Regularity theorems for nondiagonal elliptic systems

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1. Introduction

Recently the close connection between regularity properties of weak solutions of quasilinear elliptic systems of partial differential equations and Liouville-type theorems for such systems has become an object of investigation by several author's ([2], [4], [5], [7], [9]). Up to one exception results concerning regularity properties have been established first and Liouville-type theorems for the corresponding systems have been proved afterwards. Ivert [5] was the first to reverse this relation and to give a negative answer to an open question in regularity theory by using a counterexample to a Liouville-type conjecture due to Meier [8].

Concerning the other way around — to give a *positive* answer to the regularity problem for a certain class of elliptic systems under *weaker* conditions than previously known via Liouville-type theorems — nothing has come to the author's attention. It is the aim of this note to give results of this kind and show that with the same methods it is possible to prove new regularity theorems for weak solutions of quasilinear elliptic systems

$$\frac{\partial}{\partial x_\beta}\left(a_{ij}^{\alpha\beta}(x, u)\, u_{x_\alpha}^i\right) = f^j(x, u, \nabla u), \quad j = 1, \ldots, N,$$

where f may grow quadratically with respect to $|\nabla u|$ and $a_{ij}^{\alpha\beta}(x, u) = A_{ij}^{\alpha\beta}(x) + B_{ij}^{\alpha\beta}(x, u)$ is elliptic as well as $A_{ij}^{\alpha\beta}$, where $A_{ij}^{\alpha\beta}$ is continuous and $B_{ij}^{\alpha\beta}(x, u)$ only measurable, and a relative smallness condition for the measurable part is known. This condition is (in contrast to [3]) not unnecessarily strong but is — restricted to the case $A_{ij}^{\alpha\beta} = \hat{A}^{\alpha\beta}\delta_{ij}$ treated by Sperner [11] — especially independent of the ratio of eigenvalues of $\hat{A}^{\alpha\beta}$ thus improving the results given there. A concrete bound for $B_{ij}^{\alpha\beta}$ to be admissable in the general case is given too.

The results are inspired by recent work of Giaquinta—Modica [3], Kawohl [7] and Meier [9] and are essentially based on some ideas due to Campanato [1].

In the sequel we use the summation convention (repeated indices are to be summed). $|\ |$ denotes the Euclidean norm in the various spaces $\mathbf{R}^k$ and

$$B_R(x_0) = \{x | x \in \mathbf{R}^n,\ |x - x_0| < R\}, \quad B_R = B_R(0).$$

Further if $(A_{ik}^{\alpha\beta})$ with $1 \leqq i, k \leqq N$, $1 \leqq \alpha, \beta \leqq n$ is an arbitrary matrix, let $\|A_{ik}^{\alpha\beta}\|$ be the operator norm

$$\|A_{ik}^{\alpha\beta}\|^2 = \sup_{\substack{\xi \in \mathbf{R}^{nN} \\ |\xi| = 1}} \left\{\sum_{i=1}^N \sum_{\alpha=1}^N (A_{ik}^{\alpha\beta} \xi_\beta^k)^2\right\}.$$

2. Results

Theorem 1. *Let*

$$a_{ij}^{\alpha\beta}(x) = A^{\alpha\beta}\delta_{ij} + b_{ij}^{\alpha\beta}(x), \quad 1 \leqq i, j \leqq N, \ 1 \leqq \alpha, \beta \leqq n$$

with $A^{\beta\alpha} = A^{\alpha\beta} \in \mathbf{R}$ *and*

$$\lambda_0 |\xi|^2 \leqq A^{\alpha\beta} \xi_\alpha \xi_\beta \leqq \mu_0 |\xi|^2 \quad \textit{for} \quad \xi \in \mathbf{R}^n$$

with certain constants $0 < \lambda_0 \leqq \mu_0$. *Let* $b_{ij}^{\alpha\beta}$ *be measurable functions defined on* $\mathbf{R}^n$ *and* $R_0 \geqq 0$ *such that*

$$\|b_{ij}^{\alpha\beta}(x)\| \leqq \delta\lambda_0 \quad \textit{if} \quad |x| \geqq R_0$$

with

(1) $$\delta < \delta^* = \frac{c(n)}{n} \quad \textit{with} \quad c(n) \geqq \frac{2}{e} \sim 0.7357.$$

Then if $u \in H_{2,loc}^1(\mathbf{R}^n, \mathbf{R}^N) \cap L_\infty$ *is a bounded solution of*

$$\int_{\mathbf{R}^n} a_{ij}^{\alpha\beta} u_{x_\alpha}^i \varphi_{x_\beta}^j \, dx = 0 \quad \textit{for all} \quad \varphi \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^N),$$

u is a constant vector.

