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# Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$. (II)

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## Introduction

The present paper is a continuation of the paper [1]. This means that we shall try to minimize the functional

$$H(f) = \sup_x F(x, f(x), f'(x))$$

over the class $\mathfrak{J}$ of all absolutely continuous functions $f(x)$ which satisfy the boundary conditions $f(x_1) = y_1$ and $f(x_2) = y_2$.

In Chapter 1 we consider the question of the existence of a minimizing function, which was left open in [1].

In Chapter 2 some of the previous results are generalized to a wider class of functions $F(x,y,z)$: the previous condition

$$\frac{\partial F}{\partial z}\begin{cases} >0 & \text{for } z>0, \\ =0 & \text{for } z=0 \\ <0 & \text{for } z<0, \end{cases} \quad \text{is changed into} \quad \frac{\partial F}{\partial z}\begin{cases} >0 & \text{for } z>\omega(x,y), \\ =0 & \text{for } z=\omega(x,y), \\ <0 & \text{for } z<\omega(x,y). \end{cases}$$

An existence proof for absolutely minimizing functions is given in the general case.

In Chapter 3 we examine further the properties of a.s. minimals. We also consider uniqueness questions for a.s. minimals and minimizing functions. For reasons of simplicity, this chapter is restricted to the case $\omega(x,y) \equiv 0$.

In Chapter 4, finally, we give some examples in order to illustrate the previous exposition.

We shall now introduce notations for some conditions on $F(x,y,z)$. They are assumed to hold for $x \in J$ and for all real $y$ and $z$. Here $J$ is an interval which will be stated explicitly in most cases. If it is not explicitly given, then the choice of $J$ will be clear from the context.

1. $F(x,y,z)$ is continuous and $\partial F/\partial z$ exists.

2A. $$\frac{\partial F(x,y,z)}{\partial z} \text{ is } \begin{cases} >0 & \text{if } z>0, \\ =0 & \text{if } z=0, \\ <0 & \text{if } z<0. \end{cases}$$

2B. There is a continuous function $\omega(x, y)$ such that

$$\frac{\partial F(x,y,z)}{\partial z} \text{ is } \begin{cases} >0 & \text{if } z>\omega(x,y), \\ =0 & \text{if } z=\omega(x,y), \\ <0 & \text{if } z<\omega(x,y). \end{cases}$$

3. $\lim_{|z|\to\infty} F(x,y,z) = +\infty$ if $x$ and $y$ are fixed.

The conditions 1 and 3 will be assumed to hold throughout this paper, together with 2A or 2B.[1] 2A is assumed to hold in Chapter 1 and Chapter 3; in Chapter 2 we shall use the more general condition 2B. It is easy to see (using the Heine–Borel covering theorem) that

$$\lim_{|z|\to\infty} \Big( \inf_{\substack{\alpha\leqslant x\leqslant\beta \\ |y|\leqslant K}} F(x,y,z)\Big) = +\infty$$

for every compact interval $[\alpha,\beta]\subset J$ and for every $K>0$.

## Chapter 1. The question of the existence of a minimizing function

The assumptions on $F(x,y,z)$ in this chapter are mainly the same as in [1]. We shall assume that the following conditions are satisfied throughout the chapter: 1, 2A and 3. The interval $J$ will be chosen as $x_1 \leqslant x \leqslant x_2$.

**1.** We shall now give an example which shows that the existence of a minimizing function does not follow from these properties of $F(x,y,z)$.

*A counterexample.* Choose

$$F(x,y,z) = z^2/(y^2+1)^2 - \cos x - x(2\pi - x)\cos x - \varphi(y)(2\pi - \varphi(y)),$$

where $\varphi(y) = \operatorname{arctg} y + \pi/2$;

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \text{ and } \begin{cases} x_2 = 2\pi, \\ y_2 = 0. \end{cases}$$

Let $f\in\mathfrak{J}$. We have

$$H(f) \geqslant F(\pi, f(\pi), 0) = 1 + \pi^2 - \varphi(f(\pi))[2\pi - \varphi(f(\pi))] > 1,$$

since $0<\varphi(f(\pi))<\pi$. Hence $H(f)>1$ for all $f\in\mathfrak{J}$. Suppose now that $0<\delta<\pi/2$ and consider the function

[1] There is one exception: Example 5 in Chapter 4.

$$g_\delta(x)=\begin{cases}\operatorname{tg} x & \text{for } 0\leqslant x\leqslant\frac{\pi}{2}-\delta,\\ \operatorname{tg}\left(\frac{\pi}{2}-\delta\right) & \text{for } \frac{\pi}{2}-\delta\leqslant x\leqslant\frac{3\pi}{2}+\delta,\\ \operatorname{tg}(2\pi-x) & \text{for } \frac{3\pi}{2}+\delta\leqslant x\leqslant 2\pi.\end{cases}$$

It is easy to verify that

$$H(g_\delta)=F(\pi,g_\delta(\pi),0)=1+\delta^2.$$

This proves that $\inf_{f\in\mathfrak{J}} H(f)=1$ and that there is no minimizing function. Compare the remark to Theorem 1.1.

2. It follows from this example that, in order to ensure the existence of a minimizing function, we must impose suitable extra conditions on $F(x,y,z)$. We shall describe a few such conditions (they are separately sufficient), but we are not aiming at a complete discussion of the question.

In each existence proof, we shall need

**Lemma 1.1.** *Let $\{f_\nu(x)\}_1^\infty$ be a sequence of functions in $\mathfrak{J}$ such that* $\sup_{1\leqslant\nu<\infty} H(f_\nu)<\infty$ *and suppose that $f_\nu(x)\to f(x)$ uniformly for $x_1\leqslant x\leqslant x_2$. Then $f\in\mathfrak{J}$ and $H(f)\leqslant\underline{\lim}_{\nu\to\infty} H(f_\nu)$.*[1]

*Proof.* Since $|f_\nu(x)|\leqslant K_1$ and $H(f_\nu)\leqslant K_2$ for all $\nu$, it follows from the properties of $F$ that $|f_\nu'(x)|\leqslant K_3$. Hence all $f_\nu(x)$ and $f(x)$ satisfy a uniform Lipschitz condition. Clearly $f\in\mathfrak{J}$, and the rest of the proof follows easily. (Use the fact that if $f'(x_0)$ exists, then we have

$$F(x_0, f(x_0), z)\geqslant F(x_0, f(x_0), f'(x_0))$$

for all $z\geqslant f'(x_0)$ or for all $z\leqslant f'(x_0)$, together with a continuity argument.)

Let us now return to the minimization problem and introduce the notation $M_0=\inf_{f\in\mathfrak{J}} H(f)$. Suppose that $\{f_\nu\}_1^\infty$ is a minimizing sequence, i.e. $\lim_{\nu\to\infty} H(f_\nu)=M_0$. If the functions $f_\nu(x)$ are *uniformly bounded,* then it follows (as above) that they are equicontinuous, and then there is a uniformly convergent subsequence. Let the limit function be $f(x)$. It follows from the lemma that $H(f)=M_0$.

Hence, in order to prove that there is a minimizing function, it is sufficient to prove that there is a minimizing sequence of uniformly bounded functions.

In many cases, it can be seen from the function $F(x,y,0)$ that there is a minimizing function. This possibility is illustrated by the following theorem.

**Theorem 1.1.** *Suppose that there is a number $Y_1\geqslant\max(y_1,y_2)$ such that*

$$\max_{\alpha\leqslant x\leqslant\beta} F(x,Y_1,0)=\max\left(F(\alpha,Y_1,0),\ F(\beta,Y_1,0)\right)$$

*for all $\alpha$ and $\beta$ satisfying $x_1\leqslant\alpha<\beta\leqslant x_2$. Suppose also that there is a number $Y_2\leqslant\min(y_1,y_2)$ with the same property. Then there is a minimizing function.*

[1] In other words, the functional $H(f)$ is lower semi-continuous.

*Proof.* Let $f(x) \in \mathfrak{F}$ and form $g(x) = \begin{cases} Y_1 & \text{if} \quad f(x) > Y_1, \\ f(x) & \text{if} \quad Y_2 \leqslant f(x) \leqslant Y_1, \\ Y_2 & \text{if} \quad f(x) < Y_2. \end{cases}$

Clearly $H(g) \leqslant H(f)$ and the rest of the proof is obvious.

It is not difficult to find different conditions on $F(x,y,0)$ which lead to the same result. (For instance, $F(x,y,0)$ increasing in $y$ for $y \geqslant Y_1$ and all $x$, decreasing in $y$ for $y \leqslant Y_2$ and all $x$, are also sufficient conditions for the existence of a minimizing function.)

**3.** Suppose that there is a number $M > M_0$ ($M_0 = \inf_{f \in \mathfrak{F}} H(f)$) such that $H(f) \leqslant M$ implies that $\max |f(x)| \leqslant K$, where $K$ depends on $M$ but not on $f(x)$. Then, clearly, every minimizing sequence is bounded and there is a minimizing function.

