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On the existence of solutions of differential equations with constant coefficients

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1. Introduction

Let
$$P = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha \quad \left(D^\alpha = (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}\right)$$

be a linear partial differential operator in R^n with constant complex coefficients a_α (α is a multiindex $(\alpha_1, \dots, \alpha_n)$ and $|\alpha| = \sum \alpha_j$). Consider the equation

$$Pu = f \tag{1.1}$$

in an open subset Ω of R^n. Malgrange [3] has given a necessary and sufficient condition on Ω for the existence of solutions u of (1.1) for every distribution f with finite order in Ω, if the solution u shall also be a distribution of finite order in Ω. Hörmander [1] has found a corresponding condition for the general case, when f and u may be arbitrary distributions in Ω. This paper deals with the problem of finding necessary or sufficient conditions for the existence of solutions of (1.1), when f and the solution u are supposed to be distributions in Ω with finite order on certain given subsets of Ω. In particular we should obtain Hörmander's condition, when these subsets are compact, and Malgrange's condition, when they coincide with Ω. However, the results are rather incomplete, unless we also require that the order of the solution u shall depend on the order of f in a certain sense. The main part of the paper is therefore concerned with this restricted case.

I wish to thank my teacher, Professor Hörmander, for his valuable help and advice during my work on this problem.

2. Preliminaries

$C^k(\Omega)$ shall be the space of complex-valued functions in Ω with continuous derivatives of order $\leqslant k$ ($k = 0, 1, \dots$ or $+\infty$). It is a Fréchet space, if the topology is defined by all semi-norms $f \to \sup_K |D^\alpha f|$, where $|\alpha| \leqslant k$ and K is an arbitrary compact subset of Ω. If S is a subset of R^n, then $C_0^k(S)$ shall be the space of functions $\varphi \in C^k(R^n)$ with compact support contained in S.

We shall consider distributions in Ω, which are continuous linear forms on the spaces in the following definition.

Definition 2.1. *Suppose that $A=(A_k)_1^\infty$ is a non-decreasing sequence of relatively closed subsets of Ω such that every compact subset of Ω is contained in some A_k. If $M=(M_k)_1^\infty$ is a non-decreasing sequence of integers $\geqslant 0$, then $\mathcal{D}_M(\Omega, A)$ shall denote the space $C_0^\infty(\Omega)$ equipped with the topology that is defined by all semi-norms p of the form*

$$p(\varphi)=\sup_\alpha \sup_x \varrho_\alpha(x)\,|D^\alpha\varphi(x)|, \quad \varphi\in C_0^\infty(\Omega), \tag{2.1}$$

where every ϱ_α is a non-negative function in Ω, bounded on every compact subset of Ω, and where $\varrho_\alpha=0$ in A_k, if $|\alpha|>M_k$.

$\mathcal{D}_F(\Omega, A)$ shall denote the space $C_0^\infty(\Omega)$ with the topology that is defined by all semi-norms p, which satisfy the above conditions for some sequence M.

If $f\in\mathcal{D}'_M(\Omega, A)$, then f is a distribution in Ω with order $\leqslant M_k$ in A_k for every k. ($'$ denotes the topological dual space.) The converse of this implication is not always true, but we shall prove a somewhat weaker statement:

Lemma 2.2. *With A and M given as in Definition 2.1 suppose that f is a distribution in Ω, which has order $\leqslant M_k$ in an open neighbourhood Ω_k of A_k for every k. Then $f\in\mathcal{D}'_M(\Omega, A)$.*

Proof. We choose a sequence $(K_j)_1^\infty$ of compact subsets of Ω such that $K_j\nearrow\nearrow\Omega$ when $j\to\infty$. [If $(S_k)_1^\infty$ is a sequence of subsets of a set $S\subset R^n$, we write "$S_k\nearrow\nearrow S$ when $k\to\infty$" to express that $\bar{S}_k\subset S_{k+1}^\circ$ in the relative topology on S for every k and that $\bigcup_1^\infty S_k=S$.]

We can find functions $\chi_{jl}\in C_0^\infty((K_j\setminus K_{j-2})\cap(\Omega_l\setminus A_{l-1})), j, l=1, 2, \ldots,$ such that $\sum_{j,l}\chi_{jl}=1$ in Ω (partition of the unity, see [4], Chap. I, Théorème II). Here we have set $K_{-1}=K_0=A_0=\phi$. For every compact $K\subset\Omega$ we have $\chi_{jl}=0$ in K except for a finite number of indices. It follows that

$$|f(\varphi)|=|\textstyle\sum_{j,l}f(\chi_{jl}\varphi)|\leqslant\sum_{j,l}C_{jl}\sup_{|\beta|\leqslant M_l}\sup_x|D^\beta(\chi_{jl}\varphi)|, \quad \varphi\in C_0^\infty(\Omega),$$

with suitable constants C_{jl}, because $\chi_{jl}\varphi\in C_0^\infty(\Omega_l)$ and f has order $\leqslant M_l$ in Ω_l. Repeated use of Leibniz' formula for differentiation of a product and the inequality $\sum_1^\infty a_\nu\leqslant\sup_\nu 2^\nu a_\nu$ for $a_\nu\geqslant 0$ then gives the estimate

$$|f(\varphi)|\leqslant\sup_{j,l,\beta,\alpha}\sup_x \varrho_{jl\beta\alpha}|D^\alpha\varphi|, \quad \varphi\in C_0^\infty(\Omega),$$

where $\varrho_{jl\beta\alpha}$ is a constant times $|D^{\beta-\alpha}\chi_{jl}|$, if $|\beta|\leqslant M_l$ and $\alpha\leqslant\beta$, and else identically zero. The estimate can be written

$$|f(\varphi)|\leqslant\sup_\alpha\sup_x\varrho_\alpha|D^\alpha\varphi|, \quad \varphi\in C_0^\infty(\Omega), \tag{2.2}$$

where $\varrho_\alpha=\sup_{j,l,\beta}\varrho_{jl\beta\alpha}$ is a continuous non-negative function in Ω for every α, because on every compact subset of Ω the supremum in the definition of ϱ_α need only be taken over a finite number of indices. If $|\alpha|>M_k$, $|\beta|\leqslant M_l$ and $\alpha\leqslant\beta$, then $\varrho_{jl\beta\alpha}=0$ in A_k, since $k\leqslant l-1$ and $\chi_{jl}=0$ in a neighbourhood of A_{l-1}. Hence $\varrho_\alpha=0$ in A_k, if $|\alpha|>M_k$, and the right-hand side of (2.2) defines a continuous semi-norm on $\mathcal{D}_M(\Omega, A)$. This proves that $f\in\mathcal{D}'_M(\Omega, A)$.

