On convergent and divergent sequences of equilibrium distributions

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1. Introduction

We shall deal with sets belonging to an m-dimensional Euclidean space R^m and we suppose all the time that the sets are bounded Borel sets.

Let $K_n(r)$, $n = 1, 2, \ldots$, and $K(r)$ denote non-negative, non-increasing functions, defined for $r \geq 0$, which are continuous for $r > 0$. We also suppose that $K_n(r) \to K(r)$ when $n \to \infty$. The problem which we shall discuss, is of the following type. Suppose that $K_n(r)$, $n = 1, 2, \ldots$, are such that the equilibrium problem is possible for every $K_n(r)$.[1] This is, for instance, the case if every $K_n(r)$ is of the form $r^{-\alpha}$, $m - 2 \leq \alpha < m$. When is it true that the equilibrium distributions belonging to the kernels $K_n(r)$ and a certain closed set F converge, when $n \to \infty$, on the assumption that the K_n-capacity[2] of the set F is positive? By convergence we always mean convergence in the weak sence. We shall here above all deal with the case when the K-capacity of F is zero. This question is of interest because a positive answer would, to sets of capacity zero, assign a distribution of mass which would be an analogue to the equilibrium distribution for sets of positive capacity. The case when the K-capacity of F is positive is easily settled. Namely, if the K-capacity of F is positive and $K_n(r) \nearrow K(r)$ —that is $K_n(r)$ tends non-decreasingly to $K(r)$—it immediately follows that the equilibrium distributions belonging to $K_n(r)$ and the set F converge, when $n \to \infty$, towards the equilibrium distribution belonging to $K(r)$ and F. This is proved by Frostman in [2] for kernels of the type $r^{-\alpha}$ and his proof remains valid for general kernels. An assumption like $K_n(r) \nearrow K(r)$ is essential which is seen by the following counter-example.[3] There exists a closed linear set F with Hausdorff dimension[4] larger than a chosen number α_0, $\alpha_0 < 1$, such that the equilibrium distributions belonging to the kernels $r^{-\alpha}$ and F fail to converge to the equilibrium distribution belonging to $r^{-\alpha_0}$ and F, when $\alpha \to \alpha_0 + 0$—that is α tends to α_0 from above. We give a proof of this in Theorem 7.

It is possible to determine the equilibrium distribution exactly only in a few simple cases. This problem has been treated by Polya and Szegö [6], who have

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[1] For definitions see [1].
[2] As regards the definition of capacity see below.
[3] This counter-example has been shown to me by Professor Carleson.
[4] For the definition see [1], p. 90.

determined the equilibrium distribution belonging to the kernel $r^{-\alpha}$, $\alpha < 1$, when the set is, for instance, a linear interval. Their result shows that the equilibrium distributions converge to the distribution of mass which has constant density in the given interval, when $\alpha \to 1$. An analogous result is valid for a circular disk and the (3-dimensional) sphere. By mapping a linear interval isometrically on a rectifiable curve in the plane, Frostman [2] has shown, using the results of Polya and Szegö, that the equilibrium distributions belonging to $r^{-\alpha}$ and a rectifiable plane curve converge, when $\alpha \to 1$, to the distribution of mass, where the mass which is situated on an arc is proportional to the length of that arc. Another case has been treated by Lithner [3] using methods from Fourier analysis. He has proved that the equilibrium distributions belonging to $r^{-\alpha}$ and F converge, when $\alpha \to m - 0$, towards the distribution which has constant density on F, if F is a compact set in R^m which has positive m-dimensional Lebesgue measure. We shall prove (Theorem 1)—using quite other methods than those of Lithner—that this result remains valid for much more general kernels $K_n(r)$ and $K(r)$. Thus we get an analogous result even if we do not assume that the kernels are such that the equilibrium problem is possible. In this case there exists no longer an equilibrium distribution, but we still consider a (not necessarily uniquely determined) distribution of unit mass on F, μ_n, which realizes

$$\inf_{\nu \in \Gamma} \int\int_{FF} K_n(|x - y|)\, d\nu(x)\, d\nu(y),$$

where Γ is the class of all positive distributions of unit mass on F. The conclusion is then, just as before, that $\{\mu_n\}$ converges to the distribution which has constant density on F. This is the main result of the paper.

The method which we shall use in the proof of Theorem 1 can also be used to prove the following result (Theorem 2). If F is a compact set in R^m and $C_\alpha(F)$ denotes the α-capacity of F and $m(F)$ the m-dimensional Lebesgue measure of F, we have

$$\lim_{\alpha \to m - 0} \frac{C_\alpha(F)}{m - \alpha} = k\, m(F),$$

where k is a constant which only depends on the dimension m of the space R^m.[1]

The rest of the paper chiefly consists of counter-examples. The aim of these is to show that the result in Theorem 1 is, in a certain sense, the best possible. Thus there is no natural analogue to Theorem 1 for sets having Lebesgue measure zero. Due to this somewhat negative character of the rest of the paper, we shall sometimes give only a sketch of the proofs of the counter-examples. In order to keep the calculations as simple as possible we shall also as a rule give the counter-examples for linear sets.

In Theorem 3 we prove that there exists a closed linear set F with a prescribed Hausdorff dimension α_0, $0 < \alpha_0 \leqslant 1$, so that the equilibrium distributions belonging to the kernels $r^{-\alpha}$ and F do not converge when $\alpha \to \alpha_0 - 0$.

The question that is answered by Theorem 1 can in a natural way be generalized to sets having Hausdorff dimension less than 1. Given an arbitrary

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[1] Compare [2] where a similar formula is proved when F is a rectifiable plane curve.

positive number α_0, $0 < \alpha_0 < 1$, and a closed linear set F which has positive and finite α_0-dimensional Hausdorff measure, $\Lambda_{\alpha_0}(F)$, is it then true that the equilibrium distributions belonging to the kernels $r^{-\alpha}$ and the set F converge, when $\alpha \to \alpha_0 - 0$, to a distribution where the mass situated on a subset of F is proportional to the α_0-dimensional Hausdorff measure of that subset? A negative answer to this question is given in Theorem 4. It should be noticed that we must here assume that $\alpha_0 < 1$, because if $\alpha_0 = 1$ $\Lambda_{\alpha_0}(F)$ coincides with the Lebesgue measure of F, as F is a linear set, and then the answer to the question is in the affirmative according to Theorem 1. But hence there appears, in a natural way the question whether it is possible to construct a closed set F in the plane having positive and finite 1-dimensional Hausdorff measure, $0 < \Lambda_1(F) < \infty$, so that the equilibrium distributions belonging to $r^{-\alpha}$ and F do not converge to a constant times the 1-dimensional Hausdorff measure, when $\alpha \to 1 - 0$. In Theorem 5 we prove that this is possible. This result is thus an instance of the well-known fact that there exists a fundamental difference in the structure of linear and plane sets with positive and finite 1-dimensional Hausdorff measure.

The above counter-examples show that there is no result which is analogous to Theorem 1 if the set F has Lebesgue measure zero. In order to get a positive solution of the convergence problem for sets with Lebesgue measure zero, we must consequently limit ourselves to sets which satisfy some suitable condition of regularity. We shall prove (Theorem 6) that if the set is a linear Cantor[1] set, then we get a positive answer to the convergence question. On the other hand, there exist simple sets for which the answer is in the negative, which is illustrated by the fact that the sequence of equilibrium distributions does not necessarily converge if the set F is the union of two Cantor sets. We sketch a proof of this in the remark to Theorem 6.

