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# Integral representation of certain linear functionals

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## § 1. Introduction

Any addition to the already enormous literature on integral representations for abstract linear functionals on general function spaces must show good cause for its appearance. The justification submitted for the present paper consists of three assertions: (1) the exhibition of integral representations is carried out here under what must be truly minimal hypotheses; (2) the limitations inherent in any possible integral representation are clearly indicated by means of examples; (3) the results obtained have applications in the study of certain topological algebras associated with groups.

We are concerned in the present communication with the following problem. Let there be given a set $X$, a linear space $\mathfrak{F}$ of real or complex functions defined on $X$, and a linear functional I defined on $\mathfrak{F}$. Under what conditions is it possible to find a finitely (or countably) additive measure $\gamma^*$, defined on a certain family of subsets of $X$, such that

1.1 $$I(f) = \int_X f(x)\,d\gamma^*(x)$$

for all, or at least part of, the functions $f$ in $\mathfrak{F}$? A reasonably satisfactory answer to this question is contained in the present paper.

We use the following symbols and terminology. For a set $X$ and a family of subsets $\mathcal{A}$ of $X$, the symbol $\mathcal{R}(\mathcal{A})$ denotes the smallest ring of sets containing $\mathcal{A}$ (i.e., the smallest family of sets containing $\mathcal{A}$ and closed under the formation of finite unions and differences). The symbol $\mathcal{S}(\mathcal{A})$ denotes the smallest $\sigma$-ring containing $\mathcal{A}$ (i.e., the smallest ring containing $\mathcal{A}$ which is closed under the formation of countable unions). The symbol $\mathcal{H}(\mathcal{A})$ denotes the family of all subsets $Q$ of $X$ such that for some $A \varepsilon \mathcal{A}$, $Q \subset A$. For $P \subset X$, the symbol $\chi_P$ denotes the characteristic function of $P$, i.e., the function equal to 1 on $P$ and 0 on $X \cap P'$. A function $\varphi$ defined on a ring $\mathcal{B}$ of subsets of $X$ such that $0 \leqq \varphi \leqq +\infty$ is said to be a finitely additive measure on $\mathcal{B}$ if $\varphi$ is not identically $+\infty$ and if the relation $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ holds for all $A, B \varepsilon \mathcal{B}$ such that $A \cap B = 0$. A finitely additive measure on $\mathcal{B}$ is said to be countably additive if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}$, $\bigcup_{n=1}^{\infty} A_n \varepsilon \mathcal{B}$, and $A_n \cap A_m = 0$ for $n \neq m$ imply that $\varphi(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \varphi(A_n)$. If $\varphi$ is permitted to assume values in the closed interval $[-\infty, 0]$, then certain complications in the definition enter. We shall have occasion to consider negative measures only in 4.8 and 4.9, and in these cases, as will be seen, no ambiguity can occur. The symbols $R$ and $C$ designate the real and complex number systems, respectively. A family $\mathcal{A}$ of subsets of $X$ will be called an algebra ($\sigma$-algebra) if it is closed under the formation of finite (countable) unions and of complements.

## § 2. The fundamental theorem

**2.1 Definition.** Throughout the present §, let $X$ denote a fixed non-void set, and let $\mathfrak{F}$ denote a fixed set of real-valued functions defined on $X$ such that

2.1.1 $f, g \varepsilon \mathfrak{F}$ and $\alpha, \beta \varepsilon R$ imply $\alpha f + \beta g \varepsilon \mathfrak{F}$;

2.1.2 $f, g \varepsilon \mathfrak{F}$ imply $\min (f, g) \varepsilon \mathfrak{F}$;

2.1.3 $f \varepsilon \mathfrak{F}$ and $\alpha \geqq 0$ imply $\min (f, \alpha) \varepsilon \mathfrak{F}$.

We shall also consider occasionally the more stringent condition

2.1.4 the constant function $f(x) \equiv 1$ (denoted by the symbol "1") is in $\mathfrak{F}$; plainly, 2.1.4, 2.1.1, and 2.1.2 imply 2.1.3. We shall not assume that 2.1.4 holds, however, without specifically mentioning this assumption.

Assumptions 2.1.1 and 2.1.2 show that $\max (f, g) = - \min (-f, -g) \varepsilon \mathfrak{F}$ for $f, g \varepsilon \mathfrak{F}$ and that $|f| = f - 2 \min (f, 0) \varepsilon \mathfrak{F}$ for $f \varepsilon \mathfrak{F}$. Also, $f - \min (f, \alpha) = \max (f, \alpha) - \alpha$, so that $\max (f, \alpha) - \alpha \varepsilon \mathfrak{F}$ for $f \varepsilon \mathfrak{F}$ and $\alpha \geqq 0$.

**2.2 Definition.** Let $\mathcal{P}$ denote the family of all subsets of $X$ having the form $E[x; f(x) > 0]$ for $f \varepsilon \mathfrak{F}$. Let $\mathcal{Z}$ denote the family of all subsets of $X$ having the form $E[x; f(x) \geqq \alpha]$ for $f \varepsilon \mathfrak{F}$ and $\alpha > 0$.

2.3 Since $E[x; f(x) > a] = E[x; \max (f, a) - a > 0]$, we see that all sets $E[x; f(x) > a]$ for $a \geqq 0$ are in $\mathcal{P}$. It is plain from 2.2 that if $K \varepsilon \mathcal{Z}$, then $K \subset G$ for some $G \varepsilon \mathcal{P}$. We note also that $K' \cap G \varepsilon \mathcal{P}$ for all $K \varepsilon \mathcal{Z}$ and $G \varepsilon \mathcal{P}$. For, if $G = E[x; g(x) > 0]$ and $K = E[x; f(x) \geqq \alpha > 0]$, it is a routine matter to verify that $K' \cap G = E[x; \min[\alpha, g(x) + \min (f(x), \alpha)] - \min (f(x), \alpha) > 0]$.

2.4 If $G_1, G_2 \varepsilon \mathcal{P}$, and $G_i = E[x; f_i(x) > 0]$, then

$$G_1 \cup G_2 = E[x; \max (f_1(x), f_2(x)) > 0]$$

and

$$G_1 \cap G_2 = E[x; \min (f_1(x), f_2(x)) > 0].$$

Hence $G_1 \cup G_2$ and $G_1 \cap G_2 \varepsilon \mathcal{P}$. Similar computations show that if $K_1, K_2 \varepsilon \mathcal{Z}$, then $K_1 \cup K_2$ and $K_1 \cap K_2 \varepsilon \mathcal{Z}$.

2.5 Finally, we see at once from 2.2 and 2.3 that every set in $\mathcal{Z}$ is the intersection of a countable decreasing sequence of sets in $\mathcal{P}$ and that every set in $\mathcal{P}$ is the union of a countable increasing sequence of sets in $\mathcal{Z}$. Namely, $E\left[x; f(x) \geqq \alpha\right] = \bigcap_{n=1}^{\infty} E\left[x; f(x) > \alpha - \frac{1}{n}\right]$ and $E\left[x, f(x) > 0\right] = \bigcup_{n=1}^{\infty} E\left[x; f(x) \geqq \frac{1}{n}\right]$.

We next define a certain "separation" property for subsets of $X$, as follows.

**2.6 Definition.** For subsets $A, B$ of $X$, we write $A \wedge B$ if there exists a function $f \varepsilon \mathfrak{F}$ such that $f \geqq 0$, $f(x) = 1$ for $x \varepsilon A$, and $f(x) = 0$ for $x \varepsilon B$.

We now consider the functional on $\mathfrak{F}$ which it is our purpose to represent, so far as possible, by an integral.

