Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 3, 191–202
http://doi.org/10.15352/afa/06-3-16
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

MATRIX INEQUALITIES FOR THE DIFFERENCE BETWEEN ARITHMETIC MEAN AND HARMONIC MEAN

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Communicated by T. Yamazaki

Abstract. Motivated by the refinements and reverses of arithmetic-geometric mean and arithmetic-harmonic mean inequalities for scalars and matrices, in this article, we generalize the scalar and matrix inequalities for the difference between arithmetic mean and harmonic mean. In addition, relevant inequalities for the Hilbert-Schmidt norm and determinant are established.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. I stands for the identity matrix. The Hilbert-Schmidt norm (l_2 norm, Frobenius norm or Schur norm) of $A = [a_{ij}] \in M_n(\mathbb{C})$ is defined by

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \left(\operatorname{Tr} |A|^2 \right)^{1/2} = \left(\sum_{i=1}^{n} s_i^2(A) \right)^{1/2},$$

where Tr is the trace functional, $|A| = (A^*A)^{1/2}$ and $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ denote the singular values of A (i.e., the eigenvalues of positive semi-definite matrix $|A|$) arranged in non-increasing order and repeated according to multiplicity (see [8, p.341-342]). It is well-known that each unitarily invariant norm is a symmetric guage function of singular values [4, p.91], so the Hilbert-Schmidt norm is unitarily invariant.

For $a, b > 0$, $v \in [0,1]$ and $t \in \mathbb{R}$, the power mean

$$M_t(v; a, b) = \left(va^t + (1-v)b^t \right)^{1/t}, t \neq 0$$

Date: Received: Nov. 3, 2014; Revised: Jan. 23, 2015; Accepted: Feb. 12, 2015.
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2010 *Mathematics Subject Classification.* Primary 15A45; Secondary 15A60, 26E60, 47A63.
Key words and phrases. Arithmetic mean, harmonic mean, matrix inequality, positive definite matrix.

makes a path of means from the harmonic mean at $t = -1$ to the arithmetic mean at $t = 1$ via the geometric mean at $t \to 0$ and $M_0(v; a, b) = \lim_{t\to 0} M_t(v; a, b)$. If $s \le t$, then $M_s(v; a, b) \le M_t(v; a, b)$ and the two means are equal if and only if $a = b$ (see [11, p.194-196]). So

$$M_{-1}(v; a, b) \le M_0(v; a, b) \le M_1(v; a, b),$$

that is,

$$\left(va^{-1} + (1-v)b^{-1}\right)^{-1} \le a^v b^{1-v} \le va + (1-v)b. \tag{1.1}$$

Note that it is the classical arithmetic-geometric-harmonic mean inequalities and it's worthwhile to mention that the second one is the Young inequality, for more details about the refinements and reverses of the Young inequality, the reader is referred to [1, 5, 9, 14].

The following is a noncommutative matrices version of the arithmetic-geometric-harmonic mean inequalities which are the important results from [2, 3] (See also [12]): For positive definite matrices $A, B \in M_n(\mathbb{C})$ and $0 \le v \le 1$,

$$\left(vA^{-1} + (1-v)B^{-1}\right)^{-1} \le A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}} \le vA + (1-v)B. \tag{1.2}$$

For convenience, we introduce the following notations to define the weighted arithmetic mean, geometric mean and harmonic mean for scalars and matrices:

$$\begin{aligned} &A_v(a, b) = va + (1-v)b,\ H_v(a, b) = \left(va^{-1} + (1-v)b^{-1}\right)^{-1}, \\ &A\nabla_v B = vA + (1-v)B,\ A\#_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}, \\ &A!_v B = \left(vA^{-1} + (1-v)B^{-1}\right)^{-1}, \end{aligned}$$

where $a, b > 0$, $0 \le v \le 1$ and $A, B \in M_n(\mathbb{C})$ are positive definite matrices. When $v = \frac{1}{2}$, we write $A(a, b)$, $H(a, b)$, $A\nabla B$, $A\#B$ and $A!B$ for brevity, respectively. The above notations and definitions will be valid throughout the whole paper.

It is evident that the full matrix algebra of all $n \times n$ matrices with entries in the complex field is the finite-dimensional case of the C^*-algebra of all bounded linear operators on a complex separable Hilbert space. If one inequality is valid for positive invertible operators, so is valid for positive definite matrices.