A question which is related to the above is the following. Let μ_n be the distribution of the mass $1/n$ in each of the n points (not necessarily uniquely determined) which realize

$$\min_{x_\nu \in F} \frac{\displaystyle\sum_{i<j}^{1,\,\ldots,\,n} K(|x_i - x_j|)}{\dbinom{n}{2}} = K(D_n^{(K)}),$$

where F is a closed set. If the K-capacity of F is positive and the equilibrium problem is possible for $K(r)$, then it is true that $\{\mu_n\}$ converges to the equilibrium distribution belonging to $K(r)$ and F, when $n \to \infty$.[2] Does $\{\mu_n\}$ converge also if the K-capacity of F is zero? The fact that this is not necessarily the case is an immediate consequence of a result of Terasaka [8], who, in the case when $K(r) = 1/r$, constructed a closed enumerable set, such that the sequence $u_n(x) = \displaystyle\int_F K(|x - y|)\, d\mu_n(y)$, $n = 1, 2, \ldots$, does not converge for every x belonging to the complement of F. It is, however, possible to make Terasaka's

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[1] For definitions see [4], pp. 152 ff.
[2] See [1], pp. 46 ff, where this is proved for $K(r) = r^{-\alpha}$.

construction for an arbitrary kernel $K(r)$ and hence we can, to every kernel $K(r)$, find an enumerable set such that $\{\mu_n\}$ does not converge. Finally we can note that the same negative result also remains valid if instead of considering the n points which realize $K(D_n^{(K)})$ we take the n points $\{y_i\}_1^n$ which realize

$$K(R'_n) = \max_{x_i \in F} \min_{x \in F} \frac{1}{n} \sum_{i=1}^n K(|x - x_i|) = \min_{x \in F} \frac{1}{n} \sum_{i=1}^n K(|x - y_i|).$$

2. Definitions and notations

$x = (x^1, x^2, \ldots, x^m)$ denotes a point in R^m. By a closed m-dimensional interval we mean the set of points which satisfy the inequalities $a_i \leqslant x^i \leqslant b_i$, where a_i and b_i are any numbers such that $a_i < b_i$, $i = 1, 2, \ldots, m$. $S(x_0, r)$ denotes the closed sphere $|x - x_0| \leqslant r$. $I(H, \nu; F)$ denotes the energy integral

$$\iint_{FF} H(|x - y|) \, d\nu(x) \, d\nu(y).$$

The H-capacity of a set F, $C_H(F)$, we define as $C_H(F) = \{\inf_{\nu \in \Gamma} I(H, \nu; F)\}^{-1}$, where Γ is the class of all positive distributions of unit mass on F, i.e. the class of all completely additive, non-negative set functions taking the value 1 on F and vanishing outside F. Particularly, we write the α-capacity of F, i.e. the case when $H(r) = r^{-\alpha}$, as $C_\alpha(F)$.

A set E is said to be regular with respect to the distribution μ, or shorter regular μ, if μ does not distribute any mass on the boundary of E. The complement of E we denote by E' and for the m-dimensional Lebesgue measure of E we write $m(E)$. If E_1 and E_2 are two sets, we denote by $E_1 \setminus E_2$ the set of points belonging to E_1 but not to E_2. If $E_1 \supset E_2$ we write $E_1 - E_2$ instead of $E_1 \setminus E_2$.

3.

In this section we collect some inequalities which will be of constant use when we prove the lemmas required for the proof of Theorem 1. Let $H(r)$ be defined for $r \geqslant 0$, continuous for $r > 0$, non-increasing and non-negative. Let $\lim_{r \to 0} H(r) = H(0) \leqslant \infty$. Let F be a closed set of positive H-capacity. As F is closed there exists a not necessarily uniquely determined distribution, τ, of unit mass on F which realizes

$$\inf_{\nu \in \Gamma} I(H, \nu; F) = V_H(F),$$

where Γ is the class of all positive distributions of unit mass on F. We thus have

$$I(H, \tau; F) = V_H(F) = \{C_H(F)\}^{-1}.$$

We call τ a capacitary distribution belonging to $H(r)$ and F. Particularly if the equilibrium problem is possible for $H(r)$, a capacitary distribution is identical with

the equilibrium distribution and accordingly unique.[1] We now form the potential belonging to the capacitary distribution τ and the kernel $H(r)$

$$u(x) = \int_F H(|x-y|)\, d\tau(y).$$

Then the following inequalities are true:

$u(x) \geqslant V_H(F)$ for all $x \in F$ except perhaps for a set with H-capacity zero. (3.1)

$u(x) \leqslant V_H(F)$ everywhere on the support of τ. (3.2)

$u(x) \leqslant A \cdot V_H(F)$ everywhere, where A is a constant which only depends on the dimension m of the space R^m. (3.3)

The inequalities (3.1) and (3.2) follow from the fact that $u(x)$ is lower semi-continuous by using the "variation method of Gauss". For this we refer to Frostman [1], pp 35 ff. (3.3) easily follows from the fact that $H(r)$ is monotonously decreasing. This is a result of Ugaheri [9].

4.

We now prove a lemma which we shall use in the proofs of all our theorems. First, however, we collect the conditions on our functions $K_n(r)$ and $K(r)$.

$K_n(r)$, $n = 1, 2, \ldots$, and $K(r)$ are defined for $r \geqslant 0$, continuous for $r > 0$, are non-negative and non-increasing and satisfy $\lim\limits_{r \to 0} K_n(r) = K_n(0) \leqslant \infty$,

$$\lim_{r \to 0} K(r) = \infty,$$ (a)

$$\lim_{n \to \infty} K_n(r) = K(r).$$ (b)

Lemma 1. *Suppose that* $K_n(r)$, $n = 1, 2, \ldots$, *and* $K(r)$ *satisfy conditions* (a) *and* (b) *and that* F *is a closed set satisfying* $C_{K_n}(F) > 0$ *and* $C_K(F) = 0$. *Let* μ_n *be a capacitary distribution belonging to* $K_n(r)$ *and* F *and suppose that* $\{\mu_n\}$ *converges to a certain distribution* μ. *Denoting by* S *an* m-*dimensional sphere it is true that*

$$\mu(S \cap F) = \lim_{n \to \infty} \frac{C_{K_n}(S \cap F)}{C_{K_n}(F)},$$

if the sphere S *is regular* μ *and* $\mu(S) > 0$.[2]

Proof. We first prove that

$$\lim_{n \to \infty} I(K_n, \mu_n; F) = \infty.$$ (4.1)

We introduce the functions $[K_n(r)]_N$ and $[K(r)]_N$, where $[K_n(r)]_N = K_n(r)$ if $K_n(r) \leqslant N$ and $[K_n(r)] = N$ if $K_n(r) > N$, and where $[K(r)]_N$ is defined in an analogous way.

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[1] It should be observed that, according to our convention, both a capacitary distribution and the equilibrium distribution are distributions of unit mass.

[2] This lemma should be compared to the calculations in Frostman [2].

The fact that $[K_n(r)]_N$ and $[K(r)]_N$ are non-increasing and that $[K(r)]_N$ is continuous implies that $[K_n(r)]_N$ converges uniformly to $[K(r)]_N$ in every compact interval. Hence

$$\varliminf_{n\to\infty} I(K_n, \mu_n; F) \geqslant \lim_{n\to\infty} I([K_n]_N, \mu_n; F) = I([K]_N, \mu; F),$$

for all N. But since $C_K(F) = 0$ we have $\lim_{N\to\infty} I([K]_N, \mu; F) = \infty$, and so (4.1) is proved.

We now suppose that S has radius r where r is chosen so that S is regular with respect to μ and $\mu(S) > 0$. Choose a sphere S_0 with the same centre as S and with radius r_0, $r_0 > r$, so that S_0 is regular μ. Then the following estimate holds

$$\mu_n(S) \geqslant \frac{I(K_n, \mu_n; S)}{I(K_n, \mu_n; F)} \geqslant \mu_n(S) - \mu_n(S_0 - S) - \frac{{}^! K_n(r_0 - r)}{I(K_n, \mu_n; F)}. \tag{4.2}$$

This is an easy consequence of the inequalities (3.1) and (3.2). In fact, we have

$$I(K_n, \mu_n; S) \leqslant \int_S \int_F K_n(|x - y|)\, d\mu_n(x)\, d\mu_n(y) \leqslant I(K_n, \mu_n; F) \cdot \int_S d\mu_n(x),$$

according to (3.2). This gives one of the inequalities (4.2). We get the other by the following division of the energy integral

$$I(K_n, \mu_n; S) = \left(\int_S \int_F - \int_S \int_{S_0-S} - \int_S \int_{F\setminus S_0} \right) K_n(|x-y|)\, d\mu_n(x)\, d\mu_n(y) = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

$$\mathrm{I} \geqslant I(K_n, \mu_n; F) \cdot \int_S d\mu_n(x),$$

by (3.1), because μ_n does not distribute any mass on a set of K_n-capacity zero.

$$\mathrm{II} \geqslant -I(K_n, \mu_n; F) \int_{S_0 - S} d\mu_n(x),$$

according to (3.2).