**2.7 Definition.** Let I be a real-valued functional defined on $\mathfrak{F}$ such that:

2.7.1 $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for all $\alpha, \beta \varepsilon R$ and $f, g \varepsilon \mathfrak{F}$;

2.7.2 $f \geqq 0$ implies $I(f) \geqq 0$.

**2.8 Remark.** If all functions in $\mathfrak{F}$ are bounded and one other condition holds, we shall see that $I$ can be completely represented by an integral. However, our construction uses only the bounded functions in $\mathfrak{F}$, and it is perfectly possible for a functional to satisfy 2.7.1 and 2.7.2, be $\neq 0$, and vanish for all bounded functions in $\mathfrak{F}$. For example, Očan's generalized integral [14] may have this property. A simpler example is the following. Let $\mathfrak{G}$ be the set of all continuous real functions $g$ on $R$ such that $g(x) = O(x)$ $(x \to +\infty)$. Obviously, $\mathfrak{G}$ is a linear space containing all bounded continuous real functions. Let $p(g) = \overline{\lim}_{x \to +\infty} \frac{g(x)}{x}$. It is easily verified that $p(g+h) \leqq p(g) + p(h)$ and that $p(\alpha g) = \alpha p(g)$ for $\alpha \geqq 0$. On the subspace $\mathfrak{M}$ of $\mathfrak{G}$ for which $\lim_{x \to +\infty} \frac{g(x)}{x}$ exists, $p$ is a non-negative linear functional. By the Hahn-Banach theorem ([2], pp. 27—29), there exists a real-valued functional $Q$ on $\mathfrak{G}$ satisfying 2.7.1 — 2.7.2 and also the inequalities $-p(-g) \leqq Q(g) \leqq p(g)$. Thus $Q(g) = \lim_{x \to +\infty} \frac{g(x)}{x}$ whenever this limit exists. In particular, $Q = 0$ for all bounded functions in $\mathfrak{G}$.

Our problem is now to find a finitely additive measure $\gamma^*$ defined on some ring of subsets of $X$ such that $I$ is represented, so far as is possible, by an integral over $X$ with respect to $\gamma^*$. We find that the ring $\mathcal{R}(\mathcal{P})$ serves very well as the ring of sets; and we proceed to define and to study the measure $\gamma^*$.

**2.9 Definition.** Let $H$ be any set in $\mathcal{P}$. Then the set-function $\gamma(H)$ is defined as $\sup_{0 \leqq f \leqq \chi_H} I(f)$. For every subset $A$ of $X$ which is contained in some $H \varepsilon \mathcal{P}$ (i.e., which is in the ring $\mathcal{H}(\mathcal{P})$), let $\gamma^*(A)$ be defined as $\inf_{H \varepsilon \mathcal{P}, H \supset A} \gamma(H)$.

The set-function $\gamma$ is well-defined, since $0 \varepsilon \mathfrak{F}$ and $0 \leqq 0 \leqq \chi_H$. The set-function $\gamma^*$ is well-defined, in view of the restriction of its domain. As we shall see, $\gamma^*$ has some of the properties of an outer measure, and a reasonable concept of measurability can be defined in terms of it. We first list some essential facts concerning $\gamma$ and $\gamma^*$.

**2.10 Theorem.** For all sets $G, H \varepsilon \mathcal{P}$, we have:

2.10.1 $0 \leqq \gamma(G) \leqq +\infty$;

2.10.2 $G \subset H$ implies $\gamma(G) \leqq \gamma(H)$.

These facts are obvious.

**2.11 Theorem.** For all $K \varepsilon \mathfrak{Z}$, $\gamma^*(K) < +\infty$.

By 2.2, there exists a function $f \varepsilon \mathfrak{F}$ (which may be taken $\geqq 0$) and a number $\alpha > 0$ such that $K = E[x; f(x) \geqq \alpha]$. Let $\beta$ be such that $0 < \beta < \alpha$. The function $\frac{1}{\beta} \min(f, \beta)$ is equal to 1 on $G = E[x; f(x) \geqq \beta]$, which contains $K$. If $0 \leqq h \leqq \chi_G$, then $h \leqq \frac{1}{\beta} \min(f, \beta)$, and by 2.7.2, $I(h) \leqq \frac{1}{\beta} I(\min(f, \beta))$. Hence $\gamma(G)$ is $\leqq \frac{1}{\beta} I(\min(f, \beta))$, and $\gamma^*(K)$ is likewise $\leqq \frac{1}{\beta} I(\min(f, \beta)) < +\infty$.

**2.12 Theorem.** Every set $G \varepsilon \mathcal{P}$ is $\sigma$-finite.

Write $G$ as $E[x; g(x) > 0]$, for some $g \varepsilon \mathfrak{F}$. The sets $G_n = E\left[x; x(g) > \frac{1}{n}\right]$ and

$K_n = E\left[x; g(x) \geqq \frac{1}{n}\right]$ $n = 1, 2, 3, \ldots$) are clearly such that $G_n \subset K_n$, $\bigcup_{n=1}^{\infty} G_n = G$, and $\gamma(G_n) \leqq \gamma^*(K_n)$ (by 2.10.2 and 2.9). The present theorem now follows from 2.11.

**2.13 Theorem.** Let $G$ be a set in $\mathcal{P}$ such that $\gamma(G) < +\infty$. For every $\varepsilon > 0$, there exists a set $K \varepsilon \mathcal{Z}$ such that $K \subset G$ and $\gamma(K' \cap G) < \varepsilon$.

Let $f \varepsilon \mathfrak{F}$ have the properties that $0 \leqq f \leqq \chi_G$ and $I(f) + \varepsilon/3 > \gamma(G)$. Let $K = E\left[x; f(x) \geqq \frac{\varepsilon}{3}\right]$. Then $K$ is in $\mathcal{Z}$ by definition. If $g \varepsilon \mathfrak{F}$ and $0 \leqq g \leqq \chi_{K' \cap G}$, then $0 \leqq \frac{3}{3+\varepsilon}(f+g) \leqq \chi_G$, and consequently $\frac{3}{3+\varepsilon}\left[I(f) + I(g)\right] \leqq \gamma(G)$. Therefore

$$I(g) \leqq \left(1 + \frac{\varepsilon}{3}\right)\gamma(G) - I(f) \leqq \gamma(G) - I(f) + \frac{\varepsilon}{3}\gamma(G) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\gamma(G).$$

The present theorem follows at once.

Using the notation of 2.13, let $\varphi = \frac{3}{\varepsilon}\left(\min\left(f, \frac{\varepsilon}{3}\right)\right)$. Noting that $\varphi = 1$ on $K$ and $0$ on $G'$, we have:

**2.14 Theorem.** Let $G$ be in $\mathcal{P}$ and let $\gamma(G)$ be finite. Then, for every $\varepsilon > 0$, there exists $K \varepsilon \mathcal{Z}$ such that $K \subset G$, $K \wedge G'$, and $\gamma(K' \cap G) < \varepsilon$.

By a simple adaptation of the arguments used in proving 2.13 and 2.14, we establish the following result.

**2.15 Theorem.** For all $G \varepsilon \mathcal{P}$ such that $\gamma(G)$ is finite, and for every $\varepsilon > 0$, there exists a set $H \varepsilon \mathcal{P}$ such that $H \subset G$, $H \wedge G'$, and $\gamma(H) + \varepsilon > \gamma(G)$.

**2.16 Note.** If $\gamma(G) = +\infty$, then sets $K$ and $H$ analogous to those in 2.13—2.15 need not exist, as example 2.29 *infra* shows.

**2.17 Theorem.** For all $G_1$, $G_2 \varepsilon \mathcal{P}$, we have $\gamma(G_1 \cup G_2) \leqq \gamma(G_1) + \gamma(G_2)$.

