Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 4, 172–178
http://doi.org/10.15352/afa/06-4-172
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

ON CLUSTER SYSTEMS OF TENSOR PRODUCT SYSTEMS OF HILBERT SPACES

MITHUN MUKHERJEE

Communicated by B. V. R. Bhat

ABSTRACT. It is known that the spatial product of two product systems is intrinsic. Here we extend this result by analyzing subsystems of the tensor product of product systems. A relation with cluster systems as introduced by B.V.R. Bhat, M. Lindsay and M. Mukherjee, is established.

1. Introduction and preliminaries

By a product system, we mean a measurable family of Hilbert spaces $(\mathcal{E}_t)_{t>0}$ with associative identification $\mathcal{E}_s \otimes \mathcal{E}_t = \mathcal{E}_{s+t}$. Arveson [1] associated with every E_0-semigroup, a product system of Hilbert spaces. He showed that E_0-semigroups are classified by their product systems up to cocycle conjugacy. Product systems are classified as spatial and non-spatial depending on whether or not there is a unit in the product system, where a unit is a measurable family of sections $(u_s)_{s>0}$, such that $u_s \in \mathcal{E}_s, s > 0$ and $u_{s+t} = u_s \otimes u_t, s, t > 0$ under the identification. The spatial product system has an index and the index is additive with respect to the tensor product of product systems. Much of the theory has a counterpart in the theory of product system of Hilbert modules ([8], [5]). Though there is no natural tensor product in the category of product systems of Hilbert modules. To overcome this, Skeide ([11]) introduced the notion of spatial product in the category of spatial product systems of Hilbert modules for which the index is additive with respect to the spatial product.

For two product systems of Hilbert spaces $\mathcal{E} = (\mathcal{E}_t)_{t>0}$ and $\mathcal{F} = (\mathcal{F}_t)_{t>0}$ with reference units $u = (u_t)_{t>0}$ and $v = (v_t)_{t>0}$ respectively, their spatial product can be identified with the subsystem of the tensor product generated by subsystems $(u_t \otimes \mathcal{F}_t)_{t>0}$ and $(\mathcal{E}_t \otimes v_t)_{t>0}$. This is exactly the same description of the product

Date: Received: Feb. 10, 2015; Revised: Apr. 10, 2015; Accepted: Apr. 22, 2015.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 46C05.
Key words and phrases. E_0-semigroup, product system, completely positive semigroup.

systems arising from the Powers sum of two E_0-semigroups. See [10], [3]. This raises another question, namely, whether the spatial product is the tensor product or not. This has been answered in the negative sense by Powers [9].

The spatial structure of a spatial product system depends on the reference unit. Indeed, Tsirelson ([12]) showed that not all spatial product systems are transitive. I.e. there are spatial product systems in which given two normalized units, there may not exist any automorphism of the product system sending one unit to another. This immediately raises the question whether different choice of reference units yields isomorphic product systems or not. In [2], it was shown affirmatively that the spatial product of two spatial product systems is independent of the choice of the reference units. See also [3], [7].

In this paper, we show the following : given two product systems $\mathcal{E}$ and $\mathcal{F}$ and their subsystems $\mathcal{M}$ and $\mathcal{N}$ respectively, the subsystem generated by $\mathcal{E}\otimes\mathcal{N}$ and $\mathcal{M}\otimes\mathcal{F}$ is same as the subsystem generated by $\mathcal{E}\otimes\check{\mathcal{N}}$ and $\check{\mathcal{M}}\otimes\mathcal{F}$ within $\mathcal{E}\otimes\mathcal{F}$. Here $\check{\mathcal{M}}$ and $\check{\mathcal{N}}$ are respectively the cluster systems of $\mathcal{M}$ and $\mathcal{N}$ in the sense of [4]. As a special case, we have the result of [2] namely spatial products of product systems are intrinsic.

Remark 1.1. It should be noted that some of these results also follow from the theory of random sets. See Proposition 5.3, [7] for more details. See also Proposition 3.33, [6] and the identification with the cluster construction given in Theorem 27, [4]. But here we give a plain Hilbert space proof of this result.

2. Product Systems

Let us start with some definitions.