Remark 1. The proof avoids the difficult technique of [11] used by Meier for the proof of Theorem 4 in [9]. Yet the bound is independent of $\frac{\mu_0}{\lambda_0}$. $\Big($I conjecture the exact bound to be $\hat{\delta}^* = \frac{\sqrt{n-1}}{n-2}$, motivated by the example in [11]$\Big)$.

Remark 2. The estimation of $c(n)$ can in fact be slightly improved. Our method gives a $c(n)$ with $\lim_{n \to \infty} c(n) = \gamma \sim 0.8047$, $c(3) = 1.2489$, $c(4) = 1.0886$.

The combination of this theorem and the theorem 2.1 of Kawohl [7] yields

Theorem 2. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain and let*

$$a_{ij}^{\alpha\beta}(x, u) := A^{\alpha\beta}(x)\delta_{ij} + B_{ij}^{\alpha\beta}(x, u), \quad 1 \leqq i, j \leqq N, \ 1 \leqq \alpha, \beta \leqq n$$

with

$$A^{\alpha\beta} = A^{\beta\alpha} \in C(\bar{\Omega}, \mathbf{R})$$

and

$$B_{ij}^{\alpha\beta} \in C(\bar{\Omega} \times \mathbf{R}^N, \mathbf{R}).$$

Let $0 < \lambda_0 \leqq \mu_0$ *and* δ *be constants with*

$$\lambda_0 |\xi|^2 \leqq A^{\alpha\beta}(x)\xi_\alpha \xi_\beta \leqq \mu_0 |\xi|^2 \quad for \quad \xi \in \mathbf{R}^n, \quad x \in \Omega$$

and

$$\|B_{ij}^{\alpha\beta}(x, u)\| \leqq \delta\lambda_0 \quad for \quad (x, u) \in \bar{\Omega} \times \mathbf{R}^N.$$

Then every bounded weak solution $u \in H_2^1(\Omega, \mathbf{R}^N) \cap L_\infty$ *of*

$$\frac{\partial}{\partial x_\beta}\left(a_{ij}^{\alpha\beta}(x, u) u_{x_\alpha}^i\right) = 0, \quad j = 1, \ldots, N$$

is α-Hölder-continuous on compact subsets K of Ω with a certain $\alpha \in (0, 1)$ *and an a-priori-estimate*

$$\|u\|_{C^\alpha(K)} \leqq \operatorname{const}\left(\|u\|_{L_\infty}, \lambda_0, \mu_0, \delta, d(K, \partial\Omega)\right)$$

is valid, provided that

(2) $$\delta < \delta^* \quad \textit{as given by Theorem } 1.$$

Remark. The bound of [11] is of the form $\left(\frac{\lambda_0}{\mu_0}\right)^{1/2} \cdot (n^{-1/2})$. Note that the counter-example of [11] is restricted to the case $\lambda_0/\mu_0 = 1$.

In the next theorem we consider systems with a nonlinearity on the right hand side, which is of quadratic growth with respect to $|\nabla u|$.

Theorem 3. *Let* $u \in H_2^1(\Omega, \mathbf{R}^N) \cap L_\infty$ *be a bounded weak solution of*

$$\frac{\partial}{\partial x_\beta}\left(a_{ij}^{\alpha\beta}(x, u) u_{x_\alpha}^i\right) = -f_j(x, u, \nabla u), \quad j = 1, \ldots, N,$$

where f_j is measurable and satisfies

(3) $$|f(x, u, p)| \leqq a|p|^2 + b \quad \textit{with constants } a, b \geqq 0.$$

Suppose that $a_{ij}^{\alpha\beta}(x, u) = A^{\alpha\beta}(x)\delta_{ij} + B_{ij}^{\alpha\beta}(x, u)$ *with functions* $A^{\alpha\beta}$ *as in Theorem 2 above and with measurable functions* $B_{ij}^{\alpha\beta}$ *with uniformly bounded norm*

(4) $$\|B_{ij}^{\alpha\beta}(x, u)\| \leqq \delta\lambda_0.$$

Then there is an $\alpha \in (0, 1)$ *such that* u *is* α*-Hölder-continuous on compact subsets* K *of* Ω *and an a-priori-estimate*

$$\|u\|_{C_\alpha(K)} \leqq \text{const}\left(\|u\|_{L_\infty}, \lambda_0, \mu_0, \delta, a, b, d(K, \partial\Omega)\right)$$

is valid, provided that

$$\frac{\delta}{2} + \left(\frac{\delta^2}{4} + \frac{2aM}{\lambda_0}\right)^{1/2} < \delta^* \tag{5}$$

where δ^* *is the bound of Theorem* 1 *and* $M = \|u\|_{L_\infty}$.