$$\mathrm{III} \geqslant -K_n(r_0 - r).$$

This gives the other inequality (4.2). Now letting $n \to \infty$ in (4.2) we get by (4.1)

$$\mu(S) \geqslant \varlimsup_{n\to\infty} \frac{I(K_n, \mu_n; S)}{I(K_n, \mu_n; F)} \geqslant \varliminf_{n\to\infty} \frac{I(K_n, \mu_n; S)}{I(K_n, \mu_n; F)} \geqslant \mu(S) - \mu(S_0 - S).$$

If $r_0 - r$ is small $\mu(S_0 - S)$ is, however, arbitrarily small and hence we get

$$\mu(S) = \lim_{n\to\infty} \frac{I(K_n, \mu_n; S)}{I(K_n, \mu_n; F)}. \tag{4.3}$$

In an analogous way we realize that

$$\mu(S') = \lim_{n\to\infty} \frac{I(K_n, \mu_n; S')}{I(K_n, \mu_n; F)}. \tag{4.4}$$

It is easy to see that the limit in (4.3) is exactly

$$\lim_{n\to\infty} \frac{C_{K_n}(S \cap F)}{C_{K_n}(F)}.$$

Namely, if ν_n is distributed on $S \cap F$ with the same relative density as a capacitary distribution belonging to $K_n(r)$ and $S \cap F$ and with the total mass $\nu_n(S \cap F) = \mu_n(S \cap F) = \mu_n(S)$, we get

$$\lim_{n\to\infty} \frac{I(K_n, \nu_n; S \cap F)}{I(K_n, \mu_n; S)} = 1. \tag{4.5}$$

For, suppose that
$$\lim_{n\to\infty} \frac{I(K_n, \nu_n; S \cap F)}{I(K_n, \mu_n; S)} = h < 1. \tag{4.6}$$

As μ_n is a distribution of unit mass which realizes $\inf I(K_n, \nu; F)$, we get

$$1 = \frac{I(K_n, \mu_n; F)}{I(K_n, \mu_n; F)}$$

$$\leqslant \frac{I(K_n, \nu_n; S \cap F) + I(K_n, \mu_n; S') + 2 \int_S \int_{S'} K_n(|x-y|) \, d\nu_n(x) \, d\mu_n(y)}{I(K_n, \mu_n; F)}. \tag{4.7}$$

But if A is the constant which occurs in (3.3), we have

$$\int_S \int_{S'} K_n(|x-y|) \, d\nu_n(x) \, d\mu_n(y) = \left(\int_S \int_{S_0 - S} + \int_S \int_{S_0'} \right) K_n(|x-y|) \, d\nu_n(x) \, d\mu_n(y)$$

$$\leqslant A \cdot I(K_n, \mu_n; F) \cdot \{\mu_n(S)\}^{-1} \cdot \mu_n(S_0 - S) + K_n(r_0 - r)$$

which follows from the fact that

$$\sup_y \int_S K_n(|x-y|) \, d\nu_n(x) \leqslant A \cdot \{\nu_n(S)\}^{-1} \cdot I(K_n, \nu_n; S \cap F)$$

$$\leqslant A \cdot \{\nu_n(S)\}^{-1} \cdot I(K_n, \mu_n; S) \leqslant A \{\mu_n(S)\}^{-1} \cdot I(K_n, \mu_n; F),$$

where the first inequality is obtained from (3.3). If we use the estimate which we have just obtained, (4.7) gives by first letting $n \to \infty$ and then $r_0 \to r$,

$$1 \leqslant \varliminf_{n \to \infty} \frac{I(K_n, \nu_n; S \cap F)}{I(K_n, \mu_n; F)} + \lim_{n \to \infty} \frac{I(K_n, \mu_n; S')}{I(K_n, \mu_n, F)}.$$

According to (4.3), (4.4) and (4.6) this gives

$$1 \leqslant h\mu(S) + \mu(S').$$

But as $h < 1$ and $\mu(S) > 0$ we have $1 \leqslant h\mu(S) + \mu(S') < 1$, which gives a contradiction. The relation (4.5) is hence true, and this, combined with (4.3), immediately gives

$$\mu(S) = \mu(S \cap F) = \lim_{n \to \infty} \frac{C_{K_n}(S \cap F)}{C_{K_n}(F)}$$

and so the lemma is proved.

Remark. The conclusion of the lemma remains valid also if S denotes certain more general sets than spheres. For instance, for S we can choose the intersection between a sphere and an m-dimensional interval (compare Lemma 2) or the union of a finite number of m-dimensional intervals (compare Theorem 2). Clearly, the only difference in the proof in these cases is that we have to choose S_0 in a somewhat different way than above. For S_0 we choose a set, containing S, which is similar to S, and the boundary of which has a positive distance from the boundary of S.

5.

Before we can prove Theorem 1 we need some more lemmas. We first prove a lemma which is identical to Theorem 1 in the special case when F is the union of a finite number of closed intervals.

Lemma 2. *Suppose that $K_n(r)$ and $K(r)$ satisfy conditions (a) and (b), that $\int_0^1 K_n(r) r^{m-1} dr < \infty$ for $n = 1, 2, \ldots,$ and that $\int_0^1 K(r) r^{m-1} dr = \infty$. Suppose also that F is the union of a finite number of closed m-dimensional intervals. Let μ_n be a capacitary distribution belonging to $K_n(r)$ and F. Then it is true that $\mu_n \to \sigma$, when $n \to \infty$, where $\sigma(E) = m(E)/m(F)$ for every Borel set $E \subset F$.*

Proof. The conditions of the lemma guarantee that $C_{K_n}(F) > 0$ for every n and that $C_K(F) = 0$. μ_n exists as $C_{K_n}(F) > 0$ and we suppose that $\{\mu_n\}$ converges to a certain distribution μ. (If necessary, we choose a convergent subsequence.) We denote by G the interior of F.

It is easy to realize that $\mu(S(x, r)) > 0$ for every sphere $S(x, r)$ where $x \in G$ and $r > 0$. Namely, if $\mu(S(x, r)) = 0$ for some $r > 0$, we could, for large values of n, obtain a smaller value than $I(K_n, \mu_n; F)$ of the energy integral belonging to $K_n(r)$ and F by a redistribution of μ_n on F so that more mass were distributed in $S(x, r)$.[1] But this would be a contradiction to the fact that μ_n is a capacitary distribution belonging to $K_n(r)$ and F and hence we have $\mu(S(x, r)) > 0$ if $x \in G$ and $r > 0$.

We now take two spheres $S(x_1, r)$ and $S(x_2, r)$ which are contained in G. As $C_{K_n}(S(x_1, r)) = C_{K_n}(S(x_2, r))$ we have, according to Lemma 1

[1] Compare the redistribution of the mass in the proof of Lemma 4.

$$\mu(S(x_1, \ r)) = \mu(S(x_2, \ r)), \tag{5.1}$$

except possibly for an enumerable set of values r. Consequently (5.1) is valid for all values of r such that $S(x_i, \ r) \subset G$, $i = 1, 2$. Two spheres belonging to G and having the same radii are hence carrying the same quantity of mass μ, which means that on G μ is distributed with constant density. Hence it only remains to prove that the boundary $F - G$ does not carry any mass. Let the centre x_0 of the sphere $S(x_0, \ r)$ belong to $F - G$ and choose r so that $S(x_0, \ r)$ is regular with respect to μ. (If the boundary of $S(x_0, \ r)$ carries any mass, this mass is situated on $F - G$.) Choosing r small enough there is a subset of G which is congruent to $F \cap S(x_0, \ r)$ and the conclusion of Lemma 1 is, by the remark following Lemma 1, valid also for this set. This set and $F \cap S(x_0, \ r)$ hence carry the same quantity of mass. But this means that the boundary $F - G$ does not carry any mass, and thus the lemma is proved.

We now consider the interval Δ, which is determined from the inequalities $0 \leqslant x^i \leqslant a_i$, $i = 1, \ldots, m$, and a positive distribution of mass, μ, on Δ such that Δ is regular μ. We extend the domain of definition of μ by making μ periodic with periods a_i, i.e. we put

$$\mu[\omega + (n_1 a_1, \ldots, n_m a_m)] = \mu(\omega),$$

for arbitrary integers $\{n_i\}_1^m$ and intervals $\omega \subset \Delta$. $\omega + (n_1 a_1, \ldots, n_m a_m)$ denotes the interval in which ω is carried by the translation $(n_1 a_1, \ldots, n_m a_m)$. We can now define a class of distributions of mass, $\{\mu_x\}$, where μ_x denotes the distribution which arises from μ, when we translate the distribution μ by the vector x, i.e.

$$\mu_x(\omega) = \mu(\omega - x), \tag{5.2}$$

for every interval ω. Then the following lemma is valid.