Definition 2.1. A continuous tensor product system of Hilbert spaces (briefly: product system) is a measurable family $\mathcal{E} = (\mathcal{E}_t)_{t>0}$ of separable Hilbert spaces endowed with a measurable family of unitaries $V_{s,t} : \mathcal{E}_s\otimes\mathcal{E}_t \to \mathcal{E}_{s+t}$ for all $s, t > 0$, which fulfils for all $r, s, t > 0$

$$V_{r,s+t} \circ (1_{\mathcal{E}_r} \otimes V_{s,t}) = V_{r+s,t} \circ (V_{r,s} \otimes 1_{\mathcal{E}_t}).$$

Definition 2.2. A unit u of a product system is a measurable non-zero section $(u_t)_{t>0}$ through $(\mathcal{E}_t)_{t>0}$ which satisfies for all $s, t > 0$

$$V_{s,t}(u_s \otimes u_t) = u_{s+t}.$$

A unit is said to be normalized if $\|u_t\| = 1$ for all $t > 0$.

Definition 2.3. A product system $\mathcal{G}$ with associated unitaries $U_{s,t}$ is said to be a product subsystem of $\mathcal{E}$ if $\mathcal{G}_t \subset \mathcal{E}_t$ for all $t > 0$ and $U_{s,t} = V_{s,t}|_{\mathcal{G}_s\otimes\mathcal{G}_t}$ for all $s, t > 0$.

Remark 2.4. We do not make the definition of measurability more explicit throughout this paper. For a thorough discussion, see Section 7, [6].

Definition 2.5. A product system $\mathcal{E}$ is said to be spatial if $\mathcal{E}$ has a unit.

For a spatial product system $\mathcal{E}$, we denote by $\mathcal{E}^I$ to be the smallest product subsystem of $\mathcal{E}$ containing all units of $\mathcal{E}$.

Definition 2.6. A product system $\mathcal{E}$ is said to have type I if $\mathcal{E}^I = \mathcal{E}$.

Given two product subsystems $\mathcal{E}_1$ and $\mathcal{E}_2$ of a product system $\mathcal{E}$, we denote by $\mathcal{E}_1 \bigvee \mathcal{E}_2$ to be the smallest product subsystem of $\mathcal{E}$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$.

3. Roots and Cluster Systems

The following definition is adopted from [4].

Definition 3.1. Let $\mathcal{E}$ be a spatial product system and u be a unit of $\mathcal{E}$. A measurable section $(a_t)_{t>0}$ of $\mathcal{E}$ is said to be a root of u if

$$a_{s+t} = a_s \otimes u_t + u_s \otimes a_t \ , \ \langle a_t, u_t \rangle = 0 \ , \ \forall s, t > 0.$$

Note that for $t_1, t_2, \cdots, t_n > 0$ with $\sum_{i=1}^n t_i = t$, the following identity holds : $a_t = \sum_{i=1}^n y^i$, where $y^i = u_{t_1} \otimes u_{t_2} \otimes \cdots \otimes u_{t_{i-1}} \otimes a_{t_i} \otimes u_{t_{i+1}} \otimes \cdots \otimes u_{t_n}$. Also note that y^i and y^j are orthogonal for $i \neq j$. Hence $\|a_t\|^2 = \sum_{i=1}^n \|y^i\|^2$. Considering the symmetric Fock product system $\Gamma_{sym}(L^2[0,t], K)$, it is shown in Proposition 12, [4] that the roots of the vacuum unit are given by $c\chi_{[0,t]}$, $c \in K$. Note that the vacuum unit and $c\chi_{[0,t]}$, $c \in K$ generates the Fock product system and as a consequence we have the following result.

Proposition 3.2 (Corollary 15, [4])**.** *Suppose $\mathcal{E}$ is a spatial product system and u is a unit. The product system generated by the unit u and all roots of u is $\mathcal{E}^I$.*

Now we recall the notion of cluster system of a product system introduced in [4]. Suppose $(\mathcal{E}, B)$ is a product system and $(\mathcal{F}, B|_{\mathcal{F}})$ is a product subsystem. Define $\tilde{\mathcal{F}}_t$ by

$$\tilde{\mathcal{F}}_t = \overline{\text{span}}\{x \otimes y : x \in \mathcal{E}_r \ominus \mathcal{F}_r, y \in \mathcal{E}_{t-r} \ominus \mathcal{F}_{t-r}, \text{ for some } r, 0 < r < t\}.$$