It is evident that every number $M > M_0$ has this property if $\lim_{|z| \to \infty} F(x,y,z) = +\infty$ uniformly for all $x$ and $y$. But there may exist numbers $M$ with this property even if the limit is not uniform. We can describe the result of this section thus: if the roots $z_1$ and $z_2$ of the equation $F(x,y,z) = M$ do not increase too fast in absolute value when $|y| \to \infty$, then the attainable set $E(M)$ is bounded, and then there is a minimizing function. For instance, it will follow from Theorem 1.2′ that there is a minimizing function if $\lim_{|y| \to \infty} F(x,y,y) = +\infty$ uniformly in $x$.

Let us denote by $E_1(M)$ the set of points $(x,y)$ such that

a) $x_1 \leqslant x \leqslant x_2$,

b) there is an absolutely continuous function $g(t)$, joining the points $(x_1, y_1)$ and $(x,y)$, such that

$$\sup_t F(t, g(t), g'(t)) \leqslant M.$$

The set $E_2(M)$ is defined analogously, but with $(x_2, y_2)$ instead of $(x_1, y_1)$. We are interested in the set

$$E(M) = E_1(M) \cap E_2(M).$$

It is clear that $H(f) \leqslant M$ implies that $\max |f(x)| \leqslant K$ if and only if $E(M)$ is bounded, and in that case the constant $K$ is determined by $E(M)$.

We shall first examine $E(M)$ in the case $F = F(y,z)$. Later, we shall try to carry over some of the results to the general case.

Using some results from [1], we can easily prove the following theorem (concerning $\Phi_M(t)$, $\psi_M(t)$ and the integrals below, see section 2 of [1]):

**Theorem 1.2.** *Assume that $F$ is independent of $x$ and that $M > M_0$.*

A) *$E(M)$ is unbounded above (i.e. it contains points with arbitrarily large ordinates) if and only if the integrals*

$$I_1 = \int_{y_1}^{\infty} \frac{dt}{\Phi_M(t)} \quad \textit{and} \quad I_2 = \int_{y_2}^{\infty} \frac{dt}{-\psi_M(t)}$$

*are well-defined, convergent and*

$$I_1 + I_2 \leqslant x_2 - x_1. \tag{1}$$

B) *$E(M)$ is unbounded below if and only if the integrals*

$$I_3=\int_{-\infty}^{y_1}\frac{dt}{-\psi_M(t)} \quad \textit{and} \quad I_4=\int_{-\infty}^{y_2}\frac{dt}{\Phi_M(t)}$$

*are well-defined, convergent and*

$$I_3+I_4\leqslant x_2-x_1. \tag{2}$$

*Proof.* We restrict the proof to the first part of the theorem. Assume first that $E(M)$ is unbounded above. Let $(x_0,y_0)\in E(M)$ and $y_0>\max(y_1,y_2)$. It follows from Theorem 2 in [1] that

$$\int_{y_1}^{y_0}\frac{dt}{\Phi_M(t)}\leqslant x_0-x_1 \quad \text{and} \quad \int_{y_2}^{y_0}\frac{dt}{-\psi_M(t)}\leqslant x_2-x_0.$$

If we add these inequalities and let $y_0$ tend to infinity, then we have proved one part of the assertion.

Next, suppose that $I_1+I_2\leqslant x_2-x_1$. It follows from the discussion of "the attainable cone" in section 2B of [1] that $(x,y)\in E_1(M)$ if $x_1+I_1\leqslant x\leqslant x_2$ and $y\geqslant y_1$. Similarly, it follows that $(x,y)\in E_2(M)$ if $x_1\leqslant x\leqslant x_2-I_2$ and $y\geqslant y_2$.

Clearly, all points $(x_1+I_1,y)$, where $y\geqslant\max(y_1,y_2)$, belong to $E(M)$. This completes the proof of Theorem 1.2, since the reasoning in the case B is analogous.

A consequence of the theorem is that $E(M)$ is bounded if and only if neither (1) nor (2) is satisfied.

Let us now consider the general case $F=F(x,y,z)$. It can be "reduced" to the previous case simply by neglecting the dependence on $x$. This may be considered to be a crude method, but, nevertheless, it works in many cases.

The functions $\Phi_M(y)$ and $\psi_M(y)$ were defined in [1]. Clearly, we can introduce here the corresponding functions $\Phi(x,y,M)$ and $\psi(x,y,M)$. Now fix $y$ and $M$ and suppose that $U=\{x\,|\,F(x,y,0)\leqslant M\}$ is non-empty.

*Definition:*
$$\begin{cases}\alpha(y,M)=\max_{x\in U}\Phi(x,y,M),\\ \beta(y,M)=\min_{x\in U}\psi(x,y,M).\end{cases}$$

It is easy to verify that these functions are continuous and that they are monotonic in $M$, as are $\Phi_M(y)$ and $\psi_M(y)$. Finally, an argumentation, similar to the one used in the special case $F=F(y,z)$, leads to the following result:

**Theorem 1.2′.** *Assume that $M>M_0$.*

A) *If $E(M)$ is unbounded above, then the integrals*

$$I_1=\int_{y_1}^{\infty}\frac{dt}{\alpha(t,M)} \quad \textit{and} \quad I_2=\int_{y_2}^{\infty}\frac{dt}{-\beta(t,M)}$$

*are well-defined, convergent and $I_1+I_2\leqslant x_2-x_1$.*

B) *If $E(M)$ is unbounded below, then the integrals*

$$I_3 = \int_{-\infty}^{y_1} \frac{dt}{-\beta(t, M)} \quad and \quad I_4 = \int_{-\infty}^{y_2} \frac{dt}{\alpha(t, M)}$$

*are well-defined, convergent and* $I_3 + I_4 \leqslant x_2 - x_1$.

An application of this theorem is given in Example 1 in Chapter 4. Compare also the counterexample.

## Chapter 2. A more general case. Proof of the existence of absolutely minimizing functions

**1.** In this section we shall generalize some of the previous results to a more general class of functions $F(x, y, z)$.

We are not going to change the condition $\lim_{|z|\to\infty} F(x, y, z) = +\infty$, which has turned out to be very useful and which is a rather natural condition. However, the assumption that $F(x_0, y_0, z)$ has a minimum at $z = 0$ for all $(x_0, y_0)$, is more restrictive, and we shall assume instead that $F(x_0, y_0, z)$ has a minimum at $z = \omega(x_0, y_0)$, where $\omega(x, y)$ is a continuous function. Compare Example 5 in Chapter 4.

Thus, let us assume that the following conditions hold throughout this chapter: $F(x, y, z) \in C^1$; 2B and 3.

The minimization problem is meaningful, since we have

$$H(f) \geqslant \max [F(x_1, y_1, \omega(x_1, y_1)), F(x_2, y_2, \omega(x_2, y_2))]$$

for every admissible function $f(x)$.

We shall now carry over some of the results of the paper [1] and of the previous chapter to this more general case.

The analogues of the Theorems 5 and 6 in [1] are

**Theorem 5'.** *Let $f(x)$ be an admissible function such that:*

a) *$f'(x)$ is continuous on $I$ $(x_1 \leqslant x \leqslant x_2)$,*
b) *$F_z(x, f(x), f'(x)) \geqslant 0$ on $I$, or $\leqslant 0$ on $I$,*
c) *$F(x, f(x), f'(x)) = M$ on $I$.*

*Then $f(x)$ is a minimizing function*[1] *(not necessarily unique).*

**Theorem 6'.** *Suppose that $f(x) \in \mathfrak{J}$ satisfies:*

a) *$f'(x)$ is continuous on $I$ $(x_1 \leqslant x \leqslant x_2)$,*
b) *$F_z(x, f(x), f'(x)) \neq 0$ on $I$,*
c) *$F(x, f(x), f'(x)) = M$ on $I$.*

*Then $f(x)$ is a unique minimizing function.*[1]

The proofs are analogous to those of the case $\omega(x, y) \equiv 0$ and they are omitted.

Lemma 4 in [1] holds in this case without changes. The proof is simple.

[1] It also follows that $f(x)$ is an a.s. minimal on $I$.

A consequence of this lemma is that we can neglect sets of measure zero. This means that we can join two functions at a finite number of points where they are equal by taking one or the other on successive intervals. Then these points will not affect the value of the functional for the resulting function. (This situation will occur in the next section.)

Next, we make a jump to Theorem 9, which is carried over without changes:

**Theorem 9′.** *If $f(x)$ is an a.s. minimal on the interval $I$ $(x_1 \leqslant x \leqslant x_2)$, then:*

1) *$f(x) \in C^1$ on $I$,*
2) *the differential equation*

$$\frac{dF(x, f(x), f'(x))}{dx} \cdot F_z(x, f(x), f'(x)) = 0$$

*is satisfied on $I$ in the sense that if $F_z(x, f(x), f'(x)) \neq 0$ on a sub-interval $I_1 \subset I$, then $F(x, f(x), f'(x))$ is constant on $I_1$.*

We shall prove the assertions 1) and 2) under the following, weaker assumptions on $f(x)$:

I) $f(x)$ satisfies a Lipschitz condition on $[x_1, x_2]$,

II) there is a constant $K > 0$ such that if $[t_1, t_2]$ is an arbitrary sub-interval of $I$, then there are two (not necessarily different) minimizing functions $u(x)$ and $v(x)$ between $(t_1, f(t_1))$ and $(t_2, f(t_2))$ such that $|u(x)| \leqslant K$, $|v(x)| \leqslant K$ and $u(x) \leqslant f(x) \leqslant v(x)$ for $t_1 \leqslant x \leqslant t_2$.