If $\gamma(G_1)$ or $\gamma(G_2)$ is $+\infty$, the theorem is obvious. Thus we suppose that $\gamma(G_1)$ and $\gamma(G_2)$ are both finite. Let $\varepsilon$ be any positive real number. Let $K_i$ be sets in $\mathcal{Z}$ such that $K_i \subset G_i$ and $\gamma(K_i' \cap G) < \varepsilon/3$, and for which functions $\varphi_i \varepsilon \mathfrak{F}$ exist such that $0 \leqq \varphi_i \leqq 1$, $\varphi_i = 0$ on $G_i'$ and $\varphi_i = 1$ on $K_i$ $(i = 1,2)$; $\varphi_i$ and $K_i$ exist in view of 2.14. Let $f \varepsilon \mathfrak{F}$ be such that $0 \leqq f \leqq \chi_{G_1 \cup G_2}$ and $I(f) + \varepsilon/3 > \gamma(G_1 \cup G_2)$. Let $g_1 = \min(f, \varphi_1)$, and let $g_2 = \min(f - g_1, \varphi_2)$. It is clear that $g_1 + g_2 = f$ except possibly on the sets $(G_1 \cap K_1')$ and $(G_2 \cap K_2')$. On the first set, $f - (g_1 + g_2) = p_1$, say, and $0 \leqq p_1 \leqq \chi_{G_1 \cap K_1'}$. The function $f - (g_1 + g_2) - p_1 = p_2$ satisfies the inequalities $0 \leqq p_2 \leqq \chi_{G_2 \cap K_2'}$. Observing that $0 \leqq g_i \leqq \chi_{G_i}$ $(i = 1, 2)$, we have the following relations:

$$\gamma(G_1 \cup G_2) < I(f) + \frac{\varepsilon}{3} = I(g_1 + g_2 + p_1 + p_2) + \frac{\varepsilon}{3} =$$

$$I(g_1) + I(g_2) + I(p_1) + I(p_2) + \frac{\varepsilon}{3} < \gamma(G_1) + \gamma(G_2) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Since $\varepsilon$ is arbitrary, the present theorem is proved.

**2.18 Theorem.** If $G_1$, $G_2 \in \mathcal{P}$ and $G_1 \cap G_2 = 0$, then

$$\gamma(G_1 \cup G_2) = \gamma(G_1) + \gamma(G_2).$$

If $\gamma(G_1)$ or $\gamma(G_2) = +\infty$, then we have $\gamma(G_1 \cup G_2) = +\infty$, and the theorem is verified. Thus suppose that both $\gamma(G_1)$ and $\gamma(G_2)$ are finite. Let $\varepsilon$ be any positive real number, and let $f_i$ be functions in $\mathfrak{F}$ such that $0 \leqq f_i \leqq \chi_{G_i}$ and $I(f_i) + \varepsilon/2 > \gamma(G_i)$ $(i = 1, 2)$. Then $\gamma(G_1 \cup G_2) \geqq I(f_1 + f_2) > \gamma(G_1) + \gamma(G_2) - \varepsilon$. This inequality, combined with 2.17, proves the present theorem.

We turn now to the set function $\gamma^*$, listing first some obvious properties.

**2.19 Theorem.** On $\mathcal{P}$, the set-functions $\gamma$ and $\gamma^*$ coincide.

**2.20 Theorem.** For all sets $A$ and $B$ in $\mathcal{H}(\mathcal{P})$, the following relations obtain:

2.20.1 $0 \leqq \gamma^*(A) \leqq +\infty$;

2.20.2 $\gamma^*(A) \leqq \gamma^*(B)$ if $A \subset B$;

2.20.3 $\gamma^*(A \cup B) \leqq \gamma^*(A) + \gamma^*(B)$.

We next introduce measurability, in the standard way.

**2.21 Definition.** A set $A \varepsilon \mathcal{H}(\mathcal{P})$ is said to be $\gamma^*$-measurable if for every set $Q \varepsilon \mathcal{H}(\mathcal{P})$, $\gamma^*(Q) = \gamma^*(Q \cap A) + \gamma^*(Q \cap A')$.

**2.22 Theorem.** Every set in $\mathcal{P}$ is $\gamma^*$-measurable.

In view of 2.20.3, we have only to prove that $\gamma^*(Q) \geqq \gamma^*(Q \cap G) + \gamma^*(Q \cap G')$, $G$ being a set in $\mathcal{P}$ and $Q$ being a set in $\mathcal{H}(\mathcal{P})$. We may suppose that $\gamma^*(Q)$ is finite, in which case $\gamma^*(Q \cap G)$ and $\gamma^*(Q \cap G')$ are also finite, by 2.20.2. Let $\alpha$ be any positive number; let $D$ be a set in $\mathcal{P}$ such that $D \supset Q$ and $\gamma(D) < \gamma^*(Q) + \alpha$; and let $H$ be a set in $\mathcal{P}$ such that $Q \cap G \subset H \subset G \cap D$. Let $J$ be a set in $\mathcal{P}$ such that $J \subset H$, $J \wedge H'$, and $\gamma(J) + \alpha > \gamma(H)$. Let $f$ be a function in $\mathfrak{F}$ such that $f = 1$ on $J$ and 0 on $H'$, and let $K = E\,[x; f(x) \geqq \tfrac{1}{2}]$. Finally, let $P \varepsilon \mathcal{P}$ have the property that $P \supset Q \cap G'$. The set $P \cap K' \cap D$ clearly has this property as well, so we may suppose that $P \subset K' \cap D$ and that $P \cap J = 0$. Assembling the relations now established, and applying 2.18, we have $\gamma^*(Q \cap G) + \gamma^*(Q \cap G') \leqq$
$\gamma(H) + \gamma^*(Q \cap G') < \gamma(J) + \gamma^*(Q \cap G') + \alpha \leqq \gamma(J) + \gamma(P) + \alpha =$
$\gamma(J \cup P) + \alpha \leqq \gamma(D) + \alpha < \gamma^*(Q) + 2\alpha$. Since $2\alpha$ can be made arbitrarily small, we have $\gamma^*(Q \cap G) + \gamma^*(Q \cap G') \leqq \gamma^*(Q)$, and the present theorem is established.

**2.23 Theorem.** The family $\mathcal{M}(\gamma^*)$ of $\gamma^*$-measurable sets is a ring of sets containing $\mathcal{R}(\mathcal{P})$, and $\gamma^*$ is a finitely additive measure on $\mathcal{M}(\gamma^*)$.

To establish this assertion, one may refer to the proof given in [16], pp. 44—45, (4.1) and (4.4). Everything proved there except for countable additivity can be proved using only 2.20.

In a forthcoming treatment of Lebesgue measure [15], M. Riesz has employed a useful criterion for measurability, which we use in the two following theorems.

**2.24 Theorem.** A set $A \varepsilon \mathcal{H}(\mathcal{P})$ of finite $\gamma^*$-measure is $\gamma^*$-measurable if and only if for every $\varepsilon > 0$, there exists a set $G \varepsilon \mathcal{P}$ such that $G \supset A$ and $\gamma^*(G \cap A') < \varepsilon$.