Lemma 3. *Let μ be a positive distribution of mass with total mass M on $\Delta = \{x | 0 \leqslant x^i \leqslant a_i, \ i = 1, \ldots, m\}$ and let F be a closed subset of Δ having positive m-dimensional Lebesgue measure. Then there is a translation as above of the distribution μ, μ^*, so that*

$$\mu^*(F) \geqslant \frac{m(F)}{a_1 \cdot a_2 \ldots a_m} \cdot M.$$

Lemma 3 follows from the relation

$$\int_0^{a_1} \ldots \int_0^{a_m} \mu_x(F) \, dx^1 \ldots dx^m = M \cdot m(F),$$

which is easily proved by means of (5.2) for instance by introducing the characteristic function of the set F in the integral and by then using Fubini's theorem.

Using Lemma 2 and Lemma 3 we now prove

Lemma 4. *Let F_1 and F_2 be closed sets such that $F_2 \supset F_1$ and $m(F_1) > 0$. Let F_2 be the union of a finite number of m-dimensional intervals. Suppose that $K_n(r)$ and $K(r)$ satisfy the conditions (a) and (b) and that*

$$\int_0^1 K_n(r) r^{m-1} \, dr < \infty \quad and \quad \int_0^1 K(r) r^{m-1} \, dr = \infty.$$

Let μ_{in} be a capacitary distribution belonging to $K_n(r)$ and F_i, $i = 1, 2$, and suppose that $\mu_{1n} \to \sigma_1$. Then it is true that

$$\sigma_1(E) \geqslant \frac{m(E)}{m(F_2)}, \quad E \subset F_1, \quad E \text{ Borel set.}$$

Proof. Suppose that the lemma is wrong. Then there is an interval Δ such that

$$\sigma_1(\Delta) = h \cdot \frac{m(\Delta \cap F_1)}{m(F_2)}, \, h < 1 \quad \text{and} \quad m(\Delta \cap F_1) > 0. \tag{5.3}$$

The idea of the proof is as follows. $F_2 \supset F_1$ implies that $I(K_n, \mu_{1n}; F_1) \geqslant I(K_n, \mu_{2n}; F_2)$ for all n. But this indicates that if the limit distribution of $\{\mu_{2n}\}$ distributes more mass on $\Delta \cap F_1$ than the limit distribution of $\{\mu_{1n}\}$, then it should be possible to reduce the energy integral $I(K_n, \mu_{1n}; F_1)$ be redistributing the part of the mass μ_{1n} which falls in Δ so that the relative density of the mass on $\Delta \cap F_1$ coincides with that of μ_{2n}. As we cannot, however, gaurantee that $\mu_{2n}(\Delta \cap F_1) > 0$—which is required for a redistribution of the above mentioned kind—we first have to undertake a translation of the distribution μ_{2n} according to Lemma 3 so that the translated distribution distributes mass on $\Delta \cap F_1$. This translation introduces, as we shall see, faults in our estimates which we can neglect. Now we turn to the details.

We can suppose that Δ is regular with respect to the distributions σ_1 and μ_{2n}, $n = 1, 2, \dots$. We can also suppose that Δ is covered by one interval from F_2, i.e. that $\Delta \subset F_2$.

Let μ_{2n}' be the restriction of μ_{2n} to Δ. For every n we can, according to Lemma 3, find a translation, x_n, of the distribution μ_{2n}' so that the translated distribution, μ_{2n}^*, satisfies

$$\mu_{2n}^*(\Delta \cap F_1) \geqslant \mu_{2n}(\Delta) \cdot \frac{m(\Delta \cap F_1)}{m(\Delta)}.$$

This gives, by Lemma 2

$$\varlimsup_{n \to \infty} \mu_{2n}^*(\Delta \cap F_1) \geqslant \frac{m(\Delta \cap F_1)}{m(F_2)}. \tag{5.4}$$

(Actually it is true that $\lim_{n \to \infty} \mu_{2n}^*(\Delta \cap F_1) = m(\Delta \cap F_1)/m(F_2)$.) We now define a sequence $\{h_n\}_1^\infty$ by

$$\mu_{1n}(\Delta) = h_n \mu_{2n}^*(\Delta \cap F_1), \tag{5.5}$$

and introduce new distributions $\{\mu_n\}$ of unit mass on F_1.

$$\mu_n = \begin{cases} h_n \ \mu_{2n}^* & \text{on } \Delta \cap F_1 \\ \mu_{1n} & \text{on } F_1 \setminus \Delta \\ 0 & \text{on } F_1'. \end{cases}$$

To get a contradiction it is enough to prove that

$$I(K_n, \mu_n; F_1) < I(K_n, \mu_{1n}; F_1)$$

when n is sufficiently large. To prove this we shall use the inequalities

$$I(K_n, \mu_{2n}; F_2) \leqslant I(K_n, \mu_{1n}; F_1) \tag{5.6}$$

and

$$\overline{\lim_{n \to \infty}} \, h_n \leqslant h, \tag{5.7}$$

the latter of which follows from (5.3), (5.4) and (5.5).

We shall estimate the integral $I(K_n, \mu_n; F_1)$ by dividing up the domain of integration F_1 and by separating a certain rest set, $R_d = R_d^{(1)} \cup R_d^{(2)}$, $d > 0$, which is defined in the following way. We first notice that the translations by x_n, $n = 1, 2, \ldots$, can be supposed to converge to a certain translation by x^*. (If necessary, we choose a convergent subsequence.) We can also assume that one of the corners of the interval Δ is situated in the origin and that the point x^* belongs to Δ. Then $R_d^{(1)}$ shall consist of those points which are situated at a distance $\leqslant d$ from the intersection of Δ and the set which consists of the union of the $(m-1)$-dimensional planes which on one hand pass through the point x^* and where, on the other hand, each plane is parallel to one of the edge planes to Δ. $R_d^{(2)}$ shall consist of those points which are situated at a distance $\leqslant d$ from the boundary of Δ.

Put
$$u_n(x) = \int_{F_1} K_n(|x - y|) \, d\mu_n(y).$$

We shall estimate $I(K_n, \mu_n; F_1)$ by the following division.

$$I(K_n, \mu_n; F_1) = \int_{F_1} u_n(x) \, d\mu_n(x) = \left(\int_{\Delta \setminus R_d} + \int_{R_d} + \int_{(\Delta \cup R_d)'} \right) u_n(x) \, d\mu_n(y) = \text{I} + \text{II} + \text{III}.$$

$$u_n(x) = h_n \int_{\Delta \cap F_1} K_n(|x - y|) \, d\mu_{2n}^*(y) + \int_{\Delta'} K_n(|x - y|) \, d\mu_{1n}(y).$$

For x belonging to the intersection of $\Delta \setminus R_d$ and the support of μ_n we have, by 3.2, supposing n so large that $|x_n - x^*| < d/2$

$$\int_{\Delta \cap F_1} K_n(|x - y|) \, d\mu_{2n}^*(y) \leqslant I(K_n, \mu_{2n}; F_2) + K_n\left(\frac{d}{2}\right)$$

and hence, by (5.6), for a fixed d

$$u_n(x) \leqslant h_n I(K_n, \mu_{1n}; F_1) + O(1).$$

For x belonging to the intersection of $(\Delta \cup R_d)'$ and the support of μ_{1n}, we get in an analogous way

$$u_n(x) \leqslant I(K_n, \mu_{1n}; F_1) + O(1),$$

when d is fixed.

