax\|_Y / \|x\|_X : 0 \neq x \in X\}.$$

Difference schemes are described by means of the translation operator T. We consider only the two dimensional case. T_x and T_y are defined by

$$(T_x u)(\xi, \eta) = u(\xi + h, \eta), \quad (T_y u)(\xi, \eta) = u(\xi, \eta + h)$$

((ξ, η): grid points, h: grid size). T^α ($\alpha = (\alpha_x, \alpha_y)$: multi-index) denotes

$$T^\alpha = T_x^{\alpha_x} T_y^{\alpha_y}.$$

The differences with respect to the x- and y-directions are

$$\partial_x = h^{-1}(T_x - I), \quad \partial_y = h^{-1}(T_y - I) \quad (I\text{: identity}).$$

Differences of higher order are

$$\partial^\alpha = \partial_x^{\alpha_x} \partial_y^{\alpha_y} \quad (\alpha = (\alpha_x, \alpha_y)).$$

The set of grid points is Ω_h, e.g., $\Omega_h = \{(x, y) \in \Omega : x/h,\ y/h \in \mathbf{Z}\}$. $\mathscr{F}(\Omega_h)$ consists of all grid functions defined on Ω_h. In Section 2.2 we also define $\bar{\Omega}_h \supset \Omega_h$. $\mathscr{F}_0(\bar{\Omega}_h)$ is the set of grid functions u_h defined on $\bar{\Omega}_h$ with $u_h(x, y) = 0$ for $(x, y) \in \bar{\Omega}_h \setminus \Omega_h$.

2. Regularity of discrete linear boundary value problems

2.1. A general theorem

Let

(2.1) $$Lu = f \qquad (u \in X^0, f \in Y^0)$$

be a boundary value problem. Either L is a differential operator and the homogeneous boundary condition of u is incorporated into the definition of the Banach space (cf. (1.1)), or (2.1) represents the differential equation $L^\Omega u = f^\Omega$ and the boundary condition $L^\Gamma u = f^\Gamma$.

Usually, there exists a *scale* of Banach spaces $X^s, Y^s\ (s \in I)$ with $X^t \subset X^s$, $Y^t \subset Y^s$ for $t \geqq s$ so that

(2.2a) $$L\colon X^s \to Y^s \text{ is bounded for } s \in I.$$

Under suitable conditions L maps X^s *onto* Y^s:

(2.2b) $$L^{-1}\colon Y^s \to X^s \text{ is bounded for } s \in I.$$

This is the continuous regularity. Special examples are (1.2a, b): $X^s = H^{s+2}(\Omega) \cap H_0^1(\Omega)$, $Y^s = H^s(\Omega)$ and $X^s = C^{2+s}(\bar{\Omega}) \cap H_0^1(\Omega)$, $Y^s = C^s(\bar{\Omega})$, respectively. In the second case the index set I must contain no integers. For a proof of (1.2a, b) see Lions and Magenes [10] and Schauder [13] or Miranda [12].

Discretize the boundary value problem (2.1) by

(2.3) $$L_h u_h = f_h \quad (h \in H),$$

where the discretization parameter h varies in the set $H \subset (0, \infty)$ with $0 \in \bar{H}$. Eq. (2.3) may be a difference scheme or a finite element discretization. The discrete functions u_h and f_h of (2.3) belong to some vector spaces (e.g., $u_h \in \mathscr{F}_0(\bar{\Omega}_h)$, $f_h \in \mathscr{F}(\Omega_h)$, cf. Section 1.3). Endowing these vector spaces with discrete counterparts of the norm of X^s and Y^s, respectively, we obtain two scales of discrete function spaces X_h^s, Y_h^s with

$$\|\cdot\|_{X_h^s} \leqq C \|\cdot\|_{X_h^t}, \quad \|\cdot\|_{Y_h^s} \leqq C \|\cdot\|_{Y_h^t} \quad (s, t \in I, \quad s \leqq t, h \in H).$$

The discrete *regularity estimate* is

$$\|L_h^{-1}\|_{Y_h^s \to X_h^s} \leqq C \qquad \text{for all } h \in H, \tag{2.4}$$

where C is a generic constant independent of h.

The *inverse estimate* allows us to estimate finer norms by means of coarser norms:

$$\|\cdot\|_{X_h^t} \leqq Ch^{s-t} \|\cdot\|_{X_h^s} \qquad (s \leqq t, \quad h \in H). \tag{2.5}$$

This condition implies that the sets of elements of X_h^t and X_h^s coincide.

In order to compare functions $u \in X^s$ and discrete functions $u_h \in X_h^s$ we have to introduce restrictions R_h and $\tilde{R}_h$ and a prolongation P_h:

$$R_h: X^s \to X_h^s, \quad \tilde{R}_h: Y^s \to Y_h^s, \quad P_h: Y_h^s \to Y^s.$$

Assume that R_h and P_h are bounded (uniformly with respect to $h \in H$):

$$\|R_h\|_{X^s \to X_h^s} \leqq C \quad \text{for all } h \in H, \tag{2.6a}$$

$$\|P_h\|_{Y_h^s \to Y^s} \leqq C \quad \text{for all } h \in H. \tag{2.6b}$$

The product $\tilde{R}_h P_h$ maps Y_h^s into itself. For 'smooth' functions u_h, $\tilde{R}_h P_h u_h$ should approximate u_h. More precisely, the interpolation error should satisfy

$$\|\tilde{R}_h P_h - I\|_{Y_h^t \to Y_h^s} \leqq Ch^{t-s} \qquad (0 \leqq t-s \leqq \varkappa_I, \quad h \in H), \tag{2.7}$$

where I=identity and $\varkappa_I$=order of $\tilde{R}_h P_h$. Examples of P_h, R_h, $\tilde{R}_h$ are given in the following sections.

The *consistency* of the discretization L_h can be expressed by

$$\|L_h R_h - \tilde{R}_h L\|_{X^t \to Y_h^s} \leqq Ch^{t-s} \qquad (0 \leqq t-s \leqq \varkappa_C, \quad h \in H), \tag{2.8}$$

where $\varkappa_C$ denotes the order of consistency.

Note that it suffices to prove (2.7) and (2.8) for $s=t-\varkappa_I$ and $s=t-\varkappa_C$, respectively. Then (2.7), (2.8) follow for all larger s because of (2.5).

The following theorem requires a discrete regularity estimate for L_h corresponding to the spaces X_h^0, Y_h^0, and the regularity estimate (2.2b) for the continuous operator L. Then higher discrete regularity can be proved.

Theorem 2.1. *Let* $\varkappa > 0$ *and assume*

$$I \subset [0, \infty), \quad 0 \in I, \quad I \cap [t-\varkappa, t) \neq \emptyset \quad \text{for all} \quad 0 \neq t \in I. \tag{2.9}$$

Suppose

(*i*) *discrete regularity* (2.4) *for* $s=0$,

(*ii*) *continuous regularity* (2.2b) *for all* $0 \neq s \in I$.

Assume that there are P_h, R_h, $\tilde{R}_h$ *with*

(*iii*) *estimates* (2.6a, b) *for all* $0 \neq s \in I$, *(iv) estimate* (2.7) *for all* $0 \neq t \in I$, $s \in I \cap [t-\varkappa, t)$,

(v) consistency (2.8) *for all* $0 \neq t \in I$, $s \in I \cap [t-\varkappa, t)$,

(vi) inverse estimate (2.5) *for all* $s, t \in I$, $s < t$.

Then the discrete regularity estimate (2.4) *holds for all* $s \in I$.

Proof. Split L_h^{-1} into

$$L_h^{-1} = R_h L^{-1} P_h - L_h^{-1}[(L_h R_h - \tilde{R}_h L) L^{-1} P_h + (\tilde{R}_h P_h - I)].$$

Assume (2.4) for some $s \geqq 0$. Then the following estimate holds for all $t \in I \cap [s, s+\varkappa]$. The subscripts $X_h^t \to X_h^s$, $Y_h^t \to Y_h^s$, ... are abbreviated by $t \to s$:

$$\|L_h^{-1}\|_{t\to t} \leqq \|R_h\|_{t\to t}\|L^{-1}\|_{t\to t}\|P_h\|_{t\to t} + \|I\|_{s\to t}\|L_h^{-1}\|_{s\to s}[\|L_h R_h - \tilde{R}_h L\|_{t\to s}\|L^{-1}\|_{t\to t}\|P_h\|_{t\to t}$$
$$+\|\tilde{R}_h P_h - I\|_{t\to s}] \leqq C + Ch^{s-t}[Ch^{t-s} + Ch^{t-s}] \leqq C'.$$

This proves (2.4) for $I \cap [s, s+\varkappa]$. The case of general $s \in I$ follows by induction. ∎

The regularity (2.4) is a special kind of stability. Together with the consistency we obtain the following convergence estimate.

Corollary 2.1. *Let* $\gamma \leqq \varkappa_C$ *(cf.* (2.8)*). Under the conditions of Theorem 2.1 and for a right-hand side* f_h *in* (2.3) *with*

$$\|f_h - \tilde{R}_h f\|_{Y_h^s} \leqq Ch^\gamma \|f\|_{Y^{s+\gamma}} \quad (s,\ s+\gamma \in I)$$

the solution u_h *of* (2.3) *satisfies*

$$\|R_h u - u_h\|_{X_h^s} \leqq Ch^\gamma \|f\|_{Y^{s+\gamma}} \quad (s,\ s+\gamma \in I,\ u := L^{-1} f).$$

Proof. Use $R_h u - u_h = L_h^{-1}(L_h R_h - \tilde{R}_h L) L^{-1} f + L_h^{-1}(\tilde{R}_h f - f_h)$. ∎

Theorem 2.1 requires discrete regularity for $s=0$. Weakening this assumption we obtain

Corollary 2.2. *Replace assumption (i) of Theorem 2.1 by*

$$\|L_h^{-1}\|_{Y_h^0 \to X_h^{-\varepsilon}} \leqq C \qquad \textit{for all } h \in H$$

with some $\varepsilon > 0$ *and modify the assumption on I suitably. Then*

(2.10) $$\|L_h^{-1}\|_{Y_h^s \to X_h^{s-\varepsilon}} \leqq C$$

holds for all $s \in I$.

Finally we present a useful lemma about the perturbation of L_h by lower order terms.