(These assumptions are satisfied if $f(x)$ is an a.s. minimal, since we can choose $K = \max_{x \in I} |f(x)|$ and $u(x) = v(x) = f(x)$ in that case.)

The reason why we shall prove this more general result is that it will be useful in the next section.

*Proof.*[1] Consider an arbitrary point $x_0$ on the interval $x_1 \leqslant x < x_2$ and suppose that the upper and lower right-hand derivatives of $f(x)$, $\alpha$ and $\beta$, are different at $x_0$. (Clearly, they are finite.) Choose a number $\gamma \neq \omega(x_0, f(x_0))$ such that $\alpha > \gamma > \beta$. The initial-value problem $F(x, g(x), g'(x)) = \text{constant}$, $g(x_0) = f(x_0)$, $g'(x_0) = \gamma$, has a solution $\varphi(x)$ in a neighbourhood of $x_0$. Since $\alpha > \gamma > \beta$, there are points $x > x_0$, arbitrarily close to $x_0$, where $f(x) = \varphi(x)$. But since $\gamma \neq \omega(x_0, f(x_0))$, it follows from the condition II and from Theorem 6′ that $f(x) = \varphi(x)$ on $[x_0, x_0 + \delta]$ for some $\delta > 0$. The contradiction shows that $f(x)$ has a right-hand derivative. Similarly, it follows that $f(x)$ has a left-hand derivative for $x_1 < x \leqslant x_2$.

Our next step is to prove that the one-sided derivatives are equal. Consider a point $x_0$, and assume that $\alpha = f'(x_0)^+ \neq \beta = f'(x_0)^-$. Choose a number $\gamma$ between $\alpha$ and $\beta$. Clearly

$$F(x_0, f(x_0), \gamma) < \max[F(x_0, f(x_0), \alpha), F(x_0, f(x_0), \beta)].$$

Select a sequence of intervals $[p_n, q_n]$ such that

$$p_n < x_0 < q_n, \quad q_n - p_n \to 0$$

and

$$\frac{f(q_n) - f(p_n)}{q_n - p_n} = \gamma.$$

[1] The reasoning will be similar to the one used in [1], but not identically the same.

It is obvious that

$$\overline{\lim_{n\to\infty}}\ M(p_n, q_n;\ f(p_n),\ f(q_n)) \leqslant F(x_0, f(x_0), \gamma).$$

(The notation $M(...)$ was introduced in [1].) Let us assume, for example, that $\alpha > \gamma > \beta$ and $F(x_0, f(x_0), \alpha) > F(x_0, f(x_0), \gamma)$. Let the functions $u_n(x)$ correspond to the intervals $[p_n, q_n]$ according to II. It follows that $\underline{\lim}_{n\to\infty} (\sup_x u_n'(x)) \geqslant \alpha$. Thus

$$\underline{\lim_{n\to\infty}}\ H(u_n) \geqslant F(x_0, f(x_0), \alpha).$$

Consequently

$$\underline{\lim_{n\to\infty}}\ H(u_n) > \overline{\lim_{n\to\infty}}\ M(p_n, q_n;\ f(p_n), f(q_n)),$$

which is a contradiction.

The reasoning is analogous in the other cases. This proves that $f'(x)$ exists.

Suppose now that $F_z(x_0, f(x_0), f'(x_0)) \neq 0$. Then we assert that $f(x) \in C^2$ in a neighbourhood $U$ of $x_0$ and that $F(x, f(x), f'(x))$ is constant in $U$. This is proved in the same way as Theorem 8 in [1]. We omit the details.

Now the only thing left to prove is that $f'(x)$ is continuous at those points where $F_z(x, f(x), f'(x)) = 0$. Assume then that $f'(x_0) = \omega(x_0, f(x_0))$. Assume further that there are numbers $\xi_n \to x_0$, $\xi_n > x_0$ such that $f'(\xi_n) \geqslant f'(x_0) + \delta$. If the functions $u_n(x)$ correspond to the intervals $x_0 \leqslant x \leqslant \xi_n$ according to II, then we have $\sup_x u_n'(x) \geqslant f'(x_0) + \delta$. Hence $\underline{\lim}_{n\to\infty} H(u_n) \geqslant F(x_0, f(x_0), f'(x_0) + \delta)$. But this contradicts the obvious relation

$$\overline{\lim_{n\to\infty}}\ M(x_0, \xi_n;\ f(x_0), f(\xi_n)) \leqslant F(x_0, f(x_0), f'(x_0)).$$

The other cases can be treated similarly. This completes the proof.

We have thus established some important properties of a.s. minimals. But we have not examined the properties of minimizing functions in general. The reason for this is evident from Example 2 in Chapter 4.

In the previous chapter we treated the question of the existence of a minimizing function for the case $\omega(x,y) \equiv 0$. It is easy to verify that Lemma 1.1 holds in the present case. This means that if there is a minimizing sequence of uniformly bounded functions, then there is a minimizing function. In particular, this is true if $E(M)$ is bounded for some $M > M_0$.[1]

We shall not discuss generalizations of the other criteria in Chapter 1.

**2.** A problem which we have not yet discussed, is that of the existence of absolutely minimizing functions. We shall give an existence proof for the general case.

**Theorem 2.1.** *Assume that $F(x, y, z) \in C^1$ and satisfies the conditions 2 B and 3 for $X_1 \leqslant x \leqslant X_2$ and all $y, z$. (Thus $J = [X_1, X_2]$.)*

*Assume further that, for every choice of $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$, there is a minimizing function between these points, and, if there are several such functions, that they are uniformly bounded. (The bounds will depend on $\xi_1$, $\eta_1$, $\xi_2$ and $\eta_2$.)*

[1] A sufficient condition for this is that $\lim_{|z|\to\infty} F(x, y, z) = +\infty$ uniformly for all $x$ and $y$.

*Then there is an absolutely minimizing function for every minimization problem in the strip* $X_1 \leqslant x \leqslant X_2$, $-\infty < y < \infty$.

*Proof.* In order to make the proof more lucid, we shall divide it into several parts.

1) Consider the minimization problem between $(x_1, y_1)$ and $(x_2, y_2)$. Let $\mathfrak{M}$ be the class of minimizing functions, which, by assumption, is not empty.

Put

and

$$\begin{cases} u(x) = \inf\limits_{f \in \mathfrak{M}} f(x) \\ v(x) = \sup\limits_{f \in \mathfrak{M}} f(x). \end{cases}$$

It is clear that $u(x)$ and $v(x)$ belong to $\mathfrak{M}$.

Consequently there are always a smallest minimizing function $u(x)$ and a greatest minimizing function $v(x)$.

2) We shall introduce two notations in order to simplify the expressions later on.

Let $p(x)$ be a function on an interval $I$. Let $[x_1, x_2]$ be a sub-interval of $I$ and consider the minimization problem between $(x_1, p(x_1))$ and $(x_2, p(x_2))$. Let $u(x)$ be the smallest and $v(x)$ the greatest minimizing function, as above. We shall agree to say that $p(x)$ has the property $A$ on $I$ if $p(x) \geqslant u(x)$ on $[x_1, x_2]$ *for every* sub-interval $[x_1, x_2]$ of $I$ and that $p(x)$ has the property $B$ on $I$ if $p(x) \leqslant v(x)$ on $[x_1, x_2]$ *for every* sub-interval $[x_1, x_2]$ of $I$. (Compare subharmonic and superharmonic functions.)

3) Consider our minimization problem on an interval $I$ with some boundary values. It is obvious that the greatest minimizing function $v(x)$ is not less than the *greatest* minimizing function $v_1(x)$ between any two points $(t_1, v(t_1))$ and $(t_2, v(t_2))$. Similarly, the smallest minimizing function $u(x)$ is not greater than the *smallest* minimizing function $u_1(x)$ between any two points $(t_1, u(t_1))$ and $(t_2, u(t_2))$.

4) We shall now construct an a.s. minimal. *From now on, we consider a fixed minimization problem, namely between* $(x_1, y_1)$ *and* $(x_2, y_2)$. Let $G$ be the class of those minimizing functions which have the property $A$ on $[x_1, x_2]$. $G$ is not empty, since the greatest minimizing function belongs to $G$.

*We introduce the function*

$$h(x) = \inf_{g \in G} g(x)$$

*and we assert that* $h(x)$ *is an a.s. minimal.*

It is clear that $h(x)$ is a minimizing function.

5) Now let us prove that $h(x)$ has the properties $A$ and $B$.