Motivated by Furuichi's refinement of the Young inequality for positive invertible operators A, B and $v \in [0, 1]$ (see [5])

$$A\nabla_v B \ge A\#_v B + 2\min\{v, 1-v\}(A\nabla B - A\#B), \tag{1.3}$$

and Kittaneh and Manasrh's reverse Young inequality for two positive definite matrices A, B and $v \in [0, 1]$ (see [9])

$$A\nabla_v B \le A\#_v B + 2\max\{v, 1-v\}(A\nabla B - A\#B), \tag{1.4}$$

Hirzallah et al.[7] generalized the inequalities (1.3) and (1.4): For positive invertible operators A, B and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$, if $A \ge B$, $0 < p_1 \le p_2$ or $A \le B$, $0 < p_2 \le p_1$, then

$$A\nabla_{\frac{p_1}{p_1+p_2}} B \ge A\#_{\frac{p_1}{p_1+p_2}} B + \frac{4p_1p_2}{(p_1+p_2)^2}(A\nabla B - A\#B), \tag{1.5}$$

in addition, if $A \ge B$, $0 < p_2 \le p_1$ or $A \le B$, $0 < p_1 \le p_2$, then the inequality (1.5) is reversed.

Zuo et al.[14] refined the weighted arithmetic-harmonic mean inequality and extended it to two invertible positive operators A, B as follows:

$$A_v(a,b) \geq H_v(a,b) + 2\min\{v, 1-v\}\left(A(a,b) - H(a,b)\right), \tag{1.6}$$
$$A\nabla_v B \geq A!_v B + 2\min\{v, 1-v\}\left(A\nabla B - A!B\right). \tag{1.7}$$

Krnić et al.[10] presented a reverse of the inequality (1.7)

$$A\nabla_v B \leq A!_v B + 2\max\{1-v, v\}(A\nabla B - A!B). \tag{1.8}$$

According to (1.8), the following scalar inequality is valid:

$$A_v(a,b) \leq H_v(a,b) + 2\max\{1-v, v\}\left(A(a,b) - H(a,b)\right). \tag{1.9}$$

Our main task is to improve the scalar and matrix inequalities of the difference between arithmetic mean and harmonic mean. This article is organized in the following way: in Section 2, we derive several new weighted arithmetic-harmonic mean inequalities. In Section 3, we extend these inequalities proved in Section 2 from the scalars setting to a matrix-algebra setting. In Sections 4 and 5, the Hilbert-Schmidt norm and determinant inequalities for positive definite matrices are established.

2. Scalar inequalities

The first theorem is our main result about the difference between arithmetic and harmonic means which generalizes the inequalities (1.6) and (1.9).

Theorem 2.1. *Let v, τ and λ be real numbers with $0 < v < \tau < 1$ and $\lambda \geq 1$. Then*

$$\left(\frac{v}{\tau}\right)^{\lambda} < M_{v,\tau;\lambda}(a,b) < \left(\frac{1-v}{1-\tau}\right)^{\lambda} \tag{2.1}$$

hold for all positive and distinct real numbers a and b, where $M_{v,\tau;\lambda}(a,b) = \frac{A_v(a,b)^\lambda - H_v(a,b)^\lambda}{A_\tau(a,b)^\lambda - H_\tau(a,b)^\lambda}$. Moreover, $\lim\limits_{a\to 0} M_{v,\tau;\lambda}(a,b) = \left(\frac{1-v}{1-\tau}\right)^\lambda$ and $\lim\limits_{a\to\infty} M_{v,\tau;\lambda}(a,b) = \left(\frac{v}{\tau}\right)^\lambda$.

Proof. Let $\lambda \geq 1$, $0 < v < \tau < 1$ and $0 < x \neq 1$. We define

$$F(v;\lambda;x) = \frac{A_v(x,1)^\lambda - H_v(x,1)^\lambda}{v^\lambda} = \left(x + \frac{1}{v} - 1\right)^\lambda - \left(\frac{v^2}{x} + v - v^2\right)^{-\lambda},$$
$$G(v;\lambda;x) = \frac{A_v(x,1)^\lambda - H_v(x,1)^\lambda}{(1-v)^\lambda} = \left(\frac{vx}{1-v} + 1\right)^\lambda - \left(\frac{(1-v)v}{x} + (1-v)^2\right)^{-\lambda}.$$