Evidently $\mathcal{D}'_F(\Omega, A)=\bigcup_M\mathcal{D}'_M(\Omega, A)$. We shall consider two particular cases:

If all A_k are compact subsets of Ω, we shall write $\mathcal{D}(\Omega)$ instead of $\mathcal{D}_F(\Omega, A)$. (We observe that we obtain the same topology with every such A.) The dual space $\mathcal{D}'(\Omega)$ is then the space of all distributions in Ω.

When $A_k = \Omega$ for all k, we shall write $\mathcal{D}_F(\Omega)$ instead of $\mathcal{D}_F(\Omega, A)$. We observe that $\mathcal{D}'_F(\Omega)$ is the space of distributions of finite order in Ω.

For $u \in \mathcal{D}'(\Omega)$ the support of u in Ω is denoted by supp u. If S is a subset of R^n, the space of $u \in \mathcal{D}'(R^n)$, which have compact supports contained in S, shall be denoted by $\mathcal{E}'(S)$.

Similarly the singular support of $u \in \mathcal{D}'(\Omega)$ is written sing supp u, that is, sing supp u is the complement in Ω of the largest open subset Ω' of Ω such that $u \in C^\infty(\Omega')$.

For every $u \in \mathcal{D}'(\Omega)$ the distribution Pu in Ω is defined by $(Pu)(\varphi) = u(\check{P}\varphi)$, $\varphi \in C_0^\infty(\Omega)$, where $\check{P}$ is the adjoint operator of P, that is $\check{P} = \sum_{|\alpha| \leqslant m} a_\alpha(-1)^{|\alpha|} D^\alpha$.

When t is a real number, $\mathcal{H}_{(t)}$ shall be the space of temperate distributions u in R^n such that the Fourier transform $\hat{u} \in L_1^{\mathrm{loc}}(R^n)$ and

$$\|u\|_{(t)} = \left((2\pi)^{-n} \int (1 + |\xi|^2)^t |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} < +\infty$$

$\mathcal{H}_{(t)}$ is a Hilbert space with the norm $u \to \|u\|_{(t)}$. We shall also use the space $\mathcal{H}_{(t)}^{\mathrm{loc}}(\Omega)$ of distributions u in Ω such that $\varphi u \in \mathcal{H}_{(t)}$ for every $\varphi \in C_0^\infty(\Omega)$. See [2] Chapters I and II. Here we mention that $C_0^p(R^n) \subset \mathcal{H}_{(p)}$ for every non-negative integer p. Conversely we have

Lemma 2.3 (*Sobolev*). *If $u \in \mathcal{D}'(\Omega)$ and $D^\alpha u \in \mathcal{H}_{(t)}^{\mathrm{loc}}(\Omega)$ for $|\alpha| \leqslant p$, where p is an integer $\geqslant 0$, then $u \in C^r(\Omega)$, when r is an integer such that $0 \leqslant r < t + p - n/2$.*

A proof of this lemma is implicitly contained in the proof of Lemma 3.6.1 in [2].

3. A sufficient condition[1]

In this and the following sections let $A = (A_k)_1^\infty$ be a sequence of relatively closed subsets of Ω such that $A_k \nearrow \nearrow \Omega$ when $k \to \infty$ (for this notation see the proof of Lemma 2.2). Then the following theorem gives a sufficient condition for the existence of a solution $u \in \mathcal{D}'_F(\Omega, A)$ of the eqation $Pu = f$ for every $f \in \mathcal{D}'_F(\Omega, A)$. This condition, however, also implies that u can be chosen in $\mathcal{D}'_N(\Omega, A)$, if $f \in \mathcal{D}'_M(\Omega, A)$ for some M, N depending only on M.

Theorem 3.1. Suppose that

(*a*) *Ω is P-convex, that is, given a compact $K \subset \Omega$ there is a compact $K' \subset \Omega$ such that*

$$\mu \in \mathcal{E}'(\Omega), \quad \operatorname{supp} \check{P}\mu \subset K \Rightarrow \operatorname{supp} \mu \subset K',$$

(*b*) *to every integer $k > 0$ there is an integer $l > 0$ such that to every integer $j > 0$ and to every integer $r \geqslant 0$ there is an integer $s \geqslant 0$ such that*

$$\varphi \in C_0^0(A_j), \check{P}\varphi \in C^s(\complement A_k) \Rightarrow \varphi \in C^r(\complement A_l).$$

[1] After the manuscript was written I found that W. Słowikowski has obtained conditions for the solvability of linear equations in LF-spaces (Bull. Am. Math. Soc. 69: 6, 832–834 (1963)). They seem to be closely connected with the conditions obtained in this and the following section. However, his proofs have not been available to me and I have not been able to determine to which extent his results can be applied here.

(Here and in the sequel $\complement$ *denotes complement in* Ω.) *Then given a non-decreasing sequence* $M=(M_k)_1^\infty$ *of integers* $\geqslant 0$ *there is another such sequence* $N=(N_k)_1^\infty$, *for which the mapping* $\check{P}: \mathcal{D}_M(\Omega, A) \to \mathcal{D}_N(\Omega, A)$ *has a continuous inverse, which implies that* $P\mathcal{D}'_N(\Omega, A) \supset \mathcal{D}'_M(\Omega, A)$.

First we deduce a more convenient condition from (*b*).

Lemma 3.2. *Suppose that (b) in Theorem 3.1 is satisfied. Then given a non-decreasing sequence* $(r_k)_0^\infty$ *of integers* $\geqslant 0$ *there is another such sequence* $(s_k)_0^\infty$, *for which*

$$\mu \in \mathcal{E}'(\Omega),\ \check{P}\mu \in C^{s_k}(\complement A_k),\ k=0,1,\ldots, \Rightarrow \mu \in C^{r_k}(\complement A_k),\ k=0,1,\ldots, \tag{3.1}$$

where $A_0=\phi$.