I) Choose an arbitrary interval $[t_1, t_2]$ and let $u(x)$ be the smallest minimizing function between $(t_1, h(t_1))$ and $(t_2, h(t_2))$. Let $g(x) \in G$. We assert that $g(x) \geqslant u(x)$ on $t_1 \leqslant x \leqslant t_2$. If this were not true, then there would be an interval $s_1 \leqslant x \leqslant s_2$ such that $u(s_1) = g(s_1)$, $u(s_2) = g(s_2)$ and $u(x) > g(x)$ for $s_1 < x < s_2$. Let $u_1(x)$ be the smallest minimizing function between $(s_1, u(s_1))$ and $(s_2, u(s_2))$. It follows from the definition of $G$ that $g(x) \geqslant u_1(x)$. But we also have $u(x) \leqslant u_1(x)$. (See part 3 of the proof.) This gives $g(x) \geqslant u_1(x) \geqslant u(x)$ and we have a contradiction.

Thus $u(x) \leqslant g(x)$, and the inequality $u(x) \leqslant h(x)$ follows from the definition of $h(x)$.

This proves that $h(x)$ has the property $A$.

II) Assume that $h(x)$ has not the property $B$. This means that there is an interval $[t_1, t_2]$ such that the corresponding greatest minimizing function $v(x)$ does not satisfy the inequality $v(x) \geqslant h(x)$. It follows from part 3 of the proof that we may assume that $h(x) > v(x)$ for $t_1 < x < t_2$. Let us form the function

$$p(x) = \begin{cases} h(x) & \text{for} \quad x \leqslant t_1, \\ v(x) & \text{for} \quad t_1 \leqslant x \leqslant t_2, \\ h(x) & \text{for} \quad x \geqslant t_2. \end{cases}$$

If we can prove that $p(x) \in G$, then this will contradict the definition of $h(x)$, since we have $p(x) < h(x)$ on $(t_1, t_2)$. It is clear that $p(x)$ is a minimizing function on $[x_1, x_2]$, and it remains to prove that $p(x)$ has the property $A$. So let us consider an interval $[s_1, s_2]$ and let the corresponding smallest minimizing function be $u_1(x)$. We have $u_1(s_1) = p(s_1) \leqslant h(s_1)$ and $u_1(s_2) = p(s_2) \leqslant h(s_2)$. Since $h(x)$ has the property $A$, this implies that $h(x) \geqslant u_1(x)$. If $p(\xi) < u_1(\xi)$ for some $\xi \in (s_1, s_2)$, then we must have $p(\xi) = v(\xi) < u_1(\xi)$. Consequently there must be an interval $[r_1, r_2]$ such that $v(r_1) = u_1(r_1)$, $v(r_2) = u_1(r_2)$ and $v(x) < u_1(x)$ for $r_1 < x < r_2$. Let $u_2(x)$ and $v_2(x)$ correspond to this interval and these boundary values in the usual manner. Then we have

$$u_1(x) \leqslant u_2(x) \leqslant v_2(x) \leqslant v(x).$$

But this contradicts the inequality $v(x) < u_1(x)$. This proves that $h(x)$ has the property $B$.

6) It is obvious that $h(x)$ satisfies the conditions I and II which, as we have seen, guarantee that the assertions in Theorem 9′ are true. Thus $h(x) \in C^1$ and the differential equation

$$\frac{dF(x, h(x), h'(x))}{dx} \cdot F_z(x, h(x), h'(x)) = 0$$

is satisfied in the sense described there.

7) We are now in a position to prove that $h(x)$ is an a.s. minimal. Consider an arbitrary interval $t_1 \leqslant x \leqslant t_2$. Let $M_0$ be the minimum value of the functional, as usual, and put

$$M = \max_{t_1 \leqslant x \leqslant t_2} F(x, h(x), h'(x)).$$

We want to prove that $M = M_0$.

If $F(t_1, h(t_1), h'(t_1)) = M$, then the result follows at once, since $h(x)$ has the properties $A$ and $B$, and the same is true for $t_2$. So let us assume that $F(t_1, h(t_1), h'(t_1)) < M$ and that $F(t_2, h(t_2), h'(t_2)) < M$. It follows from part 6 that there is a number $\xi, t_1 < \xi < t_2$, such that $h'(\xi) = \omega(\xi, h(\xi))$ and $F(\xi, h(\xi), h'(\xi)) = M$. Let us assume that $M > M_0$.

This means that there is no minimizing function $\omega(x)$ for which $\omega(\xi) = h(\xi)$. (I.e. $(\xi, h(\xi)) \notin E(M_0)$.)

Let $K$ be the class of those minimizing functions $\omega(x)$, for which $\omega(\xi) \leqslant h(\xi)$. ($K$ is not empty, see part 5.) Put $k(x) = \sup_{\omega \in K} \omega(x)$. It is clear that $k(x) \in K$ and that $k(\xi) < h(\xi)$.

Put

$$\begin{cases} s_1 = \sup \{x \mid x < \xi, \, h(x) = k(x)\} \\ s_2 = \inf \{x \mid x > \xi, \, h(x) = k(x)\}. \end{cases}$$

and

We introduce the function

$$q(x)=\begin{cases} h(x) & \text{for} \quad x \leqslant s_1, \\ k(x) & \text{for} \quad s_1 \leqslant x \leqslant s_2, \\ h(x) & \text{for} \quad x \geqslant s_2. \end{cases}$$

If we can prove that $q(x) \in G$, then this will contradict the definition of $h(x)$, since $q(x) < h(x)$ on $(s_1, s_2)$. (Compare part 4.) *So all we have to prove is that $q(x)$ has the property $A$.* Consider then an arbitrary interval $[r_1, r_2]$ and let $u(x)$ be the smallest minimizing function between $(r_1, q(r_1))$ and $(r_2, q(r_2))$. We have $u(r_1)=q(r_1)\leqslant h(r_1)$ and $u(r_2)=q(r_2)\leqslant h(r_2)$. Since $h(x)$ has the property $A$, it follows easily that $h(x)\geqslant u(x)$ for $r_1 \leqslant x \leqslant r_2$.

So if we assume that $u(x_0) > q(x_0)$ for some $x_0$, then we must have $s_1 < x_0 < s_2$ and $q(x_0)=k(x_0)$.

Consequently there must be an interval $(\eta_1, \eta_2)$ such that $s_1 \leqslant \eta_1 < \eta_2 \leqslant s_2$, $u(\eta_1)=k(\eta_1)$, $u(\eta_2)=k(\eta_2)$ and $u(x) > k(x)$ for $\eta_1 < x < \eta_2$.

If $H(k; \eta_1, \eta_2) \leqslant H(u; \eta_1, \eta_2)$, then we get a contradiction to the definition of $u(x)$.

Suppose then that

$$H(k; \eta_1, \eta_2) > H(u; \eta_1, \eta_2).$$

Let us form the function

$$k_1(x)=\begin{cases} k(x) & \text{for} \quad x \leqslant \eta_1, \\ u(x) & \text{for} \quad \eta_1 \leqslant x \leqslant \eta_2, \\ k(x) & \text{for} \quad x \geqslant \eta_2. \end{cases}$$

Clearly, $k_1(x)$ is a minimizing function between $(t_1, h(t_1))$ and $(t_2, h(t_2))$. Further, we have $k_1(\xi) \leqslant h(\xi)$ (for, as we mentioned above, $u(x) \leqslant h(x)$ for all $x$, where $u(x)$ is defined). Hence $k_1(x) \in K$. But this contradicts the definition of $k(x)$. Hence we get a contradiction in any case.

This completes the proof of Theorem 2.1.

**3.** The method of approximating a maximum by a sequence of integral mean values is well known. In our case it means that we should consider the functional $H(f)$ as the limit of the sequence of functionals

$$H_n(f)=\left[\frac{1}{x_2-x_1}\int_{x_1}^{x_2}[F(x, f(x), f'(x))]^n\,dx\right]^{1/n}, \quad n=1, 2, 3, \ldots.$$

We have used this approach in [1] to derive the differential equation $(dF/dx)\cdot F_z=0$. It can also be used to give a different existence proof for a.s. minimals. This is very natural since a minimizing function in the calculus of variations automatically is minimizing on every sub-interval. However, we shall not carry through this proof since it is more complicated than the one already given, and since it requires stronger conditions on $F(x, y, z)$.

Another result which can be proved with this "integral method" is the following: if $f_0(x)$ is the only a.s. minimal for $H(f)$ and if $f_n(x)$ minimizes $H_n(f)$ for $n=1, 2, 3, \ldots$, then $\lim_{n\to\infty} f_n(x)=f_0(x)$ uniformly.

## Chapter 3. Further examination of absolutely minimizing functions. Uniqueness questions

For reasons of simplicity, this chapter will treat only the case $\omega(x, y) \equiv 0$. No doubt, the results can be modified for the general case.