Set $\mathcal{F}'_t = \mathcal{E}_t \ominus \tilde{\mathcal{F}}_t$. Denote by $\check{\mathcal{F}}$, the product subsystem generated by $\mathcal{F}'$. We call $\check{\mathcal{F}}$ the cluster of $\mathcal{F}$. The name 'cluster' comes from its connection to random sets ([6]) which we now describe briefly. Suppose $\mathcal{G}$ is a product subsystem of a product system $\mathcal{E}$. Then for every interval $[s,t]$, $0 < s < t < 1$, we may identify, $\mathcal{E}_1 \simeq \mathcal{E}_s \otimes \mathcal{E}_{t-s} \otimes \mathcal{E}_{1-t}$. Let $P^{\mathcal{G}}_{s,t}$, $0 \leq s \leq t \leq 1$, be the family of commuting projections in $B(\mathcal{E}_1)$ defined by

$$P^{\mathcal{G}}_{s,t} = P_{\mathcal{E}_s \otimes \mathcal{G}_{t-s} \otimes \mathcal{E}_{1-t}} = 1_{\mathcal{E}_s} \otimes P_{\mathcal{G}_{t-s}} \otimes 1_{\mathcal{E}_{1-t}},$$

where P_K denotes the projection onto the subspace K. Theorem 3.16, [6] shows that any product subsystem $\mathcal{G}$ corresponds to a unique measure type $[\mu_\eta]$ (η is a faithful state on $B(\mathcal{E}_1)$) on the closed subsets of $[0,1]$ such that the prescription

$$\chi_{\{Z : Z \cap [s,t] = \emptyset\}} \to P^{\mathcal{G}}_{s,t}, \ ((s,t) \in [0,1])$$

extends to an injective normal representation $J^{\mathcal{G}}_\eta$ of $L^\infty(\mu^{\mathcal{G}}_\eta)$ on $\mathcal{E}_1$. The mapping 'cluster' which sends a closed set to its limit points is a measurable map on this space. More precisely, for any $Z \subset [0,1]$, denote $\check{Z}$ the set of its cluster points:

$$\check{Z} = \{t \in Z : t \in \overline{Z \setminus \{t\}}\}.$$

Then from Theorem 27, [4], we have

$$J^{\mathcal{G}}_\eta(\chi_{\{Z : \check{Z} \cap [s,t] = \emptyset\}}) = P^{\check{\mathcal{G}}}_{s,t}, \ ((s,t) \in [0,1]).$$

4. Subsystems of tensor product and their relation to cluster systems

Our aim is to prove the following theorem.

Theorem 4.1. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two product systems and also suppose $\mathcal{M}$ and $\mathcal{N}$ are product subsystems of $\mathcal{E}$ and $\mathcal{F}$ respectively. Then within $\mathcal{E} \otimes \mathcal{F}$,*

$$\mathcal{E} \otimes \mathcal{N} \bigvee \mathcal{M} \otimes \mathcal{F} = \mathcal{E} \otimes \check{\mathcal{N}} \bigvee \check{\mathcal{M}} \otimes \mathcal{F}.$$

The proof we postpone to the very end, after having illustrated the immediate consequences.

Let us define inductively $\mathcal{M}^{n+1} = \check{\mathcal{M}}^n$, where $\mathcal{M}^1 = \check{\mathcal{M}}$. Denote by $\mathcal{M}^\infty = \bigvee_n \mathcal{M}^n$. Similarly for the subsystem $\mathcal{N}$. Then we have the following corollary.

Corollary 4.2. *If $\mathcal{M}^\infty = \mathcal{E}$ or $\mathcal{N}^\infty = \mathcal{F}$, then*

$$\mathcal{E} \otimes \mathcal{N} \bigvee \mathcal{M} \otimes \mathcal{F} = \mathcal{E} \otimes \mathcal{F}.$$

For product system of Hilbert spaces, the spatial product [11] can be defined as a subsystem of the tensor product in the following way. See also [2].

Definition 4.3. Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units u and v respectively. Then their spatial product is defined as

$$\mathcal{E}\ {}_u\otimes_v\ \mathcal{F} := \mathcal{E} \otimes v \bigvee u \otimes \mathcal{F} \subset \mathcal{E} \otimes \mathcal{F}.$$

The following corollary is the main result of [2].