Lemma 2.1. *Let* $\varepsilon > 0$, $\delta > 0$, $\eta := \varepsilon - \delta$. *Assume that* L_h *satisfies the discrete regularity estimate* (2.4) *for all* $s \in I = [t-\eta, t]$. *Let* l_h *be a perturbation of* L_h *with* *lower order than* L_h:

$$\|l_h\|_{X_h^s\to Y_h^{s+\delta}} \leqq C \qquad \text{for all } s+\delta\in[t-\eta,\ t+\delta].$$

Suppose that L_h+l_h *fulfils the non-optimal regularity* (2.10) *for* $s=t$. *Then* L_h+l_h *satisfies the regularity estimate* (2.4) *for* $s=t$, *too.*

We remark that $I=[t-\eta, t]$ can be replaced by $\{s=t-\eta+i\delta\in[t-\eta, t]: i \text{ integer}\}\cup\{t\}$.

Proof. By induction we show

$$\|(L_h+l_h)^{-1}\|_{Y_h^t\to X_h^{t-\varepsilon+i\delta}} \leqq C_i.$$

First observe that this holds for $i=0$ because of (2.10). Now assume the estimate is valid for some i. Using $(L_h+l_h)^{-1} = L_h^{-1}-L_h^{-1}l_h(L_h+l_h)^{-1}$ one obtains

$$\|(L_h+l_h)^{-1}\|_{t\to t-\varepsilon+(i+1)\delta} \leqq C\|L_h^{-1}\|_{t\to t}$$

$$+\|L_h^{-1}\|_{t-\varepsilon+(i+1)\delta\to t-\varepsilon+(i+1)\delta}\|l_h\|_{t-\varepsilon+i\delta\to t-\varepsilon+(i+1)\delta}\|(L_h+l_h)^{-1}\|_{t\to t-\varepsilon+i\delta} \leqq C_{i+1},$$

provided that $t-\varepsilon+(i+1)\delta<t$. After a finite number of steps $t-\varepsilon+i\delta\geqq t$ is reached and the regularity of L_h+l_h is proved. ∎

In Lemma 2.1 we needed the non-optimal regularity of L_h+l_h. This condition can be replaced by the regularity of the continuous operator $L+l$.

Lemma 2.2. *Let* $s<t$ *and assume:*

(*i*) L *and* $L+l$ *satisfy the regularity conditions* (2.2a, b) *for* s *and* t *(instead of* s *in* (2.2a, b)*)*,

(*ii*) L_h^{-1} *fulfils the regularity estimate* (2.4) *for* s *and* t *(instead of* s*)*,

(*iii*) l_h *is a term of lower order:* $\|l_h\|_{X_h^s\to Y_h^t}\leqq C$,

(*iv*) *consistency:* $\|L_hR_h-\tilde{R}_hL\|_{X^t\to Y_h^s}\leqq Ch^{t-s}$, $\|l_hR_h-\tilde{R}_hl\|_{X^t\to Y_h^s}\leqq Ch^{t-s}$,

(*v*) $f\neq 0$ *implies* $\varliminf_{h\to 0}\|\tilde{R}_hf\|_{Y_h^s}>0$ *for all* $f\in Y^s$,

(*vi*) P_h *and* $\tilde{R}_h$ *are uniformly bounded:* $\|P_h\|_{Y_h^t\to Y^t}\leqq C$, $\|\tilde{R}_h\|_{Y^s\to Y_h^s}\leqq C$,

(*vii*) *the estimate* (2.7) *holds for* $\tilde{R}_hP_h-I$,

(*viii*) *the embedding* $Y^t\subset_{\to} Y^s$ *is compact.*

Then there is h_0 *such that*

$$\|(L_h+l_h)^{-1}\|_{Y_h^r\to X_h^r} \leqq C \qquad \text{for } r=s,\ t \text{ and all } h\leqq h_0,\quad h\in H.$$

We note that the $O(h^{t-s})$ terms in (iv) and (vii) can be replaced by $o(1)$.

Proof. It suffices to prove t-regularity since as in the proof of Lemma 2.1 t-regularity implies s-regularity by using $(L_h+l_h)^{-1}=L_h^{-1}-(L_h+l_h)^{-1}l_hL_h^{-1}$.

Assume that the regularity of L_h+l_h does not hold. Then there would be a sequence $h_i\to 0$, $f_{h_i}\in X_{h_i}^t$ such that

$$\varphi_h=(L_h+l_h)L_h^{-1}f_h,\quad \|f_h\|_{Y_h^t}=1,\quad \|\varphi_h\|_{Y_h^t}\to 0\qquad (h=h_i).$$

Because of (vi) the sequence $\{P_{h_i}f_{h_i}\}$ is bounded in Y^t. By (viii) there is a subsequence $\{h_k\}$ such that $F_k:=P_{h_k}f_{h_k}$ converges in Y^s:

$$F=\lim_{k\to\infty}F_k\in Y^s.$$

The estimate

$$1=\|f_h\|_{Y_h^t}=\|\varphi_h-l_hL_h^{-1}f_h\|_{Y_h^t}\leqq\|\varphi_h\|_{Y_h^t}+C'\|f_h\|_{Y_h^s}$$

(cf. (ii), (iii)) and (vii) yield

$$\|F_k\|_{Y^s}\geqq C^{-1}\|\tilde R_hF_k\|_{Y_h^s}\geqq C^{-1}\|f_h\|_{Y_h^s}-C^{-1}\|(\tilde R_hP_h-I)f_h\|_{Y_h^s}$$

$$\geqq(CC')^{-1}(1-\|\varphi_h\|_{Y_h^t})-C''h^{t-s}\to 1/(CC')\quad\text{for } h=h_k\to 0,$$

ensuring $F\neq 0$.

In the following part we shall show $F=0$, too. This contradiction would prove the lemma. By (i) $F=0$ follows from $(L+l)L^{-1}F=0$. Hence by (v) it suffices to show $\|\tilde R_h(I+lL^{-1})F\|_{Y_h^s}\to 0$ $(h=h_k\to 0)$. Since $F_k\to F$ in Y^s, it remains to prove

$$\|\tilde R_h(I+lL^{-1})F_k\|_{Y_h^s}\to 0\quad (h=h_k\to 0).$$

But this assertion follows from

$$\tilde R_h(I+lL^{-1})F_k=\tilde R_h\{P_h(I+l_hL_h^{-1})f_h+(lL^{-1}P_h-P_hl_hL_h^{-1})f_h\}$$
$$=\tilde R_hP_h\varphi_h+\{[\tilde R_hl-l_hR_h]L^{-1}P_h+l_hL_h^{-1}[L_hR_h-\tilde R_hL]L^{-1}P_h+l_hL_h^{-1}[\tilde R_hP_h-I]$$
$$+[I-\tilde R_hP_h]l_hL_h^{-1}\}f_h\quad (h=h_k)$$

and (i-iv, vi), since the brackets [...] yield $O(h^{t-s})$. ∎

Corollary 2.3. *The condition (v) of Lemma 2.2 can be replaced by the following assumptions:*

(v_1) Y^t *dense in* Y^s,

(v_2) $\|f_h\|_{Y_h^s}\geqq\delta\|P_hf_h\|_{Y^s}$, $\delta>0$, *for all* $f_h\in Y_h^s$,

(v_3) $\|P_h\tilde R_h-I\|_{Y^t\to Y^s}\leqq Ch^{t-s}$ *(even* $o(1)$ *suffices).*

Proof. Choose $\tilde f \in Y^t$ such that $\| f-\tilde f\|_{Y^s} \leqq \varepsilon := \delta \|f\|_{Y^s}/[2(\delta+C)]$. Then one concludes from (v_1, v_2, v_3, vi) that

$$\|\tilde R_h f\|_{Y_h^s} \geqq \|\tilde R_h \tilde f\|_{Y_h^s} - \|\tilde R_h (f-\tilde f)\|_{Y_h^s} \geqq \delta \|P_h \tilde R_h \tilde f\|_{Y^s} - C\varepsilon$$

$$\geqq \delta \|\tilde f\|_{Y^s} - \|(P_h \tilde R_h - I)\tilde f\|_{Y^s} - C\varepsilon \geqq \delta \|f\|_{Y^s} - (\delta + C)\varepsilon - C' h^{t-s} \|\tilde f\|_{Y^t}$$

This estimate yields $\underline{\lim}\, \|\tilde R_h f\|_{Y_h^s} \geqq \delta \|f\|_{Y^s} - (\delta+C)\,\varepsilon = \dfrac{1}{2}\delta \|f\|_{Y^s} > 0.$ ∎

Another formulation of Lemma 2.2 is given in [18].

2.2. Difference scheme in a square

We start with the simple case of the square $\Omega = (0, 1)\times(0, 1)$. Let $h = 1/N$ and define

$$\Omega_h = \{(x, y) \in \Omega \colon x/h,\ y/h \in \mathbf{Z}\},\quad \bar\Omega_h = \{(x, y) \in \bar\Omega \colon x/h,\ y/h \in \mathbf{Z}\}.$$

Denote the grid functions defined on Ω_h by $\mathscr{F}(\Omega_h)$, and by $\mathscr{F}_0(\bar\Omega_h)$ the set of grid functions on $\bar\Omega_h$ satisfying the boundary condition: $u_h(x, y) = 0$ for $(x, y) \in \bar\Omega_h \setminus \Omega_h$.

Let L be the differential operator

$$L = a\partial^2/\partial x^2 + b\partial^2/\partial y^2 + c\partial/\partial x + d\partial/\partial y + e \tag{2.11}$$

with variable coefficients satisfying

$$\begin{gathered} a,\ b,\ c,\ d,\ e \in W^{2,\infty}(\Omega) \\ a(x, y) \geqq \varepsilon > 0,\quad b(x, y) \geqq \varepsilon > 0 \quad \text{for all } (x, y) \in \Omega. \end{gathered} \tag{2.12}$$

The boundary value problem is (1.1): $Lu = f$ (Ω), $u|_\Gamma = 0$. Therefore, we choose the following spaces:

$$X^s = \begin{cases} H_0^{1+s}(\Omega) & \text{for } s \in [-1, 0]\ s \neq -1/2, \\ H^{1+s}(\Omega) \cap H_0^1(\Omega) & \text{for } s \geqq 0 \end{cases}$$

$$Y^s = \begin{cases} \text{dual of } X^{-s} & \text{for } s \in [-1,\ -1/2), \\ H^{s-1}(\Omega) & \text{for } s \in (-1/2,\ 2),\ s \neq 1/2, \\ \{f \in H^{s-1}(\Omega) \colon f(0, 0) = f(0, 1) = f(1, 0) = f(1, 1) = 0\} & \text{for } s \in (2, 3]. \end{cases}$$

For the exceptional value $s = 2$ we define Y^s by interpolation: $Y^2 = [Y^3, Y^1]_{1/2}$ (cf. [10]).