Proof. For every $k>0$ we choose $l=l_k$ according to (*b*) and set $A'_k = A_{l_k}$. We can assume that (l_k) is strictly increasing. With $s_0 = r_{l_1-1}+n+1$ we can then successively choose integers s_k so that $s_0 \leqslant s_1 \leqslant \ldots$ and

$$\varphi \in C_0^0(A'_{k+2}),\ \check{P}\varphi \in C^{s_k}(\complement A_k) \Rightarrow \varphi \in C^{R_k}(\complement A'_k), \quad k=1,2,\ldots, \tag{3.2}$$

where $R_k = \max(r_{l_{k+1}-1}, s_{k-1}+m)$.

Assume that $\mu \in \mathcal{E}'(\Omega)$ and that $\check{P}\mu \in C^{s_k}(\complement A_k)$ for $k=0,1,\ldots$. Now $\mu = E * \check{P}\mu$, if E is a fundamental solution of $\check{P}$ ($*$ denotes convolution). E can be chosen as a distribution of order $\leqslant n+1$ in R^n ([2], Theorem 3.1.1). Hence it follows that $\mu \in C^{r_k}(R^n)$ for $k<l_1$, because $r_k+n+1 \leqslant s_0$ for these k (see the remark after Théorème XI in [4], Chap. VI). Assume that $\mu \in C^{R_k}(\complement A'_k)$ for some $k>1$ and choose $\chi \in C_0^\infty(A'_{k+1})$ with $\chi=1$ in a neighbourhood of $A'_k \cap \operatorname{supp} u$. It follows that $\check{P}(\chi\mu) \in C^{s_{k-1}}(\complement A_{k-1})$, because $\chi\mu \in C^{s_{k-1}+m}(\complement A'_k)$ and $\check{P}(\chi u) = \check{P}u$ in a neighbourhood of A'_k. Hence (3.2) gives that $\chi\mu \in C^{R_{k-1}}(\complement A'_{k-1})$, which implies that $\mu \in C^{R_{k-1}}(\complement A'_{k-1})$, since $\mu \in C^{R_{k-1}}(\complement A'_k)$ and $\chi\mu = \mu$ in a neighbourhood of A'_k. If we observe that $\operatorname{supp}\mu \subset A'_k$ and therefore $\mu \in C^\infty(\complement A'_k)$ for some k, we can now use induction on decreasing k to conclude that $\mu \in C^{R_k}(\complement A'_k)$ for $k=1,2,\ldots$. The proof is complete, because $r_l \leqslant r_{l_{k+1}-1} \leqslant R_k$ and $\complement A_l \subset \complement A'_k$, if $l_k \leqslant l < l_{k+1}$.

Proof of Theorem 3.1. $\check{P}$ is one-to-one on $\mathcal{E}'(R^n)$ so the mapping $\check{P}: \mathcal{D}_M(\Omega, A) \to \mathcal{D}_N(\Omega, A)$ has always an inverse. Continuity of the inverse means that for every continuous semi-norm p on $\mathcal{D}_M(\Omega, A)$ there is a continuous semi-norm q on $\mathcal{D}_N(\Omega, A)$ such that

$$p(\varphi) \leqslant q(\check{P}\varphi), \quad \varphi \in C_0^\infty(\Omega). \tag{3.3}$$

If we have proved this and if $f \in \mathcal{D}'_M(\Omega, A)$, we can use the Hahn–Banach theorem to extend the linear form $\check{P}\varphi \to f(\varphi)$, $\varphi \in C_0^\infty(\Omega)$, to a continuous linear form u on $\mathcal{D}_N(\Omega, A)$, that is, $u \in \mathcal{D}'_N(\Omega, A)$ and $(Pu)(\varphi) = u(\check{P}\varphi) = f(\varphi)$, $\varphi \in C_0^\infty(\Omega)$. Therefore it remains to prove the continuity of the inverse of $\check{P}: \mathcal{D}_M(\Omega, A) \to \mathcal{D}_N(\Omega, A)$ with a suitable choice of N.

Given M we set $r_k = M_{k+2}$, $k=0,1,\ldots$, and we choose a non-decreasing sequence $(s_k)_0^\infty$ of integers $\geqslant 0$ so that (3.1) is valid, which is possible according to Lemma 3.2. We now define N by $N_k = s_{k-1}+1$, $k=1,2,\ldots$.

Let p be a continuous semi-norm on $\mathcal{D}_M(\Omega, A)$. We assume that p is given as in Definition 2.1, which is sufficient when we want to prove an estimate of the form (3.3). To obtain q in (3.3) we shall use a method of successive extensions (cf. the proof of Theorem 4.5 in [1]).

Q shall be the set of semi-norms q on $\mathcal{D}_N(\Omega, A)$, which have the form

$$q(\varphi) = \sup_\alpha \sup_x \sigma_\alpha |D^\alpha \varphi|, \quad \varphi \in C_0^\infty(\Omega), \tag{3.4}$$

where every σ_α is a continuous function in Ω, satisfying

$$\sigma_\alpha > 0 \quad \text{in} \quad \Omega, \quad \text{if} \quad |\alpha| \leqslant N_1, \tag{3.5}$$

whereas, if $N_k < |\alpha| \leqslant N_{k+1}$, we have

$$\varrho_\alpha = 0 \quad \text{in} \quad A_k \quad \text{and} \quad \sigma_\alpha > 0 \quad \text{in} \quad \complement A_k, \tag{3.6}$$

$$\frac{\sigma_\alpha(x)}{\sigma_\alpha(y)} \leqslant 1 + O\left(\frac{|x-y|}{d\,(x, A_k)}\right), \quad \text{if} \quad |x-y| < d(x, A_k)/2 \quad \text{and} \quad x, y \in K \cap \complement A_k, \tag{3.7}$$

and
$$\sigma_\alpha(x) = O(d\,(x, A_k)^{|\alpha|+1}), \quad \text{if} \quad x \in K, \tag{3.8}$$

when K is a compact subset of Ω.

Q is not empty. To see this it is sufficient to observe that $\sigma_\alpha(x) = c_\alpha(x)\, d\,(x, A_k)^{|\alpha|+1}$ satisfies (3.6)–(3.8), if c_α is an arbitrary positive continuous function in Ω.