**1.** The following theorem gives us further information on the structure of absolutely minimizing functions:

**Theorem 3.1.** *Assume that $F(x, y, z)$ satisfies these conditions:*

*$F(x, y, z)$ is an analytic function of $x, y$ and $z$ in a complex neighbourhood of the set of values*

$$\begin{cases} x_1 \leqslant x \leqslant x_2 \\ y, z \text{ real}; \end{cases}$$

2A *and* 3.

*Assume further that $f(x)$ is an absolutely minimizing function on the compact interval $x_1 \leqslant x \leqslant x_2$.*

*Then:*

a) *$f(x) \in C^1$ on $[x_1, x_2]$ (already proved);*

b) *the set $\{x \mid F_z(x, f(x), f'(x)) \neq 0\}$ consists of a* ***finite*** *number of intervals (to be proved now);*

c) *the differential equation*

$$\frac{dF(x, f(x), f'(x))}{dx} \cdot F_z(x, f(x), f'(x)) = 0$$

*is satisfied in the sense that $F(x, f(x), f'(x))$ is constant on each of these intervals (already proved).*

*Proof.* In order to facilitate the further references, we divide the proof into several parts.

1) Assume that $f'(x) \neq 0$ for $p < x < q$ and that $f'(p) = f'(q) = 0$.

Then
$$\begin{cases} F_x(p, f(p), 0) \leqslant 0, \\ F_x(q, f(q), 0) \geqslant 0. \end{cases}$$

To see this, we recall that $f(x) \in C^2$ on $(p, q)$ and that $F(x, f(x), f'(x))$ is constant there (Theorem 8 in [1]). So we have

$$F_x(x, f(x), f'(x)) + F_y(\ldots)\, f'(x) + F_z(\ldots) f''(x) = 0$$

for $p < x < q$. Clearly, there must be points arbitrarily close to $p$, where $f'(x)$ and $f''(x)$ have the same sign. At such a point $F_z(\ldots) f''(x) > 0$, which gives

$$F_x(x, f(x), f'(x)) + F_y(\ldots) f'(x) < 0.$$

Since $\lim_{x \to p} f'(x) = 0$, this proves the first inequality, and the second is proved similarly (there are points arbitrarily close to $q$ where $F_z(\ldots)f''(x) < 0$, etc.).

Now suppose that there is a number $t > q$ such that $f'(t) \neq 0$. Then it follows from the above result that we can define

$$r = \inf\{x \mid x \geqslant q,\ F_x(x, f(x), 0) = 0\},$$

and it follows that $q \leqslant r < t$. If $q < r$, then we obviously have $f'(x) = 0$ for $q \leqslant x \leqslant r$ (and if $q = r$, then this is trivially true).

If there is a number $u < p$ such that $f'(u) \neq 0$, then we define

$$s = \sup\{x;\ x \leqslant p,\ F_x(x, f(x), 0) = 0\}.$$

We have $f'(x) = 0$ for $s \leqslant x \leqslant p$.

Finally, we observe that

$$\begin{cases} F_x(x, f(x), 0) \leqslant 0 & \text{for} \quad s \leqslant x \leqslant p, \\ F_x(x, f(x), 0) \geqslant 0 & \text{for} \quad q \leqslant x \leqslant r. \end{cases}$$

2) We shall give an indirect proof of the theorem. So we assume that the set $\{x \mid x_1 < x < x_2,\ f'(x) \neq 0\}$ is the union of an infinite number of disjoint open intervals. These intervals must have a limit-point and we may assume that it is $x = 0$. We may also assume that $f(0) = 0$ and that there are infinitely many such intervals in every interval $0 < x < \varepsilon$. Clearly $f'(0) = 0$ and $F_x(0, 0, 0) = 0$.

The function $F_x(x, y, 0)$ will be important for the rest of the proof. It may be identically zero or not. This gives us two cases and we shall start with the simplest of them.

3) Assume that $F_x(x,y,0) \equiv 0$. This means that we have $F(x,y,0) = F(0,y,0) = \varphi(y)$, where $\varphi(y)$ is analytic in a (complex) neighbourhood of $y = 0$.

A) $$\varphi(y) \equiv \text{constant, i.e. } F(x,y,0) \equiv \text{constant}.$$

Consider an interval $(p, q)$ as in 1). Put $t = \frac{1}{2}(p+q)$. Then we have

$$F(t, f(t), f'(t)) > F(t, f(t), 0) = F(p, f(p), 0),$$

which gives a contradiction.

B) $$\varphi(y) \not\equiv \text{constant}.$$

It follows from Theorem 10 in [1][1] that $f(x)$ is monotonic, for instance non-decreasing.

But $\varphi'(y) \neq 0$ for $0 < y < \delta$. Hence $\varphi(y)$ is strictly monotonic for $0 \leqslant y \leqslant \delta$. Consider an interval $(p,q)$ such that $0 < f(p) < f(q) < \delta$. Then we must have $\varphi(f(p)) = \varphi(f(q))$ which gives a contradiction.

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[1] It is clear that the condition on $F_x$ in Theorem 10 need only be assumed to hold for $z = 0$. In fact, the condition of Theorem 1.1 in this paper is sufficient if it holds for all $y$.

4) Now let us assume that $F_x(x, y, 0) \not\equiv 0$. Clearly $F_x(x, y, 0)$ is an analytic function and we can apply Weierstrass' preparation theorem (see [2], p. 89). In our case it says that

$$F_x(x, y, 0) = x^{\mu}\omega(x, y)\Psi(x, y)$$

in a neighbourhood $U$ of the origin. Here $\mu$ is an integer $\geqslant 0$; $\omega(x,y)$ is analytic in $U$ and $\neq 0$; $\Psi(x,y)$ is *either* $\equiv 1$ *or* of the form

$$\Psi(x, y) = y^m + A_1(x)y^{m-1} + A_2(x)y^{m-2} + \ldots + A_m(x),$$

where the functions $A_k(x)$ are analytic in a (sufficiently small) neighbourhood of $x=0$ and $A_k(0)=0$ for all $k$.

We know from 2) that $F_x(x, y, 0)=0$ at the points $(r, f(r))$ and $(s, f(s))$. Therefore we can exclude the case $\Psi \equiv 1$ and, instead, we inquire about the zeros of

$$\Psi(x,y) = y^m + A_1(x)y^{m-1} + \ldots + A_m(x).$$

Let us, for the present, replace $x$ and $y$ by the complex variables $\xi$ and $\eta$. We shall try to determine $\eta$ as a function of $\xi$ from the relation $\Psi(\xi,\eta)=0$. We are only interested in the solutions of this equation in a neighbourhood of $\xi=\eta=0$.

Consider a complex neighbourhood $V$ of $\xi=0$ and cut it along the negative real axis. Then every root of $\xi$ can be defined as a regular function in $V$, and the roots $\eta_k$ of $\Psi(\xi, \eta)=0$ may be written

$$\eta_k(\xi) = \sum_{j=1}^{\infty} C_{j,k}(\xi^{1/n})^j \quad \text{for} \quad k=1, 2, \ldots, m.$$

Concerning this expansion, see [3], p. 50; [4], pp. 98–103 and [5], Chapter XIII. (The roots can be divided into cyclic systems, and the number $n$ can be chosen as the product of the number of roots in each system.)

We may assume that $\xi^{1/n}$ is real for $\xi>0$, which gives the expressions

$$\sum_{j=1}^{\infty} C_{j,k}(x^{1/n})^j, \; k=1, 2, \ldots, m.$$

Clearly, we need only consider those series in which $C_{j,k}$ are real for all $j$. (If there are no such series, then there is nothing left to prove.) Therefore, let us suppose that

$$\eta_k(x), \quad k=1,2,\ldots,M,$$

are real for $0<x<\delta$, and that $\eta_k(x)$ are complex (not real) for $k>M$ and $0<x<\delta$.

We know from the first part of the proof that

$$F_x(r, f(r), 0) = F_x(s, f(s), 0) = 0.$$

Hence the points $(r, f(r))$ and $(s, f(s))$ must lie on the curves $y=\eta_k(x)$ if $r$ and $s$ are sufficiently close to $x=0$.

5) We shall now study in some detail the functions

$$\eta_k(x)=\sum_{j=1}^{\infty} C_{j,k}(x^{1/n})^j=g_k(x^{1/n}).$$

Here the functions $g_k(z)$ are analytic for $|z|<\delta$. Further, every $g_k(z)$ is real if $z$ is real.

Differentiation gives

$$\eta_k'(x)=g_k'(x^{1/n})\cdot\frac{1}{n}\cdot x^{\frac{1}{n}-1} \quad \text{for} \quad 0<x<\delta^n.$$

Put $t=x^{1/n}$, which gives

$$\eta_k'(x)=\frac{1}{n}\,g_k'(t)\,\frac{1}{t^{n-1}}=\varphi_k(t).$$

The function $\varphi_k(z)$ is analytic in $0<|z|<\delta$ and the singularity at $z=0$ is either removable or a pole. We have

$$\lim_{x\to+0}\eta_k'(x)=\lim_{t\to+0}\varphi_k(t),$$

and this limit must be real. It can be $+\infty$, $-\infty$, finite but $\neq 0$ and 0. Since $f'(0)=0$, we can exclude from consideration those values of $k$ for which we do *not* have $\eta_k'(0)=0$. Therefore we can assume that $\varphi_k(z)$ is analytic for $|z|<\delta$.

