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INVARIANT SUBSPACES OF COMPOSITION OPERATORS ON A HILBERT SPACE OF DIRICHLET SERIES

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ABSTRACT. In this paper, we study invariant subspaces of composition operators on the Hilbert space of Dirichlet series with square summable coefficients. The structure of invariant subspaces of a composition operator is characterized, and the strongly closed algebras generated by some composition operators with irrational symbols are shown to be reflexive. As an application, we provide a criterion for composition operators with certain symbols not to be algebraic.

1. INTRODUCTION

For H a space of analytic functions on a region $G \subseteq \mathbb{C}$ and φ an analytic map of G into itself, the composition operator C_φ is a linear operator defined by

$$C_\varphi f = f \circ \varphi, \ \ f \in H.$$

Broadly, one is interested in extracting properties of C_φ acting on H (boundedness, compactness, spectral properties, etc.) from function theoretic properties of φ and vice versa. We refer to monographs by Cowen-MacCluer [4], Shapiro [15] and Zhu [18] for general information of composition operators on the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Recently, an extensive study of composition operators was carried out on various spaces of Dirichlet series (see [1, 2, 3, 6, 12, 16]).

Let $\mathcal{H}$ be the Hilbert space of Dirichlet series with square summable coefficients:

$$\mathcal{H} = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} < \infty \right\}.$$

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By the Cauchy-Schwarz inequality, the functions in $\mathcal{H}$ are all analytic on the half-plane $\mathbb{C}_{1/2}$ (where, for $\theta \in \mathbb{R}$, $\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$ and $\mathbb{C}_+ = \mathbb{C}_0$). In the process, we also encounter the space $\mathcal{D}$ of functions f, which in some (possibly remote) half-plane admit representation by a convergent Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We also recall that a series in $\mathcal{D}$ actually converges absolutely in the half-plane one unit to the right of the half-plane of convergence.

The following comes from [6, 12], which characterizes the boundedness of composition operators on $\mathcal{H}$.

Theorem A. *An analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ determines a bounded composition operator $C_\Phi : \mathcal{H} \to \mathcal{H}$ if and only if $\Phi(s) = c_0 s + \varphi(s)$, where c_0 is a non-negative integer, and $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s} \in \mathcal{D}$ converges uniformly in $\mathbb{C}_\epsilon$ for every $\epsilon > 0$ with the following properties:*

(a) *If $c_0 = 0$, then $\varphi(\mathbb{C}_+) \subseteq \mathbb{C}_{1/2}$.*

(b) *If $c_0 \geq 1$, then either $\varphi \equiv 0$ or $\varphi(\mathbb{C}_+) \subseteq \mathbb{C}_+$.*

It is convenient to call Φ a c_0-symbol if $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is analytic and satisfies the above conditions such that C_Φ is bounded on $\mathcal{H}$.

We know one of the most famous unsolved problems in analysis is the invariant subspace problem: does every bounded linear operator on an infinite dimensional Hilbert space have a non-trivial invariant subspace? In recent decades, the invariant subspaces of composition operators acting on the Hardy-Hilbert space $H^2(\mathbb{D})$ have been extensively studied. Particularly, the authors [10] showed that every hyperbolic composition operator on $H^2(\mathbb{D})$ is cosubnormal and has universal translates. This allows us to formulate the general invariant subspace problem in terms of invariant subspaces of certain composition operators. Because of this close relationship, the invariant subspaces of invertible composition operators have been studied in [9], and Mahvidi studied the structure of invariant subspaces of a general composition operator in [8].

The purpose of this paper is to generalize these results mentioned above from the classical Hardy-Hilbert space $H^2(\mathbb{D})$ to the space $\mathcal{H}$ of Dirichlet series. Section 2 includes some background materials needed in the sequel and some basic facts on the dynamical system of c_0-symbols, which are very different from the classical case. In Section 3, we consider the structure of invariant subspaces of composition operators. The reflexivity of algebras generated by composition operators and the non-existence of non-trivial algebraic composition operators are discussed in Section 4.

2. Preliminaries

Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Recall that a closed subspace M of H is called an invariant subspace of $A \in B(H)$, if $Ax \in M$ for every $x \in M$ (i.e., $AM \subseteq M$). For $A \in B(H)$, we denote the lattice of all invariant subspaces of A by $\operatorname{Lat} A$. For $\mathcal{U} \subseteq B(H)$, we set

$$\operatorname{Lat} \mathcal{U} := \bigcap_{A \in \mathcal{U}} \operatorname{Lat} A.$$

An algebra $\mathcal{U} \subseteq B(H)$ is said to be reflexive, if for any $B \in B(H)$ the inclusion $\operatorname{Lat}\mathcal{U} \subseteq \operatorname{Lat} B$ implies $B \in \mathcal{U}$. Further, if $\mathcal{S}$ is any collection of subspaces of H, we define $\operatorname{Alg}\mathcal{S} := \{B \in B(H) : \mathcal{S} \subseteq \operatorname{Lat} B\}$. Then the algebra $\mathcal{U}$ is reflexive if and only if $\mathcal{U} = \operatorname{Alg}\operatorname{Lat}\mathcal{U}$. Reflexive algebras are associated extensively to the transitive and reductive algebra problems. For more details, see [13].

We know that $\mathcal{H}$ is a reproducing kernel Hilbert space and the reproducing kernel at w in $\mathbb{C}_{1/2}$ is given by $K_w(s) = \sum_{n=1}^{\infty} n^{-\overline{w}} n^{-s}$ for $\operatorname{Re} s > 1/2$.

If we denote by C_Φ^* the adjoint of C_Φ whenever it is bounded, then for $w \in \mathbb{C}_{1/2}$ we have

$$C_\Phi^*(K_w) = K_{\Phi(w)}.$$

Let Φ be an analytic self-map of $\mathbb{C}_{1/2}$. For all $n \in \mathbb{N}$ we denote the n-th iterate of Φ by Φ_n, that is $\Phi_n := \Phi \circ \cdots \circ \Phi$. For $n = 0$ we set $\Phi_0 := id$, the identity function of $\mathbb{C}_{1/2}$.

The followings are from [2] and [6] respectively.