Finally, for $x \in R_d$, we shall prove that

$$u_n(x) = h_n \int_{\Delta \cap F_1} K_n(|x - y|) \, d\mu_{2n}^*(y) + \int_{\Delta'} K_n(|x - y|) \, d\mu_{1n}(y) \leqslant$$

$$\leqslant h_n \cdot A \cdot 2^m \cdot I(K_n, \mu_{2n}; F_2) + A \cdot I(K_n, \mu_{1n}; F_1) = O(I(K_n, \mu_{1n}; F_1)),$$

where A is the constant in (3.3). The estimate

$$\int_{\Delta'} K_n(|x - y|) \, d\mu_{1n}(y) \leqslant A \cdot I(K_n, \mu_{1n}; F_1)$$

follows immediately from (3.3). In order to show that

$$\int_{\Delta \cap F_1} K_n(|x - y|) \, d\mu_{2n}^*(y) \leqslant A \cdot 2^m \cdot I(K_n, \mu_{2n}; F_2),$$

we proceed in the following way. Consider the domains into which Δ is subdivided by the $(m - 1)$-dimensional planes which, on the one hand, pass through the point x_n and where, on the other hand, each plane is parallel to one of the edge planes to Δ. μ_{2n}^* coincides, in each of these domains, with a translation of the part of the capacitary distribution μ_{2n} which is situated in Δ, and as there are 2^m such domains the desired inequality follows by means of (3.3).

(5.5) and the three estimates which we have obtained for $u_n(x)$ give after simplification

$$\text{I} \leqslant h_n I(K_n, \mu_{1n}; F_1) \cdot \mu_{1n}(\Delta) + O(1).$$

$$\text{II} \leqslant O(I(K_n, \mu_{1n}; F_1)) \left[\mu_{1n}(R_d \setminus \Delta) + h_n \mu_{2n}^*(R_d \cap \Delta) \right].$$

$$\text{III} \leqslant O(1) + I(K_n, \mu_{1n}; F_1) \cdot \mu_{1n}(\Delta').$$

This gives

$$\frac{I(K_n, \mu_n; F_1)}{I(K_n, \mu_{1n}; F_1)} \leqslant h_n \mu_{1n}(\Delta) + \mu_{1n}(\Delta') + O(1) \cdot \{I(K_n, \mu_{1n}; F_1)\}^{-1} +$$

$$+ O(1) \left[\mu_{1n}(R_d \setminus \Delta) + h_n \mu_{2n}^*(R_d \cap \Delta) \right].$$

By (5.7) and the fact that

$$\lim_{n\to\infty} \mu_{2n}^{*}(R_d \cap \Delta) = \frac{m(R_d \cap \Delta)}{m(F_2)},^{1}$$

we have

$$\overline{\lim_{n\to\infty}}\, \frac{I(K_n,\, \mu_n;\, F_1)}{I(K_n,\, \mu_{1n};\, F_1)} \leqslant h\sigma_1(\Delta) + \sigma_1(\Delta') + \text{const.} \cdot \left[\sigma_1(R_d \setminus \Delta) + \frac{m(R_d \cap \Delta)}{m(F_2)}\right],$$

if d is chosen so that R_d is regular σ_1. $d\to 0$ gives, as $h < 1$,

$$\overline{\lim_{n\to\infty}}\, \frac{I(K_n,\, \mu_n;\, F_1)}{I(K_n,\, \mu_{1n};\, F_1)} < 1,$$

in the case when $\sigma_1(\Delta) > 0$. Accordingly we have obtained a contradiction to (5.3) for this case. If we suppose that $\sigma_1(\Delta) = 0$ and $m(\Delta \cap F_1) > 0$, we easily obtain a contradiction by moving some mass μ_{1n} to $\Delta \cap F_1$ and there distributing it in the same way as μ_{2n}^{*} is distributed and by then doing and estimate of the energy integral which is analogous to the one we have done above. By that Lemma 4 is proved.

We are now in a position to prove Theorem 1 by using Lemma 2 and Lemma 4.

Theorem 1. *Suppose that $K_n(r)$, $n = 1, 2, \ldots,$ and $K(r)$ are defined for $r \geqslant 0$, continuous for $r > 0$, satisfy $\lim_{r\to 0} K_n(r) = K_n(0) \leqslant \infty$, are non-negative and non-increasing. Also suppose that $\lim_{n\to\infty} K_n(r) = K(r),$*

$$\int_0^1 K_n(r)\, r^{m-1}\, dr < \infty \quad and \quad \int_0^1 K(r)\, r^{m-1}\, dr = \infty.$$

Let μ_n be a capacitary distribution belonging to $K_n(r)$ and F, where F is a compact set of positive m-dimensional Lebesgue measure. Then $\{\mu_n\}$ converges weakly to the distribution σ which has constant density on F, $\sigma(E) = m(E)/m(F)$, $E \subset F$, E Borel set.

Proof. Choose a sequence $\{F_\nu\}$, where each F_ν is the union of a finite number of closed m-dimensional intervals, so that $F_1 \supset F_2 \supset \ldots \supset F_\nu \supset \ldots \supset F$ and $F = \bigcap_1^\infty F_\nu$. Let $\mu_{n\nu}$ be a capacitary distribution belonging to $K_n(r)$ and F_ν. According to Lemma 2 $\{\mu_{n\nu}\}$ converges, when $n\to\infty$, to the distribution σ_ν which has constant density on F_ν. By Lemma 4 we have $\sigma_\nu(E) \leqslant \mu(E)$ for every $E \subset F$ where μ denotes the limit distribution of a convergent subsequence to $\{\mu_n\}$. But $\{\sigma_\nu\}$ converges to the distribution σ with constant density on F. This implies $\sigma(E) \leqslant \mu(E)$, $E \subset F$. But as $\sigma(F) = \mu(F) = 1$ there must be equality, i.e.

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[1] This equality easily follows if we recall that μ_{2n}^{*} is the translation of a distribution from a sequence which converges to a constant times the Lebesgue measure and that $R_d \cap \Delta$ is regular with respect to the Lebesgue measure.

$$\mu(E) = \sigma(E) = \frac{m(E)}{m(F)},$$

and this is true for every limit distribution μ. Hence $\{\mu_n\}$ converges to σ and the theorem is proved.

6.

Lemma 1 can be proved in a generalized form which can be used to prove the following theorem

Theorem 2. *Let F be a compact set in R^m. Then we have, if $k = 2^{-1}\pi^{-m/2}\Gamma(m/2)$,*

$$\lim_{\alpha \to m-0} \frac{C_\alpha(F)}{m - \alpha} = km(F). \tag{6.1}$$

Proof. We first prowe the theorem in the case when F is a sphere, $F = S(x_0, R) = S$. For this case M. Riesz ([7], p. 16) has given an explicit formula for the equilibrium potential belonging to the kernel $r^{-\alpha}$, $m - 2 < \alpha < m$. Namely, let $dm(y)$ denote the element of volume and put

$$u(x) = \pi^{-(\frac{1}{2}m+1)} \Gamma\left(\frac{m}{2}\right) \sin\frac{\pi(m-\alpha)}{2} \int_S (R^2 - |y - x_0|^2)^{-(m-\alpha)/2} \frac{1}{|x-y|^\alpha} dm(y).$$

Then we have $u(x) = 1$ for all $x \in S$.

This formula gives

$$C_\alpha(S) = \frac{2}{\pi} \sin\frac{\pi(m-\alpha)}{2} \int_0^R (R^2 - r^2)^{-(m-\alpha)/2} r^{m-1} dr$$

$$= \frac{2}{\pi} \sin\frac{\pi(m-\alpha)}{2} R^\alpha \int_0^1 (1 - r^2)^{-(m-\alpha)/2} r^{m-1} dr.$$

Hence

$$\lim_{\alpha \to m} \frac{C_\alpha(S)}{m - \alpha} = \frac{R^m}{m} = 2^{-1} \pi^{-m/2} \Gamma\left(\frac{m}{2}\right) m(S),$$

i.e. (6.1) is true in this case.