Remarks.

1. The remarks to Theorem 1 and Theorem 2 apply here too.
2. The interest in measurable coefficients is due to the fact that the coefficients usually depend on the function u the regularity properties of which are still to be determined. So the functions $B_{ij}^{\alpha\beta}(x, u(x))$ regarded as functions in x are only measurable at the beginning.
3. As usual the constant b might be replaced by a function $b \in L_q(\Omega)$, $q > \frac{n}{2}$.
4. The same theorem (but with a more restrictive bound for the admissable variation δ) is true, if $A^{\alpha\beta}(x)$ is only measurable. See the remark at the end of the proof of Theorem 3.

The next theorem gives concrete estimates for the general case, too. Compare the analogous result in [3], Theorem 4.1.

Theorem 4. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain. Let* $u \in H_2^1(\Omega, \mathbf{R}^N) \cap L_\infty$ *be a bounded weak solution of*

$$\frac{\partial}{\partial x_\beta}\left(a_{ij}^{\alpha\beta}(x, u) u_{x_\alpha}^i\right) = f^j(x, u, \nabla u), \quad j = 1, \ldots, N, \tag{6}$$

where

$$a_{ij}^{\alpha\beta}(x, u) = A_{ij}^{\alpha\beta}(x) + B_{ij}^{\alpha\beta}(x, u)$$

with $A_{ij}^{\alpha\beta}(x) \in C(\Omega, \mathbf{R})$

$$\lambda_0 |\xi|^2 \leqq A_{ij}^{\alpha\beta}(x) \xi_\alpha^j \xi_\beta^i \leqq \mu_0 |\xi|^2 \tag{7}$$

for all $\xi \in \mathbf{R}^{nN}$, $x \in \bar{\Omega}$ *with* $0 < \lambda_0 \leqq \mu_0$. *Further let* $B_{ij}^{\alpha\beta}(x, u)$ *be measurable bounded functions with*

$$\|B_{ij}^{\alpha\beta}(x, u)\| \leqq \delta \lambda_0 \tag{8}$$

and let f_j *be measurable with*

$$|f(x, u, \nabla u)| \leqq a |\nabla u|^2 + b. \tag{9}$$

Then there are positive constants c_0, c_1 *such that for all compact subsets* $K \subset \Omega$

$$u \in C_\alpha(K) \quad \text{with a certain} \quad \alpha > 0 \quad \text{and}$$

$$\|u\|_{C_\alpha(K)} \leqq c_1\left(a, \frac{\lambda_0}{\mu_0}, b, \delta, \|u\|_{L_\infty}, n, \lambda, d(K, \partial\Omega)\right)$$

provided that

$$\frac{\delta}{2}+\left(\frac{\delta^2}{4}+\frac{aM(c_1+1)}{\lambda_0}\right)^{1/2} < \frac{2}{en} \cdot c_0^{-\left(\frac{n-2}{4}\right)}. \tag{10}$$

Here $c_0, c_1 \geqq 1$ *are given by Lemma* 5 *below.*

3. Proofs

We need some lemmata. The first one gives a precise estimate for solutions of certain constant coefficient problems.

Lemma 1. *Let* $A^{\alpha\beta}=A^{\beta\alpha}$ *be constant coefficients with*

$$\lambda_0|\xi|^2 \leqq A^{\alpha\beta}\xi_\alpha\xi_\beta \leqq \mu_0|\xi|^2 \quad \text{for all} \quad \xi \in \mathbf{R}^n, \quad 0 < \lambda_0 \leqq \mu_0.$$

Then there exists a positive definite real symmetric matrix S *such that for all solutions* $v \in H_2^1(S(B_R), \mathbf{R}^N) \cap L_\infty$ *of*

$$\int_{S(B_R)} A^{\alpha\beta}\delta_{ij}v^i_{y_\alpha}\varphi^j_{y_\beta}\,dy = 0, \quad \varphi \in C_0^\infty(S(B_R), \mathbf{R}^N)$$

we have

$$\int_{S(B_\varrho)} |S\nabla v|^2\,dy \leqq \left(\frac{\varrho}{R}\right)^n \int_{S(B_R)} |S\nabla v|^2\,dy, \quad \varrho < R. \tag{11}$$

Proof. If A denotes the matrix $(A^{\alpha\beta})$, let S be the positive definite symmetric matrix with $A=S^2=S'S$. By the transformation $y=Sx$ we see that $u^i(x)=v^i(Sx)$ is harmonic on B_R, which implies

$$\int_{B_\varrho} |\nabla u|^2\,dx \leqq \left(\frac{\varrho}{R}\right)^n \int_{B_R} |\nabla u|^2\,dx.$$

As $\nabla u^i=S\nabla v^i$ we may transform this estimate back to the y-coordinates which proves the lemma.