Lemma 2.1. *Let m be a positive integer, and $f(s) = \sum_{n=m}^{\infty} a_n n^{-s} \in \mathcal{D}$. Then $m^s f(s) \to a_m$ uniformly as $\operatorname{Re} s \to +\infty$.*

Lemma 2.2. *Let $\Phi(s) = c_0 s + \varphi(s)$ be an analytic self-map of $\mathbb{C}_+$. If $\Phi(s) \neq s + i\tau$, $\tau \in \mathbb{R}$, then there exist $\eta > 0$ and $\epsilon > 0$ such that $\Phi(\mathbb{C}_{1/2-\epsilon}) \subseteq \mathbb{C}_{1/2+\eta}$.*

We next consider the iterates of c_0-symbols, which are very different from the classical case (see [4]). Therefore, we have to use essentially different methods to study invariant subspaces of composition operators acting on $\mathcal{H}$.

The following corollary shows that for any $\lambda \in \mathbb{C}_{1/2}$, the sequence $\{\Phi_n(\lambda)\}$ never satisfies the "Blaschke" condition.

Corollary 2.3. *Let $\Phi(s) = c_0 s + \varphi(s)$ be an analytic self-map of $\mathbb{C}_+$. Then there does not exist $\lambda \in \mathbb{C}_{1/2}$ such that $\operatorname{Re}\Phi_n(\lambda) \to \frac{1}{2}$ as $n \to \infty$.*

The next concerns the dynamical system of c_0-symbols in $\mathbb{C}_{1/2}$.

Theorem 2.4. *If a c_0-symbol Φ is neither a constant nor the identity, then the set $\{\Phi_n(w)\}_{n=0}^{\infty}$ has no limit points other than the infinity point in $\mathbb{C}_{1/2}$ for each $w \in \mathbb{C}_{1/2}$.*

Proof. Since it is trivial for the case where $\Phi(s) = s + i\tau$ with $\tau \in \mathbb{R}\setminus\{0\}$, then we assume that $\Phi(s) \neq s + i\tau$ ($\tau \in \mathbb{R}$) is a c_0-symbol. Here we argue by contradiction. To this end, we assume that the set $\{\Phi_n(w_0)\}_{n=0}^{\infty}$ has a finite limit point in $\mathbb{C}_{1/2}$ for some point $w_0 \in \mathbb{C}_{1/2}$. Let $g \in \mathcal{H}$ be such that $g \perp \overline{\operatorname{span}}\{(C_\Phi^*)^n K_{w_0} : n \geq 0\}$. Then it is clear that

$$0 = \langle g, (C_\Phi^*)^n K_{w_0} \rangle = \langle g, K_{\Phi_n(w_0)} \rangle = g(\Phi_n(w_0)),$$

that is, the analytic function g has zeros $\Phi_n(w_0)$ ($n \geq 0$). Since Φ is a non-constant analytic self-map of $\mathbb{C}_{1/2}$, then $g \equiv 0$ by the identity theorem. So we have $\overline{\operatorname{span}}\{(C_\Phi^*)^n K_{w_0} : n \geq 0\} = \mathcal{H}$.

It follows from Lemma 2.2 that there exists $\eta > 0$ such that $\Phi(\mathbb{C}_{1/2}) \subseteq \mathbb{C}_{1/2+\eta}$. Then $\Phi_n(w_0) \in \mathbb{C}_{1/2+\eta}$ for all $n \geq 1$. Thus we can choose some $w_1 \in \mathbb{C}_{1/2}$ such that $1/2 < \operatorname{Re} w_1 < 1/2 + \epsilon$ for some positive number $\epsilon < \eta$ and $\operatorname{Re} w_1 \neq \operatorname{Re} w_0$.

Since $K_{w_1} \in \mathcal{H} = \overline{\operatorname{span}}\{(C_\Phi^*)^n K_{w_0} : n \geq 0\} = \overline{\operatorname{span}}\{K_{\Phi_n(w_0)} : n \geq 0\}$, then there must exist a sequence $\{g_N\}_{N\geq 0}$ with $g_N(s) = \sum_{\ell=0}^{m(N)} a_\ell^{(N)} K_{\Phi_\ell(w_0)}(s)$, such that $g_N \to K_{w_1}$ in $\mathcal{H}$ as $N \to \infty$. Note that

$$g_N(s) = \sum_{\ell=0}^{m(N)} a_\ell^{(N)} \sum_{n=1}^{\infty} n^{-\overline{\Phi_\ell(w_0)}} n^{-s} = \sum_{n=1}^{\infty} \Big(\sum_{\ell=0}^{m(N)} a_\ell^{(N)} n^{-\overline{\Phi_\ell(w_0)}} \Big) n^{-s}$$

for each $N \geq 0$. Thus we get that $\sum_{\ell=0}^{m(N)} a_\ell^{(N)} n^{-\overline{\Phi_\ell(w_0)}} \to n^{-\overline{w_1}}$, that is,

$$\sum_{\ell=0}^{m(N)} \overline{a_\ell^{(N)}} n^{w_1 - \Phi_\ell(w_0)} \to 1, \tag{2.1}$$

for all $n \geq 1$, as $N \to \infty$. Note that $\operatorname{Re} w_1 \neq \operatorname{Re} w_0$ and $\operatorname{Re} w_1 - \operatorname{Re} \Phi_\ell(w_0) < -(\eta - \epsilon)$ for all $\ell \geq 1$. Thus (2.1) is impossible. This contradiction completes the proof. □

3. Invariant subspaces

We now consider the structure of invariant subspaces of composition operators. First, we notice the following simple observation concerning the images of 1 of the adjoints of composition operators on $\mathcal{H}$.