We now consider the general case. It is clearly enough to consider the case when $m(F) > 0$. We choose a sphere $S = S(x_0, R)$ so that $S \supset F$. Let μ_α be the equilibrium distribution belonging to $r^{-\alpha}$ and S. Then $\{\mu_\alpha\}$ converges, when $\alpha \to m$, to the distribution which has constant density on S, and the following formula—which is Lemma 1 in the required generalized form—is valid

$$\varliminf_{\alpha \to m} \mu_\alpha(F) \leqslant \varliminf_{\alpha \to m} \frac{C_\alpha(F)}{C_\alpha(S)} \leqslant \varlimsup_{\alpha \to m} \frac{C_\alpha(F)}{C_\alpha(S)} \leqslant \frac{m(F)}{m(S)}. \tag{6.2}$$

In order to prove (6.2), we choose, as in the proof of Theorem 1, a sequence $\{F_n\}$ so that $S \supset F_n \supset F_{n+1} \supset F$, $\cap F_n = F$, where every F_n consists of a finite

number of intervals. According to the remark following Lemma 1 the conclusion of the lemma remains valid also in this more general situation, i.e. we have

$$\lim_{\alpha \to m} \frac{C_\alpha(F_n)}{C_\alpha(S)} = \frac{m(F_n)}{m(S)}.$$

Hence

$$\overline{\lim_{\alpha \to m}} \frac{C_\alpha(F)}{C_\alpha(S)} \leqslant \lim_{\alpha \to m} \frac{C_\alpha(F_n)}{C_\alpha(S)} = \frac{m(F_n)}{m(S)},$$

and, finally, letting $n \to \infty$, we get one of the inequalities in (6.2). In order to prove the other inequality we observe that

$$\mu_\alpha(F) \geqslant \frac{I(r^{-\alpha}, \mu_\alpha; F)}{I(r^{-\alpha}, \mu_\alpha; S)} \geqslant \frac{C_\alpha(S)}{C_\alpha(F)} \cdot \mu_\alpha^2(F),$$

and consequently we have $\mu_\alpha(F) \leqslant C_\alpha(F)/C_\alpha(S)$, which gives the other inequality in (6.2).

Our explicit formula for μ_α makes it possible to show that

$$\lim_{\alpha \to m} \mu_\alpha(F) = \frac{m(F)}{m(S)}.^{1} \qquad (6.3)$$

Namely,

$$\mu_\alpha(F) = 2^{-1} \pi^{-m/2} R^{-\alpha} \Gamma\left(\frac{m}{2}\right) \left\{ \int_0^1 (1-r^2)^{-(m-\alpha)/2} r^{m-1} dr \right\}^{-1} \int_F (R^2 - |y - x_0|^2)^{-(m-\alpha)/2} dm(y),$$

which gives $\qquad \lim_{\alpha \to m} \mu_\alpha(F) = 2^{-1} \pi^{-m/2} R^{-m} \Gamma\left(\frac{m}{2}\right) \cdot m \cdot m(F) = \frac{m(F)}{m(S)}.$

(6.2) and (6.3) give $\qquad \lim_{\alpha \to m} \frac{C_\alpha(F)}{C_\alpha(S)} = \frac{m(F)}{m(S)}.$ $\qquad (6.4)$

Hence $\qquad \lim_{\alpha \to m} \frac{C_\alpha(F)}{m - \alpha} = \lim_{\alpha \to m} \frac{C_\alpha(F)}{C_\alpha(S)} \cdot \lim_{\alpha \to m} \frac{C_\alpha(S)}{m - \alpha} = \frac{m(F)}{m(S)} \cdot km(S) = km(F),$

and (6.1) is proved also in the general case.

Remark. Theorem 1 is an easy consequence of Lemma 1 and (6.4) in the case when the kernels $K_n(r)$, $n = 1, 2, \ldots$, are all of the form $r^{-\alpha}$ and $K(r) = r^{-m}$. For general kernels $K_n(r)$ and $K(r)$ we cannot, however, prove a relation of the form (6.4) since we do not have an explicit formula for a capacitary distribution belonging to the kernel $K_n(r)$ and a sphere, and so we had to prove Theorem 1 by the more complicated method which we have used.

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¹ This formula is not a consequence of Theorem 1 as convergence in Theorem 1 means convergence in the weak sense.

7.

We now turn to the proofs of the counter-examples.

The discussion in this section is based on the following fact. To a given number a, $a < 1$, there exists a closed linear set F satisfying

$$C_\alpha(F) > 0 \quad \text{for} \quad \alpha \leqslant a$$
$$\text{and} \quad C_\alpha(F) = 0 \quad \text{for} \quad \alpha > a \qquad (7.1)$$

For F it is even possible to choose a Cantor set. To see this, we suppose that F is a Cantor set where the nth step in the construction is a set which consists of 2^n intervals where each interval has length l_n. As F has positive α-capacity if and only if $\sum_1^\infty 2^{-n} l_n^{-\alpha} < \infty$,[1] (7.1) follows from the fact that we can clearly choose $\{l_n\}$ so that

$$\sum_1^\infty 2^{-n} l_n^{-a} < \infty$$

and

$$\sum_1^\infty 2^{-n} l_n^{-(a+\varepsilon)} = \infty \quad \text{for every} \quad \varepsilon > 0.$$

Theorem 3. *To every given number α_0, $0 < \alpha_0 \leqslant 1$, there exists a closed linear set F such that $C_\alpha(F) > 0$ for $\alpha < \alpha_0$ and $C_{\alpha_0}(F) = 0$, and such that the equilibrium distributions μ_α belonging to $r^{-\alpha}$ and F do not converge when $\alpha \to \alpha_0 - 0$.*

Proof. We choose two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ where $\alpha_n \nearrow \alpha_0$, $\beta_n \nearrow \alpha_0$ and $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$. Our set F is to be the union of two sets A and B which compete for the mass of the equilibrium distributions, and where $A = \bigcup_0^\infty A_n$ and $B = \bigcup_0^\infty B_n$. We want to construct A and B so that $\{\mu_\alpha\}$ does not converge when $\alpha \to \alpha_0$ and runs through the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. We start with two closed, disjoint intervals, I and J, and we shall construct A and B in such a way that $A \subset I$ and $B \subset J$. We first show that it is enough to find A and B so that for instance

$$\frac{C_{\alpha_n}(A)}{C_{\alpha_n}(B)} \geqslant 2^{n-1} > 2^{-n} \geqslant \frac{C_{\beta_n}(A)}{C_{\beta_n}(B)}, \quad n = 1, 2, \dots . \qquad (7.2)$$

Namely, if it is true that there exists a distribution μ so that the equilibrium distributions μ_α belonging to $r^{-\alpha}$ and $F = A \cup B$ converge to μ when $\alpha \to \alpha_0$, then we have, by Lemma 1, if $\mu(A) > 0$ and $\mu(B) > 0$

$$\lim_{\alpha \to m} \frac{C_\alpha(A)}{C_\alpha(B)} = \lim_{\alpha \to m} \frac{C_\alpha(I \cap F)}{C_\alpha(J \cap F)} = \lim_{\alpha \to m} \frac{C_\alpha(I \cap F)}{C_\alpha(F)} \cdot \lim_{\alpha \to m} \frac{C_\alpha(F)}{C_\alpha(J \cap F)} = \mu(I) \cdot \{\mu(J)\}^{-1} = \frac{\mu(A)}{\mu(B)}$$

which is a contradiction to (7.2). If one of the numbers $\mu(A)$ and $\mu(B)$ is zero, we get a contradiction to (7.2) in a similar way. Consequently, it is enough to find A and B so that (7.2) holds.

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[1] See for instance [5].

We now construct A and B, $A \subset I$, $B \subset J$. We subdivide I into three equal smaller intervals. The first interval (counted from the left to the right) we denote by I_1 and we shall do the construction so that $\bigcup_2^\infty A_n \subset I_1$. The second interval is to belong to the complement of A and the third is to contain A_1. We choose A_1 in such a way that

$$C_{\alpha_1}(A_1) > 0 \quad \text{and} \quad C_\alpha(A_1) = 0 \quad \text{for} \quad \alpha > \alpha_1. \tag{7.3}$$

Starting from J we now construct J_1 in the same way, $\bigcup_2^\infty B_n \subset J_1$, and B_1 as a subset of the third subinterval of J. We want B_1 to satisfy

$$C_{\beta_1}(B_1) > 0 \quad \text{and} \quad C_\alpha(B_1) = 0 \quad \text{for} \quad \alpha > \beta_1. \tag{7.4}$$

and

$$C_{\alpha_1}(B_1) \leqslant C_{\alpha_1}(A_1) \cdot 2^{-1}. \tag{7.5}$$

This can be realized, because if (7.5) is not satisfied for a certain choice of B_1, we can replace B_1 by a subset of the third subinterval of J, a subset which is a translation of $k \cdot B_1$, $k > 0$, where $k \cdot B_1$ denotes the set consisting of the points $k \cdot x$ where $x \in B_1$, and choose k so small that $k \cdot B_1$ satisfies (7.5). Condition (7.4) is not disturbed by this.