The necessity of this condition is obvious. To prove its sufficiency, let $\varepsilon$ be an arbitrary positive number and let $A$ and $G$ be as stated in the theorem. Let $P$ be any set in $\mathcal{H}(\mathcal{P})$. Since $G$ is measurable, we have $\gamma^*(P) = \gamma^*(G \cap P) + \gamma^*(G' \cap P)$. Comparing $\gamma^*(A \cap P)$ and $\gamma^*(G \cap P)$, we find $\gamma^*(G \cap P) = \gamma^*((A \cap P) \cup (G \cap A' \cap P)) \leqq \gamma^*(A \cap P) + \gamma^*(G \cap A' \cap P) \leqq \gamma^*(A \cap P) + \gamma^*(G \cap A') < \gamma^*(A \cap P) + \varepsilon$. In a slightly different form, this is

2.24.1 $\gamma^*(A \cap P) \leqq \gamma^*(G \cap P) < \gamma^*(A \cap P) + \varepsilon.$

In like manner, we show

2.24.2 $\gamma^*(A' \cap P) - \varepsilon < \gamma^*(G' \cap P) \leqq \gamma^*(A' \cap P).$

Adding 2.24.1 and 2.24.2, we find

$$|\gamma^*(A' \cap P) + \gamma^*(A \cap P) - \gamma^*(P)| < \varepsilon,$$

which shows that $A$ is $\gamma^*$-measurable.

From 2.24, we derive a useful fact.

**2.25 Theorem.** Let $A$ be a $\gamma^*$-measurable set of finite measure. Then for every $\varepsilon > 0$, there exist functions $g$ and $h \varepsilon \mathfrak{F}$ such that $0 \leqq g \leqq h$, $B = E[x; f(x) = 0 \text{ and } h(x) > 0] \subset A$, and $\gamma^*(B' \cap A) < \varepsilon$.

Let $H \varepsilon \mathcal{P}$ be a set such that $H \supset A$ and $\gamma(H) < +\infty$. The set $A' \cap H$ is measurable and has finite measure $\gamma(H) - \gamma^*(A)$. By 2.24, there exists a set $G \varepsilon \mathcal{P}$ such that $G \supset A' \cap H$ and $\gamma^*(G \cap (A' \cap H)') < \varepsilon$. We may suppose $G \subset H$, of course. We thus have $\gamma^*(G \cap A) = \gamma^*(G \cap (A' \cap H)') < \varepsilon$, and the set $B = G' \cap H$ clearly satisfies the requirements of the theorem.

We now discuss the relation between the original functional $I$ and the integral which can be formed with the measure $\gamma^*$. We recall that a non-negative function $\varphi$ on $X$ is $\gamma^*$-measurable if $E[x; \varphi(x) > \alpha]$ is $\gamma^*$-measurable for all $\alpha > 0$; and that an arbitrary real function $\varphi$ on $X$ is $\gamma^*$-measurable if $\max(\varphi, 0)$ and $-\min(\varphi, 0)$ are $\gamma^*$-measurable. Under this definition, it is plain that all functions in $\mathfrak{F}$ are $\gamma^*$-measurable. Of the possible definitions of integral, we select the following.

**2.26 Definition.** Let $\varphi(x)$ be any non-negative, bounded, $\gamma^*$-measurable function on $X$. By the integral $\int_X \varphi(x)\, d\gamma^*(x)$, we shall mean the expression

2.26.1 $\lim_{\|\Delta\| \to 0} \Sigma_{i=1}^{n} \alpha_{i-1} \gamma^*(E[x; \alpha_{i-1} < \varphi(x) \leqq \alpha_i]).$

Here $\Delta = \{\inf \varphi(x) = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n = \sup \varphi(x)\}$, and $\|\Delta\| = \max_{1 \leqq i \leqq n} (\alpha_i - \alpha_{i-1})$. If $\varphi(x)$ is non-negative and unbounded, then we define $\int_X \varphi(x)\, d\gamma^*(x)$ as $\lim_{t \to \infty} \int_X \min(\varphi(x), t)\, d\gamma^*(x)$. If $\varphi(x)$ has variable sign, then we define $\int_X \varphi(x)\, d\gamma^*(x)$ as $\int_X \max(\varphi(x), 0)\, d\gamma^*(x) - \int_X -\min(\varphi(x), 0)\, d\gamma^*(x)$, provided that at least one of the latter integrals is non-infinite.

It is a routine matter to verify that the integral of 2.26 is a positive linear functional on the class of absolutely integrable functions. (Finite additivity of the measure $\gamma^*$ suffices to carry through all of the arguments needed here.) It is of course impossible to prove Lebesgue's theorem on term-by-term integration, since the measure $\gamma^*$ may very well fail to be countably additive. We present examples *infra* for which this theorem fails (4.5). In the very general situation considered here, it may occcur that $I(f)$ and $\int_X f(x)\,d\gamma^*(x)$ do not coincide for all functions in $\mathfrak{F}$. There are, however, always some functions in $\mathfrak{F}$ for which this coincidence does obtain.

**2.27 Theorem.** Let $f$ be a bounded function in $\mathfrak{F}$ for which $\gamma(E[x; f(x) \neq 0] < +\infty$. Then $\int_X f(x)\,d\gamma^*(x) = I(f)$.

It is obviously sufficient to consider the case $f \geqq 0$. Let $\varepsilon$ be any positive number, and let $a = \inf f$, $b = \sup f$. Let $a = \alpha_0 < \alpha_1 < \ldots < \alpha_n = b$ be a sequence of real numbers such that $\max\limits_{i=1,\ldots,n} (\alpha_i - \alpha_{i-1}) < \varepsilon$. Let $G_i = E[x; f(x) > \alpha_i]$ $(i = 0, \ldots, n-1)$. Let the function $h_i$ be defined as

$$(\alpha_i - \alpha_{i-1})^{-1}\,\{\max[\min(f, \alpha_i), \alpha_{i-1}] - \alpha_{i-1}\}, \text{ for } i = 1, 2, \ldots, n.$$

It will be noted that $h_i \varepsilon \mathfrak{F}$ and that $0 \leqq h_i \leqq \chi_{G_{i-1}}$. We also have

$$f = \Sigma_{i=1}^n (\alpha_i - \alpha_{i-1})\,h_i + \alpha_0.$$

Suppose first that $\alpha_0 = 0$. In this case, we may write

$$\begin{gathered} I(f) = I(\Sigma_{i=1}^n (\alpha_i - \alpha_{i-1})\,h_i) = \Sigma_{i=1}^n (\alpha_i - \alpha_{i-1})\,I(h_i) \leqq \Sigma_{i=1}^n (\alpha_i - \alpha_{i-1})\,\gamma(G_{i-1}) = \\ \Sigma_{i=1}^n \alpha_i[\gamma(G_{i-1}) - \gamma(G_i)] + \alpha_n\gamma(G_{n-1}) = \Sigma_{i=1}^{n-1} \alpha_i\gamma^*(G_{i-1} \cap G_i') + \alpha_{n-1}\gamma(G_{n-1}) + \\ (\alpha_n - \alpha_{n-1})\,\gamma(G_{n-1}) < \Sigma_{i=1}^n \alpha_{i-1}\gamma^*(E[x; \alpha_{i-1} < f(x) \leqq \alpha_i]) + \\ \varepsilon\gamma(G_0) \leqq \int_X f(x)\,d\gamma^*(x) + \varepsilon\gamma(G_0). \end{gathered}$$

Since $\gamma(G_0) < +\infty$, this implies that $I(f) \leqq \int_X f(x)\,d\gamma^*(x)$. The same argument can be used with a non-positive function with sup $= 0$, so that $I(-f) \leqq \int_X -f(x)\,d\gamma^*(x)$. Hence $I(f) = \int_X f(x)\,d\gamma^*(x)$.

The case $a = \inf f(x) > 0$ requires separate consideration. Here we have $a^{-1}\min(f, a) = 1$ in the class $\mathfrak{F}$, so that $I(1)$ exists. If $I(1) = 0$, then $I$ is 0 for all bounded functions in $\mathfrak{F}$, and $\gamma^* \equiv 0$. The equality $I(f) = \int_X f(x)\,d\gamma^*(x)$ is trivially verified here. Thus we may suppose that $I(1) > 0$. Writing $f$ as $\alpha_0 + \Sigma_{i=1}^n (\alpha_i - \alpha_{i-1})\,h_i$, we may now carry through the above computations with only minor changes; we omit the details. The present theorem is therefore established.