Lemma 3.1. *Let $\Phi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ be a c_0-symbol. Then the following statements hold:*

(1) *If $c_0 = 0$, then $C_\Phi^* 1 = \sum_{n=1}^{\infty} n^{-\overline{c_1}} n^{-s}$.*
(2) *If $c_0 \geq 1$, then $C_\Phi^* 1 = 1$.*

Proof. (1) If $c_0 = 0$, then $\operatorname{Re} c_1 > \frac{1}{2}$ by [6, P.319] and for each $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}$, we have

$$C_\Phi f(s) = \sum_{n=1}^{\infty} a_n n^{-c_1} n^{-\sum_{k=2}^{\infty} c_k k^{-s}} = \sum_{n=1}^{\infty} a_n n^{-c_1} \prod_{k=2}^{\infty} \Big(1 + \sum_{j=1}^{\infty} \frac{(-c_k \log n)^j}{j!} k^{-js} \Big),$$

which holds in the half-plane of absolute convergence of the series $\sum_{k=1}^{\infty} c_k k^{-s}$. Thus

$$\langle C_\Phi^* 1, f \rangle = \langle 1, C_\Phi f \rangle = \overline{C_\Phi f(+\infty)} = \sum_{n=1}^{\infty} \overline{a_n} n^{-\overline{c_1}} = \langle \sum_{n=1}^{\infty} n^{-\overline{c_1}} n^{-s}, f \rangle.$$

By the arbitrariness of $f \in \mathcal{H}$, we get that $C_\Phi^* 1 = \sum_{n=1}^{\infty} n^{-\overline{c_1}} n^{-s}$.

(2) If $c_0 \geq 1$, then

$$\langle C_\Phi^* 1, n^{-s} \rangle = \langle 1, n^{-\Phi(s)} \rangle = 0$$

for all $n \geq 2$. Thus (2) is immediately obtained. □

The following concerns the lattices of two composition operators.

Theorem 3.2. *Let $\Phi(s) = c_0^{(1)} s + \sum_{k=1}^{\infty} c_k^{(1)} k^{-s}$ and $\Psi(s) = c_0^{(2)} s + \sum_{k=1}^{\infty} c_k^{(2)} k^{-s}$ be c_0-symbols. If $\operatorname{Lat} C_\Phi \subseteq \operatorname{Lat} C_\Psi$, then the following statements hold:*

(1) *If $c_0^{(1)} = 0$, then $\Psi(\alpha) = \alpha$, where α is the fixed point of Φ.*
(2) *If $c_0^{(1)} \geq 1$, then $c_0^{(2)} \geq 1$.*

Proof. (2) follows easily from Lemma 3.1, we only need to prove (1). Suppose that $c_0^{(1)} = 0$ and α is the fixed point of Φ in $\mathbb{C}_{1/2}$. If $\operatorname{Lat} C_\Phi \subseteq \operatorname{Lat} C_\Psi$, then $\operatorname{Lat} C_\Phi^* \subseteq \operatorname{Lat} C_\Psi^*$. Note that $\operatorname{span}\{K_\alpha\} \in \operatorname{Lat} C_\Phi^*$, then

$$K_{\Psi(\alpha)} = C_\Psi^* K_\alpha \in \operatorname{span}\{K_\alpha\}.$$

If $\Psi(\alpha) \neq \alpha$, then K_α and $K_{\Psi(\alpha)}$ are linearly independent. So $\Psi(\alpha) = \alpha$. □

We also consider the images of two composition operators.

Theorem 3.3. *Let $\Phi(s) = c_0^{(1)} s + \sum_{k=1}^\infty c_k^{(1)} k^{-s}$ and $\Psi(s) = c_0^{(2)} s + \sum_{k=1}^\infty c_k^{(2)} k^{-s}$ be c_0-symbols such that $C_\Phi \mathcal{H} \subseteq C_\Psi \mathcal{H}$. Then the following statements hold:*
(1) If $c_0^{(1)} = 0$ and either $c_2^{(1)}$ or $c_3^{(1)}$ is not zero, then $c_0^{(2)} \leq 1$.
(2) If $c_0^{(1)} \geq 1$, then $c_0^{(2)} \leq c_0^{(1)}$.

Proof. (1) Let $c_0^{(1)} = 0$ and assume that either $c_2^{(1)}$ or $c_3^{(1)}$ is not zero. By way of contradiction, we assume that $c_0^{(2)} > 1$, then $n^{c_0^{(2)}} \geq n^2 \geq 2^2$ for all $n \geq 2$. Thus

$$\langle \ell^{-s}, C_\Psi(n^{-s})\rangle = \langle \ell^{-s}, n^{-c_0^{(2)}s} n^{-c_1^{(2)}} n^{-\sum_{k=2}^\infty c_k^{(2)} k^{-s}}\rangle = 0$$

for all $n \geq 1$ and $\ell = 2$ or 3. That is $\ell^{-s} \in (C_\Psi\mathcal{H})^\perp$, then $\ell^{-s} \in (C_\Phi\mathcal{H})^\perp$, for $\ell = 2$ or 3. However,

$$\langle \ell^{-s}, C_\Phi(n^{-s})\rangle = \langle \ell^{-s}, n^{-c_1^{(1)}} n^{-\sum_{k=2}^\infty c_k^{(1)} k^{-s}}\rangle = -\overline{n^{-c_1^{(1)}} c_\ell^{(1)} \log n} \neq 0$$

for all $n > 1$ and some $\ell \in \{2, 3\}$, which implies that $\ell^{-s} \notin (C_\Phi\mathcal{H})^\perp$ for $\ell = 2$ or 3. This contradiction implies that $c_0^{(2)} \leq 1$.

(2) If $c_0^{(1)} \geq 1$ and $c_0^{(2)} > c_0^{(1)}$, then $n^{c_0^{(2)}} > n^{c_0^{(1)}} \geq 2^{c_0^{(1)}}$ for all $n \geq 2$. Note that

$$\langle 2^{-c_0^{(1)}s}, C_\Psi(n^{-s})\rangle = \langle 2^{-c_0^{(1)}s}, n^{-c_0^{(2)}s} n^{-c_1^{(2)}} n^{-\sum_{k=2}^\infty c_k^{(2)} k^{-s}}\rangle = 0$$

for all $n \geq 1$, and

$$\langle 2^{-c_0^{(1)}s}, C_\Phi(2^{-s})\rangle = \langle 2^{-c_0^{(1)}s}, 2^{-c_0^{(1)}s} 2^{-c_1^{(1)}} 2^{-\sum_{k=2}^\infty c_k^{(1)} k^{-s}}\rangle = \overline{2^{-c_1^{(1)}}} \neq 0.$$

Since the inclusion $C_\Phi\mathcal{H} \subseteq C_\Psi\mathcal{H}$ means that $(C_\Psi\mathcal{H})^\perp \subseteq (C_\Phi\mathcal{H})^\perp$, then the previous two equations leads to a contradiction. So $c_0^{(2)} \leq c_0^{(1)}$, which completes the proof. □

Recall that an operator is reductive if every invariant subspace for the operator is reducing. The following characterizes the reductive composition operators on $\mathcal{H}$.