Note that $H_0^t(\Omega) = H^t(\Omega)$ for $t \in [0, 1/2]$ and $H^0(\Omega) = L^2(\Omega)$.

Lemma 2.3. *Assume that* $\lambda = 0$ *is not an eigenvalue of* L. *Then* (2.2a) *and the continuous regularity* (2.2b) *hold for* $s \in I := [-1, 3] \setminus \{-1/2, 1/2\}$.

Proof. For $|s|\leqq 1$, $|s|\neq 1/2$ use the result of Kadlec [8] and interpolation. The proof for $s>1, s\in I$ is given in the appendix of [7]. ∎

Discretize (1.1) by $L_h u_h = f_h$ with

$$\text{(2.13)}\quad L_h = aT_x^{-1}\partial_x^2 + bT_y^{-1}\partial_y^2 + \frac{c}{2}(I+T_x^{-1})\partial_x + \frac{d}{2}(I+T_y^{-1})\partial_y + e \quad (h^{-1}\in\mathbf{Z}).$$

X_h^s and Y_h^s are the vector spaces $\mathscr{F}_0(\bar{\Omega}_h)$ and $\mathscr{F}(\Omega_h)$, respectively. For simplicity we define the norms only for the integers $s=k\in\{0,1,2,3\}$. We denote $[\sum_{|\alpha|=j}\sum_P|\partial^\alpha g_h|^2]^{1/2}$ by $|g_h|_{j,S_h}$, where $\partial^\alpha g_h(P)$ involves only values of g_h belonging to S_h. Set

$$\|u_h\|_{X_h^k} = \left[\sum_{j=0}^{k+1}|u_h|_{j,\bar{\Omega}_h}^2\right]^{1/2} \qquad (k=0,1,\ldots),$$

$$\|f_h\|_{Y_h^k} = \left[\sum_{j=0}^{k-1}|f_h|_{j,\Omega_h}^2\right]^{1/2} \qquad (k=1),$$

$$\|f_h\|_{Y_h^0} = \sup\{h^2|\sum_{P\in\Omega_h} f_h(P)\,u_h(P)|:\ \|u_h\|_{X_h^0}=1\} \qquad (k=0).$$

For $k=3$ $f\in Y^3$ satisfies $f(0,0)=\ldots=0$. This property cannot be translated into $f_h(0,0)=\ldots=0$ since $(0,0)\notin\Omega_h$. Therefore define

$$\bar{f}(0,0) = 2f(h,h) - f(2h,2h)$$

with analogous definitions for $\bar{f}(0,1)$, $\bar{f}(1,0)$, $\bar{f}(1,1)$. Then we set

$$\|f\|_{Y_h^k} = \left[\sum_{j=0}^{k-1}|f_h|_{j,\Omega_h}^2 + h^{4-2k}(|\bar{f}(0,0)|^2+|\bar{f}(0,1)|^2+|\bar{f}(1,0)|^2+|\bar{f}(1,1)|^2)\right]^{1/2} \qquad (k=2,3).$$

Theorem 2.2. *Let L_h be the difference operator* (2.13) *in the square Ω_h with coefficients satisfying* (2.12). *Assume l_2-stability* (1.5). *Then the discrete regularity estimate* (2.4) *holds for $s=0,1,2,3$. In particular for $s=3$ one obtains* (1.6) *with $\hat{H}_h^2(\Omega_h):=Y_h^3$. The regularity can be extended to $s\in I$ (cf. Lemma 2.3) if the norms of Y_h^s, Y_h^t are suitably defined.*

Proof. Define R_h and $\tilde{R}_h$ by

$$(R_h u)(x,y) = h^{-2}\iint_{|x-\xi|,\,|y-\eta|\leqq h/2} u(\xi,n)\,d\xi\,d\eta$$

for $(x,y)\in\Omega_h$. (2.6a) holds for $s\in\{1,3\}$. The construction of prolongations P_h is described by Aubin [2]. Special care is needed to satisfy $P_h u_h=0$ at the corners of Ω. Thanks to the definition of Y_h^3 the estimates (2.6b) ($s\in\{1,3\}$) and (2.7) ($s,t\in\{0,1,3\}$, $s\leqq t\leqq s+2$) can be fulfilled. Obviously, (2.8) is valid with consistency order $\varkappa_C=2$, i.e., for $s=0, t=1$, and $s=1, t=3$. Also (2.5) is trivial. Now apply Theorem 2.1 with $I=\{0,1,3\}$, $\varkappa=2$. The regularity for $s=2$ follows by interpolation. ∎

2.3. Difference scheme in a square, continued

This section contains the proof of regularity with respect to Hölder spaces. We will use Lemma 2.1 rather than Theorem 2.1.

The following spaces X_h^s and Y_h^s correspond to $C^s(\bar{\Omega})$ with zero boundary condition and to a subspace of $C^{s-2}(\bar{\Omega})$, respectively. s varies in $I=(2,3)$. The norms are

$$\|u_h\|_{X_h^{k+\lambda}} = \sum_{|\alpha|\le k} |\partial^\alpha u_h|_{0,\bar{\Omega}_h} + \sum_{|\alpha|=k} |\partial^\alpha u_h|_{\lambda,\bar{\Omega}_h}, \quad u_h \in \mathscr{F}_0(\bar{\Omega}_h)$$

$$\|f_h\|_{Y_h^{2+\lambda}} = |f_h|_{0,\Omega_h} + |f_h|_{\lambda,\Omega_h} + h^{-\lambda}[|f_h(h,h)| + |f_h(1-h,h)| + |f(h,1-h)|$$

$$+|f(1-h,1-h)|],$$

where $|\partial^\alpha v_h|_{0,S_h}$ is the maximum of all $\partial^\alpha v_h(P)$ with P such that $(\partial^\alpha v_h)(P)$ involves only $v_h(R)$ with $R\in S_h$. $|\partial_h^\alpha v_h|_{\lambda,S_h}$ is the maximum of all $|\partial^\alpha v_h(P)-\partial^\alpha v_h(Q)|/[\text{distance }(P,Q)]^\lambda$ with P and Q as above.

We consider the same difference scheme as in Section 2.2 and show (1.7).

Theorem 2.3. *Let L_h be the scheme* (2.13) *with coefficients $a, b, c, d, e\in C^{2+\lambda}(\bar{\Omega})$, $\lambda\in(0,1)$. Assume l_2-stability* (1.5). *Then the discrete regularity estimate* (2.4) *holds with X_h^s and Y_h^s, $s=2+\lambda$, as defined above (hence* (1.7) *with $\alpha=\lambda$, $C_h^{2+\alpha}=X_h^{2+\alpha}$, $\hat{C}_h^\alpha=Y_h^{2+\alpha}$).*

Proof. (i) In the first step we show that without loss of generality the coefficients c and d may be taken to be zero. Set $l_h=\frac{1}{2}[c(I+T_x^{-1})\partial_x+d(I+T_y^{-1})\partial_y]+e-\sigma$ and $\tilde{L}_h=L_h-l_h$. For σ large enough $\tilde{L}_h$ is also l_2-stable. Let $H_h^t=Y_h^{t+1}$, $H_{0,h}^t=X_h^{t-1}$ with X_h^τ, Y_h^τ from Section 2.2. According to the comment following Theorem 2.2, the norms of X_h^τ, Y_h^τ can also be defined for nonintegers τ (cf. [6]). Then Theorem 2.2 yields

$$\|\tilde{L}_h^{-1}\|_{H_h^\lambda\to H_{0,h}^{2+\lambda}} \le C.$$

The discrete analogues of the embeddings $C^\lambda(\Omega)\hookrightarrow H^\lambda(\Omega)$, $H^{2+\lambda}(\Omega)\hookrightarrow C^{1+\lambda}(\Omega)$ are

$$\|\cdot\|_{H_h^\lambda} \le C\|\cdot\|_{Y_h^{2+\lambda}}, \quad \|\cdot\|_{X_h^{1+\lambda}} \le C\|\cdot\|_{H_{0,h}^{2+\lambda}}.$$

Combining the three inequalities we obtain

$$\|\tilde{L}_h^{-1}\|_{Y_h^{2+\lambda}\to Y_h^{1+\lambda}} \le C \qquad (\lambda=s-2).$$

Obviously, $l_h\colon X_h^{1+\lambda}\to Y_h^{2+\lambda}$ is uniformly bounded. Note that the estimate of $h^{-\lambda}[|f_h(h,h)|+\dots]$ follows from the zero boundary condition $u_h\in\mathscr{F}_0(\bar{\Omega}_h)$. Applying

Lemma 2.1 with $I=\{2+\lambda\}$, $\varepsilon=\delta=1$, one obtains that X_h^s-regularity of $\tilde{L}_h$ implies X_h^s-regularity of L_h. In the following we write L_h instead of $\tilde{L}_h$.

(ii) Define $f_h(P)=0$ at $P\in\bar{\Omega}_h\setminus\Omega_h$ and extend the function by reflection: $f_h(x,y)=-f_h(-x,y)=-f_h(x,-y)$, $f_h(1-x,y)=-f_h(1+x,y), \dots$ for $(x,y)\in\Omega_h$. Let $\hat{\Omega}_h=\{(x,y)\in(-1,2)\times(-1,2),\ x/h,\ y/h\in\mathbf{Z}\}$ be the extended domain of f_h. Obviously,

$$\|f_h\|_{C_h^\lambda(\hat{\Omega}_h)}=\|f_h\|_{C_h^\lambda(\Omega_h)} \tag{2.14}$$

holds, where $\|f_h\|_{C_h^\lambda(S_h)}=|f_h|_{0,S_h}+|f_h|_{\lambda,S_h}$. The solution u_h is to be extended in the same way, whereas the coefficients a,b are extended symmetrically: $a(-x,y)=$ $=a(x,y)$, etc. Note that $L_hu_h=f_h$ holds for the extended domain $\hat{\Omega}_h$. The interior Schauder regularity proved by Thomée [16] yields

$$\|u_h\|_{X_h^{2+\lambda}}=\|u_h\|_{C_h^{2+\lambda}(\Omega_h)}\leqq C[|u_h|_{0,\hat{\Omega}_h}+\|f_h\|_{C_h^\lambda(\hat{\Omega}_h)}]\leqq C'\|f_h\|_{C_h^\lambda(\Omega_h)} \tag{2.15}$$

thanks to (2.14) and

$$|u_h|_{0,\hat{\Omega}_h}\leqq C\|u_h\|_{H_h^{2+\lambda}(\hat{\Omega}_h)}\leqq C'\|f_h\|_{H_h^\lambda(\hat{\Omega}_h)}\leqq C''\|f_h\|_{C_h^\lambda(\hat{\Omega}_h)}.$$

Note that the needed estimate of [16] requires only $a,b\in C^\lambda(\mathbf{R}^2)$ as fulfilled in our situation.