Lemma 2. *Let* $u, v \in L_2(\Omega_R, \mathbf{R}^N)$, $A, B, C \in \mathbf{R}^+$, *such that*

i) $\int_{\Omega_R} v(u-v)\,dx = 0,$

ii) $\int_{\Omega_R} |u-v|^2\,dx \leqq A^2\int_{\Omega_R} |u|^2\,dx + C,$

and suppose that for a subset Ω_ϱ of Ω_R

iii) $\int_{\Omega_\varrho} |v|^2\,dx \leqq B^2 \int_{\Omega_R} |v|^2\,dx.$

Then we have

$$\int_{\Omega_\varrho} |u|^2\,dx \leqq \left(A\sqrt{1-B^2}+B\sqrt{1-A^2}\right)^2 \int_{\Omega_R} |u|^2\,dx + \left(1+\frac{1}{A}\right)C \tag{12}$$

provided $A^2+B^2 \leqq 1$.

Proof. We have for $\alpha \in [0, 1]$, $\varepsilon > 0$

$$\int_{\Omega\varrho} |u|^2\,dx$$

$$= \int_{\Omega_\varrho} |v|^2\,dx + \int_{\Omega_\varrho} |u-v|^2\,dx + 2\alpha \int_{\Omega_\varrho} v(u-v)\,dx - 2(1-\alpha)\int_{\Omega_R \setminus \Omega_\varrho} v(u-v)\,dx$$

$$\leqq \left(1+\frac{\alpha^2}{\varepsilon}\right)\int_{\Omega_\varrho} |v|^2\,dx + \frac{(1-\alpha)^2}{\varepsilon+1}\int_{\Omega_R\setminus\Omega_\varrho} |v|^2\,dx + (\varepsilon+1)\int_{\Omega_R} |u-v|^2\,dx$$

$$\leqq (\varepsilon+1)\int_{\Omega_R} |u-v|^2\,dx + \left[\left(\frac{\alpha^2}{\varepsilon}+1-\frac{(1-\alpha)^2}{\varepsilon+1}\right)B^2 + \frac{(1-\alpha)^2}{\varepsilon+1}\right]\int_{\Omega_R} |v|^2\,dx.$$

Now because of i) we have

$$\int_{\Omega_R} |v|^2\,dx = \int_{\Omega_R} |u|^2\,dx - \int_{\Omega_R} |u-v|^2\,dx.$$

This gives

$$\int_{\Omega_\varrho} |u|^2\,dx \leqq \left(A^2(\varepsilon+1-D)+D\right)\int_{\Omega_R} |u|^2\,dx + (\varepsilon+1-D)C$$

with

$$D = \left(\frac{\alpha^2}{\varepsilon}+1-\frac{(1-\alpha)^2}{\varepsilon+1}\right)B^2 + \frac{(1-\alpha)^2}{\varepsilon+1}$$

provided that

$$D \leqq \varepsilon+1.$$

Now let $\alpha = \dfrac{b}{a+b} \in [0, 1]$ with

$$a = \frac{B^2}{\varepsilon}, \quad b = \frac{1-B^2}{\varepsilon+1}.$$

This implies

$$D = B^2 + \frac{ab}{a+b} = B^2 + \frac{B^2(1-B^2)}{B^2+\varepsilon} \leqq 1.$$

Inserting we have

$$\int_{\Omega_\varrho} |u|^2\,dx \leqq K_1 \int_{\Omega_R} |u|^2\,dx + K_2$$

with

$$K_1 = A^2(\varepsilon+B^2)+A^2(1-B^2)+B^2(1-A^2)+\frac{(1-A^2)\,B^2(1-B^2)}{B^2+\varepsilon}.$$

So choose $\varepsilon=\frac{B}{A}\sqrt{1-A^2}\sqrt{1-B^2}-B^2\leqq\frac{1}{A}$ which yields the desired estimate. As we may assume that $A^2+B^2<1$, we see that $\varepsilon>0$ indeed.

As we need a sharpened version of an inequality due to Campanato ([1], Lemma 6.I.), we give another proof which seems not only more elementary to us, but provides us also with the idea for proving regularity theorems.