Theorem 3.4. *Let $\Phi(s) = c_0 s + \sum_{k=1}^\infty c_k k^{-s}$ be a c_0-symbol. Then $C_\Phi : \mathcal{H} \to \mathcal{H}$ is reductive if and only if $\Phi(s) = s + c_1$ with $\operatorname{Re} c_1 \geq 0$.*

Proof. We first claim that $c_0 \neq 0$. To this end, by way of contradiction, we assume that $c_0 = 0$. It is clear that $\mathbb{C} \in \operatorname{Lat} C_\Phi$, which implies that $C_\Phi^* 1 \in \mathbb{C}$ by the reduction of C_Φ. Hence $\langle C_\Phi^* 1, n^{-s}\rangle = 0$ for all $n \geq 2$. But $\langle C_\Phi^* 1, n^{-s}\rangle = n^{-\overline{c_1}} \neq 0$ for each $n \geq 2$ by Lemma 3.1. This contradiction gives the claim.

Next we claim that $c_0 = 1$. If this is not the case, then $c_0 > 1$. Then $2^{c_0} > 2$. It is clear that $\mathcal{L}_n := \overline{\text{span}}\{k^{-s} : k \geq n\} \in \text{Lat}\, C_\Phi$, and then $\mathcal{L}_n \in \text{Lat}\, C_\Phi^*$ for all $n \geq 1$ by the reduction of C_Φ. But

$$\langle C_\Phi^* 2^{-c_0 s}, 2^{-s}\rangle = \langle 2^{-c_0 s}, 2^{-\Phi(s)}\rangle = 2^{-\overline{c_1}} \neq 0,$$

which implies that $C_\Phi^* 2^{-c_0 s} \notin \mathcal{L}_{2^{c_0}}$. This contradicts to $\mathcal{L}_{2^{c_0}} \in \text{Lat}\, C_\Phi^*$, then $c_0 = 1$.

So we can suppose that $\Phi(s) = s + c_1 + \sum_{k=2}^\infty c_k k^{-s}$. Since $\mathcal{L}_n \in \text{Lat}\, C_\Phi$, and then $\mathcal{L}_n \in \text{Lat}\, C_\Phi^*$ for all $n \geq 1$. Note that $\mathcal{L}_n^\perp = \text{span}\,\{1, 2^{-s}, \cdots, (n-1)^{-s}\}$ for $n \geq 2$, and $\mathcal{H}^\perp = \{0\}$. So $C_\Phi(m^{-s}) \in \text{span}\,\{1, 2^{-s}, \cdots, (n-1)^{-s}\}$ for each $n \geq 2$ and $m \in \{1, 2, \cdots, n-1\}$. Note that

$$\begin{aligned} C_\Phi(m^{-s}) &= m^{-\Phi(s)} = m^{-c_1} m^{-s} m^{-\sum_{k=2}^\infty c_k k^{-s}} \\ &= m^{-c_1} m^{-s} \prod_{k=2}^\infty \left(1 + \sum_{j=1}^\infty \frac{(-c_k \log m)^j}{j!} k^{-js}\right), \end{aligned}$$

which holds in the half-plane of absolute convergence of the series $\sum_{k=1}^\infty c_k k^{-s}$. So $c_k = 0$ for all $k \geq 2$. Consequently, $\Phi(s) = s + c_1$ with $\text{Re}\, c_1 \geq 0$.

Conversely, assume that $\Phi(s) = s + c_1$ with $\text{Re}\, c_1 \geq 0$. Then C_Φ is normal. If $\text{Re}\, c_1 > 0$, then C_Φ is compact by [5], and hence every invariant subspace of C_Φ contains at least one of eigenvectors. It follows from [17] that C_Φ is reductive. If $\text{Re}\, c_1 = 0$, then C_Φ is a unitary, and $\{n^{-s}\}_{n=1}^\infty$ are eigenvectors of C_Φ. By [17, Theorem 6], there exists a sequence of polynomials $p_n(z)$ such that $p_n(C_\Phi)$ converges strongly to C_Φ^*. Hence C_Φ is reductive. □

If a non-constant function $f \in \mathcal{H}$ is an eigenvector of a composition operator, then the 2-dimensional subspace $\text{span}\,\{1, f\}$ must be invariant under it. The following actually characterizes 2-dimensional invariant subspaces containing the constants of composition operators.

Theorem 3.5. *Let $\Phi(s) = c_0 s + \sum_{k=1}^\infty c_k k^{-s}$ be a c_0-symbol. If there exists a non-constant function f such that* $\text{span}\,\{1, f\}$ *is invariant under C_Φ, then $c_0 \leq 1$. Moreover, we have:*

(1) If $c_0 = 0$, then $f - f(\alpha)$ is an eigenvector of C_Φ with respect to the eigenvalue $(\Phi'(\alpha))^k$ for some integer $k \geq 1$, where α is the fixed point of Φ in $\mathbb{C}_{1/2}$.

(2) If $c_0 = 1$, then $f - f(+\infty)$ is an eigenvector of C_Φ with respect to the eigenvalue ℓ^{-c_1} for some integer $\ell \geq 2$.

Proof. Let $c_0 = 0$ and α be the fixed point of Φ. We claim that $g := f - f(\alpha)$ is an eigenvector of C_Φ with respect to the eigenvalue $(\Phi'(\alpha))^k$ for some integer $k \geq 1$. It is obvious that g is also non-constant and $g \in \text{span}\,\{1, f\}$. Since $\text{span}\,\{1, f\}$ is invariant under C_Φ, we have $C_\Phi g = c + dg$ for some $c, d \in \mathbb{C}$. Since $g(\alpha) = 0$, then we have $c = 0$. Hence $C_\Phi g = dg$, i.e., $f - f(\alpha)$ is an eigenvector of C_Φ with respect to the eigenvalue d. It is obvious that $d = (\Phi'(\alpha))^k$ for some integer $k \geq 1$ by [3, Theorem 4].