Corollary 4.4. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems with normalized units u and v respectively. Then*

$$\mathcal{E}\ {}_u\otimes_v\ \mathcal{F} = (\mathcal{E} \otimes \mathcal{F}^I) \bigvee (\mathcal{E}^I \otimes \mathcal{F})$$

Proof. It is enough to show $\mathcal{E}^I \subset \check{u}$. For any root a of u, it is easy to see that $a \in \check{u}$. Now the result follows from Proposition 3.2. □

For each $t \in \mathbb{R}_+$, we set

$$J_t = \{(t_1, t_2, \ldots, t_n) : t_i > 0, \sum_{i=1}^{n} t_i = t, n \geq 1\}.$$

For $\mathbf{s} = (s_1, s_2, \ldots, s_m) \in J_s$, and $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in J_t$ we define $\mathbf{s} \smile \mathbf{t} := (s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_n) \in J_{s+t}$. On J_t, define a partial order $\mathbf{t} \geq \mathbf{s} = (s_1, s_2, \ldots, s_m)$ if for each i, $(1 \leq i \leq m)$ there exists (unique) $\mathbf{s}_i \in J_{s_i}$ such that $\mathbf{t} = \mathbf{s}_1 \smile \mathbf{s}_2 \smile \cdots \smile \mathbf{s}_m$. The order relation $\geq$ makes J_t a directed set, i.e. given $\mathbf{s}, \mathbf{t} \in J_t$, there exists $\mathbf{r} \in J_t$ such that $\mathbf{r} \geq \mathbf{s}, \mathbf{t}$.

The key of the proof of our main theorem is the following lemma.

Lemma 4.5. *Suppose $(\mathcal{E}, W)$ is a product system and $\mathcal{F}$ is a product subsystem of $(\mathcal{E}, W)$. Set $\mathcal{X}_t = \mathcal{F}'_t \ominus \mathcal{F}_t$, $t > 0$. Then*

$$\mathcal{F}_s \otimes \mathcal{X}_t \oplus \mathcal{X}_s \otimes \mathcal{F}_t = \mathcal{X}_{s+t}.$$

Proof. For $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in J_t$, we denote $\mathcal{A}_{\mathbf{t}} = \bigoplus_{i=1}^{n} \mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_i}^{\perp} \otimes \cdots \otimes \mathcal{F}_{t_n}$.

Claim 1: $\mathcal{X}_t = \bigcap_{\mathbf{t} \in J_t} \mathcal{A}_{\mathbf{t}}$.

Proof of claim 1: Suppose $z \in \mathcal{E}_r \ominus \mathcal{F}_r$ and $w \in \mathcal{E}_{t-r} \ominus \mathcal{F}_{t-r}$ for some $0 < r < t$. Then $z \otimes w \in \mathcal{A}_{\mathbf{t}}^{\perp}$ where $\mathbf{t} = (r, t-r) \in J_t$. Therefore $\tilde{\mathcal{F}}_t$ is orthogonal to $\bigcap_{\mathbf{t} \in J_t} \mathcal{A}_{\mathbf{t}}$. We get the right hand side is a subspace of $\mathcal{F}_t'$. Also note that for each $\mathbf{t} \in J_t$, $\mathcal{A}_{\mathbf{t}}$ is orthogonal to $\mathcal{F}_t$. Hence $\mathcal{X}_t \supset \bigcap_{\mathbf{t} \in J_t} \mathcal{A}_{\mathbf{t}}$. Now for the reverse inclusion, note that $\mathcal{X}_t = \bigcap_{0<r<t} \mathcal{B}_r$, where $\mathcal{B}_r = \mathcal{F}_r \otimes \mathcal{F}_{t-r}^{\perp} \oplus \mathcal{F}_r^{\perp} \oplus \mathcal{F}_{t-r}$. For any $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in J_t$, with $n \geq 2$, set $r_1 = t_1, r_2 = t_1 + t_2, \cdots, r_{n-1} = t_1 + t_2 + \cdots + t_{n-1}$. Then the claim will follow if we show that $\bigcap_{i=1}^{n-1} \mathcal{B}_{r_i} = \mathcal{A}_{\mathbf{t}}$. To show that we will use induction on n. For $n = 2$, the result is obvious. Assume the result is true for $n = k \geq 2$. Suppose $\mathbf{t} = (t_1, t_2, \cdots, t_{k+1}) \in J_t$ be an arbitrary element. Set $\mathbf{t}' = (t_1, t_2, \cdots, t_{k+1})$ be an arbitrary element. Then $\mathbf{t}' \in J_t$. By the induction hypothesis, $\bigcap_{i=1}^{k-1} \mathcal{B}_{r_i} = \mathcal{A}_{\mathbf{t}'}$. Then