The function $F_{\lambda,x}(v) = F(v;\lambda;x)$ on $(0,1)$ is differentiable and its partial differentiation at v can be expressed as

$$\begin{aligned}\frac{\partial}{\partial v}F(v;\lambda;x) &= \frac{\lambda}{v^{\lambda+1}H_v(x,1)^{1-\lambda}}\left[H_v(x,1)^2\left(\frac{2v}{x}+1-2v\right)-\left(\frac{A_v(x,1)}{H_v(x,1)}\right)^{\lambda-1}\right]\\ &\le \frac{\lambda}{v^{\lambda+1}H_v(x,1)^{1-\lambda}}\left[H_v(x,1)^2\left(\frac{2v}{x}+1-2v\right)-1\right]\\ &= \frac{\lambda}{v^{\lambda+1}H_v(x,1)^{1-\lambda}}\cdot\frac{-v^2\left(x^{-1}-1\right)^2}{\left(vx^{-1}+(1-v)\right)^2}\\ &< 0.\end{aligned}$$

This implies that $F(v;\lambda;x)$ is strictly decreasing with respect to v. Hence, for $0 < v < \tau < 1$, we obtain

$$\frac{A_\tau(x,1)^\lambda - H_\tau(x,1)^\lambda}{\tau^\lambda} < \frac{A_v(x,1)^\lambda - H_v(x,1)^\lambda}{v^\lambda}. \tag{2.2}$$

The partial differentiation of $G(v;\lambda;x)$ at v yields

$$\begin{aligned}&\frac{\partial}{\partial v}G(v;\lambda;x)\\ &= \frac{\lambda x}{(1-v)^{\lambda+1}H_v(x,1)^{1-\lambda}}\left[\left(\frac{A_v(x,1)}{H_v(x,1)}\right)^{\lambda-1}+H_v(x,1)^2\left(\frac{1-2v}{x^2}+\frac{2v-2}{x}\right)\right]\\ &\ge \frac{\lambda x}{(1-v)^{\lambda+1}H_v(x,1)^{1-\lambda}}\left[1+H_v(x,1)^2\left(\frac{1-2v}{x^2}+\frac{2v-2}{x}\right)\right]\\ &= \frac{\lambda x}{(1-v)^{\lambda+1}H_v(x,1)^{1-\lambda}}\cdot\frac{(1-v)^2\left(x^{-1}-1\right)^2}{\left(vx^{-1}+(1-v)\right)^2}\\ &> 0.\end{aligned}$$

Thus, $G(v;\lambda;x)$ is strictly increasing with respect to v. For $0 < v < \tau < 1$, we obtain

$$\frac{A_\tau(x,1)^\lambda - H_\tau(x,1)^\lambda}{(1-\tau)^\lambda} > \frac{A_v(x,1)^\lambda - H_v(x,1)^\lambda}{(1-v)^\lambda}. \tag{2.3}$$

Next, taking $x = a/b$ in (2.2) and (2.3) and multiplying both sides by b^λ, we obtain (2.1).

Further, we have

$$\begin{aligned}M_{v,\tau;\lambda}(a,b) &= \frac{A_v(a,b)^\lambda - H_v(a,b)^\lambda}{A_\tau(a,b)^\lambda - H_\tau(a,b)^\lambda}\\ &= \frac{(va+(1-v)b)^\lambda - (va^{-1}+(1-v)b^{-1})^{-\lambda}}{(\tau a+(1-\tau)b)^\lambda - (\tau a^{-1}+(1-\tau)b^{-1})^{-\lambda}}\\ &= \frac{(va+(1-v)b)^\lambda - [ab(vb+(1-v)a)^{-1}]^\lambda}{(\tau a+(1-\tau)b)^\lambda - [ab(\tau b+(1-\tau)a)^{-1}]^\lambda},\end{aligned}$$

which satisfies

$$\lim_{a\to 0} M_{v,\tau;\lambda}(a,b) = \frac{((1-v)b)^{\lambda}-0}{((1-\tau)b)^{\lambda}-0} = \left(\frac{1-v}{1-\tau}\right)^{\lambda}, \tag{2.4}$$

and the representation

$$M_{v,\tau;\lambda}(a,b) = \frac{A_v(1,\frac{b}{a})^{\lambda} - H_v(1,\frac{b}{a})^{\lambda}}{A_\tau(1,\frac{b}{a})^{\lambda} - H_\tau(1,\frac{b}{a})^{\lambda}}$$

leads to

$$\lim_{a\to\infty} M_{v,\tau;\lambda}(a,b) = \frac{v^{\lambda}-0}{\tau^{\lambda}-0} = \left(\frac{v}{\tau}\right)^{\lambda}. \tag{2.5}$$

The limit relations (2.4) and (2.5) reveal that the upper and lower bounds given in (2.1) are sharp. □

Remark 2.2. If $\lambda = 1$ and $\tau = 1/2$ in the inequalities (2.1), then

$$2v(A(a,b)-H(a,b)) \le A_v(a,b) - H_v(a,b) \le 2(1-v)(A(a,b)-H(a,b))$$

hold for $0 < v < 1/2$, which are equivalent to (1.6) and (1.9), respectively.