According to (a) we can choose sequences $(K_j)_0^\infty$ and $(K_j')_0^\infty$ of compact subsets of Ω so that $K_0 = K_1 = K_0' = K_1' = \phi$, $K_j \nearrow \nearrow \Omega$ and $K_j' \nearrow \nearrow \Omega$ when $j \to \infty$ and

$$\mu \in \mathcal{E}'(\Omega), \operatorname{supp} \check{P}\mu \subset K_j \Rightarrow \operatorname{supp} \mu \subset K_j', \, j = 0, 1, \ldots. \tag{3.9}$$

The main step in the proof is the following lemma (cf. Lemma 4.1 in [1]).

Lemma 3.3. *With the previous notations let q be a semi-norm $\in Q$ such that for some integer $j > 0$*

$$p(\varphi) \leqslant q(\check{P}\varphi), \quad \varphi \in C_0^\infty(K_j'). \tag{3.10}$$

Then given $\varepsilon > 0$ one can find a semi-norm $q' \in Q$ such that

$$p(\varphi) \leqslant q'(\check{P}\varphi), \quad \varphi \in C_0^\infty(K_{j+1}')$$

and $q' = (1+\varepsilon)\, q$ on $C_0^\infty(K_{j-1})$.

Proof of Theorem 3.1, continued. Suppose for a moment that Lemma 3.3 is proved. We then choose numbers $\varepsilon_j > 0, j = 1, 2, \ldots,$ such that $\prod_1^\infty (1 + \varepsilon_j) < +\infty$. For any $q_1 \in Q$ we have $p(\varphi) \leqslant q_1(\check{P}\varphi)$, when $\varphi \in C_0^\infty(K_1')$, since $K_1' = \phi$. Using Lemma 3.3 we can therefore successively find semi-norms $q_j \in Q$ such that (3.10) is fulfilled with $q = q_j$ and $q_{j+1} = (1 + \varepsilon_j)\, q_j$ on $C_0^\infty(K_{j-1})$ for every $j > 0$. It follows that $q(\varphi) = \lim_{j\to\infty} q_j(\varphi)$ exists for every $\varphi \in C_0^\infty(\Omega)$, because $q_j(\varphi) = \prod_{i+1}^{j-1}(1 + \varepsilon_k)\, q_{i+1}(\varphi)$, if $\operatorname{supp} \varphi \subset K_i$ and $j > i + 1$. It is obvious that q is a continuous semi-norm on $\mathcal{D}_N(\Omega, A)$, satisfying (3.3). Thus it only remains to prove Lemma 3.3.

Proof of Lemma 3.3. Let $\mathfrak{J}$ be the space of $\mu \in \mathcal{E}'(K_{j+1}')$ such that $\check{P}\mu \in C^{s_k}(\complement A_k)$ for every $k \geqslant 0$. $\mathfrak{J}$ is a Fréchet space, if the topology is defined by all semi-norms $\mu \to \sup_K |D^\alpha \check{P}\mu|$, where K is an arbitrary compact subset of Ω such that $K \subset \complement A_k$ for some k with $|\alpha| \leqslant s_k$. From (3.1) it follows that $\mu \in C^{r_k}(\complement A_k)$ for every $k \geqslant 0$, if $\mu \in \mathfrak{J}$. In this way natural mappings $\mathfrak{J} \to C^{r_k}(\complement A_k)$ are defined, and they are continuous for every $k \geqslant 0$ in virtue of the closed graph theorem.

Now assume that the lemma is not true. This means that we can find a sequence $(\varphi_\nu)_1^\infty$ in $C_0^\infty(K_{j+1}')$ such that

$$p(\varphi_\nu) > 1+\varepsilon \quad \text{and} \quad q(\check{P}\varphi_\nu) \leqslant 1, \quad \nu = 1, 2, \ldots, \tag{3.11}$$

and
$$\check{P}\varphi_\nu \to 0 \quad \text{in} \quad C^0(\complement K_{j-1}) \quad \text{when} \quad \nu \to \infty. \tag{3.12}$$

For to every compact $K \subset \complement K_{j-1}$ and every constant C we can find a semi-norm $q' \in \boldsymbol{Q}$ such that $q' \geqslant (1+\varepsilon)q$ with equality on $C_0^\infty(K_{j-1})$ and $q'(\varphi) \geqslant C \sup_K |\varphi|$, $\varphi \in C_0^\infty(\Omega)$.

q has the form (3.4), where every σ_α is continuous and satisfies (3.5)–(3.8). Using the continuity of σ_α and (3.5) or (3.6) we conclude from the second inequality in (3.11) that $(\check{P}\varphi_\nu)$ is a bounded sequence in $C^{N_k+1}(\complement A_k)$ for every $k \geqslant 0$. Hence Ascoli's theorem gives a subsequence $(\psi_\nu)_1^\infty$ from (φ_ν) such that $(\check{P}\psi_\nu)$ converges in $C^{N_k+1-1}(\complement A_k)$ for every $k \geqslant 0$ when $\nu \to \infty$. But $N_{k+1} - 1 = s_k$ so this means that (ψ_ν) is a Cauchy sequence in $\mathfrak{F}$, hence that it converges to an element ψ in $\mathfrak{F}$. Then the continuity of the natural mappings $\mathfrak{F} \to C^{r_k}(\complement A_k)$ shows that $\psi_\nu \to \psi$ in $C^{r_k}(\complement A_k)$ for every $k \geqslant 0$ when $\nu \to \infty$. Now $\check{P}\psi = \lim \check{P}\psi_\nu = 0$ in $\complement K_{j-1}$ in view of (3.12), so (3.9) gives that $\operatorname{supp} \psi \subset K'_{j-1}$.