**2.28 Theorem.** For every non-negative bounded function $f \varepsilon \mathfrak{F}$, the inequality $\int_X f(x)\,d\gamma^*(x) \leqq I(f)$ obtains.

Unless $\inf f = 0$, there is nothing to prove. For positive real numbers $\alpha < \sup f$, let $K_\alpha = E[x; f(x) \geqq \alpha]$; and let

2.28.1 $$g_\alpha = \min\left\{f, \frac{2\sup f}{\alpha}\left[\max\left(\min(f, \alpha), \frac{\alpha}{2}\right) - \frac{\alpha}{2}\right]\right\}.$$

Since $g_a = 0$ on $K'_{\frac{a}{2}}$, and since $\gamma^*(K_{\frac{a}{2}})$ is finite (2.11), 2.27 shows that

$$I(g_a) = \int_X g_a(x)\,d\gamma^*(x).$$

We also have $f = g_a$ on $K_a$ and $0 \leqq g_a \leqq f$ everywhere. Thus the inequalities $0 \leqq f\chi_{K_a} \leqq g_a \leqq f$ obtain, and hence also $\int_X f(x)\,\chi_{K_a}(x)\,d\gamma^*(x) \leqq \int_X g_a(x)\,d\gamma^*(x) = I(g_a) \leqq I(f)$.

From 2.26, it is clear that $\lim_{a\to 0}\int_X f(x)\,\chi_{K_a}(x)\,d\gamma^*(x) = \int_X f(x)\,d\gamma^*(x)$, and this proves the present theorem.

**2.29 Remark.** The inequality 2.28 cannot be replaced by an equality, as the following example shows. Let $\mathfrak{H}$ consist of all continuous real functions $f$ on $[0, +\infty)$ such that $f = O(x^{-1})$ $(x \to +\infty)$. Let $p(f) = \overline{\lim_{x\to\infty}}\, xf(x)$. It is clear that $p(f+g) \leqq p(f) + p(g)$ and that $p(\alpha f) = \alpha p(f)$ for $\alpha \geqq 0$. Therefore, as in 2.8, we infer that there exists a linear functional $W$ on $\mathfrak{H}$ such that $-p(-f) = \underline{\lim_{x\to\infty}}\, xf(x) \leqq W(f) \leqq p(f)$. For this functional, the corresponding measure $\omega^*$ has the property that all open sets are measurable, that $\omega^* = 0$ for all bounded sets, and that $\omega((a, +\infty)) = +\infty$ for all $a > 0$. For the function $(1+x)^{-1}$, for example, we have $\int_X (1+x)^{-1}\,d\omega^*(x) = 0$ and $W((1+x)^{-1}) = 1$.

**2.30 Remark.** For unbounded $f$ in $\mathfrak{F}$, $I(f)$ may fail to be equal to $\int_X f(x)\,d\gamma^*(x)$, even though $X$ has total measure finite. For $f \geqq 0$, it is easy to see that $I(f) \geqq \int_X f(x)\,d\gamma^*(x)$, but as in 2.29, one can produce an example where equality fails. Consider the space $\mathfrak{G}$ and the linear functional $Q$ of 2.8. Let $Q_1(g) = Q(f) + f(0)$. Then $Q_1$ satisfies 2.7 and also $Q_1(1) = 1$; a measure $\sigma_1^*$ can be defined by 2.9 for $Q_1$ so that $\sigma_1^*$ satisfies 2.20. It is easy to see that all subsets of $R$ are $\sigma_1^*$-measurable and that $\sigma_1^*(A) = \chi_A(0)$. The function $f(x) = x$ has the two properties that $Q_1(x) = 1$ and that

$$\int_R \min(x, t)\,d\sigma_1^*(x) = 0 \text{ for all } t \geqq 0. \text{ Therefore } \int_R x\,d\sigma_1^*(x) = 0,$$

and we see that $Q_1$ fails to be representable by the standard integral. Indeed, if we consider $Q$ alone, we find that there is no finitely additive measure $\varrho$, $0 \leqq \varrho \leqq +\infty$, for which all functions in $\mathfrak{G}$ are measurable and for which $Q(g) = \int_R g(x)\,d\varrho(x)$ for all $g \in \mathfrak{G}$. The reason for this, of course, is that if $0 < \varrho(A) \leqq +\infty$ for some subset $A$ of $R$, then $\int_R 1\,d\varrho(x) \geqq \varrho(A)$; and $Q(1) = 0$.

**2.31 Remark.** A countably additive measure $\bar{\gamma}$ could be obtained from our $\gamma$ of 2.9 by defining $\bar{\gamma}(Q)$ for every $Q \in \mathcal{H}(\mathcal{S}(\mathcal{P}))$ as $\inf \Sigma_{n=1}^{\infty} \gamma(G_n)$, taken over all $\{G_n\}_{n=1}^{\infty}$ such that $Q \subset \bigcup_{n=1}^{\infty} G_n$. In this way, we make our integration theory simpler, but quite useless for present purposes. As one can easily see from example 4.5, $\bar{\gamma}$ may be $\equiv 0$ for non-zero $I$.

**2.32 Remark.** Examples 2.29 and 2.30 show the limitations of our present methods of obtaining integral representations. However, if we try to find integral representations of any kind at all for functionals on function spaces satisfying 2.7 and 2.1, so that all functions in $\mathfrak{F}$ are measurable, we find upon a little reflection that $\gamma$ is our only choice for sets in $\mathcal{P}$ and that $\gamma^*$ is the largest possible outer measure on the ring $\mathcal{H}(\mathcal{P})$. Hence the results of the present are the best possible: unless we radically alter the notion of integral to be employed.

## § 3. Various special conditions

Other writers on the topic of integral representation have ordinarily added certain conditions to 2.7.1 and 2.7.2 which ensure that the measure $\gamma^*$ defined by 2.9 shall be countably additive on the family of $\gamma^*$-measurable sets. We first present a condition (3.2), equivalent to countable additivity, which appears to be simpler than many of those found elsewhere and which is in consonance with the point of view adopted here. This condition, it will be noted, is exactly the condition $(L)$ of DANIELL ([3], p. 280), and is of course well known. Throughout the present §, we make the assumption

3.1 $$\gamma(E[f(x)>0]) < +\infty \text{ for all } f \varepsilon \mathfrak{F}.$$

We now list two conditions which may be imposed on $I$.

3.2 Let $\{f_n\}_{n=1}^{\infty}$ be a decreasing sequence of bounded functions in $\mathfrak{F}$ with pointwise limit 0. Then $\lim\limits_{n\to\infty} I(f_n) = 0$.

3.3 Let $\{f_n\}_{n=1}^{\infty}$ be an arbitrary decreasing sequence of functions in $\mathfrak{F}$ with limit 0. Then $\lim\limits_{n\to\infty} I(f_n) = 0$.

**3.4 Theorem.** If condition 3.1 holds, then 3.2 is both necessary and sufficient for $\gamma^*$ to be countably additive on the ring $\mathcal{M}(\gamma^*)$.