(iii) Let $f_h\in\mathscr{F}(\Omega_h)$ and define f_h at $P\in\bar{\Omega}_h\setminus\Omega_h$ by $f_h(0,y)=f_h(h,y)$, $f_h(1,y)=$ $f_h(1-h,y), \dots$, except at the corners where we set $f_h(0,0)=f_h(1,0)=f_h(0,1)=$ $f_h(1,1)=0$. We have

$$\|f_h\|_{C_h^\lambda(\bar{\Omega}_h)}=\|f_h\|_{Y_h^{2+\lambda}}. \tag{2.16}$$

Piece-wise linear interpolation of $f_h(0,\nu h)$, $0\leqq\nu\leqq 1/h$, gives a function $g_1\in C^\lambda(I)$, $I=(0,1)$, with $\|g_1\|_{C^\lambda(I)}\leqq\|f_h\|_{C_h^\lambda(\bar{\Omega}_h)}$ and $g_1(0)=g_1(1)=0$. Extend $g_1/\alpha(0,\cdot)$ to a 2-periodic function g with $g(-t)=-g(t)$. The function

$$G(x,y)=c_0x\int_{-\infty}^{\infty}\exp(-\sqrt{1+(y-\eta)^2/x^2})\,g(\eta)\,d\eta$$

with $c_0=1/\int_{-\infty}^{\infty}\exp(-\sqrt{1+t^2})\,dt$ satisfies

$$G(0,y)=G(x,0)=G(x,1)=0,$$

$$G\in C^{2+\lambda}(\bar{\Omega}),\quad \|G\|_{C^{2+\lambda}(\Omega)}\leqq C\|g\|_{C^\lambda(\mathbf{R})}, \tag{2.17}$$

$$G_{xx}(0,y)=g(y).$$

Choose $\chi\in C^\infty(\mathbf{R})$ with $\chi(y)=1$ for $y\leqq 1/3$, $\chi(y)=0$ for $y\geqq 2/3$ and define $u_1(x,y)=G(x,y)\,\chi(x)$. Using (2.17) and

$$\|g\|_{C^\lambda(\mathbf{R})}\leqq C\|g_1\|_{C^\lambda(I)}\leqq C'\|f_h\|_{C_h^\lambda(\bar{\Omega}_h)}=C'\|f_h\|_{Y_h^{2+\lambda}}$$

we obtain

$$\|u_1\|_{C^{2+\lambda}(\Omega_h)} \leqq C\|f_h\|_{Y_h^{2+\lambda}}.$$

Since the restriction $u_{1,h}$ of u_1 to the grid points of $\bar{\Omega}_h$ belongs to $\mathscr{F}_0(\bar{\Omega}_h)$, the estimate

$$\|u_{1,h}\|_{X_h^{2+\lambda}} \leqq C\|f_h\|_{Y_h^{2+\lambda}} \tag{2.18}$$

holds. Set $f_{1,h}=L_h u_{1,h}\in\mathscr{F}(\Omega_h)$. Obviously, (2.18) implies $\|f_{1,h}\|_{Y_h^{2+\lambda}}\leqq C\|f_h\|_{Y_h^{2+\lambda}}$. In addition the third part of (2.17) proves

$$|f_{1,h}(h, \nu h)-f_h(h, \nu h)| \leqq Ch^\lambda \|f_h\|_{Y_h^{2+\lambda}}, \tag{2.19}$$

while $f_{1,h}(x, 0)=f_{1,h}(x, 1)=f_{1,h}(1, y)=0$ implies

$$|f_{1,h}(x, h)|,\quad |f_{1,h}(x, 1-h)|,\quad |f_{1,h}(1-h, y)| \leqq Ch^\lambda\|f_h\|_{Y_h^{2+\lambda}}. \tag{2.20}$$

Analogously, $f_{j,h}$ $(j=2, 3, 4)$ can be defined so that (2.19) holds for $x=1-h$ or $y=h$ or $y=1-h$, respectively. By virtue of (2.19/20) the function $f_{0,h}=f_h-\sum_{j=1}^4 f_{j,h}$ extended to $\mathscr{F}_0(\bar{\Omega}_h)$ as in (ii) satisfies

$$\|f_{0,h}\|_{C_h^\lambda(\Omega_h)} \leqq C\|f_h\|_{Y_h^{2+\lambda}}.$$

Hence, the solution of $L_h u_{0,h}=f_{0,h}$ can be estimated by

$$\|u_{0,h}\|_{X_h^{2+\lambda}} \leqq C\|f_h\|_{Y_h^{2+\lambda}}$$

(cf. (2.15)). The proof is concluded by noting that $u_h=\sum_{j=0}^4 u_{j,h}$ and using (2.18). ∎

2.4. Difference schemes in a general domain

In the following we assume $\Omega\subset\mathbf{R}^2$ to be a domain with smooth boundary. In this case the continuous regularity is well-known. However, the analysis of the difference scheme is more difficult, since the discretization is irregular at points near the boundary. We illustrate the application of Theorem 2.1 by special examples.

2.4.1. Shortley—Weller scheme

Poisson's equation $-\Delta u=f(\Omega)$, $u=0(\Gamma)$ can be discretized by the Shortley—Weller scheme (cf. [6], [11, p. 203]). $(L_h u)(P)$ is the usual five-point formula if all neighbours $(x\pm h, y)$, $(x, y\pm h)$ of $P=(x, y)$ belong to $\Omega_h=\{(x, y)\in\Omega\colon x/h,\ y/h\in\mathbf{Z}\}$. Otherwise the second derivative is discretized more generally. E.g. in the case of $(x, y)\in\Omega_h$, $(x+h, y)\in\Omega_h$, $(x-\varkappa h, y)\in\Gamma=\partial\Omega$ $(0<\varkappa\leqq 1)$ the derivative $-u_{xx}$ is approximated by

(2.21)
$$-u_{xx}(x,y)\approx h^{-2}\left[\frac{2}{\varkappa}u(x,y)-\frac{2}{\varkappa(1+\varkappa)}u(x-\varkappa h,y)-\frac{2}{1+\varkappa}u(x+h,y)\right],$$

where $u(x-\varkappa h, y)=0$ because of the boundary condition. If $P\in\Omega_h$ and $Q=P+(0,h)\in\Omega_h$ are grid points, we neglect a possible part of the boundary Γ between these points. Hence, neighbours with respect to the grid are also neighbours with respect to the discretization.

The norms of $X_h^0=H_h^1(\bar\Omega_h)$ and $X_h^1=H_h^2(\bar\Omega_h)$ must be defined carefully. If the norm of $H_h^2(\bar\Omega_h)$ also involves differences of the form (2.21), then the inverse estimate (2.5) holds with C depending on the minimum of all $\varkappa$. Since $\varkappa$ may become arbitrarily small, the inverse estimate (2.5) is not valid.

It is easy to define the norms of L_h^2 and H_h^1:

$$\|u_h\|_{L_h^2(\Omega_h)}=\{h^2\textstyle\sum_{P\in\Omega_h}|u_h(P)|^2\}^{1/2},$$

$$\|u_h\|_{H_h^1(\bar\Omega_h)}=\{\|u_h\|^2_{L_h^2(\Omega_h)}+\textstyle\sum_{P\in G_h}\sum_{i=1,2}|\partial_i u_h|^2\}^{1/2},$$

where $G_h=\{(x,y)\in\mathbf{R}^2\colon x/h,\ y/h \text{ integers}\}$ is the infinite grid. ∂_i $(i=1,2)$ are the first differences: $\partial_1=\partial_x$, $\partial_2=\partial_y$ (cf. Section 1). Here, the grid function u_h is extended by zero on $G_h\setminus\Omega_h$. The norm of $H_h^{-1}(\bar\Omega_h)$ is the dual norm

$$\|u_h\|_{H_h^{-1}(\bar\Omega_h)}=\sup\{h^2|\textstyle\sum_{P\in\Omega_h}u_h(P)\bar v_h(P)|\colon \|v_h\|_{H_h^1(\bar\Omega_h)}=1\}.$$

The extension by zero cannot be used for $H_h^2(\bar\Omega_h)$, since this space is the discrete analogue of $H^2(\Omega)\cap H_0^1(\Omega)$ and not of $H_0^2(\Omega)$. We must use differences of values at points P_i with dist $(P_i, P_j)\geqq h$ in order to satisfy the inverse estimate (2.5). Let $\bar\Omega_h$ be the set of all points $P=(x,y)$ with $P\in\Omega_h$ or $P\in\Gamma$ and either x/h or y/h being an integer. $\bar\Omega_h$ differs from Ω_h by the set

$$\Gamma_h=\bar\Omega_h\setminus\Omega_h$$

containing the intersection points of the lines $x=\nu h$ and $y=\mu h$ with the boundary Γ. $P\in\bar\Omega_h$ are the points involved in the difference formula (2.21). The second x-difference at $(x,y)\in\Omega_h$ can be defined by

$$(D_{xx}u)(x,y)$$
$$=\begin{cases} h^{-2}[u(x+h,y)-2u(x,y)+u(x-h,y)] & \text{if } (x,y),(x\pm h,y)\in\bar\Omega_h,\\ h^{-2}\left[\dfrac{2u(x+\varkappa h,y)}{(1+\varkappa)(2+\varkappa)}-\dfrac{2u(x-h,y)}{1+\varkappa}+\dfrac{2u(x-2h,y)}{2+\varkappa}\right] & \\ \qquad\qquad \text{if } (x-h,y),\ (x-2h,y)\in\bar\Omega_h,\ (x+\varkappa h,y)\in\Gamma_h \end{cases}$$

and by a similar expression in the case of $(x+h,\ y),\ (x+2h,\ y)\in\bar{\Omega}_h,\ (x-\varkappa h,\ y)\in\Gamma_h$. The distances of the points are h and $(1+\varkappa)h$ and not h and $\varkappa h$ as in (2.21). This is necessary to ensure the inverse estimate (2.5). $D_{yy}u$ is defined analogously. The description of the mixed difference $D_{xy}u$ at a point near the boundary usually involves more than four grid points. E.g., D_{xy} can be defined by a difference formula using the six grid points $(x\pm h,\ y),\ (x+h,\ y+h),\ (x,y+h),\ (x,y+2h)\in\bar{\Omega}_h,\ (x,y-\varkappa h)\in\Gamma_h$. Then the norm of $H_h^2(\bar{\Omega}_h)$ reads as

$$\|u_h\|_{H_h^2(\bar{\Omega}_h)} = \left\{\|u_h\|^2_{H_h^1(\bar{\Omega}_h)} + h^2 \sum_{P\in\Omega_h} \left(|D_{xx}u_h(P)|^2 + |D_{yy}u_h(P)|^2\right) + |D_{yx}u_h(P)|^2\right\}^{\frac{1}{2}}.$$

The following theorem establishes the H_h^2-regularity of the Shortley—Weller difference scheme.