Let $c_0 \geq 1$ and define the analytic function $\widehat{f} := f - f(+\infty)$. Then $\widehat{f} \in \text{span}\,\{1, f\}$ is also non-constant. Since $\text{span}\,\{1, f\}$ is invariant under C_Φ, we have

$C_\Phi \widehat{f}(s) = a + b\widehat{f}(s)$ for some constants $a, b \in \mathbb{C}$. Letting $\operatorname{Re} s \to +\infty$ and noting that $\Phi(+\infty) = +\infty$ for $c_0 \geq 1$, we can get $a = 0$ by the fact $\widehat{f}(+\infty) = 0$. Hence $C_\Phi \widehat{f}(s) = b\widehat{f}(s)$, that is, $\widehat{f} = f - f(+\infty)$ is an eigenvector of C_Φ with respect to the eigenvalue b.

We now exclude that $c_0 > 1$. Indeed, if $c_0 > 1$, then $b \in \{0, 1\}$ by [3, Theorem 4] and $C_\Phi \widehat{f}(s) = b\widehat{f}(s)$ with $\widehat{f}(+\infty) = 0$. But this is impossible because $\widehat{f}$ is neither a constant function or zero, which completes the proof. □

By the previous theorem, we get that: if Φ is a c_0-symbol with $c_0 > 1$, then C_Φ has no 2-dimensional invariant subspaces containing the constants. It is interesting to find other finite dimensional invariant subspaces of C_Φ. The following shows the non-existence of non-trivial finite dimensional invariant subspaces.

Theorem 3.6. *If $\Phi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ is a c_0-symbol with $c_0 > 1$, then C_Φ has no non-trivial finite dimensional invariant subspaces other than the constants.*

Proof. Let $\mathcal{M}$ be a non-trivial finite dimensional invariant subspace of C_Φ other than the constants. We first show that $\mathcal{M}$ must contain the constants. According to Theorem 4 in [3], the point spectrum of C_Φ is $\{1\}$ and a computation shows that the eigenvectors corresponding to eigenvalue 1 should be the constants. Since $\mathcal{M}$ is finite dimensional, $C_\Phi|_{\mathcal{M}}$ has an eigenvector. So 1 is in $\mathcal{M}$. By Lemma 3.1, $\mathbb{C} \in \operatorname{Lat} C_\Phi^*$. So $\mathcal{N} = \mathcal{M} \cap \mathbb{C}^\perp$ is a finite dimensional invariant subspace for C_Φ, which implies that $C_\Phi|_{\mathcal{M}}$ has an eigenvalue other than 1. This yields a contradiction since the point spectrum of C_Φ is $\{1\}$. □

4. Reflexivity

In the sequel, we define P_n for $n \geq 1$ on $\mathcal{H}$ as follows: $(P_n f)(s) = a_n n^{-s}$ for $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{H}$. It is clear that P_1 is the orthogonal projection from $\mathcal{H}$ onto $\mathbb{C}$, so we will also use $P_\mathbb{C}$ in place of P_1. We begin with the following interesting fact.

Theorem 4.1. *Let Φ be a c_0-symbol with $c_0 > 1$. If a strongly closed unital algebra of operators on $\mathcal{H}$ contains C_Φ, then $P_\mathbb{C}$ is in the algebra.*

Proof. Let $\Phi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ be a c_0-symbol with $c_0 > 1$, and let $\mathcal{U}$ be a strongly closed unital algebra of operators on $\mathcal{H}$ containing C_Φ.
Fix any $m \geq 1$ and non-zero element $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{H}$, and set

$$\mathcal{K}_m = \operatorname{span}\{1, 2^{-s}, \cdots, m^{-s}\}.$$

We claim that $(C_\Phi^m - P_\mathbb{C})f \perp \mathcal{K}_m$. In fact, it follows from [6] that $\Phi_m(s) = c_0^m s + \sum_{k=1}^{\infty} c_k^{(m)} k^{-s}$ with $c_k^{(m)} \in \mathbb{C}$ $(k \geq 1)$ for each $m \geq 2$ and $c_1^{(m)} = c_1 \sum_{i=0}^{m-1} c_0^i$, where the series $\sum_{k=1}^{\infty} c_k^{(m)} k^{-s}$ converges uniformly in some (possibly remote) half-plane. Because

$$(C_\Phi^m - P_\mathbb{C})f(s) = \sum_{k=2}^{\infty} a_k k^{-\Phi_m(s)},$$

and $n^{c_0^m} \geq c_0^m > m$ for $n \geq 2$ and $c_0 > 1$, the expression of $(C_\Phi^m - P_\mathbb{C})f(s)$ does not contain any k^{-s} with $k \leq m$. Thus $(C_\Phi^m - P_\mathbb{C})f \perp \mathcal{K}_m$ for each $m \geq 1$

Since $\overline{\cup_m \mathcal{K}_m} = \mathcal{H}$ and $(C_\Phi^m - P_\mathbb{C})f \perp \mathcal{K}_m$, $(C_\Phi^m - P_\mathbb{C})f$ converges weakly to 0 in $\mathcal{H}$. Thanks to Mazur's theorem (see [14, Theorem 3.13]), there exists a sequence $\{h_N\}$ of convex combinations of $(C_\Phi^m - P_\mathbb{C})f$ such that $h_N \to 0$ in $\mathcal{H}$ as $N \to \infty$, i.e., $h_N(s) = \sum_{i=1}^{k_N} r_{N_i}(C_\Phi^{N_i} - P_\mathbb{C})f(s)$, $r_{N_i} \geq 0$, $\sum_{i=1}^{k_N} r_{N_i} = 1$. Then

$$\sum_{i=1}^{k_N} r_{N_i} C_\Phi^{N_i} f(s) \to P_\mathbb{C} f(s), \text{ as } N \to \infty,$$

in the norm of $\mathcal{H}$. By the arbitrariness of f, $P_\mathbb{C}$ is in the algebra $\mathcal{U}$. □

The following gives a characterization of invariant subspaces of composition operators for c_0-symbols with $c_0 > 1$.