Lemma 3. *Let* $\varphi\colon \mathbf{R}^+\to\mathbf{R}^+$ *be a nonnegative, nondecreasing function, such that there are nonnegative constants* a, b, φ_0, γ *with*

(i) $\varphi(\varrho)\leqq\left[a\left(\frac{\varrho}{R}\right)^\gamma+b\right]\varphi(R)+\varphi_0$ *for all* $\varrho<R<\infty$, $\frac{\varrho}{R}<\theta$ *with a fixed* $\theta\leqq 1$,

(ii) $\sup_{R\geqq 1}\varphi(R)R^{\varepsilon-\gamma}<\infty$ *for a certain* $\varepsilon\in(0,\gamma)$.

Then we have

$$\sup_R \varphi(R)\leqq\varphi_0\cdot 2(b^*-b)^{-1}<\infty \tag{13}$$

provided that

$$b<b^*:=\frac{\varepsilon a}{\gamma-\varepsilon}\min\left\{\theta^\gamma,\left(\frac{1-\frac{\varepsilon}{\gamma}}{a}\right)^{\gamma/\varepsilon}\right\}. \tag{14}$$

Remark. b^* cannot be increased, since otherwise $\varphi(\varrho):=\varrho^{\gamma-\varepsilon}$ would furnish a counterexample, if $a\theta^\varepsilon>1-\frac{\varepsilon}{\gamma}$.

Proof. Let us suppose that (13) is false. Then for $R\geqq R_2$ we would have

$$\varphi(\varrho)\leqq\left[a\left(\frac{\varrho}{R}\right)^\gamma+\hat b\right]\varphi(R)\quad\text{with } \hat b>b,\ \text{but } \hat b \text{ still less than } b^*.$$

Now let $A=\sup_{R\geqq R_2}\varphi(R)R^{\varepsilon-\gamma}$, which is nonnegative and finite by (ii). This implies

$$\varphi(\varrho)\varrho^{\varepsilon-\gamma}\leqq\left[a\left(\frac{\varrho}{R}\right)^\varepsilon+\hat b\left(\frac{\varrho}{R}\right)^{\varepsilon-\gamma}\right]\varphi(R)R^{\varepsilon-\gamma}$$

$$\leqq A\left[a\left(\frac{\varrho}{R}\right)^\varepsilon+\hat b\left(\frac{\varrho}{R}\right)^{\varepsilon-\gamma}\right]\quad\text{for}\quad \varrho<R\theta$$

which in turn gives

$$0\leqq A\leqq A\inf_{S>\theta^{-1}}(aS^{-\varepsilon}+\hat b S^{\gamma-\varepsilon}).$$

If the factor on the right hand side is smaller than 1, we would have $A=0$ and hence $\varphi\equiv 0$, a contradiction. A computation shows that this amounts to the condition on b^* as stated in the lemma.

Lemma 4. *Let $a_{ij}^{\alpha\beta}(x)$ be bounded measurable coefficients on $\mathbf{R}^n$ and $\lambda>0$ with*

$$a_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j\geqq\lambda|\xi|^2 \quad \textit{for} \quad \xi\in\mathbf{R}^{nN}.$$

If $u\in H^1_{2,loc}(\mathbf{R}^n,\mathbf{R}^N)\cap L_\infty$ is a solution of

$$\int_{\mathbf{R}^n} a_{ij}^{\alpha\beta}u^i_{x_\alpha}\varphi^j_{x_\beta}\,dx\leqq\int_{\mathbf{R}^n} a|\varphi|\,|\nabla u|^2\,dx$$

for all $\varphi\in C_0^\infty(\mathbf{R}^n,\mathbf{R}^N)$ with $a\|u\|_{L_\infty}<\lambda$, we have

$$\sup_R\left\{R^{2-n}\int_{B_R}|\nabla u|^2\,dx\right\}<\infty.$$

Proof. A test with $\varphi=u\cdot\eta^2$ yields the well known result, see e.g. [9] for the necessary estimations.

Proof of Theorem 1. The method of this proof is essentially due to Meier [9]. Let us suppose $R_0=0$. Let S be given by Lemma 1. For an arbitrary $R>0$ let $v\in H_2^1(S(B_R),\mathbf{R}^N)\cap L_\infty$ be the solution of

$$\text{(15)}\qquad \frac{\partial}{\partial x_\beta}(A^{\alpha\beta}\delta_{ij}v^i_{x_\alpha})=0,\quad j=1,\dots,N \text{ on } S(B_R)$$

$$v-u\in\mathring{H}_2^1(S(B_R),\mathbf{R}^N).$$

Set $w:=v-u$. We have $\|w\|_{L_\infty}\leqq 2\|u\|_{L_\infty}$. For all testfunctions φ we have

$$\int_{S(B_R)}A^{\alpha\beta}\delta_{ij}w^i_{x_\alpha}\varphi^j_{x_\beta}\,dx=\int_{S(B_R)}b_{ij}^{\alpha\beta}u^i_{x_\alpha}\varphi^j_{x_\beta}\,dx.$$