If $\lambda = 2$ and $\tau = 1/2$ in the inequalities (2.1), then

$$\begin{aligned} 4v^2(A(a,b)^2 - H(a,b)^2) &\le A_v(a,b)^2 - H_v(a,b)^2 \\ &\le 4(1-v)^2(A(a,b)^2 - H(a,b)^2) \end{aligned} \tag{2.6}$$

hold for $0 < v < 1/2$.

Next, we establish another type of upper and lower bounds for the difference of $A_v(a,b)$ and $H_v(a,b)$. We need the following lemma (see [1]).

Lemma 2.3. *Let $v \in (0,1)$ and $f : [a,b] \to \mathbb{R}$ be twice differentiable with $-\infty < m \le f''(x) \le M < +\infty$ for all $x \in (a,b)$. Then*

$$\begin{aligned} \frac{v(1-v)}{2}(b-a)^2 m &\le vf(a) + (1-v)f(b) - f(va + (1-v)b) \\ &\le \frac{v(1-v)}{2}(b-a)^2 M. \end{aligned}$$

The factor $v(1-v)$ is the best possible.

Theorem 2.4. *Let $v \in (0,1)$ and $a, b > 0$ with $a < b$. Then we have*

$$v(1-v)\left(1-\frac{a}{b}\right)^2 a \le A_v(a,b) - H_v(a,b) \le v(1-v)\left(1-\frac{b}{a}\right)^2 b. \tag{2.7}$$

Proof. Taking $f(x) = 1/x$ in Lemma 2.3, then we have

$$\begin{aligned} v(1-v)\,(b-a)^2\,\frac{1}{b^3} &\le va^{-1} + (1-v)b^{-1} - (va + (1-v)b)^{-1} \\ &\le v(1-v)\,(b-a)^2\,\frac{1}{a^3}. \end{aligned}$$

Replace a by b^{-1} and b by a^{-1} in the above inequalities, respectively,

$$v(1-v)\left(b^{-1}-a^{-1}\right)^2 a^3 \le vb+(1-v)a-(vb^{-1}+(1-v)a^{-1})^{-1} \le v(1-v)\left(b^{-1}-a^{-1}\right)^2 b^3,$$

then we can obtain the desired result by replacing v by $1-v$.

Setting $t = b/a > 1$, then (2.7) leads to

$$v(1-v) \le g_v(t) \le v(1-v)\frac{1}{t},$$

where

$$g_v(t) = \frac{v+(1-v)t-(v+(1-v)t^{-1})^{-1}}{(1-t)^2}.$$

Since $\lim_{t\to 1} g_v(t) = v(1-v)$, the limit reveals that the factor $v(1-v)$ is sharp. $\square$

3. Matrix inequalities

In this section, we begin with the following lemma which is based on the inequalities (2.1) and the spectral theorem for Hermitian matrices.

Lemma 3.1. *Let $Q \in M_n(\mathbb{C})$ be positive definite and v, τ be real numbers with $0 < v \le \tau < 1$. Then*

$$\frac{v}{\tau}(I\nabla_\tau Q - I!_\tau Q) \le I\nabla_v Q - I!_v Q \le \frac{1-v}{1-\tau}(I\nabla_\tau Q - I!_\tau Q). \tag{3.1}$$

Proof. By the spectral theorem (see [8, Theorem 2.5.6]), there exists a unitary matrix U such that $Q = UDU^*$, where $D = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n)$ with the eigenvalues $\mu_i > 0$, $i = 1, 2, \cdots, n$ of Q. Applying (2.1) with $\lambda = 1$, we have

$$\frac{v}{\tau}(1\nabla_\tau \mu_i - 1!_\tau \mu_i) \le 1\nabla_v \mu_i - 1!_v \mu_i \le \frac{1-v}{1-\tau}(1\nabla_\tau \mu_i - 1!_\tau \mu_i).$$

For diagonal matrix D, the above inequality can be written as

$$\frac{v}{\tau}(I\nabla_\tau D - I!_\tau D) \le I\nabla_v D - I!_v D \le \frac{1-v}{1-\tau}(I\nabla_\tau D - I!_\tau D).$$

Using the fact that any $*$-conjugation preserves the Löewner partial order between Hermitian matrices (see [8, Theorem 7.7.2]), we obtain (3.1) by applying the $*$-conjugation $\bullet \mapsto U \bullet U^*$ to the identity matrix I and the diagonal matrix D. $\square$

The next theorem generalize the inequalities (1.7) and (1.8).