By subdividing I_1 into three equal smaller intervals, we get I_2, $\bigcup_3^\infty A_n \subset I_2$, and we construct A_2 analogously. The condition (7.3) is replaced by

$$C_{\alpha_2}(A_2) > 0 \quad \text{and} \quad C_\alpha(A_2) = 0 \quad \text{for} \quad \alpha > \alpha_2$$

and

$$C_{\beta_1}(A_2) \leqslant C_{\beta_1}(B_1) \cdot 2^{-2}.$$

The construction is now continued in the same way. The conditions on A_n are

$$C_{\alpha_n}(A_n) > 0 \quad \text{and} \quad C_\alpha(A_n) = 0 \quad \text{for} \quad \alpha > \alpha_n$$

and

$$C_{\beta_k}(A_n) \leqslant C_{\beta_k}(B_k) \cdot 2^{-n}, \quad k = 1, 2, \ldots, n-1.$$

The conditions on B_n are

$$C_{\beta_n}(B_n) > 0 \quad \text{and} \quad C_\alpha(B_n) = 0 \quad \text{for} \quad \alpha > \beta_n$$

and

$$C_{\alpha_k}(B_n) \leqslant C_{\alpha_k}(A_k) \cdot 2^{-n}, \quad k = 1, 2, \ldots, n.$$

Finally we choose the left endpoints of I and J as A_0 and B_0 respectively, which guarantees that A and B will be closed.

We now have $\quad\quad\quad C_{\alpha_n}(A) \geqslant C_{\alpha_n}(A_n)$

and
$$C_{\alpha_n}(B) \leqslant \sum_{\nu=0}^{\infty} C_{\alpha_n}(B_\nu) = \sum_{\nu=n}^{\infty} C_{\alpha_n}(B_\nu) \leqslant \sum_{\nu=n}^{\infty} C_{\alpha_n}(A_n) \cdot 2^{-\nu}$$

$$= C_{\alpha_n}(A_n) \cdot 2^{-n+1} \leqslant C_{\alpha_n}(A) \cdot 2^{-n+1}.$$

This implies
$$\frac{C_{\alpha_n}(A)}{C_{\alpha_n}(B)} \geqslant 2^{n-1}, \quad n = 1, 2, \ldots,$$

which is one of the inequalities in (7.2). In exactly the same way we get

$$\frac{C_{\beta_n}(A)}{C_{\beta_n}(B)} \leqslant 2^{-n}, \quad n = 1, 2, \ldots,$$

which is the other inequality, and hence Theorem 3 is proved.

8.

By doing a construction similar to the one we have used in the proof of Theorem 3 we also get the counter-example which is formulated in Theorem 4. However, we first have to prove the existence of sets having somewhat different qualities than those which have been formulated in (7.1). We do this in the following lemma.

Lemma 5. *Suppose that two numbers l and α_0 are given, where $l > 0$ and $0 < \alpha_0 < 1$. Then there exists a constant $r(l, \alpha_0)$ which only depends on l and α_0 so that, to any given positive number α_1 satisfying $\alpha_1 < \alpha_0$, it is possible to construct a closed linear set F satisfying*

F belongs to an interval with length l. (8.1)

$C_{\alpha_1}(F) > r(l, \alpha_0).$ (8.2)

The Hausdorff dimension of F is less than α_0. (8.3)

Proof. The set F shall be a generalized Cantor set, $F = \bigcap_{1}^{\infty} F_n$. We start with a closed interval ω with length l. Let k be an arbitrary positive number and a an arbitrary positive integer. We divide the interval ω into $2a - 1$ subintervals with lengths

$$l \cdot (a + ak - k)^{-1}, \ kl \cdot (a + ak - k)^{-1}, \ l \cdot (a + ak - k)^{-1}, \ \ldots kl \cdot (a + ak - k)^{-1},$$
$$l \cdot (a + ak - k)^{-1}$$

counted from the left to the right, and separate the a (closed) intervals with lengths $l(a + ak - k)^{-1}$. The union of the separated intervals constitutes F_1. Every interval belonging to F_1 is now subdivided in an analogous way into $2a - 1$ subintervals and a intervals are separated. In this way we get F_2 which consists of a^2 intervals with length $l(a + ak - k)^{-2}$ each. In the nth step of this procedure we obtain F_n which hence consists of a^n intervals with length $l(a + ak - k)^{-n}$ each.

We begin by showing that the Hausdorff dimension of F is

$$\beta = \frac{\log a}{\log (a + ak - k)}.^{1}$$

It is immediately realized that the Hausdorff dimension of F cannot be larger than β and in order to realize that it is equal to β, we can, for instance, construct a bounded γ-potential, $\gamma < \beta$, belonging to a distribution of unit mass on F. To do this we introduce the set functions $\{\mu_n\}_1^\infty$, where μ_n is the unit mass uniformly distributed on F_n. $\{\mu_n\}$ converges to a distribution of the unit mass on F and we get the following estimate, if $l_n = l(a + ak - k)^{-n}$,

$$\int_F \frac{1}{|x - y|^\gamma} \, d\mu(y) \leqslant \lim_{n \to \infty} \int_{F_n} \frac{1}{|x - y|^\gamma} \, d\mu_n(y)$$

$$< \sum_{n=0}^\infty \frac{2}{l_n \cdot a^n} \int_0^{l_n/2} r^{-\gamma} \, dr = \frac{2^\gamma}{l^\gamma(1 - \gamma)} \sum_{n=0}^\infty \frac{(a + ak - k)^{n \cdot \gamma}}{a^n}$$

$$= \frac{2^\gamma}{l^\gamma(1 - \gamma)} \cdot \frac{a}{a - (a + ak - k)^\gamma} < \infty,$$

if $\gamma < \log a / \log (a + ak - k) = \beta$. Hence the Hausdorff dimension of F is β.

The estimate which we have done also gives us the following inequality, if $\gamma < \beta$,

$$C_\gamma(F) > \frac{l^\gamma(1 - \gamma)}{2^\gamma} \cdot \frac{a - (a + ak - k)^\gamma}{a} = M(\gamma).$$

Hence the lemma follows if we can find a constant $r(l, \alpha_0)$ so that

$$\beta < \alpha_0 \tag{8.4}$$

and

$$M(\alpha_1) \geqslant r(l, \alpha_0), \tag{8.5}$$

where α_1 is the number which is given in the lemma, $\alpha_1 < \alpha_0$. However, a simple calculation shows that we can find a constant $r(l, \alpha_0)$ only depending on l and α_0 so that (8.4) and (8.5) are satisfied if k and a are chosen in a suitable way and large enough.[2] Lemma 5 is hence proved and we can now show

Theorem 4. *Let α_0 be an arbitrary number satisfying $0 < \alpha_0 < 1$. Then there exists a closed linear set F with positive and finite α_0-dimensional Hausdorff measure, $0 < \Lambda_{\alpha_0}(F) < \infty$, so that the equilibrium distributions μ_α belonging to the kernels $r^{-\alpha}$ and F do not converge, when $\alpha \to \alpha_0 - 0$ to a distribution where the mass situated on a subset of F is proportional to the α_0-dimensional Hausdorff measure of that subset.*

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[1] This is a consequence of a theorem by Ohtsuka [5] on the capacity of generalized Cantor sets. We give, however, a short direct proof which will also permit us to estimate $C_\gamma(F)$, $\gamma < \beta$.

[2] The choice of k and a will naturally depend on α_1. For $r(l, \alpha_0)$ we can for instance choose any number which is smaller than $[l^{\alpha_0}(1 - \alpha_0)]/2^{\alpha_0}$.

Proof. Again F shall be the union of two sets A and B, $F = A \cup B$, where for B we choose an arbitrary compact set having positive and finite α_0-dimensional Hausdorff measure, $0 < \Lambda_{\alpha_0}(B) < \infty$, and where A shall be constructed so that $\Lambda_{\alpha_0}(A) = 0$. In order to construct A we choose disjoint intervals ω_i with lengths l_i, $i = 1, 2, \ldots$, so that $\sum_1^\infty l_i < \infty$, and $\bigcup_1^\infty \omega_i$ is at a positive distance from B. We also assume that the choice is such that the intervals ω_i converge to one point. We shall construct A so that $A = \bigcup_0^\infty A_i$, where A_i is closed and $A_i \subset \omega_i$ for $i = 1, 2, \ldots$. For A_0 we choose the point to which the intervals ω_i converge which makes A closed.