Suppose that $\gamma^*$ is countably additive. Then $\gamma^*$ admits a unique countably additive extension $\gamma^{**}$ over the $\sigma$-ring $\mathcal{S}(\mathcal{M}(\gamma^*))$ ([5], p. 54). For functions $\varphi$ measurable $\gamma^*$, the integrals of $\varphi$ with respect to $\gamma^*$ and $\gamma^{**}$ are equal. Under assumption 3.1, we have by 2.27 $I(f_n) = \int\limits_X f_n(x)\,d\gamma^*(x)$ for all $f_n$ mentioned in 3.2. Since $f_1 \varepsilon L_1(\gamma^{**})$, we apply a classical theorem ([16] p. 28) to infer that

$$0 = \lim_{n\to\infty} \int\limits_X f_n(x)\,d\gamma^{**}(x) = \lim_{n\to\infty} \int\limits_X f_n(x)\,d\gamma^*(x) = \lim_{n\to\infty} I(f_n).$$

To prove the sufficiency of 3.2, we note that 3.1 implies, in view of the very definition of $\gamma^*$, that $\gamma^*$ is a finite-valued, although possibly unbounded outer measure on $\mathcal{H}(\mathcal{P})$. This fact enables us to replace the condition of countable additivity by

3.4.1 $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}(\gamma^*)$, $A_1 \supset A_2 \supset \ldots$, and $\bigcap_{n=1}^{\infty} A_n = 0$ imply $\lim\limits_{n\to\infty} \gamma^*(A_n) = 0$.

Verification of 3.4.1 requires several steps. First, suppose that $\{G_n\}_{n=1}^{\infty} \subset \mathcal{P}$ is a decreasing sequence of sets with intersection 0 such that $G_n \wedge G'_{n-1}$ $(n = 2, 3, \ldots)$. Let $h_n \varepsilon \mathfrak{F}$ have the properties that $0 \leqq h_n \leqq 1$, $h_n = 0$ on $G'_{n-1}$ and $h_n = 1$ on $G_n$; let $p_n \varepsilon \mathfrak{F}$ have the properties that $0 \leqq p_n \leqq \chi_{G_n}$ and $I(p_n) + n^{-1} > \gamma(G_n)$; let $g_n = \max(p_n, h_n)$ $(n = 1, 2, 3, \ldots)$. Then $I(g_n) + n^{-1} > \gamma(G_n)$ and the $g_n$ are a decreasing, bounded sequence with limit 0. Thus, by 3.2, $\lim\limits_{n\to\infty} I(g_n) = 0$. It follows that $\lim\limits_{n\to\infty} \gamma(G_n) = 0$.

Next, let $\{G_n\}_{n=0}^{\infty} \subset \mathscr{P}$ be any decreasing sequence of sets with intersection 0. Let $\varepsilon$ be an arbitrary positive number, and let $\{\alpha_n\}_{n=1}^{\infty}$ be any sequence of positive numbers such that $\Sigma_{n=1}^{\infty} \alpha_n < \varepsilon$. Let $H_1 \varepsilon \mathscr{P}$ have the properties that $H_1 \subset G_0$, $H_1 \wedge G_0'$, and $\gamma(H_1) + \alpha_1 > \gamma(G_0)$ (as shown possible by 2.15). Then clearly $\gamma^*(H_1' \cap G_0) < \alpha_1$. Considering the set $G_1$, we have

$$\gamma(G_1) = \gamma(H_1 \cap G_1) + \gamma^*(H_1' \cap G_1) \leqq \gamma(H_1 \cap G_1) + \gamma^*(H_1' \cap G_0) <$$

$\gamma(H_1 \cap G_1) + \alpha_1$. Also, it is clear that $H_1 \cap G_1 \wedge G_0'$. Let $H_2 \varepsilon \mathscr{P}$ have the properties that $H_2 \subset H_1 \cap G_1$, $H_2 \wedge (H_1 \cap G_1)'$, and $\gamma^*(H_2' \cap H_1 \cap G_1) < \alpha_2$.
Then we have $\gamma(G_2) =$

$$\gamma(H_2 \cap G_2) + \gamma^*(H_2' \cap H_1 \cap G_2) + \gamma^*(H_1' \cap G_2) \leqq$$

$$\gamma(H_2 \cap G_2) + \gamma^*(H_2' \cap H_1 \cap G_1) + \gamma^*(H_1' \cap G_0) <$$

$$\gamma(H_2 \cap G_2) + \alpha_2 + \alpha_1. \text{ Also, } H_2 \cap G_2 \wedge (H_1 \cap G_1)'.$$

Carrying out an obvious finite induction, we find that there exists a sequence $H_1 \supset H_2 \supset \ldots \supset H_n \supset \ldots$ of sets in $\mathscr{P}$ such that

$$H_n \cap G_n \wedge (H_{n-1} \cap G_{n-1})' \text{ and } \gamma(G_n) < \gamma(H_n \cap G_n) + \alpha_n + \ldots + \alpha_1.$$

It follows at once that $\lim\limits_{n\to\infty} \gamma(G_n) \leqq \lim\limits_{n\to\infty} \gamma(H_n \cap G_n) + \Sigma_{n=1}^{\infty} \alpha_n = 0 + \Sigma_{n=1}^{\infty} \alpha_n < \varepsilon$, and that $\lim\limits_{n\to\infty} \gamma(G_n) = 0$.

Next, let $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence of sets such that $\cap_{n=1}^{\infty} K_n = 0$, where each $K_n$ has the form $E[f_n > 0] \cap E[g_n = 0]$ for some $f_n, g_n \varepsilon \mathfrak{F}$, and $0 \leqq g_n \leqq f_n$. It is easy to see that $K_n = \cap_{m=1}^{\infty} E[x; \min(m^{-1}, f_n + g_n) - g_n > 0]$, i.e., that $K_n$ is the intersection of a countable decreasing sequence of sets in $\mathscr{P}$. Thus we have $K_n = \cap_{m=1}^{\infty} G_{n,m}$, with $G_{n,m} \varepsilon \mathscr{P}$. By renumbering, we can write $\{G_{n,m}\}_{n,m=1}^{\infty}$ as $\{H_k\}_{k=1}^{\infty}$. Let $J_p = H_1 \cap \ldots \cap H_p$ $(p = 1, 2, 3, \ldots)$. Then clearly $J_1 \supset J_2 \supset \ldots$ and $\cap_{p=1}^{\infty} J_p = \cap_{n=1}^{\infty} K_n = 0$. By the above, we have $\lim\limits_{p\to\infty} \gamma(J_p) = 0$. However, every $J_p$ contains some $K_n$, say $K_{n(p)}$, and it follows that $\lim\limits_{p\to\infty} \gamma^*(K_{n(p)}) = 0$. If $\overline{\lim\limits_{p\to\infty}}\, n(p) = \infty$, then we have $\lim\limits_{n\to\infty} \gamma^*(K_n) = 0$. If $\overline{\lim\limits_{p\to\infty}}\, n(p) = n_0 < \infty$, then $\gamma^*(K_{n_0}) = 0$.

Finally, suppose that $\{A_n\}_{n=0}^{\infty}$ is any decreasing sequence of $\gamma^*$-measurable sets with intersection 0. By 2.25, there exists for every $C \varepsilon \mathscr{M}(\gamma^*)$ a set $K$ of the kind described in the last paragraph such that $K \subset C$ and $\gamma^*(C \cap K')$ is arbitrarily small. This fact enables us to apply the argument of the last paragraph but one, with only notational changes, and we conclude that $\lim\limits_{n\to\infty} \gamma^*(A_n) = 0$. Thus $\gamma^*$ is countably additive on $\mathscr{M}(\gamma^*)$.