$$
\begin{aligned}
\bigcap_{i=1}^{k} \mathcal{B}_{r_i} &= \mathcal{A}_{\mathbf{t}'} \cap \mathcal{B}_{r_k} \\
&= [\bigoplus_{i=1}^{k-1} \mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_i}^{\perp} \otimes \cdots \otimes \mathcal{F}_{t_k+t_{k+1}} \bigoplus \mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_{k-1}} \otimes \mathcal{F}_{t_k+t_{k+1}}^{\perp}] \\
&\quad \bigcap [\mathcal{F}_{t_1+t_2+\cdots+t_k} \otimes \mathcal{F}_{t_{k+1}}^{\perp} \bigoplus \mathcal{F}_{t_1+t_2+\cdots+t_k}^{\perp} \otimes \mathcal{F}_{t_{k+1}}].
\end{aligned}
$$

Decomposing further $\mathcal{F}_{t_k+t_{k+1}}^{\perp}$ and $\mathcal{F}_{t_1+t_2+\cdots+t_k}^{\perp}$, we see that only terms which will survive is of the form $\mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_i}^{\perp} \otimes \cdots \otimes \mathcal{F}_{t_{k+1}}$, $1 \leq i \leq k+1$. Hence $\bigcap_{i=1}^{k} \mathcal{B}_{r_i} = \bigoplus_{i=1}^{k+1} \mathcal{F}_{t_1} \otimes \cdots \otimes \mathcal{F}_{t_i}^{\perp} \otimes \cdots \otimes \mathcal{F}_{t_{k+1}} = \mathcal{A}_{\mathbf{t}}$.

Claim 2 : For $\mathbf{s} \leq \mathbf{t}$, $\mathcal{A}_{\mathbf{s}} \supset \mathcal{A}_{\mathbf{t}}$.

Proof of claim 2: For $\mathbf{s} = (s_1, s_2, \cdots, s_n)$, $\mathbf{t} = \mathbf{s}_1 \smile \cdots \smile \mathbf{s}_n$. We note that $\mathcal{A}_{\mathbf{t}} = \bigoplus_{i=1}^{n} \mathcal{F}_{s_1} \otimes \cdots \otimes \mathcal{A}_{\mathbf{s}_i} \otimes \cdots \otimes \mathcal{F}_{s_n}$ and $\mathcal{A}_{\mathbf{s}} = \bigoplus_{i=1}^{n} \mathcal{F}_{s_1} \otimes \cdots \otimes \mathcal{F}_{s_i}^{\perp} \otimes \cdots \otimes \mathcal{F}_{s_n}$. As $\mathcal{F}_{s_i}^{\perp} \supset \mathcal{A}_{\mathbf{s}_i}$ for all $i = 1, 2, \cdots, n$, this proves the claim.

It follows that $\{\mathcal{A}_{\mathbf{t}} : \mathbf{t} \in J_t\}$ forms an inverse system under inclusion maps and we have $\mathcal{X}_t = \varprojlim_{J_t} \mathcal{A}_{\mathbf{t}}$.

For $\mathbf{s} \in J_s$, $\mathbf{t} \in J_t$, we observe that $\mathcal{F}_s \otimes \mathcal{A}_{\mathbf{t}} \;\oplus\; \mathcal{A}_{\mathbf{s}} \otimes \mathcal{F}_t = \mathcal{A}_{\mathbf{s} \smile \mathbf{t}}$. As inverse limit passes through taking tensor product and direct sums with other subspaces we get,