A consequence of this is that there is a $\delta_1>0$ such that $\varphi_k(z)\neq 0$ for $0<|z|<\delta_1$, unless $\varphi_k(z)\equiv 0$. From the relation $\eta_k'(x)=\varphi_k(x^{1/n})$, we can infer a corresponding result for $\eta_k'(x)$.

Now let us compare the functions $\eta_k(x)=g_k(t)$ and $\eta_l(x)=g_l(t)$. The functions $g_k(z)$ and $g_l(z)$ are analytic and not identical. Hence there must exist a $\delta_2>0$ such that

$$g_k(z)\neq g_l(z) \quad \text{for} \quad 0<|z|<\delta_2.$$

This gives a corresponding result for $\eta_k(x)$ and $\eta_l(x)$.

We have found that $\eta_k(x)=g_k(x^{1/n})$ and $\eta_k'(x)=\varphi_k(x^{1/n})$, where $g_k(z)$ and $\varphi_k(z)$ are analytic for $|z|<\delta$. Let us write $x^{1/n}=t$, as above. This gives

$$F(x,\eta_k(x),0)=F(t^n,g_k(t),0)=u(t)$$

and

$$F(x,\eta_k(x),\eta_k'(x))=F(t^n,g_k(t),\varphi_k(t))=v(t).$$

Clearly, the functions $u(z)$ and $v(z)$ are analytic in a neighbourhood of $z=0$. Hence there is a $\delta_3>0$ such that each of the functions $F(x,\eta_k(x),0)$ and $F(x,\eta_k(x),\eta_k'(x))$ is either increasing, constant or decreasing on the whole interval $0\leqslant x\leqslant\delta_3$.

We have excluded those functions $\eta_k(x)$, for which we did not have $\eta_k'(0)=0$. After a renumbering, we have $\eta_k(x)$, $k=1,2,\ldots,P$, left.

If we collect our results, then we see that there exist numbers $\alpha>0$ and $\nu_0$ such that:

a) $\eta_k(x)\neq\eta_l(x)$ if $0<x\leqslant\alpha$ and $k\neq l$. This means that one of the functions $\eta_k(x)$ is smallest and one is greatest on the whole interval $0<x\leqslant\alpha$.

b) $\eta_k(x)\in C^1$ on $0\leqslant x\leqslant\alpha$ and $\eta_k(0)=\eta_k'(0)=0$. If $\eta_k(x)\not\equiv 0$, then $\eta_k'(x)\neq 0$ for $0<x\leqslant\alpha$.

c) If we have $p$, $q$, $r$ and $s$ as in 1) and $0<s<r\leqslant\alpha$, then $f(r)=\eta_k(r)$ for some $k\leqslant P$ and $f(s)=\eta_{k_1}(s)$ for some $k_1\leqslant P$.

d) Every function $F(x, \eta_k(x), 0)$ is either constant or strictly monotonic on $0 \leqslant x \leqslant \alpha$. The same is true for $F(x, \eta_k(x), \eta_k'(x))$ $(1 \leqslant k \leqslant P)$.

6) We are interested only in those functions $\eta_k(x)$ for which one of the relations under c) above really occurs on the interval $0 < x \leqslant \alpha$. If we exclude the others, then we get (after a renumbering) the functions $\eta_k(x)$, $k = 1, 2, \ldots, N$, left. This does not affect the validity of the summary a)–d).

If $N = 1$ and $\eta_1(x) \equiv 0$, then we get a contradiction at once and there is nothing left to prove.

If this is not the case, then the greatest of the functions $\eta_k(x)$ is $>0$ (for $0 < x \leqslant \alpha$) or the smallest is $<0$.

Let us choose the first case and let $\eta_N(x)$ be the function in question. Then $\eta_N(x) > 0$ and $\eta_N'(x) > 0$ on $0 < x \leqslant \alpha$.

It follows from our choice of $\eta_N(x)$ that there is an interval $(p_0, q_0)$ such that $f(r_0) = \eta_N(r_0)$. We have thus

$$f(q_0) = f(r_0) = \eta_N(r_0); \quad f'(q_0) = 0$$

and $\eta_N'(x) > 0$.

Hence $f(x) > \eta_N(x)$ for $q_0 - \delta < x < q_0$. But we also have $f(p_0) = f(s_0) \leqslant \eta_N(s_0)$ (owing to our choice of $\eta_N(x)$) and $f'(p_0) = 0$. Hence $f(x) < \eta_N(x)$ for $p_0 < x < p_0 + \delta_1$. Consequently there must be a $\xi$, $p_0 < \xi < q_0$, such that $f(\xi) = \eta_N(\xi)$ and $f'(\xi) \geqslant \eta_N'(\xi) > 0$.

Now assume first that $F(x, \eta_N(x), 0)$ is constant or decreasing. Then we have

$$F(\xi, \eta_N(\xi), 0) \geqslant F(r_0, \eta_N(r_0), 0) \geqslant F(q_0, f(q_0), 0).$$

(See the first part of the proof.) This gives

$$F(\xi, f(\xi), f'(\xi)) > F(q_0, f(q_0), 0),$$

which is a contradiction.

Assume then that $F(x, \eta_N(x), 0)$ is increasing. An obvious consequence of this is that $F(x, \eta_N(x), \eta_N'(x))$ is increasing. Let us now consider the minimization problem between the points $x_1 = y_1 = 0$ and $x_2 = \xi$, $y_2 = f(\xi) = \eta_N(\xi)$. Since $f(x)$ is an absolute minimal, we have

$$M_0 = H(f) \geqslant F(\xi, f(\xi), f'(\xi)).$$

But $F(x, \eta_N, \eta_N')$ is increasing, which gives

$$M_0 \leqslant H(\eta_N) = F(\xi, \eta_N(\xi), \eta_N'(\xi)) \leqslant H(f) = M_0.$$

Thus $H(\eta_N) = M_0$ and $\eta_N(x)$ is also a minimizing function.

Choose a function $\phi(x) \in C^1$ on $0 \leqslant x \leqslant \xi$ such that $\phi(0) = \phi(\xi) = 0$ and $\phi'(\xi) < 0$. Form the function $g(x) = \eta_N(x) + \lambda\phi(x)$, where $\lambda > 0$ is a parameter. We have

$$F(x, g(x), g'(x)) = F(x, \eta_N(x), \eta_N'(x)) + \lambda(a(x)\phi(x) + b(x)\phi'(x)) + R(x, \lambda).$$

Here $a(x) = F_y(x, \eta_N(x), \eta_N'(x))$ and $b(x) = F_z(x, \eta_N(x), \eta_N'(x))$ and $|R(x, \lambda)| \leqslant C\lambda^2$ where $C$ is independent of $x$ and $\lambda$, if $0 < \lambda < 1$.

Now, recalling the fact that $F(x, \eta_N, \eta_N')$ takes its maximum only at $x = \xi$ and that $F_z(\xi, \eta_N(\xi), \eta_N'(\xi)) > 0$, it is easy to verify that $H(g) < H(\eta_N)$, if $\lambda$ is small enough. This gives $H(g) < M_0$ which is a contradiction. This completes the proof.

2. This section will treat briefly the question of uniqueness for a.s. minimals. It will be shown by examples that there may be several a.s. minimals for a given minimization problem. A few sufficient conditions for uniqueness will also be given.

Choose
$$F(x, y, z) = \frac{2}{(1 + x^2 + y^2)^2} + z^2,$$
$$x_1 = -2, \quad y_1 = 0, \quad x_2 = 2 \quad \text{and} \quad y_2 = 0.$$

It follows from Theorem 2.1 that there is an a.s. minimal $f(x)$. Clearly, $g(x) = -f(-x)$ is also an a.s. minimal. Now, if there is a unique a.s. minimal, then we get $-f(-x) \equiv f(x)$, i.e. $f(0) = 0$. Consequently $M_0 \geqslant F(0,0,0) = 2$.

Put
$$h(x) = \begin{cases} 2 - x & \text{for} \quad x \geqslant 0, \\ 2 + x & \text{for} \quad x \leqslant 0. \end{cases}$$

Clearly $H(h) = 2/9 + 1 < 2$.

The contradiction shows that there is no unique a.s. minimal. (This situation can be described thus: every minimizing function must evade the maximum at $x = y = 0$, which, owing to the symmetry, leads to non-uniqueness.)

Put $F(x, y, z) = y^2 + z^2$ and let $f_0(x)$ be defined as in Example 3 in [1]. We know that all functions $pf_0(x+q)$ ($p$, $q$ are constants) are a.s. minimals. It is evident that there are infinitely many a.s. minimals if $y_2 = -y_1 \neq 0$ and $x_2 - x_1 > \pi$.