Corollary 4.2. *Let Φ be a c_0-symbol with $c_0 > 1$. If $\mathcal{M}$ is an invariant subspace of C_Φ, then either $\mathbb{C} \subseteq \mathcal{M}$ or $\mathbb{C} \subseteq \mathcal{M}^\perp$.*

Proof. Fix any $f \in \mathcal{M}$. By Theorem 4.1, we know that $C_\Phi^n f$ converges weakly to $P_\mathbb{C} f$. Since $\mathcal{M} \in \operatorname{Lat} C_\Phi$, then $P_\mathbb{C} f \in \mathcal{M}$. If $\mathcal{M}$ does not contains the constants, then $P_\mathbb{C} f = 0$. By the arbitrariness of $f \in \mathcal{M}$, we have $\mathbb{C} \subseteq \mathcal{M}^\perp$. So the proof is complete. □

Before proceeding to the reflexivity results, we introduce some notation and one useful lemma. If $\Phi(s) = s + i\tau$ with $\tau \in \mathbb{R}$ such that $(\frac{n}{m})^{i\tau}$ are not roots of unity for all integers $n \neq m$ and $n, m \geq 1$, then we call Φ an irrational c_0-symbol. For example, $\Phi(s) = s + i2\pi$ is an irrational c_0-symbol. If x is a real number, let $\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. Namely, $\{x\}$ denotes the fractional part of x. We recall the following remarkable fact from [11]:

Lemma 4.3. If θ is an irrational number and f is continuous on $[0, 1]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\{k\theta\}) = \int_0^1 f(\alpha)\, d\alpha.$$

Motivated by Lemma 2.1 and Theorem 2.2 in [10], we give the following two results.

Theorem 4.4. *If a strongly closed unital algebra of operators on $\mathcal{H}$ contains a composition operator with an irrational c_0-symbol, then the algebra contains P_n for $n = 1, 2, \cdots$.*

Proof. Fix an $n \geq 1$ and let $\Phi(s) = s + i\tau$ with $\tau \in \mathbb{R}$ such that $(\frac{p}{q})^{i\tau}$ are not roots of unity for all integers $p, q \geq 1$ with $p \neq q$. Let $\mathcal{U}$ be a strongly closed unital algebra of operators on $\mathcal{H}$ that contains C_Φ, $C_n = n^{i\tau} C_\Phi$ and $A_k = \frac{1}{k}(C_n + C_n^2 + \cdots + C_n^k)$. Since C_Φ is a unitary, by the mean ergodic theorem and uniform boundedness principle, $\{A_k - P_n\}_{k=1}^\infty$ is bounded. It is clear that $E := \{f \in \mathcal{H} : \lim_{k\to\infty}(A_k - P_n)f = 0\}$ is a closed subspace of $\mathcal{H}$. We now claim that $m^{-s} \in E$ for each $m \geq 1$. In fact, we note that

$$A_k m^{-s} = \frac{1}{k}(n^{i\tau} m^{-s-i\tau} + n^{i2\tau} m^{-s-i2\tau} + \cdots + n^{ik\tau} m^{-s-ik\tau}) = \frac{1}{k}\big[(\frac{n}{m})^{i\tau} + (\frac{n}{m})^{i2\tau} + \cdots + (\frac{n}{m})^{ik\tau}\big] m^{-s}.$$

If $m = n$, then $A_k n^{-s} = n^{-s}$, and thus $n^{-s} \in E$. If $m \neq n$, then $\theta := (\tau \log(n/m))/2\pi$ is irrational by our hypothesis. Thus by Lemma 4.3 for θ and $f(\alpha) = e^{i\alpha 2\pi}$, we have

$$\lim_{k\to\infty} \frac{1}{k}\big[(\frac{n}{m})^{i\tau} + (\frac{n}{m})^{i2\tau} + \cdots + (\frac{n}{m})^{ik\tau}\big] = 0,$$

which implies that $m^{-s} \in E$. Consequently, E contains m^{-s} for every $m \geq 1$. Thus $E = \mathcal{H}$, which is equivalent to say P_n is the strong limit of $\{A_k\}$. Since A_k is in the algebra $\mathcal{U}$ for each $k \geq 1$, it follows that P_n is in the algebra for $n = 1, 2, \cdots$. □

Theorem 4.5. *If Φ is an irrational c_0-symbol, then every strongly closed unital algebra of operators on $\mathcal{H}$ containing C_Φ is reflexive.*

Proof. Let $\mathcal{U}$ be any strongly closed unital algebra of operators on $\mathcal{H}$ containing C_Φ. Then $\mathcal{U}$ contains P_n for $n = 1, 2, \cdots$ by Theorem 4.4. Suppose $B \in B(\mathcal{H})$ such that $\operatorname{Lat}\mathcal{U} \subseteq \operatorname{Lat} B$. For arbitrary n, Bn^{-s} is in the cyclic subspace of $\mathcal{U}$ determined by n^{-s}, and thus there is a sequence $\{A_k\}$ in $\mathcal{U}$ such that $Bn^{-s} = \lim_{k\to\infty} A_k n^{-s}$. It follows that BP_n is the strong limit of $\{A_k P_n\}_{k=1}^{\infty}$, and hence $BP_n \in \mathcal{U}$.

Let σ_k be the k-th Cesàro mean of the series $\sum_{m=1}^{\infty} P_m$, i.e., $\sigma_k = \sum_{m=1}^{k} \frac{k+1-m}{k+1} P_m$. Let $\widetilde{P}_N$ be the N-th partial sum of $\sum_{m=1}^{\infty} P_m$, i.e., $\widetilde{P}_N = \sum_{m=1}^{N} P_m$. It is clear that all σ_k ($k \geq 1$) and $\widetilde{P}_N$ ($N \geq 1$) are in $\mathcal{U}$. Note that $\lim_{k\to\infty} \sigma_k P = P$ for any polynomial P with respect to $\{n^{-s}\}_{n=1}^{\infty}$. Also for any fixed $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{H}$, and $\epsilon > 0$, it is trivial that there is an integer N_0 such that for each $N \geq N_0$, we have $\|\widetilde{P}_N f - f\| < \epsilon/2$. Further we can find an integer k_N such that $\|\sigma_{k_N} \widetilde{P}_N f - \widetilde{P}_N f\| < \epsilon/2$. Therefore $\sigma_{k_N} \widetilde{P}_N f$ converges to f in $\mathcal{H}$ as $N \to \infty$. It follows that $B\sigma_{k_N} \widetilde{P}_N$ converges strongly to B in $\mathcal{H}$. Since $B\sigma_{k_N} \in \mathcal{U}$ and $\widetilde{P}_N \in \mathcal{U}$, then $B\sigma_{k_N} \widetilde{P}_N \in \mathcal{U}$. And thus $B \in \mathcal{U}$, which completes the proof. □

Next we give a strongly closed unital algebra of operators on $\mathcal{H}$ generated by certain sets of composition operators such that the non-trivial common invariant subspace of the algebra is $\mathbb{C}$. For convenience, we will denote by *Prime* the set of all prime numbers.