$$\begin{aligned}
\mathcal{F}_s \otimes \mathcal{X}_t \oplus \mathcal{X}_s \otimes \mathcal{F}_t &= \varprojlim_{J_t} \mathcal{F}_s \otimes A_{\mathbf{t}} \;\oplus\; \varprojlim_{J_s} \mathcal{A}_{\mathbf{s}} \otimes \mathcal{F}_t \\
&= \varprojlim_{J_s \smile J_t} \; \mathcal{F}_s \otimes \mathcal{A}_{\mathbf{t}} \;\oplus\; \mathcal{A}_{\mathbf{s}} \otimes \mathcal{F}_t \\
&= \varprojlim_{J_s \smile J_t} \; \mathcal{A}_{\mathbf{s}\smile\mathbf{t}} \\
&= \mathcal{X}_{s+t}. \qquad (4.1)
\end{aligned}$$

In the last step of Eq. (4.1), we use that $J_s \smile J_t := \{\mathbf{s} \smile \mathbf{t} : \mathbf{s} \in J_s, \mathbf{t} \in J_t\}$ is a cofinal subset of J_{s+t}, i.e. given $\mathbf{r} \in J_{s+t}$, there are $\mathbf{s} \in J_s$ and $\mathbf{t} \in J_t$ such that $\mathbf{s} \smile \mathbf{t} \geq \mathbf{r}$.

□

Proof of Theorem 4.1 : It is enough to prove $\mathcal{E} \otimes \check{\mathcal{N}} \subset \mathcal{E} \otimes \mathcal{N} \bigvee \mathcal{M} \otimes \mathcal{F}$. By symmetry, the result follows. Fix the time point $t = 1$. It is enough to show that for $z \in \mathcal{E}_1$, and for $\eta \in \mathcal{Y}_1 := \mathcal{N}_1' \ominus \mathcal{N}_1$, $z \otimes \eta \in ((\mathcal{E} \otimes \mathcal{N}) \bigvee (\mathcal{M} \otimes \mathcal{F}))_1$. For other time point, proof goes identically. From Proposition 3.18, [6], we know that $(s,t) \to P^{\mathcal{M}}_{s,t}$ is jointly SOT continuous and the following holds : for every $x \in \mathcal{E}_1$, $\|P^{\mathcal{M}}_{s,s+\epsilon}x - x\| \to 0$ and $\|P^{\mathcal{M}}_{t-\epsilon,t}x - x\| \to 0$ as $\epsilon \downarrow 0$. So in the compact simplex $\{0 \leq s \leq t \leq 1\}$, it is uniformly continuous. i.e. for every $x \in \mathcal{E}_1$, $\|P^{\mathcal{M}}_{s,t}x - x\| \to 0$ as $(t-s) \to 0$. For $n \geq 1$, we have $P^{\mathcal{M}}_{\frac{(i-1)}{n},\frac{i}{n}} = 1_{\mathcal{E}_{\frac{1}{n}}} \otimes \cdots \otimes 1_{\mathcal{E}_{\frac{1}{n}}} \otimes P_{\mathcal{M}_{\frac{1}{n}}} \otimes 1_{\mathcal{E}_{\frac{1}{n}}} \otimes \cdots \otimes 1_{\mathcal{E}_{\frac{1}{n}}}$, where $P_{\mathcal{M}_{\frac{1}{n}}}$ is on the i-th place. Let $\epsilon > 0$ be given. Choose N such that for all $n \geq N$, $\|z - P^{\mathcal{M}}_{\frac{i-1}{n},\frac{i}{n}} z\| < \frac{\epsilon}{\|\eta\|}$, for every $i = 1, 2, \cdots, n$. From Lemma 4.5, the following decomposition holds : $\mathcal{Y}_1 = \oplus_{i=1}^n \mathcal{Z}_i$, where $\mathcal{Z}_i = \mathcal{N}_{\frac{1}{n}} \otimes \mathcal{N}_{\frac{1}{n}} \otimes \cdots \otimes \mathcal{Y}_{\frac{1}{n}} \otimes \cdots \otimes \mathcal{N}_{\frac{1}{n}}$ with $\mathcal{Y}_{\frac{1}{n}}$ is on the i-th place. Let $\eta = \oplus_i \eta_i$ be the corresponding (orthogonal) decomposition. Note that η_i is in the closed linear span of elementary tensors of the form $P = p^1 \otimes p^2 \otimes \cdots \otimes q \otimes \cdots \otimes p^n$ with $p^j \in N_{\frac{1}{n}}$ for $j \neq i$ and $q \in \mathcal{Y}_{\frac{1}{n}}$. Also $P^{\mathcal{M}}_{\frac{i-1}{n},\frac{i}{n}} z$ is in the closed linear span of elementary tensors of the form $R = w^1 \otimes w^2 \otimes \cdots \otimes v \otimes \cdots \otimes w^n$ with $w^j \in \mathcal{E}_{\frac{1}{n}}$ for $j \neq i$ and $v \in \mathcal{M}_{\frac{1}{n}}$. Now note that