As $\Lambda_{\alpha_0}(B) < \infty$ we have $C_{\alpha_0}(B) = 0$ which implies

$$\lim_{\alpha \to \alpha_0} C_\alpha(B) = 0.$$

Hence we can choose a sequence $\{\alpha_i\}_1^\infty$, $\alpha_i \nearrow \alpha_0$ so that for instance

$$C_{\alpha_i}(B) < 2^{-1} \cdot r(l_i, \alpha_0), \quad i = 1, 2, \ldots,$$

where $r(l_i, \alpha_0)$ is the constant which occurs in Lemma 5. The lemma then guarantees that we can construct A_i so that the conditions which we have formulated above are valid and so that

$$C_{\alpha_i}(A_i) > r(l_i, \alpha_0), \quad i = 1, 2, \ldots.$$

The last two inequalities imply that

$$C_{\alpha_i}(A) > 2 \cdot C_{\alpha_i}(B), \quad i = 1, 2, \ldots.$$

But this means, according to Lemma 1, that the limit distribution of every convergent subsequence of $\{\mu_{\alpha_i}\}$ distributes mass on A. As $\Lambda_{\alpha_i}(A) = 0$ we can hence conclude that $\{\mu_\alpha\}$ does not converge to a distribution where the mass situated on a subset of F is proportional to the α_0-dimensional Hausdorff measure of that subset.

9.

We shall now prove that there is a correspondence to Theorem 4 even when $\alpha_0 = 1$. As has been pointed out in the introduction, we have to use plane sets to get such a correspondence.

Theorem 5. *There exists a closed set F in the plane having positive and finite 1-dimensional Hausdorff measure, $\Lambda_1(F)$, so that the equilibrium distributions belonging to the kernels $r^{-\alpha}$ and F do not converge, when $\alpha \to 1 - 0$, to a distribution where the mass situated on a subset of F is proportional to the 1-dimensional Hausdorff measure of that subset.*

The proof of this is analogous to the proof of Theorem 4. The difference is that instead of considering intervals ω_i with lengths l_i, we consider squares ω_i with sides of lengths l_i, $i = 1, 2, \ldots$. In order to be able to construct the sets A_i we also have to reformulate Lemma 5 for plane sets and the case $\alpha_0 = 1$.

Lemma 5′. *To a given positive number l there exists a constant $r(l)$ so that, to any given positive number α_1 satisfying $\alpha_1 < 1$, it is possible to construct a closed plane set F satisfying*

$$F \text{ belongs to a square with side of length } l. \tag{9.1}$$

$$C_{\alpha_1}(F) > r(l). \tag{9.2}$$

The Hausdorff dimension of F is less than 1. $\tag{9.3}$

Proof of Lemma 5′. Again we construct F as a generalized Cantor set, $F = \bigcap_1^\infty F_n$. We start with a square ω with side of length l and with numbers k and a as before. We devide ω into subsquares by dividing every side of ω into $2a - 1$ subintervals having the same lengths as before, i.e. the same lengths as in the proof of Lemma 5. We obtain F_1 by separating the a^2 squares with area $l^2 \cdot (a + ak - k)^{-2}$. Generally F_n consists of a^{2n} squares with area $l^2 \cdot (a + ak - k)^{-2n}$ each.

By doing calculations similar to those in the proof of Lemma 5, we can show that F has Hausdorff dimension

$$\beta = \frac{2 \log a}{\log (a + ak - k)} .$$

The estimate of $C_\gamma(F)$ becomes

$$C_\gamma(F) > \frac{l^\gamma (2 - \gamma)}{2^{\gamma+1}} \cdot \frac{a^2 - (a + ak - k)^\gamma}{a^2} = M(\gamma),$$

if $\gamma < \beta$. This finally gives that we can obtain

$$\beta < 1 \text{ and } M(\alpha_1) \geqslant r(l),$$

by choosing k, a and $r(l)$ suitably. By this Lemma 5′ and hence also Theorem 5 is proved.

Remark. Lemma 5′ can of course be formulated for an arbitrary number α_0 satisfying $0 < \alpha_0 < 2$ and a closed plane set F. This implies that Theorem 5 can also be formulated for sets F having positive and finite α_0-dimensional Hausdorff measure, where $\alpha_0 < 2$. Similar extensions of the counter-examples to higher dimensions are of course also possible.

10.

The above counter-examples show that the set F has to satisfy suitable conditions of regularity in order to give convergence of the sequence of equilibrium distributions in the case when the Lebesgue measure of F is zero. As an example we prove the simple theorem that the Cantor sets are regular enough to give convergence.

Theorem 6. *Let F be a linear Cantor set, $F = \bigcap_1^\infty F_\nu$, where F_ν consists of 2^ν intervals with length l_ν each. Call the intervals which constitute F_ν $\omega_{i\nu}$, $i = 1, 2, \ldots, 2^\nu$. Let $K_n(r)$ and $K(r)$ satisfy the conditions (a) and (b) and suppose that $C_{K_n}(F) > 0$ and $C_K(F) = 0$. Then it is true that a sequence of capacitary distributions, $\{\mu_n\}$, where μ_n belongs to $K_n(r)$ and F, converges, when $n \to \infty$, to the distribution which distributes as much mass on $\omega_{i\nu}$ as on $\omega_{j\nu}$, for all i and j.*

As to the proof we only observe that

$$C_{K_n}(F \cap \omega_{i\nu}) = C_{K_n}(F \cap \omega_{j\nu})$$

for all i, j, ν and n and hence the theorem can be proved in a similar way as Lemma 2.

Remark. There exist sets having a very simple structure for which the question of convergence is answered in the negative, at least for certain types of kernels. This is for instance the case for the set $F = A \cup B$ where $A \cap B = \emptyset$ and where $A = \cap A_\nu$ and $B = \cap B_\nu$ are the Cantor sets which are obtained from intervals of lengths 1 and 2 respectively by letting the sets A_ν and B_ν consist of 2^ν intervals with lengths $4^{-\nu}$ and $2 \cdot 4^{-\nu}$ respectively. To this set F it is possible to construct a sequence of kernels $\{K_n(r)\}$ such that $C_{K_n}(F) > 0$, $n = 1, 2, \ldots$, and a kernel $K(r)$ such that $C_K(F) = 0$, kernels which all are such that the equilibrium problem is possible and conditions (a) and (b) are satisfied and such that the equilibrium distributions μ_n belonging to $K_n(r)$ and F do not converge when $n \to \infty$. The idea of the construction is to choose the kernel $K_n(r)$ piecewise linear for $n = 1, 2, \ldots$, which makes it possible approximatively to determine the equilibrium distribution μ_n belonging to $K_n(r)$ and F. The choice of different lengths of the intervals building up A_ν and B_ν respectively, $\nu = 1, 2, \ldots$, also makes it possible to choose $K_n(r)$ so that the ability of A to compete for the mass μ_n is maximized for certain n and the ability of B is maximized for other values of n. In this way it is possible to arrange so that the sequence of equilibrium distributions does not converge.

11.

Finally we give a counter-example of a somewhat different kind.

Theorem 7. *Let α_0 be any positive number satisfying $\alpha_0 < 1$. There exists a closed linear set F with Hausdorff dimension larger than α_0 which is of such a nature that the equilibrium distributions μ_α belonging to $r^{-\alpha}$ and F do not converge towards the equilibrium distribution μ_{α_0} belonging to $r^{-\alpha_0}$ and F when $\alpha \to \alpha_0 + 0$.*

Proof. Let F be the union of two closed disjoint sets F_1 and F_2, $F = F_1 \cup F_2$, $F_1 \cap F_2 = \emptyset$, where F_1 is chosen so that the Hausdorff dimension of F_1 is larger than α_0 and F_2 is chosen so that $C_{\alpha_0}(F_2) > 0$ but $C_\alpha(F_2) = 0$ for $\alpha > \alpha_0$. Then we have $\mu_\alpha(F_2) = 0$ if $\alpha > \alpha_0$ and $\mu_{\alpha_0}(F_2) > 0$. The conclusion that $\mu_{\alpha_0}(F_2) > 0$ is a consequence of the maximum principle. Namely, since $C_{\alpha_0}(F_2) > 0$ there is a point in F_2 where the equilibrium potential $\int_F |x - y|^{-\alpha_0} d\mu_{\alpha_0}(y)$ takes its maximum value and this would be a contradiction to the maximum principle if $\mu_{\alpha_0}(F_2) = 0$. Hence $\mu_{\alpha_0}(F_2) > 0$ and thus $\{\mu_\alpha\}$ does not converge to μ_{α_0} and Theorem 7 is proved.

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