With a non-negative function $\chi \in C_0^\infty(R^n)$ such that $\chi(x) = 0$, when $|x| \geqslant 1$, and $\int \chi\, dx = 1$ we set $\chi_\delta(x) = \delta^{-n}\chi(x/\delta)$ for $\delta > 0$. We have $\psi * \chi_\delta \in C_0^\infty(K'_j)$ and $\psi_\nu * \chi_\delta \in C_0^\infty(K'_{j+2})$ for all ν, when δ is sufficiently small. Furthermore $\psi * \chi_\delta \to \psi$ in $C^{r_k}(\complement A_k)$, $k = 0, 1, \ldots$, when $\delta \to +0$. Since $\operatorname{supp} \varrho_\alpha \cap K'_{j+1}$ (with ϱ_α given in Definition 2.1) is a compact subset of Ω, when $|\alpha| \leqslant M_2 = r_0$, and of $\complement A_k$, when $M_{k+1} < |\alpha| \leqslant M_{k+2} = r_k$, we obtain therefore

$$\begin{aligned}\lim_{\delta\to+0} p(\psi * \chi_\delta) &= \lim_{\delta\to+0} \sup_\alpha \sup_x \varrho_\alpha |D^\alpha \psi * \chi_\delta| = \sup_\alpha \sup_{\varrho_\alpha(x)>0} \varrho_\alpha(x) |D^\alpha \psi(x)| \\ &= \lim_{\nu\to\infty} \sup_\alpha \sup_x \varrho_\alpha |D^\alpha \psi_\nu| \geqslant 1+\varepsilon \end{aligned} \tag{3.13}$$

because we need only take the suprema over those finitely many α for which $\operatorname{supp} \varrho_\alpha \cap K'_{j+1} \neq \phi$. The last inequality in (3.13) follows from (3.11).

We shall also prove that

$$\limsup_{\delta\to+0} q(\check{P}\psi * \chi_\delta) \leqslant 1, \tag{3.14}$$

which together with (3.13) contradicts (3.10), since $\psi * \varphi_\delta \in C_0^\infty(K'_j)$ for all sufficiently small $\delta > 0$. This will complete the proof of the lemma.

To prove (3.14) we observe that $D^\alpha \check{P}\psi_\nu \to D^\alpha \check{P}\psi$ in the weak topology of $\mathcal{D}'(R^n)$ for every α when $\nu \to \infty$. This implies that for fixed $x \in \Omega$

$$\begin{aligned}\sigma_\alpha(x) |D^\alpha(\check{P}\psi * \chi_\delta)(x)| &= \lim_{\nu\to\infty} \sigma_\alpha(x) |(D^\alpha \check{P}\psi_\nu * \chi_\delta)(x)| \\ &\leqslant \liminf_{\nu\to\infty} \sigma_\alpha(x) \int |D^\alpha \check{P}\psi_\nu(y) \chi_\delta(x-y)|\, dy \\ &\leqslant \liminf_{\nu\to\infty} \sup_{|x-y|\leqslant\delta} \frac{\sigma_\alpha(x)}{\sigma_\alpha(y)} \sup_y \sigma_\alpha(y) |D^\alpha \check{P}\psi_\nu(y)| \leqslant \sup_{|x-y|\leqslant\delta} \frac{\sigma_\alpha(x)}{\sigma_\alpha(y)}, \end{aligned} \tag{3.15}$$

because $\int |\chi_\delta|\, dy = \int \chi_\delta\, dy = 1$ and $q(\check{P}\psi_\nu) \leqslant 1$. If $N_k < |\alpha| \leqslant N_{k+1}$, then (3.15) is valid only when $d(x, A_k) > \delta$.

If $|\alpha| \leqslant N_1$, it follows from the continuity of σ_α and from (3.5) that

$$\sup_{|x-y|\leqslant\delta}\frac{\sigma_\alpha(x)}{\sigma_\alpha(y)}\to 1 \quad \text{uniformly for} \quad x\in K_j' \quad \text{when} \quad \delta\to+0. \tag{3.16}$$

When $N_k<|\alpha|\leqslant N_{k+1}$, we obtain from (3.7) an estimate

$$\sup_{|x-y|\leqslant\delta}\frac{\sigma_\alpha(x)}{\sigma_\alpha(y)}\leqslant 1+\frac{C}{t}, \quad \text{if} \quad x\in K_j', d(x,A_k)>t\delta \quad \text{and} \quad t>2, \tag{3.17}$$

for sufficiently small $\delta>0$ with C independent of δ and t, whereas (3.8) gives (with C' independent of δ and t)

$$\sigma_\alpha(x)\,|(D^\alpha\check{P}\psi*\chi_\delta)(x)|\leqslant\sigma_\alpha(x)\int|\check{P}\psi(x-y)\,D^\alpha\chi_\delta(y)|\,dy$$

$$=\sigma_\alpha(x)\,\delta^{-|\alpha|}\int|\check{P}\psi(x-\delta y)\,D^\alpha\chi(y)|\,dy\leqslant C'\,t^{|\alpha|+1}\delta,$$

if $$x\in K_j' \quad \text{and} \quad d(x,A_k)\leqslant t\delta. \tag{3.18}$$

We now take the supremum of $\sigma_\alpha(x)|D^\alpha(\check{P}\psi*\chi_\delta)(x)|$ over all $x\in K_j'$ and over those finitely many α for which $\operatorname{supp}\sigma_\alpha\cap K_j'\neq\phi$. Then (3.14) follows, if we use (3.15)–(3.18). The proof is complete.

Corollary 3.4. *If (a) and (b) in Theorem 3.1 are satisfied, then the mapping* $\check{P}\mathcal{D}_F(\Omega,A)\to\mathcal{D}_F(\Omega,A)$ *has a continuous inverse and* $P\mathcal{D}_F'(\Omega,A)=\mathcal{D}_F'(\Omega,A)$.

Applying this corollary to the case when $A_k=\Omega$ for every k we obtain Malgrange's existence theorem, if we observe that condition (b) is always satisfied in this case:

Corollary 3.5. (*Malgrange* [3].) *If* Ω *is P-convex, then* $P\mathcal{D}_F'(\Omega)=\mathcal{D}_F'(\Omega)$.

We observe that in this case the proof of Theorem 3.1 is essentially the proof of Theorem 5.4 in [1].

4. Some necessary conditions

In this section we shall show that the sufficient conditions in Theorem 3.1 are also necessary for the conclusions to be true.

Theorem 4.1. *Suppose that for every non-decreasing sequence M of integers* $\geqslant 0$ *there is another such sequence N for which* $P\mathcal{D}_N'(\Omega,A)\supset\mathcal{D}_M'(\Omega,A)$. *Then (a) and (b) in Theorem 3.1 are true.*

Proof. (a) follows from Theorem 3.5.4 in [2], since the hypothesis implies that $P\mathcal{D}'(\Omega)\supset C^\infty(\Omega)$.