For $\varphi=w$, we get

$$\int_{S(B_R)}|S\nabla w|^2\,dx\leqq\delta\lambda_0\int_{S(B_R)}|\nabla u|\,|\nabla w|\,dx\leqq\delta\int_{S(B_R)}|S\nabla u|\,|S\nabla w|\,dx$$

and

$$\text{(16)}\qquad \int_{S(B_R)}|S\nabla w|^2\,dx\leqq\delta^2\int_{S(B_R)}|S\nabla u|^2\,dx.$$

Note that w depends on R. Now using (11), (15) and (16) we apply Lemma 2 and get by the estimate (12)

$$\int_{S(B_\varrho)}|S\nabla u|^2\,dx\leqq\left(\delta\sqrt{1-\left(\frac{\varrho}{R}\right)^n}+\left(\frac{\varrho}{R}\right)^{n/2}\sqrt{1-\delta^2}\right)^2\int_{S(B_R)}|S\nabla u|^2\,dx$$

if $\delta^2+\left(\dfrac{\varrho}{R}\right)^n\leqq 1.$

Now we apply Lemma 3 to the function

$$\varphi(\varrho) = \left(\int_{S(B_\varrho)} |S\nabla u|^2 \, dx\right)^{1/2}, \quad \text{with} \quad \gamma = \frac{n}{2}, \; a = \sqrt{1-\delta^2}$$

$\varepsilon=1$, $\varphi_0=0$ and $b=\delta$ to conclude that $\varphi\equiv 0$, since the condition on b is fulfilled and $\varphi(\varrho)\varrho^{1-(n/2)}$ remains bounded by Lemma 4. As $\varphi\equiv 0$, u has to be a constant, which proves the theorem. The slight refinement given by Remark 2 will become clear in the proof of Theorem 3. The case $R_0>0$ is similar; we would conclude that φ remains bounded, which gives the same result.

Proofs of Theorems 2 and 3. As stated, a combination of [7], Theorem 2.1 and of our first theorem proves Theorem 2. But the nonhomogeneous case cannot be treated this way. We proceed as follows. Let $x_0\in\Omega$ be arbitrary, for simplicity we may suppose $x_0=0$. Let $R_0\leqq 1$ such that $B_{2R_0}\subset\Omega$. By a variant of Lemma 4, we have

$$\int_{B_\varrho} |\nabla u|^2 \, dx \leqq c_3 \varrho^{n-2} \quad \text{for all} \quad \varrho \leqq R_0, \tag{17}$$

where c_3 depends on the parameters, but not on ϱ. We start with a similar procedure as in the proof of Theorem 1. Let $S^2=(A^{\alpha\beta}(0))$ as in Lemma 1 and $R\leqq R_1\leqq R_0$ such that $S(B_{R_1})\subset\Omega$. Let $v^R\in H_2^1(S(B_R), \mathbf{R}^N)\cap L_\infty$ be the solution of

$$\frac{\partial}{\partial x_\beta}\left(A^{\alpha\beta}(0)\delta_{ij} v^{R,i}_{x_\alpha}\right) = 0, \quad j=1,\dots,N \text{ on } S(B_R),$$

$$v^R - u \in \mathring{H}_2^1(S(B_R), \mathbf{R}^N).$$

Set $w^R := v^R - u$. We have $\|w^R\|_{L_\infty}\leqq 2M = 2\|u\|_{L_\infty}$ by the maximum principle. For all testfunctions φ we have

$$\begin{aligned}
&\int_{S(B_R)} A^{\alpha\beta}(0)\delta_{ij} w^{R,i}_{x_\alpha}\varphi^j_{x_\beta}\,dx \\
&= \int_{S(B_R)} b^{\alpha\beta}_{ij} u^i_{x_\alpha}\varphi^j_{x_\beta}\,dx + \int_{S(B_R)} f^j\varphi^j\,dx \\
&+ \int_{S(B_R)} \left(A^{\alpha\beta}(x) - A^{\alpha\beta}(0)\right)\delta_{ij} u^i_{x_\alpha}\varphi^j_{x_\beta}\,dx.
\end{aligned}$$

Using the continuity of $A^{\alpha\beta}$, we get for $\varphi = w^R$ the estimate

$$\begin{aligned}
\int_{S(B_R)} |S\nabla w^R|^2\,dx &\leqq \delta\lambda_0 \int_{S(B_R)} |\nabla u|\,|\nabla w^R|\,dx \\
&+ \frac{2aM}{\lambda_0}\int_{S(B_R)} \lambda_0 |\nabla u|^2\,dx + \lambda_0\varepsilon(R)\int_{S(B_R)} |\nabla u|\,|\nabla w^R|\,dx + c_4\cdot R^n
\end{aligned} \tag{18}$$

with $\lim_{R\to 0}\varepsilon(R)=0$.