Theorem 4.6. *Let Φ be an irrational c_0-symbol, and $\Gamma(s) = c_1 + \sum_{p\in Prime} c_p p^{-s}$ a c_0-symbol with $c_p \neq 0$. Then the strongly closed unital algebra of operators on $\mathcal{H}$ generated by C_Φ and C_Γ is* $\operatorname{Alg}\{\{0\}, \mathbb{C}, \mathcal{H}\}$.

Proof. Let $\mathcal{U}$ denote the strongly closed unital algebra of operators on $\mathcal{H}$ generated by C_Φ and C_Γ, and let $\mathcal{M}$ be an arbitrary invariant subspace of $\mathcal{U}$. If $\mathcal{M} \neq \{0\}$, then we claim that $\mathcal{M}$ must contain the constants $\mathbb{C}$. In fact, since $\mathcal{M} \neq \{0\}$, then there is a non-zero function $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{M}$. If $a_1 \neq 0$, then $a_1 = P_1 f(s) \in \mathcal{M}$, i.e., $\mathbb{C} \subseteq \mathcal{M}$. If $a_1 = 0$, then $f(s) = \sum_{k=\ell}^{\infty} a_k k^{-s}$ for $a_\ell \neq 0$ and $\ell \geq 2$. Since $\mathcal{M}$ is invariant under P_ℓ, then $\ell^{-s} \in \mathcal{M}$. Due to

$$\begin{aligned} C_\Gamma \ell^{-s} &= \ell^{-\Gamma(s)} = \ell^{-c_1} \ell^{-\sum_{p\in Prime} c_p p^{-s}} \\ &= \ell^{-c_1} \prod_{p\in Prime} \Big(1 + \sum_{j=1}^{\infty} \frac{(-c_p \log \ell)^j}{j!} p^{-js}\Big), \end{aligned}$$

we have $P_1 C_\Gamma \ell^{-s} = \ell^{-c_1} \neq 0$. Again $C_\Gamma \mathcal{M} \subseteq \mathcal{M}$ implies that $P_1 C_\Gamma \mathcal{M} \subseteq P_1 \mathcal{M} \subseteq \mathcal{M}$. Therefore $\ell^{-c_1} \in \mathcal{M}$, which implies the claim.

Moreover, if $\mathcal{M} \neq \mathbb{C}$, then there exists some $n_0 \geq 2$ such that $n_0^{-s} \in \mathcal{M}$. Since $P_n \in \mathcal{U}$ for every $n \geq 1$ by Theorem 4.4 and $C_\Gamma \mathcal{M} \subseteq \mathcal{M}$, then $P_n C_\Gamma n_0^{-s} \in \mathcal{M}$ for all $n \geq 1$. We know that each positive integer n can be uniquely expressed as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell},$$

where $p_1 < p_2 < \cdots < p_\ell \in Prime$ and $\alpha_1, \alpha_2, \cdots, \alpha_\ell$ are positive integers. Because

$$C_\Gamma n_0^{-s} = n_0^{-c_1} \prod_{p\in Prime} \Big(1 + \sum_{j=1}^{\infty} \frac{(-c_p \log n_0)^j}{j!} p^{-js}\Big), \tag{4.1}$$

we can get the coefficient of n^{-s} in (4.1) as follows

$$a_n := n_0^{-c_1} \frac{(-c_{p_1} \log n_0)^{\alpha_1}}{\alpha_1!} \frac{(-c_{p_2} \log n_0)^{\alpha_2}}{\alpha_2!} \cdots \frac{(-c_{p_\ell} \log n_0)^{\alpha_\ell}}{\alpha_\ell!}.$$

It is clear that $a_n \neq 0$. Note that $P_n C_\Gamma n_0^{-s} = a_n n^{-s}$, which gives that $n^{-s} \in \mathcal{M}$. Thus $\operatorname{span}\{n^{-s} : n \geq 1\} \subseteq \mathcal{M}$ and $\mathcal{M} = \mathcal{H}$. We now get that $\operatorname{Lat}\mathcal{U} = \{\{0\}, \mathbb{C}, \mathcal{H}\}$. By Theorem 4.5, we know that the algebra $\mathcal{U}$ is reflexive. This immediately gives that $\mathcal{U} = \operatorname{Alg}\operatorname{Lat}\mathcal{U} = \operatorname{Alg}\{\{0\}, \mathbb{C}, \mathcal{H}\}$, which completes the proof. □

Interestingly, the following shows that the subalgebra of $B(\mathcal{H})$ generated by certain sets of composition operators and their adjoints is $B(\mathcal{H})$.

Corollary 4.7. *Let Φ be an irrational c_0-symbol, and let $\Gamma(s) = \sum_{k=1}^{\infty} c_k k^{-s}$ be a c_0-symbol. Then the strongly closed unital algebra of operators on $\mathcal{H}$ generated by C_Φ, C_Γ and their adjoints equals $B(\mathcal{H})$.*

Proof. Let $\mathcal{U}$ be the strongly closed unital algebra of operators on $\mathcal{H}$ generated by C_Φ, C_Γ and their adjoints, and let $M \neq \{0\}$ be any invariant subspace of $\mathcal{U}$. By the proof of Theorem 4.6, we know that $\mathbb{C} \subseteq M$. Since M is invariant under C_Γ^*, $C_\Gamma^* 1 \in M$. Again $C_\Gamma^* 1 = \sum_{n=1}^{\infty} n^{-\overline{c_1}} n^{-s}$ by Lemma 3.1 and $P_n C_\Gamma^* 1 \in M$ for all $n \geq 1$ by Theorem 4.4, we have $n^{-s} \in M$ for all $n \geq 1$, i.e., $\mathcal{H} = M$. Therefore $\operatorname{Lat}\mathcal{U} = \{\{0\}, \mathcal{H}\}$. Corollary 8.5 in [13] now gives the desired assertion $\mathcal{U} = B(\mathcal{H})$. □

Algebraic composition operators on Hardy and Bergman spaces were studied by many authors. Especially, Mahvidi showed that the non-existence of non-trivial algebraic composition operators on $H^2(\mathbb{D})$ by using their invariant subspaces in [8]. At the end of this section we extend Mahvidi's result to the Dirichlet series case.