Theorem 2.4. *Suppose that $\Omega\subset\mathbf{R}^2$ is a bounded domain with the uniform C^2-regularity property (cf.* [1, *p.* 67]*). Then the Shortley—Weller scheme satisfies the regularity estimate*

$$\|L_h^{-1}\|_{L_h^2(\Omega)\to H_h^2(\bar{\Omega}_h)} \leqq C.$$

Proof. We want to apply Theorem 2.1 with $I=\{0,1\}$, $\varkappa=1$:

$$Y_h^0 = H_h^{-1}(\bar{\Omega}_h),\quad Y_h^1 = L_h^2(\Omega_h),\quad X_h^0 = H_h^1(\bar{\Omega}_h),\quad X_h^1 = H_h^2(\bar{\Omega}_h).$$

According to the suppositions (i)—(vi) of Theorem 2.1 the proof consists of six steps.

Step 1. Discrete regularity for $s=0$. This result is contained in [6], but it can also be obtained directly by estimating the scalar product $\langle u_h, L_hu_h\rangle$. Let $L_h=L_h^x+L_h^y$, where L_h^x and L_h^y are the differences with respect to x and y. Extending u_h by zero outside of Ω_h, we obtain

$$\langle u_h, L_h^xu_h\rangle = h^2\sum_{P\in\Omega_h} u_h(P)(L_h^xu_h)(P) = h^2\sum_{P\in G_h}|\partial_xu_h|^2$$

$$+h^2\Sigma_1\partial_xu_h(Q)\left\{\left[\frac{2}{\varkappa(1+\varkappa)}-1\right]\partial_xu_h(Q)+\left[1-\frac{2}{1+\varkappa}\right]\partial_xu_h(P)\right\}+h^2\Sigma_2[\dots],$$

where the sum Σ_1 is taken over all $P\in\Omega_h$ with $P+(h,0)\in\Omega_h$ and $Q=P-(h,0)\notin\Omega_h$. Σ_2 is a similar expression for the case $P,\ P-(h,0)\in\Omega_h$ and $P+(h,0)\notin\Omega_h$. $\varkappa=\varkappa(P)\in(0,1]$ is the number defined in (2.21). The inequality $2ab\geqq-\lambda^2a^2-b^2/\lambda^2$ yields

$$h^2\Sigma_1[\dots]+h^2\Sigma_2[\dots]\geqq-\frac{1}{25}h^2\sum_{P\in G_h}|\partial_xu_h|^2.$$

This estimate and the analogous one for L_h^y imply

$$\langle u_h, L_hu_h\rangle \geqq 0.96h^2\sum_{\substack{P\in G_h\\ i=1,2}}|\partial_iu_h|^2.$$

Since Ω is bounded, the right-hand side is the square of a norm equivalent to $|\cdot|_{H_h^1(\bar\Omega_h)}$. The inequality $\langle u_h, L_h u_h\rangle \geqq c|u_h|^2_{H_h^1(\bar\Omega_h)}$ with $c>0$ for all $u_h \in H_h^1(\bar\Omega_h)$ proves the desired H_h^1-regularity.

Step 2. Continuous regularity for $s=1$. See e.g., Theorem 37,I of Miranda [12].

Step 3. A restriction satisfying $\|R_h\|_{H^2\cap H_0^1\to H_h^2} \leqq C$ has to be defined. Let $u \in H^2(\Omega)\cap H_0^1(\Omega)$. There is a continuous extension operator E: $H^2(\Omega)\to H^2(\mathbf{R}^2)$ (cf. Adams [1, p. 84]) yielding $\tilde u = Eu$. Define a provisional grid function $\tilde u_h$ by the mean value

$$\tilde u_h(P) = \int_{B_h(P)} \tilde u(x, y)\,dx\,dy \Big/ \int_{B_h(P)} dx\,dy,\ B_h(P) = \{(x, y)\colon \|P-(x, y)\| \leqq h\}$$

for $P \in \bar\Omega_h$. The construction of $\tilde u_h$ implies

$$\|\tilde u_h\|_{H_h^2(\bar\Omega_h)} \leqq C\|u\|_{H^2(\mathbf{R}^2)} \leqq C'\|u\|_{H^2(\Omega)}.$$

Unfortunately, $\tilde u_h(P)$ does *not* satisfy $\tilde u_h(P)=0$ at points $P\in\Gamma_h = \bar\Omega_h \setminus \Omega_h$ on the boundary. Therefore, $R_h u$ is the following modification of $\tilde u_h$:

$$(R_h u)(P) = \begin{cases} \tilde u_h(P) & \text{if } P\in\Omega_h\setminus\gamma_h \\ 0 & \text{if } P\in\Gamma_h \\ \text{solution of } (L_h R_h u)(P) = 0 & \text{if } P\in\gamma_h \end{cases}$$

where

$$\gamma_h = \{P\in\Omega_h\colon \text{ not all neighbours of } P \text{ belong to } \Omega_h\}$$

is the set of points near the boundary.

The difference $\delta_h = \tilde u_h - R_h u$ satisfies $\delta_h = \tilde u_h$ on Γ_h, $L_h\delta_h = L_h\tilde u_h$ on γ_h, $\delta_h = 0$ otherwise. Split δ_h into $\delta_h^1 + \delta_h^2$, where

$$\delta_h^1 = \tilde u_h(\Gamma_h),\ L_h\delta_h^1 = 0(\gamma_h),\ \delta_h^1 = \delta_h^2 = 0(\Omega_h\setminus\gamma_h),\ \delta_h^2 = 0(\Gamma_h),\ L_h\delta_h^2 = L_h\tilde u_h(\gamma_h).$$

It can be shown that $|\delta_h^1(P)| = |\tilde u_h(P)|$ is bounded by $Ch\|u\|_{H^2(B_h(P))}$ for $P\in\Gamma_h$. The strong diagonal dominance of the matrix L_h restricted to the near boundary points γ_h implies

$$\|\delta_h^1\|_{L_h^2(\Omega_h)} \leqq C\big[h^2 \textstyle\sum_{P\in\Gamma_h} |\delta_h^1(P)|^2\big]^{1/2} \leqq C' h^2 \big[\sum_{P\in\Gamma_h} \|\tilde u\|^2_{H^2(B_h(P))}\big]^{1/2} \leqq C'' h^2 \|\tilde u\|_{H^2(\mathbf{R}^2)}$$

Estimating differences by integrals of derivatives we obtain

$$|L_h\delta_h^2|(P) = |L_h\tilde u_h|(P) \leqq Ch^{-1}\|u\|_{H^2(B_{2h}(P))} \quad \text{for} \quad P\in\gamma_h.$$

The strong diagonal dominance again shows that:

$$\|\delta_h^2\|_{L_h^2(\Omega_h)} \leqq Ch^2\|(L_h\tilde{u}_h)|_{\gamma_h}\|_{L_h^2(\Omega_h)}$$

$$= Ch^2\big[\textstyle\sum_{P\in\gamma_h} h^2|L_h\tilde{u}_h(P)|^2\big]^{1/2} \leqq C'h^2\big[\textstyle\sum_{P\in\gamma_h}\|\tilde{u}\|^2_{H^2(B_{2h}(P))}\big]^{1/2} \leqq C''h^2\|\tilde{u}\|_{H^2(\mathbf{R}^2)}.$$

Hence, the grid function δ_h satisfies

$$\|\delta_h\|_{H_h^2(\bar{\Omega}_h)} \leqq Ch^{-2}\|\delta_h\|_{L_h^2(\Omega_h)} \leqq C'\|\tilde{u}\|_{H^2(\mathbf{R}^2)} \leqq C''\|u\|_{H^2(\Omega)}.$$

Here, we used the fact that the inverse estimate $\|\cdot\|_{H_h^2}\leqq Ch^{-2}\|\cdot\|_{L_h^2}$ holds because of the definition of the norm of H_h^2.

The estimates of $\tilde{u}_h$ and δ_h imply

$$\|R_hu\|_{H_h^2(\bar{\Omega}_h)} \leqq \|\tilde{u}\|_{H_h^2(\bar{\Omega}_h)}+\|\delta_h\|_{H_h^2(\bar{\Omega}_h)} \leqq C\|u\|_{H^2(\Omega)}.$$

Step 4. The estimate $\|\tilde{R}_hP_h-I\|_{L_h^2\to H_h^{-1}}\leqq Ch$ has to be proved for a suitable choice of P_h and $\tilde{R}_h$. Let P_h be the piece-wise constant prolongation $(P_hu_h)(x, y)=u_h(Q)$ with $Q=(x_Q, y_Q)$ if $x_Q-\frac{h}{2}<x\leqq x_Q+\frac{h}{2}$, $y_Q-\frac{h}{2}<y\leqq y_Q+\frac{h}{2}$, and let $\tilde{R}_hu$ be defined by

$$(\tilde{R}_hu)(P) = \begin{cases}\int_{B_{h/2}(P)} \tilde{u}(x, y)\,dx\,dy\Big/\int_{B_{h/2}(P)} dx\,dy & \text{if } P\in\Omega_h\diagdown\gamma_h. \\ 0 & \text{if } P\in\gamma_h.\end{cases}$$

where $\tilde{u}=Eu$ and $\gamma_h\subset\Omega_h$ are defined in the preceding Step 3.