By virtue of $\lambda_0|z|^2 \leqq |Sz|^2$ and of Young's inequality we arrive at

$$\int_{S(B_R)} |S\nabla w^R|^2 \, dx \leqq \hat{\delta}^2 \int_{S(B_R)} |S\nabla u|^2 \, dx + c_5 \cdot R^n \tag{19}$$

with

$$\hat{\delta} = \frac{\delta}{2} + \left(\frac{\delta^2}{4} + \frac{2aM}{\lambda_0}\right)^{1/2} + \hat{\varepsilon}(R_0).$$

Note that $\hat{\varepsilon}(R_0)$ can be made sufficiently small by choosing R_0 small enough. As above we may use Lemma 2 to get

$$\int_{S(B_\varrho)} |S\nabla u|^2 \, dx \leqq \left(\hat{\delta}\sqrt{1-\left(\frac{\varrho}{R}\right)^n} + \left(\frac{\varrho}{R}\right)^{n/2} \sqrt{1-\hat{\delta}^2}\right)^2 \int_{S(B_R)} |S\nabla u|^2 \, dx + c_6 \cdot R^n \tag{20}$$

provided $\left(\frac{\varrho}{R}\right)^n + \hat{\delta}^2 \leqq 1$.

Now consider the function

$$\psi(\varrho) = \varrho^{(2-n)/2} \left(\int_{S(B_\varrho)} |S\nabla u|^2 \, dx\right)^{1/2}.$$

This function is finite by (17) and bounded independently of ϱ. Let $R = \varrho \cdot s_0$ with a number $s_0, s_0^n \geqq (1-\hat{\delta}^2)^{-1}$, which will be determined in the following. Then by (20) we have

$$\psi\left(\frac{R}{s_0}\right) \leqq \gamma(s_0)\psi(R) + c_7(s_0)R \tag{21}$$

with

$$\gamma(s_0) = \frac{1}{s_0}\left(\hat{\delta}(s_0^n - 1)^{1/2} + (1-\hat{\delta}^2)^{1/2}\right).$$

Choose s_0 as to minimize $\gamma(s_0)$ subject to the restriction $s_0^n \geqq (1-\hat{\delta}^2)^{-1}$. This leads to

$$(s_0^n - 1)^{1/2}(1-\hat{\delta}^2)^{1/2} = \hat{\delta}\left(1 + \left(\frac{n}{2} - 1\right)s_0^n\right)$$

which in turn gives

$$W := \frac{\hat{\delta}^2}{2} s_0^n (n-2)^2 = 1 - (n-1)\hat{\delta}^2 + \sqrt{(1-(n-1)\hat{\delta}^2)^2 - \hat{\delta}^2(n-2)^2}$$

if $\hat{\delta} \leqq \frac{1}{n-1}$. It is not hard to check that the restriction is fulfilled. Using this value of s_0 we get

$$\gamma^n = \gamma^n(s_0) = \frac{\hat{\delta}^2(n-2)^2 \cdot \left(1 + \frac{W}{n-2}\right)^n}{2W(1-\hat{\delta}^2)^{n/2}}. \tag{22}$$

If $\hat{\delta}$ is so small as to make $\gamma<1$, the usual iteration procedure of (21) leads to

$$\psi(\varrho) \leqq c_8\big(\hat{\delta}, \sup_{R\leqq R_1} \psi(R), c_7(s_0)\big)\left(\frac{\varrho}{R_1}\right)^{\alpha}, \quad \varrho \leqq R_1, \tag{23}$$

with a positive constant α.