Theorem 4.8. *Let $\Phi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ be a c_0-symbol. Then on $\mathcal{H}$ the following statements hold.*

(1) *If $c_0 = 0$ and $\{\Phi'(\alpha)^n\}_{n=0}^{\infty}$ is infinite for the fixed point α of Φ in $\mathbb{C}_{1/2}$, then C_Φ is not algebraic.*

(2) *If $c_0 = 1$ and $c_1 \neq 0$, then C_Φ is not algebraic.*

(3) *If $c_0 > 1$, then C_Φ is not algebraic.*

Proof. (1) If C_Φ is algebraic on $\mathcal{H}$, then so is C_Φ^*. Thus there exists a polynomial $P(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0$ such that $P(C_\Phi^*) = 0$ on $\mathcal{H}$. So

$$(C_\Phi^*)^n K_\alpha^{[m]} + p_{n-1}(C_\Phi^*)^{n-1} K_\alpha^{[m]} + \cdots + p_1 C_\Phi^* K_\alpha^{[m]} + p_0 K_\alpha^{[m]} = 0 \qquad (4.2)$$

for each $m \geq 0$, where $K_\alpha^{[m]}$ denotes the m-th derivative of K_α. Since the coefficient of $K_\alpha^{[m]}$ in the expansion of the left hand side of the equation (4.2) is

$$\big([\Phi'(\alpha)]^m\big)^n + p_{n-1}\big([\Phi'(\alpha)]^m\big)^{n-1} + \cdots + p_1[\Phi'(\alpha)]^m + p_0$$

for each $m \geq 0$. So $[\Phi'(\alpha)]^m$ for all $m \geq 0$ are the roots of the equation $P(z) = 0$. But the polynomial $P(z)$ has at most n zeros, which leads to a contradiction. Therefore C_Φ is not algebraic.

(2) If $c_0 = 1$, then $\Phi_n(s) = s + nc_1 + \sum_{k=2}^{\infty} c_k^{(n)} k^{-s}$ with some $c_k^{(n)} \in \mathbb{C}$ ($k \geq 2$) for each $n \geq 2$, where the series $\sum_{k=2}^{\infty} c_k^{(n)} k^{-s}$ converges uniformly in some (possibly remote) half-plane. Suppose that there is a polynomial $P(z) = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0$ such that $P(C_\Phi) = 0$ on $\mathcal{H}$, then

$$m^{-\Phi_n(s)} + p_{n-1} m^{-\Phi_{n-1}(s)} + \cdots + p_1 m^{-\Phi(s)} + p_0 m^{-s} = 0. \qquad (4.3)$$

Note that

$$m^{-\Phi_j(s)} = m^{-s} m^{-jc_1} \prod_{k=2}^{\infty} \left(1 + \sum_{i=1}^{\infty} \frac{(-c_k^{(j)} \log m)^i}{i!} k^{-is}\right)$$

in some (possibly remote) half-plane for $m \geq 1$ and $j \geq 1$. Considering the coefficient of m^{-s} in the left hand side of (4.3), we get

$$m^{-nc_1} + p_{n-1} m^{-(n-1)c_1} + \cdots + p_1 m^{-c_1} + p_0 = 0,$$

that is, m^{-c_1} is a root of $P(z) = 0$. Then by the arbitrariness of m, the polynomial P has zeros $1, 2^{-c_1}, 3^{-c_1}, \cdots, m^{-c_1}, \cdots$. Note that, if $\operatorname{Re} c_1 \neq 0$, then m^{-c_1} are mutually distinct for all $m \geq 1$; if $\operatorname{Re} c_1 = 0$ but $\operatorname{Im} c_1 \neq 0$, then $m^{-c_1} = m^{-i\operatorname{Im} c_1} = e^{-i\operatorname{Im} c_1 \log m}$ are dense in the unit circle, by Lemma 3.5 in [7]. In short, the polynomial P has infinitely many zeros whenever $c_1 \neq 0$. This contradiction implies that C_Φ is not algebraic.

(3) If $c_0 > 1$, by the proof of Theorem 4.1, $(C_\Phi^*)^n f$ converges weakly to $P_{\mathbb{C}} f$ in $\mathcal{H}$. Thus

$$\langle C_{\Phi_n}^* K_w, g\rangle \to \langle P_{\mathbb{C}} K_w, g\rangle$$

for each fixed $w \in \mathbb{C}_{1/2}$ and $g \in \mathcal{H}$, as $n \to \infty$. Note that $P_{\mathbb{C}} K_w(s) = 1$. Thus

$$m^{-\overline{\Phi_n(w)}} = \langle K_{\Phi_n(w)}, m^{-s}\rangle \to \langle 1, m^{-s}\rangle = \begin{cases} 1, & m = 1 \\ 0, & m > 1 \end{cases}$$

as $n \to \infty$. Therefore the sequence $\{\mathrm{Re}\,\Phi_n(w)\}_{n=0}^{\infty}$ tends to $+\infty$. If C_Φ on $\mathcal{H}$ is algebraic, then so is C_Φ^*. Hence, all cyclic subspaces of C_Φ^* are finite dimension. But the cyclic subspace $\mathrm{span}\,\{(C_\Phi^*)^n K_w : n \geq 0\} = \mathrm{span}\,\{K_{\Phi_n(w)} : n \geq 0\}$ of C_Φ^* is infinite dimension, because $\{K_w\}$ are linearly independent for $w \in \mathbb{C}_{1/2}$ with distinct real parts. This contradiction completes the proof. □

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