To prove (b) we observe that given a non-decreasing sequence $M=(M_l)_1^\infty$ of integers $\geqslant 0$ we can define a distribution $f\in\mathcal{D}_M'(\Omega,A)$ by $f(\varphi)=D^\alpha\varphi(x)$, $\varphi\in C_0^\infty(\Omega)$, if $x\in\complement A_l$ and $|\alpha|\leqslant M_{l+1}$ for some l. From the hypothesis it follows that $D^\alpha\varphi(x)=(Pu)(\varphi)=u(\check{P}\varphi)$, $\varphi\in C_0^\infty(\Omega)$, for some distribution $u\in\mathcal{D}_N'(\Omega,A)$, where $N=(N_l)_1^\infty$ is a non-decreasing sequence of integers $\geqslant 0$, depending only on M. Hence we obtain

$$|D^\alpha\varphi(x)|\leqslant q(\check{P}\varphi), \quad \varphi\in C_0^\infty(\Omega), \tag{4.1}$$

if $x \in \complement A_l$ and $|\alpha| \leqslant M_{l+1}$, with a continuous semi-norm q on $\mathcal{D}_N(\Omega, A)$, possibly depending on x and α.

Assume that (b) is not true. This means that there is an integer $k > 0$ such that for every integer $l > 0$ there are integers $j_l > 0$ and $r_l \geqslant 0$ such that for every integer $s \geqslant 0$ we can find functions φ_{ls} satisfying

$$\varphi_{ls} \in C_0^0(A_{j_l}) \quad \text{and} \quad \check{P}\varphi_{ls} \in C^s(\complement A_k) \quad \text{but} \quad \varphi_{ls} \notin C^{r_l}(\complement A_l), \qquad l = 1, 2, \ldots, \quad s = 0, 1, \ldots. \tag{4.2}$$

We choose functions $\chi_{ls} \in C_0^\infty(\mathring{A}_{k+1})$ with $\chi_{ls} = 1$ in a neighbourhood of $A_k \cap \operatorname{supp} \varphi_{ls}$ and open balls ω_{ls} with centers at the origin such that

$$\operatorname{supp} \chi_{ls} + \omega_{ls} \subset A_{k+1} \tag{4.3}$$

and

$$\operatorname{supp} \varphi_{ls} + \omega_{ls} \subset A_{j_l+1}. \tag{4.4}$$

We observe that $\varphi_{ls} * \varphi \in C_0^\infty(A_{j_l+1})$, if $\varphi \in C_0^\infty(\omega_{ls})$, according to (4.4). Hence (4.1) gives an estimate

$$\begin{aligned} |(D^\alpha \varphi_{ls} * \varphi)(x)| &\leqslant q(\check{P}\varphi_{ls} * \varphi) \\ &\leqslant C'_{\alpha,x} \sup_{|\beta| \leqslant N_{k+1}} \sup |D^\beta \psi'_{ls} * \varphi| + C''_{\alpha,x} \sup_{|\beta| \leqslant N_{j_l+1}} \sup |D^\beta \psi''_{ls} * \varphi|, \quad \varphi \in C_0^\infty(\omega_{ls}), \end{aligned} \tag{4.5}$$

where $\psi'_{ls} = \chi_{ls}\check{P}\varphi_{ls}$ and $\psi''_{ls} = \check{P}\varphi_{ls} - \psi'_{ls}$ and where we have used that $\psi'_{ls} * \varphi \in C_0^\infty(A_{k+1})$ in view of (4.3). Now $\psi_{ls} \in C_0^0(R^n) \subset \mathcal{H}_{(0)}$, so we have $D^\beta \psi'_{ls} \in \mathcal{H}_{(-m-N_{k+1})}$, if $|\beta| \leqslant N_{k+1}$. Furthermore $\psi''_{ls} \in C_0^s(R^n) \subset \mathcal{H}_{(s)}$, which implies that $D^\beta \psi''_{ls} \in \mathcal{H}_{(s-N_{j_l+1})} \subset \mathcal{H}_{(-m-N_{k+1})}$, if $|\beta| \leqslant N_{j_l+1}$ and $s \geqslant N_{j_l+1} - m - N_{k+1}$. For such s we can therefore obtain the following estimate from (4.5)

$$|(D^\alpha \psi_{ls} * \varphi)(x)| \leqslant C_{\alpha,x} \|\varphi\|_{(m+N_{k+1})}, \quad \varphi \in C_0^\infty(\omega_{ls}), \tag{4.6}$$

because $\mathcal{H}_{(t)}$ and $\mathcal{H}_{(-t)}$ are dual spaces. Now we can use the same argument as in the proof of Theorem 3.6.3 in [2] to prove that (4.6) implies

$$D^\alpha \varphi_{ls} \in \mathcal{H}^{\mathrm{loc}}_{(-m-N_{k+1})}(x + \omega_{ls}).$$

Since this is valid for $x \in \complement A_l$ and $|\alpha| \leqslant M_{l+1}$, Lemma 2.3 shows that

$$\varphi_{ls} \in C^{r_l}(x + \omega_{ls}) \quad \text{for} \quad x \in \complement A_l, \tag{4.7}$$

if $r_l < M_{l+1} - m - N_{k+1} - n/2$, which is fulfilled for sufficiently large l, if we have chosen M so that $M_{l+1} - r_l \to +\infty$. But (4.7) contradicts (4.2), so the assumption is false and (b) is true.

By the same argument we can also prove

Theorem 4.2. *Suppose that $\check{P}: \mathcal{D}_F(\Omega, A) \to \mathcal{D}_F(\Omega, A)$ has a continuous inverse. Then (a) and (b) in Theorem 3.1 are satisfied.*

Proof. (a) follows in the same way as in Theorem 4.1, if we observe that the hypothesis implies that $P\mathcal{D}'_F(\Omega, A) = \mathcal{D}'_F(\Omega, A)$ (cf. the proof of Theorem 3.1).