Note once more that $\sup_{R\leqq R_1} \psi(R)$ is bounded by a constant, depending only on $d(x_0, \partial\Omega)$ and the parameters of the system. So we have arrived at the well known Morrey-condition for α-Hölder-continuity

$$\int_{B_\varrho} |\nabla u|^2\, dx \leqq c_9 \cdot \varrho^{n-2+2\alpha} \tag{24}$$

uniformly on compact subsets of Ω. We still have to check the necessary conditions for $\hat{\delta}$ in (22). Now

$$\begin{aligned}
\gamma(s_0) &\leqq \min_s \left\{\hat{\delta} s^{(n/2)-1} + (1-\hat{\delta}^2)^{1/2} s^{-1}\right\} \\
&= \left(1+\frac{1}{\frac{n}{2}-1}\right)\left(\hat{\delta}\left(\frac{n}{2}-1\right)\right)^{2/n} (1-\hat{\delta}^2)^{n-2/2n} \\
&< \left(\left(1+\frac{1}{\frac{n}{2}-1}\right)^{(n/2)-1} \cdot \frac{\hat{\delta} n}{2}\right)^{2/n} \\
&< \left(e\frac{\hat{\delta} n}{2}\right)^{2/n} \leqq 1
\end{aligned}$$

if $\hat{\delta} \leqq \dfrac{2}{en}$ shows the sufficiency of the condition. On the other hand, inserting $\hat{\delta} = \dfrac{1}{n-1}$ in (22), we would get

$$\gamma^n \geqq \frac{\left(1+\frac{1}{n-1}\right)^{n-1}}{2} \geqq 1.$$

This implies that $\gamma<1$ if and only if $\hat{\delta} < \dfrac{c(n)}{n}$ with $\lim_{n\to\infty} c(n) = \mu$ with

$$\frac{\mu^2}{2} e^{1+\sqrt{1-\mu^2}} = 1+\sqrt{1-\mu^2}$$

the only solution of which in $[0, 1]$ is $\mu \sim 0.80474$.

Concerning the remark at the end of Theorem 3, it is enough to observe that we may consider the function $\hat{v}^R$ with $\hat{v}^R - u \in \mathring{H}_2^1(B_R, \mathbf{R}^N)$ and

$$\frac{\partial}{\partial x_\beta}\left(A^{\alpha\beta}(x)\hat{v}_{x_\alpha}^{R,i}\right) = 0, \quad i = 1, \dots, N,$$

as the auxiliary function. By using Hole-filling technique and Green's function we arrive at

$$\int_{B_\varrho} |\nabla v^R|^2 \, dx \leqq K_0 \left(\frac{\varrho}{R}\right)^{n-2+2\beta} \int_{B_R} |\nabla v^R|^2 \, dx$$

with a certain $\beta > 0$ which may be used instead of Lemma 1.

The proof of Theorem 4 proceeds in exactly the same way. The only difference consists in using the following Lemma 5 instead of Lemma 1.

Lemma 5. *Let $A_{ij}^{\alpha\beta}$ be constant coefficients with*

$$\lambda_0 |\xi|^2 \leqq A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \quad \textit{for all} \quad \xi \in \mathbf{R}^{nN},$$

$$\|A_{ij}^{\alpha\beta}\| \leqq \mu_0,$$

and $u \in H_2^1(\Omega, \mathbf{R}^N) \cap L_\infty$ with $|u| \leqq 1$. Let $\Omega_\varrho \subset \Omega$ be a family of subsets with $\Omega_{\varrho_1} \subset \Omega_{\varrho_2}$ if $\varrho_1 \leqq \varrho_2$ and $B_{a\varrho}(x_0) \subset \Omega_\varrho \subset B_{b\varrho}(x_0)$ for $\varrho > 0$ with constants $0 < a < b$. Let v be a solution of

$$\frac{\partial}{\partial x_\beta}\left(A_{ij}^{\alpha\beta} v_{x_\alpha}^i\right) = 0, \quad j = 1, \dots, N,$$

on Ω_R and $v - u \in \mathring{H}_2^1(\Omega_R, \mathbf{R}^N)$. Then there are constants $c_0, c_1 \geqq 1$ such that for $\varrho < R$

$$\int_{\Omega_\varrho} A_{ij}^{\alpha\beta} v_{x_\alpha}^i v_{x_\beta}^j \, dx \leqq c_0 \left(\frac{\varrho}{R}\right)^n \int_{\Omega_R} A_{ij}^{\alpha\beta} v_{x_\alpha}^i v_{x_\beta}^j \, dx$$

and $|v| \leqq c_1$ on Ω_R.

Proof. See e.g. [1]. The freedom in the choice of Ω_ϱ allows sometimes to get the best constants $c_0 = c_1 = 1$ as in Lemma 1.

4. Concluding remarks

A smallness condition on the measurable part is necessary for a regularity theorem as can be seen by the counterexample of [11]. The best what may be conjectured seems to be that in the case of Theorem 3 the solutions are Hölder-continuous if $\delta_3^* < \left(1 - \frac{aM}{\lambda_0}\right)\frac{\sqrt{n-1}}{n-2}$.

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