Let $v_h\in H_h^1(\bar{\Omega}_h)$ and $u_h\in L^2(\Omega_h)$. Split v_h into $v_h^1+v_h^2$ with $v_h^1=v_{h|\gamma_h}$ (restriction to γ_h) and $v_h^2=v_h-v_h^1$. The definitions of P_h and $\tilde{R}_h$ yield

$$|\langle v_h, [\tilde{R}_hP_h-I]u_h\rangle| = |\langle v_h^1, [\tilde{R}_hP_h-I]u_h\rangle| = |\langle v_h^1, u_h\rangle| \leqq \|v_h^1\|_{L_h^2(\Omega_h)}\|u_h\|_{L_h^2(\Omega_h)}.$$

Using $\|v_h^1\|_{L_h^2(\Omega_h)}\leqq Ch\|v_h\|_{H_h^1(\bar{\Omega}_h)}$ (cf. [6, Lemma 2.2]) we finish the proof of the desired estimate.

Step 5. Consistency $\|L_hR_h-\tilde{R}_hL\|_{H^2\cap H_0^1\to H_h^{-1}}\leqq Ch$. Let $v_h\in H_h^1(\bar{\Omega}_h)$ and $u\in H^2(\Omega)\cap H_0^1(\Omega)$ be arbitrary, extend u to $\tilde{u}=Eu\in H^2(\mathbf{R}^2)$ and set

$$\tilde{v}_h(P) = v_h(P) \quad\text{for}\quad P\in\Omega_h\diagdown\gamma_h,\quad v_h(P) = 0 \quad\text{otherwise.}$$

The new functions satisfy

$$\|\tilde{v}_h\|_{H_h^1(G_h)} \leqq C\|v_h\|_{H_h^1(\bar{\Omega}_h)},\quad \|\tilde{u}\|_{H^2(\mathbf{R}^2)} \leqq C\|u\|_{H^2(\Omega)}.$$

Let $G_h=\{(x, y)\in\mathbf{R}^2\colon x/h,\ y/h \text{ integers}\}$ be the indefinite grid in $\mathbf{R}^2$ and define restrictions $\hat{R}_h$ and $\hat{\tilde{R}}_h$ on the grid G_h in the same way as R_h and $\tilde{R}_h$, resp., are defined in $\Omega_h\diagdown\gamma_h$. Furthermore, denote the five-point formula in G_h by $\hat{L}_h$, while $L=-\Delta$

is the negative Laplacian in $\mathbf{R}^2$. The first term of

$$\langle v_h, [L_h R_h - \tilde{R}_h L]u\rangle_{L_h^2(\Omega)} = \langle v_h, [\hat{L}_h \hat{R}_h - \hat{\tilde{R}}_h L]\hat{u}\rangle_{L_h^2(G_h)} + \langle v_h - \tilde{v}_h, [L_h R_h - \tilde{R}_h L]\tilde{u}\rangle_{L_h^2(\Omega)}$$

can be analysed by Fourier techniques yielding the bound $Ch\|v_h\|_{H_h^1}\|u\|_{H^2}$. The support of $v_h - \tilde{v}_h$ is γ_h. Since $L_h R_h \tilde{u}$ as well as $\tilde{R}_h L\tilde{u}$ vanish on γ_h, we obtain

$$\langle v_h, [L_h R_h - \tilde{R}_h L]u\rangle_{L_h^2(\Omega_h)} = \langle \tilde{v}_h, [\hat{L}_h \hat{R}_h - \hat{\tilde{R}}_h L]\hat{u}\rangle_{L_h^2(G_h)} + \langle v|_{\gamma_h'}, L_h(R_h - \tilde{R}_h)\tilde{u}\rangle_{L_h^2(\Omega_h)},$$

where $v_{h|\gamma_h'}$ is the restriction of v_h to $\gamma_h' = \{P \in \Omega_h\colon\ P \text{ neighbour of } \gamma_h\}$. By

$$\langle v_h|_{\gamma_h'}, L_h(R_h - \tilde{R}_h)\tilde{u}\rangle \leqq \|v_h|_{\gamma_h'}\|_{L_h^2}\|L_h(R_h - \tilde{R}_h)\tilde{u}\|_{L_h^2} \leqq Ch\|v_h\|_{H_h^1}\|u\|_{H^2}$$

the estimates result in

$$|\langle v_h, [L_h R_h - \tilde{R}_h L]u\rangle| \leqq Ch\|v_h\|_{H_h^1(\bar{\Omega}_h)}\|u\|_{H^2(\Omega)}.$$

Hence, the consistency condition is proved.

Step 6. The inverse estimate $\|\cdot\|_{H_h^2} \leqq Ch^{-1}\|\cdot\|_{H_h^1}$ holds by definition of the norms. Since all suppositions of Theorem 1.1 are fulfilled, the H_h^2-regularity is valid for the Shortley—Weller scheme L_h. ∎

2.4.2. Inhomogeneous boundary conditions

Discretize the boundary value problem

$$-\Delta u = f\ (\Omega), \quad u = g\ (\Gamma)$$

by the Shortley—Weller scheme with $u_h(P) = g_h(P)$ for $P \in \Gamma$. The right-hand sides f_h and g_h are obtained by suitable restrictions: $f_h = R_h^\Omega f$, $g_h = R_h^\Gamma g$. Here R_h^Γ: $H^{3/2}(\Gamma) \to H_h^{3/2}(\Gamma_h)$ can be defined as follows: $(R_h^\Gamma g)(P) = (\pi h^2)^{-1} \int_{K(P)} (Eg)(\xi, \eta)\, d\xi\, d\eta$, where $K(P) = \{Q \in \mathbf{R}^2\colon \|Q - P\| \leqq h\}$ and E: $H^{3/2}(\Gamma) \to H^2(\mathbf{R}^2)$ a suitable extension.

We define $Y_h^1 = L_h^2(\Omega_h) \times H_h^{3/2}(\Gamma_h)$, where $\Gamma_h = \bar{\Omega}_h \setminus \Omega_h$ is the set of boundary points involved in (2.21). The norm of $H_h^{3/2}(\Gamma_h)$ is

$$\|g_h\|_{H_h^{3/2}(L_h)} = \inf\{\|v_h\|_{H_h^2(\bar{\Omega}_h)}\colon\ v_{h|\Gamma_h} = g_h\}.$$

Proposition 2.1. *Let $H_h^2(\bar{\Omega}_h)$ be defined as above (without $u_h = 0$ on $\bar{\Omega}_h \setminus \Omega_h$). Then H_h^2-regularity holds for the inhomogeneous Shortley—Weller scheme:*

$$\|u_h\|_{H_h^2(\bar{\Omega}_h)} \leqq C(\|f_h\|_{L_h^2(\Omega_h)} + \|g_h\|_{H_h^{3/2}(\Gamma_h)}).$$

Proof. Choose $v_h \in H_h^2(\bar{\Omega}_h)$ with $g_h = v_{h|\Gamma_h}$ and $\|g_h\|_{H^3_{h/2}(\Gamma_h)} = \|v_h\|_{H_h^2(\bar{\Omega}_h)}$. Define w_h by $w_h(P)=v_h(P)$ except for those $P \in \Omega_h$ corresponding to irregular discretizations. Here we determine w_h from $(L_h w_h)(P)=0$. w_h satisfies $w_h = g_h(\Gamma_h)$ and $\|L_h w_h\|_{L_h^2(\Omega_h)} \leqq C\|w_h\|_{H_h^2(\bar{\Omega}_h)} \leqq C'\|v_h\|_{H_h^2(\bar{\Omega}_h)} = C'\|g_h\|_{H_h^{3/2}(\Omega_h)}$. The application of Theorem 2.4 for the right-hand side $\tilde{f}_h = f_h - L_h w_h$ yields the desired estimate. ▮

2.4.3. Discretization by composed meshes

As a last example we discuss an unusual discretization: a difference scheme on composed meshes as proposed by Starius [15]. Assume that the boundary Γ of Ω is sufficiently smooth. Let Ω_i $(i=1,2,3)$ be subregions of Ω with boundaries Γ_i (cf. Fig. 1). Assume that a given

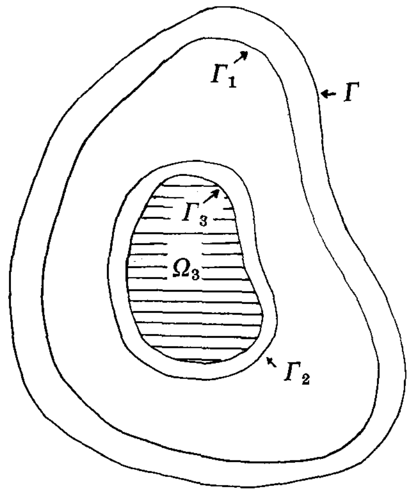

Fig. 1

transformation maps the annular strip $\Omega \setminus \Omega_3$ between Γ_3 and Γ into a rectangle R. The inverse transformation maps a regular square grid of R into a curved grid Ω_h^A of $\Omega \setminus \Omega_3$. Let $\Omega_h^B \subset \Omega$ be a usual square grid. The boundary value problem (1.1),

$$Lu = f \ (\Omega), \qquad u|_\Gamma = 0 \ (\Gamma)$$

with a second order differential operator L with smooth coefficients (a more general boundary condition $Bu|_\Gamma = g$ is also possible) is discretized by

(2.22a) $$L_h^A u_h^A = f_h^A \ (\mathring{\Omega}_h^A), \quad u^A_{h|\Gamma \cap \Omega_h^A} = 0$$

on the curved mesh Ω_h^A and by

(2.22b) $$L_h^B u_h^B = f_h^B \quad (\mathring{\Omega}_h^B)$$

on the square grid Ω_h^B. Here, $\mathring{\Omega}_h^A$ and $\mathring{\Omega}_h^B$ denote the *interior* points of Ω_h^A, Ω_h^B: P is an interior point of Ω_h^A, if (2.22a) evaluated at P involves only $u_h^A(Q)$ with $Q\in\Omega_h^A$. The non-interior points of $\Omega_h^A\setminus\mathring{\Omega}_h^A$ belong either to Γ (then $u_h^A=0$ by (2.22a)) or to Γ_3. Let Π^B be a prolongation (interpolation) of grid functions defined on Ω_h^B to functions defined on Ω. Set

(2.22c) $$u_h^A(P) = (\Pi^B u_h^B)(P) \quad \text{for} \quad P\in\Omega_h^A\cap\Gamma_3.$$

Similarly define

(2.22d) $$u_h^B(P) = (\Pi^A u_h^A)(P) \quad \text{for} \quad P\in\Omega_h^B\setminus\mathring{\Omega}_h^B.$$

We assume that (2.22c) involves only values of $u_h^B(Q)$ for $Q\in\Omega_2\supset\Omega_3$, while (2.22d) involves only $u_h^A(Q)$ for $Q\in\Omega\setminus\Omega_1$.