**3.5 Note.** As we shall see below (4.4), the family $\mathscr{M}(\gamma^*)$ need not be a $\sigma$-ring even when $\gamma^*$ is countably additive. However, if the bounded part of $\mathfrak{F}$ is closed under the formation of uniform limits, we have the equality $\cup_{n=1}^{\infty} \cdot E[x; f_n(x) > 0] = E[x; \Sigma_{n=1}^{\infty} 2^{-n} \min(f_n(x), 1) > 0]$, for non-negative $f_n$; thus $\mathscr{P}$ is closed under the formation of countable unions. In this case, if $\gamma^*$ is countably additive, we find that $\gamma(\cup_{n=1}^{\infty} G_n) \leqq \Sigma_{n=1}^{\infty} \gamma(G_n)$ for all $\{G_n\}_{n=1}^{\infty} \subset \mathscr{P}$; and routine calculations show that $\mathscr{M}(\gamma^*)$ is a $\sigma$-ring.

There remain two possible properties of $I$ and $\gamma^*$ that deserve attention:

3.6 For all bounded $f \varepsilon \mathfrak{F}$, $I(f) = \int\limits_X f(x)\,d\gamma^*(x)$.

3.7 For all $f \varepsilon \mathfrak{F}$, $I(f) = \int\limits_X f(x)\,d\gamma^*(x)$.

3.8 We have not succeeded in finding reasonable necessary and sufficient conditions, expressed in terms of $I$ alone, for the validity of 3.6 and 3.7. Condition 3.1 is sufficient for 3.6, but not necessary, as the example $L_1(-\infty, +\infty)$ with Lebesgue integration shows. Note, however, that conditions 3.3 and 3.6 together are necessary and sufficient for 3.7, as one can easily prove.

## § 4. Examples

The theory expounded in §§ 2—3 finds, as is natural, applications to many of the standard examples of linear functionals. In addition to these, we obtain integral representations for functionals which apparently have not been discussed in print heretofore.

4.1 Perhaps the best-known example is that afforded by a locally compact Hausdorff space $T$ taken as $X$ and the class $\mathfrak{C}_{\infty\infty}(T, R)$ taken as $\mathfrak{F}$. ($\mathfrak{C}_{\infty\infty}(T, R)$) denotes the set of all continuous real functions on $T$ such that $(E[x; |f(x)| > 0])^-$ is compact; "$^-$" denotes the closure operator in $T$.) It is an elementary exercise to show that for every set $G = E[x; |f(x)| > 0]$ $(f \varepsilon \mathfrak{C}_{\infty\infty}(T, R))$ there exists a function $c \varepsilon \mathfrak{C}_{\infty\infty}(T, R)$ such that $c = 1$ on $G$; hence $\gamma(G) \leqq I(|c|)$, and condition 3.1 obtains. Next, it is clear that all functions in $\mathfrak{C}_{\infty\infty}(T, R)$ are bounded. It is also elementary to show that a sequence $\{f_n\}_{n=1}^{\infty}$ as in 3.2 converges uniformly, and hence that 3.2 is satisfied. Hence $\gamma^*$ is countably additive; and (as we show in 4.3) the family $\mathcal{M}(\gamma^*)$ is a $\sigma$-ring. Hence the standard representation theorem follows.

4.2 Let $X$ be the space $T$ of 4.1 and let $\mathfrak{F}$ be the space of function $\mathfrak{C}_{\infty}(T, R)$ obtained by completing $\mathfrak{C}_{\infty\infty}(T, R)$ in the uniform metric. Functions in $\mathfrak{C}_{\infty}(T, R)$ are just those continuous real $f$ such that $E[x; |f(x)| \geqq \alpha]$ is compact for every $\alpha > 0$. A positive linear functional on $\mathfrak{C}_{\infty}(T, R)$ is easily identified with a functional $I$ on $\mathfrak{C}_{\infty\infty}(T, R)$ such that $I(f)$ is bounded for all $f$ such that $0 \leqq f \leqq 1$. Hence 4.1 applies, and one obtains an integral representation for $I$ in terms of a measure $\gamma^*$ which is both countably additive and bounded on the domain $\mathcal{M}(\gamma^*)$. As the writer has pointed out in another communication [7], one can define a measure, call it $\gamma^\dagger$, which coincides with $\gamma^*$ on $\mathcal{M}(\gamma^*)$ and for which $\mathcal{M}(\gamma^\dagger)$ is a $\sigma$-algebra of sets.

4.3 One case of some interest is that in which $X$ is a topological space, say $T$; $\mathfrak{F}$ is any linear space $\mathfrak{S}$ of continuous functions satisfying 2.1 such that $E[x; |f(x)| \geqq \alpha]$ is *countably* compact for all $f \varepsilon \mathfrak{S}$ and $\alpha > 0$; and $\gamma(H)$ is finite for all $H \varepsilon \mathcal{P}$. In this case, we infer quickly that $\gamma^*$ is countably additive. For, consider any set $H \varepsilon \mathcal{P}$ and any family $\{G_n\}_{n=1}^{\infty} \subset \mathcal{P}$ such that $H \subset \bigcup_{n=1}^{\infty} G_n$. By 2.14, there is a set $K = E[x; f(x) \geqq \alpha > 0]$ such that $K \subset H$ and $\gamma^*(K) + \varepsilon > \gamma(H)$. We have $K \subset \bigcup_{n=1}^{\infty} G_n$, and hence $K \subset \bigcup_{n=1}^{N} G_n$ for some positive integer $N$. It follows from 2.17 that

$$\gamma(H) - \varepsilon < \gamma^*(K) \leqq \gamma\left(\bigcup_{n=1}^{N} G_n\right) \leqq \Sigma_{n=1}^{N} \gamma(G_n) \leqq \Sigma_{n=1}^{\infty} \gamma(G_n).$$

Example 2.29 shows that this may fail if $\gamma(H)$ is allowed to be infinite.

4.4 Another example which is quite elementary but not devoid of interest is that provided by $X = [0,1)$, $\mathfrak{F}$ = all finite linear combinations of functions

$\chi_{[\alpha,\beta)}$ (we denote this function space by $\mathfrak{C}(0,1)$), and $I$ any positive linear functional. Here our definition of $\gamma^*$ may not produce a $\sigma$-ring $\mathcal{M}(\gamma^*)$ even if $\gamma^*$ is countably additive. For $I(\chi_{[\alpha,\beta)}) = \beta - \alpha$, the outer measure $\gamma^*$ is exactly Jordan content, and for this set-function, any countable dense subset of $[0,1)$ is manifestly non-measurable. Note that $\mathcal{M}(\gamma^*)$ is, however, an algebra. By taking $X = R$ and $\mathfrak{F}$ = all finite linear combinations of characteristic functions $\chi_{[\alpha,\beta)}$, $-\infty < \alpha < \beta < +\infty$ (which function-space we denote by $\mathfrak{C}(-\infty, +\infty)$), we obtain an example where $\mathcal{M}(\gamma^*)$ is not an algebra at all.

In both of the examples $\mathfrak{C}(0,1)$ and $\mathfrak{C}(-\infty, +\infty)$, the functionals $L_t(f) = f(t-0)$ are particularly interesting. (The corresponding measures $\lambda_t$ have been discussed elsewhere by the writer [9].) We note here that the family of sets $\mathcal{M}(\lambda^*{}_t)$ is characterized as follows. A bounded set $A$ is $\lambda_t^*$-measurable if and only if there is an $\varepsilon > 0$ such that $(t-\varepsilon, t) \subset A$ or there is an $\varepsilon > 0$ such that $(t-\varepsilon, t) \cap A = 0$. Hence $\mathcal{M}(\lambda_t^*)$ is never closed under the formation of countable unions.