To prove (b) we shall show that given a non-decreasing sequence $M = (M_l)_1^\infty$ of integers $\geqslant 0$ there is another such sequence N for which (4.1) is valid, if $x \in \complement A_l$

and $|\alpha|\leqslant M_{l+1}$, with some continuous semi-norm q on $\mathcal{D}_N(\Omega, A)$. Therefore we define a continuous semi-norm p on $\mathcal{D}_M(\Omega, A)$ by

$$p(\varphi)=\sup_{\alpha}\sup_{x}\varrho_\alpha|D^\alpha\varphi|,\quad \varphi\in C_0^\infty(\Omega),$$

where $\varrho_\alpha=1$ in Ω, if $|\alpha|\leqslant M_1$, and where $\varrho_\alpha=0$ in A_l and $=1$ in $\complement A_l$, if $M_l<|\alpha|\leqslant M_{l+1}$. By the hypothesis we can find a continuous semi-norm q on $\mathcal{D}_F(\Omega, A)$ such that $p(\varphi)\leqslant q(\check{P}\varphi)$ for $\varphi\in C_0^\infty(\Omega)$. Hence (4.1) follows with this q, if $x\in\complement A_l$ and $|\alpha|\leqslant M_{l+1}$ for some l. But q is also continuous on $\mathcal{D}_N(\Omega, A)$ for some non-decreasing sequence N of integers $\geqslant 0$. Thus we can use the rest of the proof of Theorem 4.1 to prove (*b*) here too.

It has not been possible for me to decide whether (*b*) in Theorem 3.1 is necessary also for $P\mathcal{D}'_F(\Omega, A)=\mathcal{D}'_F(\Omega, A)$ to be valid. We only have the following weaker theorem. It is a simple extension of Theorem 3.6.3 in [2] and it can be proved by the same argument as in [2] with only slight modifications and re-arrangements.

Theorem 4.3. *Suppose that* $P\mathcal{D}'_F(\Omega, A)=\mathcal{D}'_F(\Omega, A)$: *Then*

(*a*) Ω *is P-convex,*

(*b*) *to every integer* $k>0$ *there is an integer* $l>0$ *such that*

$$\mu\in\mathcal{E}'(\Omega),\ \text{sing supp}\ \check{P}\mu\subset A_k\Rightarrow\text{sing supp}\ \mu\subset A_l.$$

On the other hand, I have not been able to see if (*a*) and (*b*) in the last theorem are also sufficient for $P\mathcal{D}'_F(\Omega, A)=\mathcal{D}'_F(\Omega, A)$. In particular cases, however, a somewhat stronger form of these conditions is in fact sufficient because they imply the conditions of Theorem 3.1. This will be studied in the next section.

5. Existence theorems in the spaces $\mathcal{D}'_F(\Omega;\omega)$

With a relatively open subset ω of the boundary $\partial\Omega$ of Ω we make the following

Definition 5.1. $\mathcal{D}_F(\Omega;\omega)$ *shall be the space* $\mathcal{D}_F(\Omega, A)$, *if* $A=(A_k)_1^\infty$ *is a sequence of relatively closed subsets of* Ω *such that* $\bar{A}_k$ *is compact for every k and* $\bar{A}_k\nearrow\nearrow\Omega\cup\omega$ *when* $k\to\infty$.

We observe that every such sequence A gives the same topology on $\mathcal{D}_F(\Omega;\omega)$, so the definition has a sense. We also see that $\mathcal{D}'_F(\Omega;\omega)$ is the space of distributions in Ω with finite order in $\Omega\cap K$ for every compact $K\subset\Omega\cup\omega$.

With the notations of Definition 5.1 we obtain the following particular case of Theorem 4.3:

Theorem 5.2. *Suppose that* $P\mathcal{D}'_F(\Omega;\omega)=\mathcal{D}'_F(\Omega;\omega)$. *Then*

(*a*) Ω *is P-convex,*

(*b*) *to every compact* $K\subset\Omega\cup\omega$ *there is a compact* $K'\subset\Omega\cup\omega$ *such that*

$$\mu\in\mathcal{E}'(\Omega),\ \text{sing supp}\ \check{P}\mu\subset K\Rightarrow\text{sing supp}\ \mu\subset K'.$$

To obtain sufficient conditions we strengthen condition (*b*):

Theorem 5.3. *Suppose that*

(*a*) Ω *is P-conxex,*

(*b*) *to every compact* $K \subset \Omega \cup \omega$ *there is a compact* $K' \subset \Omega \cup \omega$ *such that*

$$\mu \in \mathcal{E}'(\Omega \cup \omega), \text{sing supp } \check{P}\mu \subset K \Rightarrow \text{sing supp } \mu \subset K'. \tag{5.1}$$

Then $P\mathcal{D}'_F(\Omega; \omega) = \mathcal{D}'_F(\Omega; \omega)$.

Proof. $\mathcal{D}_F(\Omega; \omega) = \mathcal{D}_F(\Omega, A)$, if A is chosen as in Definition 5.1. We shall show that (*b*) in Theorem 3.1 is satisfied with this sequence A. Hence the theorem follows from Corollary 3.4.

Given k we choose a compact $K' \subset \Omega \subset \omega$ such that (5.1) is fulfilled with $K = \bar{A}_k$ and after that l so that K' is contained in the interior of $\bar{A}_l$ in the relative topology on $\Omega \cup \omega$.

Now let j be a positive integer and let $\mathfrak{J}$ be the space of functions $\psi \in C_0^0(\bar{A}_{j+1})$ for which sing supp $\check{P}\psi \subset K$. $\mathfrak{J}$ is a Fréchet space with the topology defined by the semi-norm $\psi \to \sup |\psi|$ and the semi-norms $\psi \to \sup_L |D^\alpha \check{P}\psi|$, where L is an arbitrary compact subset of $R^n \setminus K$. Now (5.1) implies that $\psi \in C^\infty(R^n \setminus K')$, if $\psi \in \mathfrak{J}$, and the closed graph theorem shows that this natural mapping of $\mathfrak{J}$ into $C^\infty(R^n \setminus K')$ is continuous. Hence given an integer $r \geq 0$ and a compact $L' \subset R^n \setminus K'$ we can find an integer $s \geq 0$, a compact $L \subset R^n \setminus K$ and a constant C so that

$$\sup_{|\alpha| \leq r} \sup_{L'} |D^\alpha \psi| \leq C(\sup_{|\alpha| \leq s} \sup_{L} |D^\alpha \check{P}\psi| + \sup |\psi|), \quad \psi \in \mathfrak{J}. \tag{5.2}$$