It follows from this example that there may be several a.s. minimals even if $F(x, y, z)$ is convex in $y$ and $z$. In the calculus of variations, this condition leads to uniqueness (for all boundary values). It follows from this example that the condition $(y_1 - C)(y_2 - C) \geqslant 0$ in Theorem 3.3 cannot be omitted.

These examples show that extra conditions on $F(x, y, z)$ or on the boundary values must be added to our usual ones in order to secure uniqueness. Two ways to do this are shown by the following theorems:

**Theorem 3.2.** *Assume that $F(x,y,z)$ satisfies the conditions of Theorem 2.1, with $\omega(x, y) \equiv 0$. Then, as we know from that theorem, there is at least one a.s. minimal for every choice of $(x_1, y_1)$ and $(x_2, y_2)$.*

*Assume now that*
$$\frac{\partial F(x, y, 0)}{\partial x} \neq 0 \quad \textit{for all } x \textit{ and } y.$$

*Then there is a* UNIQUE *a.s. minimal for every choice of $(x_1, y_1)$ and $(x_2, y_2)$.*

*Proof.* We assume that $F_x(x,y,0) > 0$. We may also assume that $y_1 < y_2$ (the case $y_1 > y_2$ is analogous and the case $y_1 = y_2$ is trivial). Let $f(x)$ be an a.s. minimal. If $f'(x) > 0$ for $x_1 \leqslant x \leqslant x_2$, then the assertion follows from the Theorems 6 and 9 in [1]. If $f'(x) = 0$ for some $x$, then it follows from part 1 of the proof of Theorem 3.1 that there is a number $\xi$, $x_1 < \xi \leqslant x_2$, such that
$$f'(x) \text{ is} \begin{cases} > 0 & \text{for} \quad x_1 \leqslant x < \xi, \\ = 0 & \text{for} \quad \xi \leqslant x \leqslant x_2. \end{cases}$$

If $g(x)$ is a different a.s. minimal, then the same reasoning holds for $g(x)$. Let $\xi'$ correspond to $g(x)$. Assume that $\xi' < \xi$. This gives $g'(x_1) > f'(x_1)$ and

$$M_1 = F(x_1, y_1, f'(x_1)) < F(x_1, y_1, g'(x)) = M_2.$$

But we also have

$$M_1 = F(\xi, y_2, 0) > F(\xi', y_2, 0) = M_2.$$

The contradiction proves the theorem.

**Theorem 3.3.** *Assume that $F(x,y,z)$ satisfies these conditions:*

*$F(x,y,z)$ is analytic, as in Theorem 3.1;*

*2A and 3;*

*the condition concerning existence and boundedness of minimizing functions, which was given in Theorem 2.1;*

$$\frac{\partial F(x,y,z)}{\partial y} \text{ is } \begin{cases} >0 & \text{if } y > C, \\ =0 & \text{if } y = C, \\ <0 & \text{if } y < C, \end{cases}$$

*($C$ is a constant);*

$$F(x, \lambda y_1 + (1-\lambda) y_2, \lambda z_1 + (1-\lambda) z_2) \leqslant \lambda F(x, y_1, z_1) + (1-\lambda) F(x, y_2, z_2)$$

*for all $x$, $y_1$, $y_2$, $z_1$ and $z_2$ and for all $\lambda$ such that $0 < \lambda < 1$, equality holds if and only if $y_1 = y_2$ and $z_1 = z_2$.*

*We consider the minimization problem between $(x_1, y_1)$ and $(x_2, y_2)$. We assume that the boundary values satisfy the inequality*

$$(y_1 - C)(y_2 - C) \geqslant 0.$$

*Then there is a* UNIQUE *a.s. minimal between $(x_1, y_1)$ and $(x_2, y_2)$.*

The proof is omitted, since it is rather laborious and since the result is not used in this paper.

**3.** In the previous section, we discussed the uniqueness question for a.s. minimals. We shall now briefly consider the same question for minimizing functions. The following theorem shows that those functions $F(x,y,z)$, for which every minimization problem has a unique solution, constitute a very "small" class.

**Theorem 3.4.** *Let $F(x, y, z)$ satisfy the following conditions for $X_1 \leqslant x \leqslant X_2$ and all $y, z$:*
*$F(x, y, z) \in C^2$;*
*2A and 3;*
*the condition concerning existence and boundedness of minimizing functions which was given in Theorem 2.1.*

*We consider the minimization problem between $(x_1, y)$ and $(x_2, y_2)$ $(X_1 \leqslant x_1 < x_2 \leqslant X_2)$.*

*Then there is a* UNIQUE *minimizing function for every choice of $x_1$, $y_1$, $x_2$ and $y_2$ if and only if*

$$F(x, y, 0) \equiv constant.$$

*Proof.* 1) Suppose that $F(x, y, 0) \equiv$ constant. Consider the minimization problem between $(x_1, y_1)$ and $(x_2, y_2)$. We know that there is an a.s. minimal $f(x)$ and we know that $f(x) \in C^1$.

If $f'(x) \neq 0$ for $x_1 \leqslant x \leqslant x_2$ then it follows from Theorem 8 and Theorem 6 in [1] that $f(x)$ is a unique minimizing function.

If there is an $\alpha$ such that $f'(\alpha) = 0$, then $f'(x) \equiv 0$. In order to see that, suppose that $f'(\beta) \neq 0$ for some $\beta > \alpha$. Put

$$\gamma = \sup\{x \mid x < \beta, f'(x) = 0\}.$$

Then we get

$$F(\beta, f(\beta), f'(\beta)) > F(\beta, f(\beta), 0) = F(\gamma, f(\gamma), 0).$$

But this contradicts the relation $F(x, f, f') =$ constant which holds on every interval where $f'(x) \neq 0$.

So we have $f(x) \equiv y_1 (= y_2)$. If $g(x)$ is an admissible function and $g'(\xi) \neq 0$ for some $\xi$, then we get

$$H(g) \geqslant F(\xi, g(\xi), g'(\xi)) > F(\xi, g(\xi), 0) = H(f),$$

and we have proved one half of the theorem.

2) Suppose now that there is a unique minimizing function for every choice of $x_1$, $x_2$, $y_1$ and $y_2$.

If $F_x(x_1, y_1, 0) \neq 0$, then we choose $y_2 = y_1$ and $x_2$ such that $F_x(x, y_1, 0) \neq 0$ for $x_1 \leqslant x \leqslant x_2$. Clearly, $f(x) \equiv y_1$ is a minimizing function, but not the only one. This proves that $F_x(x, y, 0) \equiv 0$.

If $F_y(x_1, y_1, 0) \neq 0$, then we choose $y_2 = y_1$ and $x_2$ such that $F_y(x, y_1, 0) \neq 0$ for $x_1 \leqslant x \leqslant x_2$. We shall prove that $f(x) \equiv y_1$ is not the only minimizing function.

It is no restriction to assume that $x_1 = y_1 = 0$ and $F_y(0,0,0) < 0$. Consider the function $y = \alpha x^2$ where $\alpha > 0$ is a constant. Taylors formula gives

$$\begin{aligned} F(x, \alpha x^2, 2\alpha x) = F(x, 0, 0) &+ \alpha x^2 F_y(x, 0, 0) \\ &+ \tfrac{1}{2}(\alpha^2 x^4 F_{yy}(x, \theta\cdot\alpha x^2, \theta\cdot 2\alpha x) \\ &+ 2\cdot\alpha x^2\cdot 2\alpha x F_{yz}(\ldots) + 4\alpha^2 x^2 F_{zz}(\ldots)). \end{aligned}$$

If we consider a suitable neighbourhood of the origin and assume that $0 < \alpha < 1$, then $F_y(x,0,0) < -K < 0$, and the modulus of the expression in brackets is not greater than $K_1\cdot\alpha^2\cdot x^2$, where $K$ and $K_1$ are independent of $x$ and $\alpha$.

This means that

$$F(x, \alpha x^2, 2\alpha x) \leqslant F(x, 0, 0) - K\cdot\alpha x^2 + K_1 \alpha^2 x^2 = F(x, 0, 0) - \alpha x^2(K - \alpha K_1).$$

If we choose $\alpha$ so small that $K - \alpha K_1 > 0$, then we have $F(x, \alpha x^2, 2\alpha x) \leqslant F(x,0,0)$ for $0 \leqslant x \leqslant \delta$, for some $\delta > 0$.

A similar construction can be carried through for $(x_2, y_2)$, and the rest of the proof is obvious.

Therefore, if $F(x, y, 0)$ is not constant, then we must impose conditions on the boundary values in order to secure uniqueness. The following theorem illustrates this possibility.