4.5 Let $X$ be any uniform space, say $W$, and let $\mathfrak{F}$ be the set $\mathfrak{U}_B$ of all bounded uniformly continuous real functions on $W$. Theorem 2.27 gives us an integral representation valid for all functions in $\mathfrak{U}_B$, since $1 \varepsilon \mathfrak{U}_B$, and all sets have outer measure not exceeding $I(1)$. Since $\mathfrak{U}_B$ is closed under the formation of sums, products, and uniform limits, is provided with a norm $\|f\| = \sup_{w \varepsilon W} |f(w)|$ such that $\|f^2\| = \|f\|^2$, and has the property that $(1+f^2)^{-1}$ exists for all $f \varepsilon \mathfrak{U}_B$, one may apply a theorem of GEL'FAND [4] to show that $\mathfrak{U}_B$ is realizable as the algebra of all continuous real functions on its space of maximal ideals, which we denote as $\Upsilon W$. This space contains $W$ as an open dense subspace; and permits us to study $I$ from a different point of view. As a functional on the space $\mathfrak{C}(\Upsilon W, R)$, $I$ has an integral representation with respect to a countably additive measure, say $\tilde{\gamma}$, on the Borel sets of $\Upsilon W$. As an open subset of $\Upsilon W$, $W$ is always $\tilde{\gamma}$-measurable. One may then ask, what is the relation between $\gamma^*$ and $\tilde{\gamma}$? To exclude pathological cases, let us now suppose that $W$ is the union of an increasing sequence of compact supspaces. Then we find: $\gamma^*$ is countably additive if and only if $\tilde{\gamma}(\Upsilon W \cap W') = 0$; and in this case, $\gamma^*$ and $\tilde{\gamma}$ can be identified with each other in an obvious way. On the other hand, $\gamma^*$ may be purely finitely additive [18], and this occurs if and only if $\tilde{\gamma}(W) = 0$.

4.6 If $X$ is any topological space, say $T$, and $\mathfrak{F} = \mathfrak{C}_B(T, R)$ = all bounded continuous real functions on $T$, and $I$ is any functional on $\mathfrak{C}_B(T, R)$ satisfying 2.7, then we can apply 2.27 and the fact that $1 \varepsilon \mathfrak{C}_B(T, R)$ to obtain an integral representation valid for all $f \varepsilon \mathfrak{C}_B(T, R)$.

4.7 Let $T$ be as in 4.6, and let $\mathfrak{F}$ be the space $\mathfrak{C}(T, R)$ consisting of *all* continuous real functions on $T$; let $I$ be any functional as in 2.7. The writer has shown elsewhere that $I$ has a representation, valid for all $f \varepsilon \mathfrak{C}(T, R)$, as an integral with respect to a finite-valued, countably additive measure $\gamma^*$ [6]. This fact, which is not obvious from the definition of $\gamma^*$, may be combined with 3.4 and 3.8 to assert that for any decreasing sequence $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{C}(T, R)$ with limit 0, $\lim_{n\to\infty} I(f_n) = 0$.

4.8 Hypothesis 2.7.2, which makes the arguments in § 2 much simpler than they would otherwise be, can be weakened for an arbitrary $X$ and $\mathfrak{F}$ as in 2.1, by admitting also those functionals $I$ which can be written as the difference of two functionals satisfying 2.7. If $I(f)$ is bounded for all $f$ in any bounded set of functions in $\mathfrak{F}$, then $I$ can be so written: $I = I^+ - I^-$, where $I^+(f) =$

$\sup_{0 \leq g \leq f} I(g)$ (for $f \geq 0$, and $I^+(f) = I^+(\max(f, 0)) - I^+(-\min(f, 0))$ for arbitrary $f$) and $I^- = I^+ - I$. The measure $\gamma$ on $\mathcal{P}$ corresponding to $I$ has the form $\gamma^+ - \gamma^-$, where $\gamma^+(G)$ is defined from $I^+$ and $\gamma^-(G)$ is defined from $I^-$. It is plain that $\gamma$ is well-defined, in that $\gamma^+(G) = +\infty$, $\gamma^-(G) = -\infty$, can never occur for $G \varepsilon \mathcal{P}$. Defining $\gamma^*$ as $(\gamma^+)^* - (\gamma^-)^*$, we obtain a set-function on $\mathcal{H}(\mathcal{P})$; and integral representation goes through as before.

4.9 Let $X$ be any set, and let $\mathfrak{F}$ be any *complex* linear space of complex functions defined on $X$ such that for $f \varepsilon \mathfrak{F}$, the complex conjugate $\bar{f}$ is also in $\mathfrak{F}$ and such that the set of real functions in $\mathfrak{F}$ satisfies 2.1. Let $I$ be any complex-valued complex-linear functional defined on $\mathfrak{F}$. Then any possible representation of $I$ as an integral with respect to a complex-valued finitely additive measure can be obtained by the following well-known reduction. Let $f = \varphi + i\psi$ be an element of $\mathfrak{F}$, where $\varphi$ and $\psi$ are real. Then $I(\varphi) = I_1(\varphi) + iI_2(\varphi)$, where $I_1$ and $I_2$ are, as one sees immediately, real functionals satisfying 2.7.1. For a pure imaginary $i\psi$, we have $I(i\psi) = iI(\psi) = iI_1(\psi) - I_2(\psi)$. Thus $I$ is completely determined by its real and imaginary parts applied to real functions in $\mathfrak{F}$, and we have the general formula $I(\varphi + i\psi) = I_1(\varphi) - I_2(\psi) + i(I_1(\psi) + I_2(\varphi))$. $I_1$ and $I_2$ may be quite arbitrary linear functionals on the real part of $\mathfrak{F}$. If $I_1$ and $I_2$ are as in 4.8, they may be written as differences $I_1^+ - I_1^-$ and $I_2^+ - I_2^-$, so that $I = (I_1^+ - I_1^-) + i(I_2^+ - I_2^-)$, and any integral representation possible for $I$ can be built up from integral representations for the four non-negative functionals just listed.

## § 5. Relations with other results

Beyond the early work of F. Riesz and J. Radon, which is of only historical interest in the present context, there are a number of results which bear a close connection to those set forth here and which deserve mention.

5.1 The fundamental paper of Daniell [4] can be connected with the present treatment only by showing that summability in his sense is equivalent to integrability with respect to $\gamma^*$, in the sense that a function on $X$ is integrable with respect to $\gamma^*$ if and only if it is summable, and that the processes of integration and summation yield the same value whenever they are applicable. All this, of course, is under assumption 3.2, which forms an essential part of Daniell's hypotheses. (Actually, the proof of this equivalence has been carried out by H. S. Zuckerman and the writer, for the case in which $X$ is a locally compact Hausdorff space and $\mathfrak{F}$ is the set of all continuous real functions vanishing outside of compact sets [8].) So far as the writer knows, no one has studied this equivalence problem for the general case.

5.2 The four notes recently published by M. H. Stone [17] on abstract integration start from our hypotheses and an additional assumption (I (3)) which is clearly sufficient to establish countable additivity. The correlative note of McShane [12], which is, to be sure, subsumed under [17], IV, imposes yet stronger conditions to ensure countable additivity. Note also a far-reaching generalization recently announced by McShane [13], still with hypotheses to ensure countable additivity.

5.3 The encyclopedic paper of A. D. Alexandroff [1] stands in a very close relation to our results, as he assumes beyond our hypotheses 2.1 and 2.7 only that all functions to be considered are bounded and that $1 \varepsilon \mathfrak{F}$. However, his theorems are for the most part couched in terms of cumbersome quasi-topological spaces, and his main results can be readily inferred from ours. Nevertheless, he has anticipated BOURBAKI's integral ([17], IV) in his study of "real charges" and has proved a number of interesting theorems regarding weak convergence of finitely additive measures (Ch. IV and Ch. V).

5.4 One should also mention the paper of A. MARKOV [11], which deals with the case of the bounded continuous real functions on a topological space satisfying the axiom of normality but not necessarily any separation axiom. His results on integral representation are completely subsumed under our example 4.6.

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