Assume that $\varphi \in C_0^0(A_j)$ and $\check{P}\varphi \in C^s(\complement A_k)$. With χ_δ defined for $\delta > 0$ as in the proof of Lemma 3.3 we form the regularizations $\varphi * \chi_\delta$. Now $\varphi * \chi_\delta \to \varphi$ in $C^0(R^n)$ and $\check{P}\varphi * \chi_\delta \to \check{P}\varphi$ in $C^s(R^n \setminus K)$ when $\delta \to +0$. Since $\varphi * \chi_\delta \in C_0^\infty(\bar{A}_{j+1})$ for all sufficiently small $\delta > 0$, we can therefore use (5.2) with $\psi = \varphi * \chi_\delta - \varphi * \chi_{\delta'}$ to prove that $\varphi * \chi_\delta \to \varphi$ in $C^r(L'^0)$ when $\delta \to +\infty$. But this shows that $\varphi \in C^r(\complement A_l)$, if we have chosen L' so that $A_j \setminus A_l \subset L'^0$. Hence (*b*) in Theorem 3.1 is satisfied and the proof is complete.

Remark. In the proof we have only used that (5.1) is valid for $\mu \in C_0^0(\Omega \cup \omega)$.

We shall consider two particular cases of these theorems.

First we observe that $\mathcal{D}_F(\Omega; \phi) = \mathcal{D}(\Omega)$. Therefore we obtain the following corollary from Theorem 5.2 and Theorem 5.3.

Corollary 5.4 (*Hörmander* [*1*]). *A necessary and sufficient condition for* $P\mathcal{D}'(\Omega) = \mathcal{D}'(\Omega)$ *is that* Ω *is strongly P-convex, that is,*

(*a*) Ω *is P-convex,*

(*b*) *to every compact* $K \subset \Omega$ *there is another compact* $K' \subset \Omega$ *such that*

$$\mu \in \mathcal{E}'(\Omega) \text{ sing supp } \check{P}\mu \subset K \Rightarrow \text{sing supp } \mu \subset K'.$$

Our proofs, however, give somewhat more. For if $A = (A_k)_1^\infty$ is a sequence of compact subsets of Ω such that $A_k \nearrow \nearrow \Omega$ when $k \to \infty$, we have proved that (*a*) and (*b*) in Corollary 5.4 imply that $P\mathcal{D}'_N(\Omega, A) \supset \mathcal{D}'_M(\Omega, A)$, where M is an arbitrary non-decreasing sequence of integers ≥ 0 and N another such sequence, depending only on M. But the proofs do not show any exact dependence.

Another particular case is obtained with $\omega = \partial\Omega$. Then (*b*) in Theorem 5.3 is always satisfied, for the convex hulls of sing supp μ and sing supp $\check{P}\mu$ are identical for every $\mu \in \mathcal{E}'(R^n)$ ([2], Theorem 3.6.1). Hence we get

Corollary 5.5. *A necessary and sufficient condition for $P\mathcal{D}'_F(\Omega; \partial\Omega) = \mathcal{D}'_F(\Omega; \partial\Omega)$ is that Ω is P-convex.*

Here we observe that $\mathcal{D}'_F(\Omega; \partial\Omega)$ is the space of distributions in Ω, which have finite order in every bounded subset of Ω (cf. the remark after Lemma 4.2 in [1]).

Finally we give some geometric conditions corresponding to conditions in [2], section 3.7.

We suppose that Ω has a C^2-boundary, that is, to every $x^0 \in \partial\Omega$ there is an open neighbourhood U of x^0 and a real-valued function $\psi \in C^2(U)$ such that $U \cap \Omega = \{x \in U; \psi(x) < 0\}$ and grad $\psi \neq 0$ in U. Then $\partial\Omega$ is said to be *strictly pseudo-convex* at x^0 with respect to P, if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial x_j \partial x_k}(x^0)\, P_m^{(j)}(\xi)\, \overline{P_m^{(k)}(\xi)} > 0,$$

when
$$0 \neq \xi \in R^n, P_m(\xi) = 0 \quad \text{and} \quad \sum_1^n \frac{\partial \psi}{\partial x_j}(x^0)\, P_m^{(j)}(\xi) = 0,$$

where $P_m(\xi) = \sum_{|\alpha|=m} a_\alpha \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and $P_m^{(j)}(\xi) = \partial P_m(\xi)/\partial \xi_j$ (m is the order of P). In particular this condition implies that P has no multiple real characteristics, and if $P_m(\xi)$ has real coefficients, it means that $\partial\Omega$ has a positive outer normal curvature at x^0 in every tangential direction, which is bicharacteristic with respect to P.

Using the same arguments as in the proofs of Theorems 3.7.6 and 3.7.5 in [2] one can then prove

Theorem 5.6. *If $\partial\Omega$ is strictly pseudo-convex at x^0 with respect to P for every $x^0 \in \partial\Omega \setminus \omega$, then (b) in Theorem 5.3 is satisfied.*

And conversely

Theorem 5.7. *If P_m has real coefficients and to every compact $K \subset \Omega$ there is a compact $K' \subset \Omega \cup \omega$ such that*

$$\mu \in \mathcal{E}'(\Omega), \text{sing supp } \check{P}\mu \subset K \Rightarrow \text{sing supp } \mu \subset K',$$

then $\partial\Omega$ has non-negative outer normal curvature at $x^0 \in \partial\Omega \setminus \omega$ in every tangential direction, which is bicharacteristic with respect to P.

Theorem 5.6 together with Theorem 3.7.4 in [2] gives a sufficient geometrical condition for $P\mathcal{D}'_F(\Omega; \omega) = \mathcal{D}'_F(\Omega; \omega)$ at least when P_m has real coefficients.

A final observation: When $n = 2$ and P has no multiple real characteristics, every P-convex open set Ω in R^2 also satisfies (b) in Theorem 5.3. In particular it is strongly P-convex. This follows, if we use Theorem 3.7.2 in [2] and observe that in this case every non-characteristic C^2-surface is strongly pseudo-convex, so that Theorem 8.8.1 in [2] can be used in a similar way as Holmgren's uniqueness theorem.

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