By (2.22a—d) the solution $u_h=(u_h^A, u_h^B)$ is determined. For the sake of consistency we define f_h^A from f_h^B by:

$$f_h^A = \Pi^B f^B_{h|\Omega_h^A}.$$

The discrete spaces $L_h^2(\Omega_h^B)$ and $H_h^s(\Omega_h^B\cap\Omega_2)$ are defined as usual. For the definition of $H_h^s(\Omega_h^A)$, use the differences with respect to the transformed (rectangular) grid.

Proposition 2.2. *Let* $s\geqq 0$. *Assume that*

(i) the scheme (2.22a—d) *is* l_2*-stable, i.e.,*

$$\|u_h^A\|_{L_h^2(\Omega_h^A)}+\|u_h^B\|_{L_h^2(\Omega_h^B)} \leqq C\|f_h^B\|_{L_h^2(L_h^B)},$$

(ii) L_h^A *and* L_h^B *are elliptic (cf.* [17]*),*

(iii) distance $(\Gamma_1, \Gamma_2)>\varepsilon$, *with* ε *independent of* h,

(iv) the interpolation Π^B *is sufficiently accurate,*

(v) the coefficients of L, L_h^A, L_h^B, *the boundary* Γ, *and the transformation of the strip* $\Omega\setminus\Omega_3$ *into* R *are smooth enough.*

Then regularity holds in the following form:

$$\|u_h^A\|_{H_h^{s+2}(\Omega_h^A)}+\|u_h^B\|_{H_h^{s+2}(\Omega_h^B\cap\Omega_2)} \leqq C\|f_h^B\|_{H_h^s(\Omega_h^B)}.$$

Note that the regions Ω_h^A and $\Omega_h^B\cap\Omega_2$ overlap.

Proof. The interior regularity of L_h^B yields

$$\|u_h^B\|_{H_h^{s+2}(\Omega_h^B\cap\Omega_2)} \leqq C'(\|u_h^B\|_{L_h^2(\Omega_h^B)}+\|f_h^B\|_{H_h^s(\Omega_h^B)}) \leqq C\|f_h^B\|_{H_h^s(\Omega_h^B)}.$$

(cf. Thomée and Westergren [17]). By the assumption on Π^B the boundary values (2.22c) of u_h^A at Γ_3 can be estimated with respect to $H_h^{s+3/2}(\Omega_h^A\cap\Gamma_3)$ (in the sense of

Section 2.4.2) by $C\|f_h^B\|_{H_h^s(\Omega_h^B)}$. Considerations similar to those of Sections 2.2 and 2.4.2 show

$$\|u_h^A\|_{H_h^{s+2}(\Omega_h^A)} \leqq C'(\|u_h^A\|_{L_h^2(\Omega_h^A)}+\|f_h^A\|_{H_h^s(\Omega_h^A)}+\|f_h^B\|_{H_h^s(\Omega_h^A)}) \leqq C\|f_h^B\|_{H_h^s(\Omega_h^B)}. \quad \blacksquare$$

Remark. An analogous regularity estimate holds for Hölder spaces C_h^s.

3. Regularity of discrete nonlinear boundary value problems

3.1. Main theorems

We want to show that under suitable assumptions the discrete solution of the nonlinear problem is as regular as the solution u^* of the continuous boundary value problem

$$\mathscr{L}(u^*) = 0. \tag{3.1}$$

Denote the discretization of (3.1) by

$$\mathscr{L}_h(u_h) = 0. \tag{3.2}$$

Assume $u^*\in X^t$ and define $u_h^*=R_h u^*\in X_h^t$ [cf. (2.6a)]. The consistency order of $\mathscr{L}_h$ is $\varkappa$ if

$$\|\mathscr{L}_h(u_h^*)\|_{Y_h^s} \leqq Ch^{\min(\varkappa, t-s)} \quad (s \leqq t). \tag{3.3}$$

The derivative of $\mathscr{L}_h$ is denoted by L_h:

$$L_h(v_h) = \partial\mathscr{L}_h(v_h)/\partial v_h.$$

Assume that L_h satisfies the Lipschitz condition

$$\|L_h(v_h)-L_h(w_h)\|_{X_h^s\to Y_h^s} \leqq Ch^{-\lambda}\|v_h-w_h\|_{X_h^s} \quad \text{for all} \quad v_h, w_h\in K_{h,s}^\mu(r), \tag{3.4}$$

where

$$K_{h,s}^\mu(r) = \{v_h\in X_h^s\colon \|v_h-u_h^*\|_{X_h^s} \leqq rh^\mu\}.$$

The following result guarantees the existence of a discrete solution of (3.2):

Theorem 3.1. *Let* $u^*\in X^t$ *be a solution of* (3.1). *Assume* (3.3), (3.4), *and*

$$\|L_h^{-1}(u_h^*)\|_{Y_h^s\to X_h^s} \leqq Ch^{-\varrho} \tag{3.5}$$

for some s, λ, μ, ϱ *with*

$$\min(\varkappa, t-s) > \max(\lambda+2\varrho, \mu+\varrho).$$

Then there exists $h_0>0$ *so that for all* $h<h_0$ *the discrete problem* (3.2) *has a solution* $u_h\in K_{h,s}^\mu(r)$.

Note that for $\varrho>0$ (3.5) follows from the (non-)optimal regularity (2.10) with $\varepsilon=\varrho$ and (2.8).

Proof. Apply the Newton—Kantorovič theorem (cf. Meis and Marcowitz [11, p. 282ff]). The iteration

$$u_h^0 = u_h^*, \quad u_h^{i+1} = u_h^i - L_h^{-1}(u_h^*)\mathscr{L}_h(u_h^i)$$

converges to $u_h \in K_{h,s}^{\mu}(r)$ if

(3.6) $$C_N h^{\varepsilon} \leqq 1/2,$$

where $\varepsilon = \min(\varkappa, t-s) - \max(\lambda+2\varrho, \mu+\varrho)$. C_N is determined by the constants involved in (3.3—5). Therefore, Theorem 3.1 is proved with $h_0=(2C_N)^{-1/\varepsilon}$. ∎

The next theorem proves the discrete regularity of u_h:

Theorem 3.2. *Let* $u^* \in X^t$ *be a solution of* (3.1). *Suppose that there is some* s *such that the following conditions hold:*

(i) discrete regularity estimate (2.4) *for* $L_h = L_h(u_h^*)$, *i.e.*, (3.5) *for* $\varrho=0$,
(ii) consistency (3.3) *with* $\varkappa \geqq t-s$,
(iii) Lipschitz condition (3.4) *for all* $\lambda=\mu$ *in some interval* $[\mu_1, \mu_2)$, *where* $\mu_2 = t-s$ *and* μ_1 *arbitrary with* $\mu_1<\mu_2$,
(iv) inverse estimate (2.8),
(v) $\|u_h^*\|_{X_h^t} \leqq C$.

Then for $h \leqq h_0$ *(h_0 sufficiently small) there is a solution of the discrete equation* (3.2) *with*

(3.7) $$\|u_h\|_{X_h^t} \leqq C \quad \textit{for all} \quad h \in H \cap (0, h_0].$$

Proof. Let $h\in H\cap(0,h_0)$. Set $\mu=\mu_2-\varepsilon(h)$, where $\varepsilon(h) = -\log(2C_N)/\log h$ with C_N as in (3.6). By virtue of Theorem 3.1 we have $u_h \in K_{h,s}^{\mu}(r)$. Hence $\|u_h-u_h^*\|_{X_h^s} \leqq$ $\leqq rh^{\mu}$. The assumptions (iv) and (v) imply

$$\|u_h\|_{X_h^t} \leqq \|u_h^*\|_{X_h^t} + \|u_h - u_h^*\|_{X_h^t} \leqq C' + C''h^{s-t+\mu} = C' + C''h^{\varepsilon} = C' + C''/(2C_N) = C.$$

Since the right-hand side is independent of h, (3.7) is proved. ∎

In Theorem 3.2 μ varies, while s is fixed. The same result can be obtained if $\mu<\varkappa$ is fixed and s varies in $[t-\mu-\eta, t-\mu)$, $n>0$ *arbitrary.*

Corollary 3.1. *In the case of a non-optimal estimate* (3.5) $\varrho>0$, *the estimate* (3.7) *requires (ii)—(v) with* $\lambda+\varrho=\mu\in[\mu_1, \mu_2]$.

Our main interest is the regularity of u_h. Usually, one is more interested in convergence:

Corollary 3.2. *Assume* (3.3), (3.4), (3.5) *with* $s=t-\varkappa$, $\varrho=0$, $\lambda\leqq u$, *for all* $\mu\in(\varkappa-\eta,\ \varkappa)$, $\eta>0$ *arbitrary. Then the estimate*

$$\|u_h-u_h^*\|_{X_h^{t-\varkappa}}\leqq Ch^{\varkappa}\quad (h\leqq h_0)$$

holds.

Proof. Set $\mu=\mu(h)=\varkappa-\varepsilon(h)$, $\varepsilon(h)$ as in the proof of Theorem 3.1. Theorem 3.1 implies $\|u_h-u_h^*\|_{X_h^{t-\varkappa}}\leqq Ch^{\mu}=2CC_N h^{\varkappa}$.
An application to the stationary Navier—Stokes equations is given in [18].