**Theorem 3.5.** *Assume that $F(x, y, z)$ satisfies these conditions:*

*$F(x, y, z) \in C^1$;*

*2A and 3;*

*the condition concerning existence and boundedness of minimizing functions which was given in Theorem 2.1;*

$$\textit{either } \frac{\partial F(x, y, 0)}{\partial x} \geqslant 0 \textit{ for all } (x, y) \textit{ or } \leqslant 0 \textit{ for all } (x, y).$$

*Let us denote the admissible linear function by $l(x)$. Finally, we assume that*

$$\min_{x_1 \leqslant x \leqslant x_2} F(x, l(x), l'(x)) > \max_{(x, y) \in K} F(x, y, 0),$$

*where $K$ is the set*
$$\begin{cases} x_1 \leqslant x \leqslant x_2 \\ y_1 \leqslant y \leqslant y_2 (y_1 \geqslant y \geqslant y_2). \end{cases}$$

*Then there is a unique minimizing function $f(x)$. Moreover, $f'(x) \neq 0$ for $x_1 \leqslant x \leqslant x_2$.*

*Proof.* We know that there is an a.s. minimal $f(x) \in C^1$. If $f'(x) = 0$ for some $x$, then it follows from the Theorems 9 and 10 (with a trivial change) in [1] that

$$M_0 \leqslant \max_{(x, y) \in K} F(x, y, 0).$$

But it was proved in Lemma 5 in [1] that $M_0 \geqslant \min_{x_1 \leqslant x \leqslant x_2} F(x, l(x), l'(x))$. So the assumption that $f'(x) = 0$ for some $x$ leads to a contradiction. Finally, it follows from Theorem 6 in [1] that $f(x)$ is a unique minimizing function.

*Remark.* It is easy to find new results of the same type by using new estimates. It should be mentioned here that the inequality

$$M_0 \geqslant \inf_{x_1 \leqslant x \leqslant x_2} F(x, g(x), g'(x))$$

holds for every admissible monotonic function $g(x) \in C^1$ (compare Lemma 5 in [1]). This gives

$$M_0 \geqslant \sup_{g(x)} [\inf_{x_1 \leqslant x \leqslant x_2} F(x, g(x), g'(x))],$$

where the supremum is taken over all such functions.

It is also clear that the condition on $F_x$ in the theorem can be replaced by the condition in Theorem 1.1 (in this case assumed to hold for all $y$).

## Chapter 4. Comments and examples

In this chapter, we shall illustrate some of the theorems by means of examples.

*Example 1.* Suppose that

$$F(x, y, z) = G(x, y) + Ay^2 + By + z^2,$$

where $G(x,y)$ is continuous and bounded from below and $A, B$ are constants.

Is there a minimizing function? We shall apply Theorem 1.2′. Formal calculation gives

$$\begin{cases} \Phi(x, y, M) = +(M - G(x, y) - Ay^2 - By)^{1/2}, \\ \Psi(x, y, M) = -(M - G(x, y) - Ay^2 - By)^{1/2}. \end{cases}$$

If $\alpha(y, M)$ is defined for $y \geqslant y_1$, then we see from the expression for $\Phi(x, y, M)$ that $\alpha(y, M) = O(y)$, when $y \to +\infty$. Hence the integral $I_1$ in Theorem 1.2′ cannot exist finite. It follows in the same way that none of the integrals $I_2$, $I_3$ and $I_4$ can exist finite.

So we can conclude that $E(M)$ is bounded for all $M$. Consequently there is a minimizing function for every choice of $x_1$, $x_2$, $y_1$ and $y_2$.

*Example 2.* We have not studied the properties of minimizing functions in general, we have only studied the special cases of a.s. minimals and unique minimizing functions. The reason for this is clear from the following example:

Choose $F(x, y, y') = x + y'^2$, $x_1 = y_1 = 0$, $x_2 = 1$ and $y_2 = 0$. This gives $M_0 \geqslant F(x_2, y_2, 0) = 1$, and if $g(x) \equiv 0$, then $H(g) = 1$.

Thus we have $M_0 = 1$. It follows that if $f(x)$ is an admissible function, then it is a minimizing function if and only if $|f'(x)| \leqslant \sqrt{1-x}$ a.e.

Clearly, the fact that $f(x)$ is a minimizing function implies very little about $f(x)$. This motivates the introduction of a.s. minimals.

On the other hand, suppose that we have a minimizing function $g(x)$ *which belongs to* $C^1$. Then the local variation method, which was used in the end of the proof of Theorem 3.1, can be applied to $g(x)$. This method works also in the general case of variable $\omega(x, y)$. For instance, it can be used to derive the following result: if

$$F_z(x, g(x), g'(x)) \neq 0$$

for all $x$, then $F(x, g(x), g'(x)) = M_0$ for all $x$. (This is mentioned in [6] with a sketch of a proof.) This result leads us once again to the differential equation $(dF/dx) \cdot F_z = 0$ for a.s. minimals.

*Example 3.* We have proved in Chapter 2 that if $F(x, y, z)$ satisfies certain conditions, then there is an a.s. minimal for every choice of the boundary values, and we have also proved that every a.s. minimal belongs to $C^1$.

*Consequently, there is a minimizing function in* $C^1$.

However, there need not exist a minimizing function in $C^2$, as can be seen from the following example:

Put $F(x, y, y') = y'^2 - \cos^2 x$, $x_1 = y_1 = 0$, $x_2 = n\pi$ and $y_2 = 2n$. We assert that

$$f(x) = \int_0^x |\cos t| \, dt$$

is the only minimizing function. Clearly, $f'(x) \geqslant 0$ and $F(x, f(x), f'(x)) \equiv 0$. If $H(g) \leqslant 0$, then it follows that $g'(x) \leqslant f'(x)$ a.e. This implies that $g(x) \equiv f(x)$, which proves our assertion.

Finally, it is easy to verify that $f''(x)$ has a jump at $x = \pi/2 + k \cdot \pi$, $k = 0, 1, 2, \ldots, n-1$.

*Example 4.* Let us give a complete treatment of the minimization problem if $F(x, y, y') = -x + y'^2$ and with arbitrary boundary values.

The case $y_1 = y_2$ is trivial: $f(x) \equiv y_1$ is the only a.s. minimal (see Theorem 10 in [1]) but not the only minimizing function.

So let us assume $y_1 \neq y_2$. Let $f(x)$ be an a.s. minimal (Theorem 2.1). Since $F_x \equiv -1$, it follows (from part 1 of the proof of Theorem 3.1) that there are two possibilities:

A) $f'(x) \neq 0$ for $x_1 \leqslant x \leqslant x_2$,

B) there is a number $\xi$, $x_1 \leqslant \xi < x_2$, such that $f'(x)$ is $=0$ for $x_1 \leqslant x \leqslant \xi$ and $\neq 0$ for $\xi < x \leqslant x_2$.

Consider the equation $-x + (f'(x))^2 = \alpha$. It gives

$$f(x) = \pm \tfrac{2}{3}(\alpha + x)^{3/2} + \beta.$$

If we introduce the function

$$\varphi(t) = \begin{cases} \tfrac{2}{3}t^{3/2} & \text{for } t \geqslant 0, \\ 0 & \text{for } t \leqslant 0. \end{cases}$$

then it follows that we have

$$f(x) = \pm \varphi(x + \alpha) + \beta \quad \text{for} \quad x_1 \leqslant x \leqslant x_2.$$

Clearly, the sign and the constants $\alpha$ and $\beta$ are uniquely determined from the boundary conditions. Consequently, there is a unique a.s. minimal. (This can also be seen from Theorem 3.2.)

It is also clear that $f(x)$ is the only minimizing function if and only if $x_1 + \alpha \geqslant 0$, and this holds if and only if

$$|y_2 - y_1| \geqslant \tfrac{2}{3}(x_2 - x_1)^{3/2}. \tag{1}$$

(The reader might study the dependence of $\varphi(x_2 + \alpha) - \varphi(x_1 + \alpha)$ on $\alpha$.)

We have thus proved that *there is always a unique a.s. minimal*, and that *there is a unique minimizing function if and only if the inequality (1) holds.*

It may be interesting to compare and see what Theorem 3.5 gives in this case. The theorem says that there is a unique minimizing function if

$$-x_2 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 > -x_1,$$

i.e. if

$$|y_2 - y_1| > (x_2 - x_1)^{3/2},$$

which is not too far from the condition (1).

*Example 5.* So far, we have assumed that $\lim_{|z| \to \infty} F(x, y, z) = +\infty$, that $\varphi(z) = F(x_0, y_0, z)$ has a single minimum, at $z = \omega(x_0, y_0)$, and that $\varphi(z)$ is monotonic for $z > \omega(x_0, y_0)$ as well as for $z < \omega(x_0, y_0)$. Now, let us retain the first condition but allow $\varphi(z)$ to have several minima. This example will illustrate such a case. Choose

$$F(x, y, z) = y^2 + (z^2 - 1)^2,$$

$$x_1 = y_1 = 0, \quad x_2 = 1 \quad \text{and} \quad y_2 = 0.$$

Clearly, $H(f)>0$ for every admissible $f(x)$. But there is a sequence $\{f_n(x)\}_1^\infty$, tending to zero uniformly, such that $f_n'(x)$ takes only the values $\pm 1$. Hence $\lim_{n\to\infty} H(f_n)=0$.

*Consequently, there is no minimizing function in spite of the fact that $E(M)$ is bounded for every $M$.* Thus $H(f)$ is no longer lower semi-continuous.

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