$$\begin{aligned}
R \otimes P &= (w^1 \otimes \cdots \otimes v \otimes \cdots \otimes w^n) \otimes (p^1 \otimes \cdots \otimes q \otimes \cdots \otimes p^n) \\
&= (w^1 \otimes p^1) \otimes \cdots \otimes (v \otimes q) \otimes \cdots \otimes (w^n \otimes p^n) \\
&\in ((\mathcal{E} \otimes \mathcal{N}) \bigvee (\mathcal{M} \otimes \mathcal{F}))_1.
\end{aligned}$$

This gives us $P^{\mathcal{M}}_{\frac{i-1}{n},\frac{i}{n}} z \otimes \eta_i \in ((\mathcal{E} \otimes \mathcal{N}) \bigvee (\mathcal{M} \otimes \mathcal{F}))_1$. Now

$$\begin{aligned}
\|z \otimes \eta - \sum_{i=1}^n P^{\mathcal{M}}_{\frac{i-1}{n},\frac{i}{n}} z \otimes \eta_i\|^2 &= \sum_{i=1}^n \|(z - P^{\mathcal{M}}_{\frac{i-1}{n},\frac{i}{n}} z) \otimes \eta_i\|^2 \\
&< \sum_{i=1}^n \frac{\epsilon^2 \|\eta_i\|^2}{\|\eta\|^2} \\
&< \epsilon^2.
\end{aligned}$$

The result follows as the subspace is closed. □

Acknowledgement. I thank Professor B.V. Rajarama Bhat for several useful discussions on the subject. I thank the anonymous referee for valuable suggestions to improve the general appearance of the paper. I also thank DST-Inspire (IFA-13 MA-20) for financial support.

References

1. W. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc. **80** (1989), no. 409.
2. B.V.R. Bhat, V. Liebscher, M. Mukherjee and M. Skeide, *The spatial product of Arveson systems is intrinsic*, J. Funct. Anal. **260** (2011), no. 2, 566–573.
3. B.V.R. Bhat, V. Liebscher and M. Skeide, *A problem of powers and the product of spatial product systems*, Quantum probability and related topics, QP–PQ: Quantum Probab. White Noise Anal., vol. 23, World Sci. Publ., Hackensack, NJ, 2008, 93–106.
4. B.V.R. Bhat, M. Lindsay and M. Mukherjee, *Additive units of product system*, arxiv:1501.07675v1.
5. B.V.R. Bhat and M. Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), no. 4, 519–575.
6. V. Liebscher, *Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces*, Mem. Amer. Math. Soc. **199** (2009), no. 930.
7. ______, *The relation of spatial and tensor product of Arveson systems –the random set point of view*, arXiv: 1409.2801v1.
8. P.S. Muhly and B. Solel, *Quantum Markov processes (correspondences and dilations)*, Internat. J. Math. **13** (2002), no. 8, 863–906.
9. R.T. Powers, *Addition of spatial E_0-semigroups*, Operator algebras, quantization, and noncommutative geometry, Contemp. Math., vol. 365, Amer. Math. Soc., Providence, RI, 2004, pp. 281–298.
10. *Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules*, Advances in quantum dynamics (South Hadley, MA, 2002), Contemp. Math., vol. 335, Amer. Math. Soc., Providence, RI, 2003, pp. 253–262.
11. M. Skeide, *The index of (white) noises and their product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), no. 4, 617–655.
12. B. Tsirelson, *On automorphisms of type II Arveson systems (probabilistic approach)*, New York J. Math. **14** (2008), 539–576.

Department of mathematics and statistics, Indian Institute of Science Education and Research Kolkata, Mohanpur-741 246, India.

E-mail address: mithun.mukherjee@iiserkol.ac.in