On the Hölder continuity of monotone extremals in the "borderline case"

Peter Lindqvist

1. Introduction

The Hölder continuity for the solutions of quasilinear elliptic differential equations of the second order has been proved e.g. by Serrin [14, Theorem 8, p. 269] and Ladyzhenskaya & Ural'tseva [5, Theorem 1.1, p. 251]. These advanced proofs give, however, no explicit information about the Hölder exponent, but for certain equations (and systems of equations) the "hole-filling technique" of Widman [15] gives a more precise result in this direction. In the important "borderline case" simple direct proofs are known; c.f. Morrey [10, Theorem 4.3.1, p. 105].

The purpose of this paper is to give a quite elementary proof of the Hölder continuity for the monotone (free) extremals of certain variational integrals in the "borderline case". By the way we shall obtain a relevant lower bound for the Hölder exponent. For the proof of our main result, Theorem 2.9, the well known oscillation lemma of Gehring [2, Lemma 1] and Mostow [11, Lemma 4.1] combined with a good estimate of Dirichlet's integral over a ball play a central rôle.

2. Statement of the problem

We consider variational integrals of the form

(2.1) $$I(u) = \int_G F(x, \nabla u(x))\, dx$$

where $G \subset R^n$ is a domain. The integrand $F\colon G \times R^n \mapsto [0, \infty)$ is supposed to satisfy the following three conditions:

(2.2) *measurability*. The mapping $x \mapsto F(x, \nabla u(x))$ is measurable for each fixed function u in Sobolev's space $W_n^1(G)$.

(2.3) *convexity*. For a.e. fixed $x \in G$ the mapping $w \mapsto F(x, w)$ is convex.

(2.4) *growth condition*. There exist such constants $\beta \geqq \alpha > 0$ that the inequality

$$\alpha |w|^n \leqq F(x, w) \leqq \beta |w|^n$$

holds for a.e. $x \in G$ and for all $w \in R^n$.

It is worth noting that nothing is assumed about the existence of Euler's equation corresponding to (2.1).

2.5 *Remark*. 1°) The assumption (2.2) of measurability can be described more explicitly as a condition imposed on F; c.f. Reshetnyak [12]. 2°) Using an approximation of the integrand and applying the theory of Serrin or that of Ladyzhenskaya & Ural'tseva on Euler's equations for the approximative integrals, Granlund [3] has proved the Hölder continuity of the extremals, when the growth condition (2.4) is valid in the form $\alpha |w|^p \leqq F(x, w) \leqq \beta |w|^p$ for some fixed p, $1 < p \leqq n$.

2.6. *Definition*. We say that $u \in W^1_{n,\mathrm{loc}}(G) \cap C(G)$ is a free extremal of (2.1) if

$$\int_D F(x, \nabla u(x))\, dx \leqq \int_D F(x, \nabla v(x))\, dx \quad (\bar{D} \subset G) \tag{2.7}$$

for every domain D with $\bar{D} \subset G$ and for every function v in the class

$$\mathscr{F}_u(D) = \{v \in C(\bar{D}) \cap W^1_n(D) | v - u \in W^1_{n,0}(D)\}. \tag{2.8}$$

The space $W^1_{n,0}(D)$ is the closure in $W^1_n(D)$ of the space $C_0^\infty(D)$ of infinitely many times differentiable functions with compact support in D. If u is an extremal (in the ordinary sense), it is obviously also a free extremal. It is well known that every (free) extremal u of (2.1) is monotone (in the sense of Lebesgue), i.e. that

$$\sup_{x \in D} u(x) = \sup_{x \in \partial D} u(x), \quad \inf_{x \in D} u(x) = \inf_{x \in \partial D} u(x)$$

for all domains D, $\bar{D} \subset G$. (For a simple proof of this fact, see e.g. Granlund [4, Lemma 2.3].) The existence of extremals is considered e.g. by Martio [7].

We are now in a position to state our main result.

2.9. Theorem. *If u is a free extremal of* (2.1) *and if $B_\varrho \subset B_L$ are concentric balls with radii ϱ and L respectively, $\bar{B}_L \subset G$, then*

$$\operatorname*{osc}_{B_\varrho} u \leqq e \left(\frac{\varrho}{L}\right)^{\varkappa} \operatorname*{osc}_{B_L} u \tag{2.10}$$

where e is Neper's number and where

$$\frac{1}{\varkappa} = e n^2 \omega_{n-1}^{1/n} A_n^{1/n} (n-1)^{(1-n)/n} (\beta/\alpha)^{1/n}$$

A_n being the constant in (3.10) *and ω_{n-1} the area of the unit sphere in R^n.*

Proof. The proof is given in Chapter 4.