3.2. First example: Discrete Hölder spaces

Consider the general nonlinear equation

(3.8) $$\mathscr{L}(u)\equiv\varphi(x,y,u,u_x,u_y,u_{xx},u_{yy})=0(\Omega),\quad u=0(\Gamma)$$

in the square $\Omega=(0,1)\times(0,1)$ and assume that the solution u^* of (3.8) belongs to the Hölder space $C^{2+\lambda}(\bar\Omega)$ for some $\lambda\in(0,1)$. This implies $\varphi(x,y,0,0,0,0,0)=0$ at the corners $(x,y)=(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. Therefore we choose

$$X^{2+\lambda}=\{u\in C^{2+\lambda}(\bar\Omega)\colon u|_\Gamma=0\},$$
$$Y^{2+\lambda}=\{f\in C^{\lambda}(\bar\Omega)\colon f(0,0)=f(0,1)=f(1,0)=f(1,1)=0\}$$

for $\lambda\in I=(0,1)$.
A suitable discretization is

(3.9)
$$\mathscr{L}_h(u_h)=\varphi(x,y,u_h,1/2(I+T_x^{-1})\partial_x u_h,1/2(I+T_y^{-1})\partial_y u_h,T_x^{-1}\partial_x^2 u_h,T_y^{-1}\partial_y^2 u_h)=0.$$

The discrete spaces $X_h^s=C_h^s$, Y_h^s can be defined as in Section 2.3. The derivative at $u_h^*=u^*|_{\bar\Omega_h}$ is (2.13) with

$$a^*(x,y)=\varphi_{u_{xx}}(x,y,u_h^*(x,y),1/2(I+T_x^{-1})\partial_x u_h^*,\ldots),$$
$$b^*=\varphi_{u_{yy}},\quad c^*=\varphi_{u_x},\quad d^*=\varphi_{u_y},\quad e^*=\varphi_u.$$

Define L_h by (2.13) with

$$a(x,y)=\varphi_{u_{xx}}(x,y,u^*(x,y),u_x^*(x,y),u_y^*(x,y),u_{xx}^*(x,y),u_{yy}^*(x,y)),$$

and $b,c,d,e,$ analogously.

Theorem 3.3. *Let* $u^*\in C^{2+\lambda}(\bar\Omega)$ *be a solution of* (3.8). *Assume*
(i) $a(x,y)$, $b(x,y)\geqq\varepsilon>0$,
(ii) L_h *defined by* a,b,c,d,e *is* l_2*-stable (cf.* (1.5)*),*

(iii) a, b, c, d, e are uniformly Lipschitz continuous in U, where $U \subset \bar{\Omega} \times \mathbf{R}^5$ is a neighbourhood of $\{(x, y, u^*(x, y), u_x^*, u_y^*, u_{xx}^*, u_{yy}^*) : (x, y) \in \bar{\Omega}\}$.
Then for h sufficiently small ($h < h_0$) there is a solution u_h of (3.9) with

$$\|u_h\|_{C_h^{2+\lambda}} \leqq C.$$

Proof. Apply Theorems 3.2 and 2.3. ∎

3.3. Second example: Discrete Sobolev spaces

We consider the same problem as Lapin [9]:

$$\mathscr{L}(u) \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x, u, \frac{\partial u}{\partial x_i}\right) + a_0(x, u, \operatorname{grad} u) = 0 \tag{3.10}$$

in $\Omega = (0, 1)^n = \{x \in \mathbf{R}^n : 0 < x_i < 1 \text{ for } 1 \leqq i \leqq n\}$ and $u = 0$ on the boundary. The discretization may be as in [9] or

$$\begin{aligned} \mathscr{L}_h(u_h) \equiv & -\sum_{i=1}^{n} T_i^{-1} \partial_i a_i\left(x + \frac{h}{2} e_i, 1/2(I + T_i) u_h, \partial_i u_h\right) \\ & + a_0(x, u_h, 1/2(I+T_1)\partial_1 u_h, \ldots, 1/2(I+T_n)\partial_n u_h) = 0, \end{aligned} \tag{3.11}$$

where $e_i = i$-th unit vector, $T_i = T_{x_i}$ and $\partial_i = \partial_{x_i}$. Lapin requires almost $u \in C^4(\bar{\Omega})$ and restricts the dimension by $n \leqq 3$. We show that the weaker assumptions $u \in H^4(\Omega)$ and $n \leqq 5$ yield the same result:

Theorem 3.4. *Let $u^* \in H^4(\Omega) \cap H_0^1(\Omega)$ be a solution of (3.10) with $n \leqq 5$. Then the solution u_h of (3.11) exist and*

$$\|u_h\|_{H_h^4(\Omega_h)} \leqq C, \quad \|u_h - R_h u^*\|_{H_h^2(\Omega_h)} \leqq Ch^2 \quad (R_h \textit{ suitable})$$

(H_h^s discrete counterpart of $H^s(\Omega)$, cf. Section 2) holds under the following assumptions:

$$h \leqq h_0, \; a_i \in W^{3,\infty}(U), \; a_0 \in W^{2,\infty}(U),$$
$$\partial a_i(x, u^*, u_{x_i}^*)/\partial u_{x_i} \geqq \varepsilon > 0, \quad 1 \leqq i \leqq n,$$

where U is a neighbourhood of $\{(x, u^*, \operatorname{grad} u^*) : x \in \Omega\}$.

Proof (sketched). (i) Let $u_h^* = R_h u^* \in H_h^4(\Omega_h)$ and let $u^{**} = I_h u_h^* \in C^4(\Omega)$ be an interpolating function: $u^{**}|_{\Omega_h} = u_h^*$, $u^{**}|_\Gamma = 0$. For a suitable R_h and I_h we have

$$\begin{aligned} & \|u^* - u^{**}\|_{H^2(\Omega)} \leqq Ch^2 \|u^*\|_{H^4(\Omega)}, \\ & |(D^\alpha u^{**})(x)| \leqq Ch^{-n/p} \|u^*\|_{W^{|\alpha|,p}(K_x)} \quad (|\alpha| \leqq 4, \; 2 \leqq p \leqq \infty), \end{aligned} \tag{3.12}$$

with $K_x = \{y \in \Omega : \|x - y\| \leqq C_K h\}$ for some C_K.

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Wolfgang Hackbusch
Institut für Informatik
Christian-Albrechts-Universität Kiel
Olshausenstr. 40
D—2300 Kiel 1
Germany

(ii) The dicrete regularity (3.5) ($\varrho=0$) of L_h follows from Theorem 2.2 in the case of $n=2$. But Theorem 2.2 can also be extended to $n>2$.

(iii) (3.3) is to be proved for $Y_h^s=L_h^2(\Omega_h)$, $\min(\varkappa, t-s)=2$. It suffices to estimate

$$(3.13) \qquad T_i^{-1}\partial_i a_i\left(x+\frac{h}{2}e_i,\ 1/2(I+T_i)u^{**},\ \partial_i u^{**}\right)-\frac{\partial}{\partial x_i}a_i\left(x,\ u^{**},\ \frac{\partial u^{**}}{\partial x_i}\right)\Big|_{\Omega_h},$$

$$(3.14) \qquad \frac{\partial}{\partial x_i}a_i\left(x,\ u^{**},\ \frac{\partial u^{**}}{\partial x_i}\right)\Big|_{\Omega_h}-\tilde{R}_h\frac{\partial}{\partial x_i}a_i\left(x,\ u^*,\ \frac{\partial u^*}{\partial x_i}\right)$$

and similar differences for a_0. Taylor expansion of the left term of (3.13) shows

$$(3.13)=h^2 O(C+|u^{**}_{x_ix_i}|^3+|u^{**}_{x_ix_ix_i}u^{**}_{x_ix_i}|+|u^{**}_{x_ix_ix_ix_i}|),$$

where the derivatives are evaluated at $x+\vartheta he_i$, $|\vartheta|\leqq 1$. Here we used $\|u^{**}\|_{W^{1,\infty}(\Omega)}\leqq\|u^*\|_{H^4(\Omega)}$. By virtue of (3.12) the estimate

$$|u^{**}_{x_ix_i}|\leqq Ch^{-n/6}\|u^*\|_{W^{3,6}(K_x)}$$

holds ($p=6$). Summing over Ω_h we obtain

$$\||u^{**}_{x_ix_i}|^3\|^2_{L_h^2(\Omega_h)}=h^n\sum_{x\in\Omega_h}|u^{**}_{x_ix_i}(x+\vartheta(x)he_i)|^6\leqq C'\|u^*\|^6_{W^{2,6}(\Omega)}\leqq C''\|u^*\|^6_{H^4(\Omega)}\leqq C,$$

since $L^p(\Omega)\subset H^2(\Omega)$ for $2\leqq p\leqq 10$, $n\leqq 5$ (cf. Adams [1]). Using $L^q(\Omega)\subset H^1(\Omega)$ ($2\leqq q\leqq 10/3$, $n\leqq 5$) for $q=3$, we are able to estimate

$$\||u^{**}_{x_ix_ix_i}||u^{**}_{x_ix_i}|\|_{L_h^2(\Omega)} \quad\text{by}\quad \|u^*\|_{W^{3,3}(\Omega)}\|u^*\|_{W^{2,6}(\Omega)}\leqq C\|u^*\|^2_{H^4(\Omega)}.$$

The obvious inequality $\||u^{**}_{x_ix_ix_ix_i}|\|_{L_h^2(\Omega_h)}\leqq C\|u^*\|_{H^4(\Omega)}$ and (3.12) imply $(3.13)=O(h^2)$. A similar estimate can be obtained for (3.14).

(iv) We have to prove (3.4) for $Y_h^s=H_h^{s-1}(\Omega_h)$, $X_h^s=H_h^{s+1}(\Omega_h)$, $s=1$. For $s=1$ (3.4) becomes

$$(3.15) \qquad \|[L_h(v_h)-L_h(w_h)]u_h\|_{L_h^2(\Omega_h)}\leqq Ch^{-\lambda}\|v_h-w_h\|_{H_h^2(\Omega_h)}\|u_h\|_{H_h^2(\Omega_h)}.$$

provided that $v_h,\ w_h\in K^\mu_{h,1}(r)$. A rough estimate gives

$$\begin{aligned}\|[L_h(v_h)-L_h(w_h)]u_h\|_{L_h^2(\Omega_h)}&\leqq C\{\|v_h-w_h\|_{W_h^{1,\infty}}\|u_h\|_{H_h^2}\\&+\|v_h-w_h\|_{H_h^2}\|u_h\|_{W_h^{1,\infty}}+\|v_h\|_{H_h^2}\|v-w\|_{W_h^{1,\infty}}\|u_h\|_{W_h^{1,\infty}}\}\\&\leqq Ch^{-\frac{n-2}{n}-\varepsilon}\left(1+h^{\mu-\frac{n-2}{n}-\varepsilon}\right)\|v_h-w_h\|_{H_h^2(\Omega_h)}\|u_h\|_{H_h^2(\Omega_h)},\end{aligned}$$

if $v_h,\ w_h\in K^\mu_{h,1}(r)$ with $\mu\geqq\varepsilon+\dfrac{n-2}{n}$, $\varepsilon>0$ arbitrary. Hence, (3.15) [i.e., (3.4) with $s=1$] holds for all $\mu=\lambda\in\left[\varepsilon+\dfrac{n-2}{2},\ 2\right]$. Note that this interval is nonempty since $n\leqq 5$.

(v) Theorem 3.2 and Corollary 3.2 yield Theorem 3.4. ∎