Various conditions of local Hölder continuity can easily be derived from Theorem 2.9. As a simple consequence of (2.10) we obtain Liouville's theorem:

2.11. Theorem. (Liouville) *If* $u\colon R^n \to R$ *is a free extremal in* R^n *of* (2.1) *and if*

$$\lim_{x\to\infty} |u(x)|/|x|^\varkappa = 0,$$

$\varkappa$ *being the exponent in* (2.10), *then* u *is constant.*

Proof. If $B_L \subset R^n$, then $\lim_{L\to\infty} \operatorname{osc}_{B_L} u/L^\varkappa = 0$ by the assumption, since u is monotone. Thus the desired conclusion follows from (2.10).

3. Preliminary estimates

Dirichlet's integral of a free extremal can be estimated in the following way.

3.1. Lemma. *If* u *is a free extremal of* (2.1), *then the inequality*

$$(3.2) \qquad \int_{B_r} |\nabla u|^n\, dm \leqq n^n(\beta/\alpha) \operatorname*{osc}_{B_R}{}^{\!n}\, u\, \omega_{n-1} \left(\log \frac{R}{r}\right)^{1-n}$$

is valid for all concentric balls $B_r \subset B_R$, $\bar{B}_R \subset G$.

Proof. Suppose that $\zeta \in C_0^\infty(B_R)$ is a test function, $0 \leqq \zeta \leqq 1$, $\zeta | B_r = 1$, $0 < r < R$; $\bar{B}_R \subset G$. The function

$$v = u - \zeta^n u$$

is in the class $\mathscr{F}_u(B_R)$ and has the generalized derivative

$$\nabla v = (1-\zeta^n)\nabla u - n\zeta^{n-1} u \nabla \zeta.$$

The assumptions (2.3) and (2.4) give

$$(3.3) \qquad F(x, \nabla v(x)) \leqq \big(1-\zeta^n(x)\big) F\big(x, \nabla u(x)\big) + \beta n^n |u(x)|^n |\nabla \zeta(x)|^n$$

for a.e. $x \in G$. As u is minimizing the integral (2.1), we have by (2.7) and (3.3)

$$\int_{B_R} F(x, \nabla u(x))\, dx \leqq \int_{B_R} F(x, \nabla v(x))\, dx$$

$$\leqq \int_{B_R} \big(1-\zeta^n(x)\big) F\big(x, \nabla u(x)\big)\, dx + \beta n^n \int_{B_R} |u|^n |\nabla \zeta|^n\, dm,$$

and so

$$\int_{B_r} F(x, \nabla u(x))\,dx \leqq \beta n^n \int_{B_R} |u|^n |\nabla\zeta|^n\,dm. \tag{3.4}$$

If u is a free extremal, so is $u-\inf_{B_R} u$, and since $|u-\inf_{B_R} u|^n \leqq \operatorname{osc}^n_{B_R} u$ in B_R, we get

$$\int_{B_r} F(x, \nabla u(x))\,dx \leqq \beta n^n \operatorname*{osc}^n_{B_R} u \int_{B_R} |\nabla\zeta|^n\,dm. \tag{3.5}$$

Taking the infimum over all admissible ζ we obtain the capacity of the condenser $(B_R, \bar{B}_r)$:

$$\inf_{\zeta} \int_{B_R} |\nabla\zeta|^n\,dm = \omega_{n-1}\left(\log\frac{R}{r}\right)^{1-n}. \tag{3.6}$$

The desired result now follows from (3.5), (3.6), and the first inequality in (2.4).

3.7 *Remark.* Using Lindqvist [6, Appendix] in the estimation of (3.4) with u replaced by $u-\inf_{B_r} u$, we get the sharper bound

$$\int_{B_r} |\nabla u|^n\,dm \leqq n^n(\beta/\alpha)\left[\int_r^R \frac{dt}{t(\operatorname*{osc}_{B_t} u)^{n/(n-1)}}\right]^{1-n} \omega_{n-1}. \tag{3.8}$$

The following estimate, valid for monotone functions, is closely related to Morrey's lemma [10, Theorem 3.5.2] for Hölder continuity.

3.9. Lemma. (Gehring—Mostow) *If the function* $u \in C(G) \cap W^1_{n,\mathrm{loc}}(G)$ *is monotone, then*

$$\operatorname*{osc}^n_{B_\varrho} u \log\frac{r}{\varrho} \leqq A_n \int_{B_r} |\nabla u|^n\,dm \tag{3.10}$$

for all concentric balls $B_\varrho \subset B_r$, $\bar{B}_r \subset G$. *Here the constant* A_n *depends only on the dimension n.*

Proof. The inequality follows from the oscillation lemma proved by Mostow [11, Lemma 4.1] and Gehring [2, Lemma 1].

3.11 *Remark.* The optimal constants in (3.10) are $A_2 = \pi$,

$$A_3 = \frac{1}{\pi}\left(\int_0^\infty (t+t^3)^{-(1/2)}\,dt\right)^2, \dots,$$

$$A_n = \frac{2}{\omega_{n-2}}\left(\int_0^\infty [(1+t^2)\,t^{n-2}]^{1/(1-n)}\,dt\right)^{n-1}.$$

4. Proof of the Hölder continuity

If u is a free extremal of (2.1), then Lemma 3.1 and Lemma 3.9 give the estimate

$$\operatorname*{osc}_{B_\varrho}{}^n u \leqq \frac{A_n n^n \omega_{n-1}(\beta/\alpha)}{\log\frac{r}{\varrho}\left(\log\frac{R}{r}\right)^{n-1}} \operatorname*{osc}_{B_R}{}^n u \tag{4.1}$$

for all concentric balls $B_\varrho \subset B_r \subset B_R$, $\bar{B}_R \subset G$. Obviously (4.1) is optimal for $r=(R\varrho^{n-1})^{1/n}$; $\varrho<(R\varrho^{n-1})^{1/n}<R$ if $\varrho<R$. Thus we have obtained the following result.

4.2. Proposition. *If u is a free extremal of* (2.1), *then*

$$\operatorname*{osc}_{B_\varrho} u \leqq n^2(A_n\omega_{n-1}\beta/\alpha)^{1/n}(n-1)^{(1-n)/n}\left(\log\frac{R}{\varrho}\right)^{-1} \operatorname*{osc}_{B_R} u \tag{4.3}$$

for all concentric balls $B_\varrho \subset B_R$, $\bar{B}_R \subset G$.

Actually, Proposition 4.2 contains all information needed for the proof of Theorem 2.9.

Proof of Theorem 2.9. Let us iterate (4.3). Denote therefore $R/\varrho=\lambda>1$ and consider all pairs of subsequent radii in $\varrho, \lambda\varrho, \lambda^2\varrho, \ldots, \lambda^\nu\varrho$. The iteration gives

$$\operatorname*{osc}_{B_\varrho} u \leqq \left(\frac{K}{\log\lambda}\right)^\nu \operatorname*{osc}_{B_{\lambda^\nu\varrho}} u \tag{4.4}$$

where

$$K = n^2(A_n\omega_{n-1})^{1/n}(\beta/\alpha)^{1/n}(n-1)^{(1-n)/n}.$$

Let us write $L=\lambda^\nu\varrho$. With this notation (4.4) takes the form

$$\operatorname*{osc}_{B_\varrho} u \leqq \left(\frac{\varrho}{L}\right)^{[\log(\log\lambda/K)]/\log\lambda} \operatorname*{osc}_{B_L} u. \tag{4.5}$$

Choosing $\log\lambda=eK$ we get

$$\operatorname*{osc}_{B_\varrho} u \leqq \left(\frac{\varrho}{L}\right)^{1/eK} \operatorname*{osc}_{B_L} u. \tag{4.6}$$

The validity of (4.6) is limited by the restrictions $\lambda=e^{eK}$ and $L=\lambda^\nu\varrho$ (ν is a natural number). Removing these restrictions we finally get

$$\operatorname*{osc}_{B_\varrho} u \leqq e\left(\frac{\varrho}{L}\right)^{1/eK} \operatorname*{osc}_{B_L} u$$

provided $\bar{B}_L \subset G$. This is the desired result.

4.7 *Remark.* If $f=(f_1, f_2, \ldots, f_n)\colon G\to R^n$ is quasiregular, then each coordinate function $f_1, f_2, \ldots, f_n$ is, according to Reshetnyak [13], a free extremal of a variational integral of the type (2.1), the integrand satisfying the conditions (2.2), (2.3), and (2.4). Explicitly $\alpha=1/K_0(f)$, $\beta=K_I(f)$ in (2.4), $K_0(f)$ and $K_I(f)$ being the outer and the inner dilatations of f respectively (the dilatations are considered in the sense of Martio, Rickman, and Väisälä [8]). The best possible Hölder exponent is $K_I(f)^{1/(1-n)}$ in this special case; c.f. Martio, Rickman, and Väisälä [9, Theorem 3.2]. In the two-dimensional case a simple proof is given by Finn and Serrin [1].

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Received October 22, 1979

Peter Lindqvist
Matematiska Institutionen
Tekniska Högskolan i Helsingfors
SF-02150 ESBO 15
Finland