Theorem 3.2. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If v and τ are two real numbers with $0 < v \le \tau < 1$, then*

$$\frac{v}{\tau}(A\nabla_\tau B - A!_\tau B) \le A\nabla_v B - A!_v B \le \frac{1-v}{1-\tau}(A\nabla_\tau B - A!_\tau B). \tag{3.2}$$

Proof. The matrices $A^{-1/2}$ and $A^{1/2}$ are positive definite under the condition that A is positive definite. The result follows from putting $Q = A^{-1/2}BA^{-1/2}$ in Lemma 3.1 and applying the $*$-conjugation $\bullet \mapsto A^{1/2} \bullet A^{1/2}$ to it. $\square$

Corollary 3.3. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If v is a real number with $0 < v \le 1/2$, then*

$$2v(A\nabla B - A!B) \le A\nabla_v B - A!_v B \le 2(1-v)(A\nabla B - A!B). \tag{3.3}$$

Proof. Taking $\tau = 1/2$ in the inequalities (3.2). □

Note that the inequalities (3.3) are equivalent to (1.7) and (1.8), respectively.

Based on Theorem 2.4, we establish another type of upper bound of the difference of arithmetic mean and harmonic mean for positive definite matrices which is only related to one argument.

Theorem 3.4. *Let $A, B \in M_n(\mathbb{C})$ be positive definite with $0 < mI \le A \le B \le MI$. If v is a real number with $0 \le v \le 1$, then*

$$A\nabla_v B - A!_v B \le v(1-v)\left(1 - \frac{M}{m}\right)^2 B. \tag{3.4}$$

Proof. Let $t = b/a > 1$ in the second inequality of (2.7). Then for $0 \le v \le 1$ we have

$$v + (1-v)t - (v + (1-v)t^{-1})^{-1} \le v(1-v)\left(1-t\right)^2 t.$$

Thus we have the following inequality

$$v + (1-v)T - (v + (1-v)T^{-1})^{-1} \le v(1-v) \max_{1 \le t \le M/m} \left(1-t\right)^2 T,$$

for the positive definite matrix T with $I \le T \le M/mI$. Since $I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le M/mI$, putting $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality, we deduce

$$\begin{aligned} v + (1-v)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - \left(v + (1-v)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-1}\right)^{-1} \\ \le v(1-v)\left(1 - \frac{M}{m}\right)^2 A^{-\frac{1}{2}}BA^{-\frac{1}{2}}. \end{aligned}$$

Multiplying the both sides by $A^{\frac{1}{2}}$, we obtain the inequality (3.4). □

4. Hilbert-Schmidt norm inequalities

Based on the refinements and reverses of the Young inequality, Hirzallah and Kittaneh [6] and Kittaneh and Manasrah [9] had shown that if $A, B, X \in M_n(\mathbb{C})$ with two positive definite matrices A and B, then for $v \in [0,1]$,

$$\min\{v^2, (1-v)^2\} \left\|AX - XB\right\|_F^2 \le \left\|vAX + (1-v)XB\right\|_F^2 - \left\|A^v XB^{1-v}\right\|_F^2,$$
$$\max\{v^2, (1-v)^2\} \left\|AX - XB\right\|_F^2 \ge \left\|vAX + (1-v)XB\right\|_F^2 - \left\|A^v XB^{1-v}\right\|_F^2.$$

From the above inequalities, it is evident that the following inequality holds:

$$\left\|vAX + (1-v)XB\right\|_F^2 \ge \left\|A^v XB^{1-v}\right\|_F^2,$$

and the authors of [13] pointed out that if A and B are positive semidefinite and $X \in M_n(\mathbb{C})$, then for unitarily invariant norm, $|||A^v XB^{1-v}||| \le |||vAX + (1-v)XB|||$ does not holds in general.

Inspired by the above Hilbert-Schmidt norm versions of the improved Young inequalities, we derive the following theorem about the difference-type inequalities between arithmetic and harmonic means for the Hilbert-Schmidt norm by the inequalities (2.1) with $\lambda = 2$.

Theorem 4.1. *Let A, B, $X \in M_n(\mathbb{C})$ such that A and B are positive definite. If v and τ are two real numbers with $0 < v \le \tau < 1$, then*

$$\left(\frac{v}{\tau}\right)^2 \le \frac{\|\mathbb{A}_v(A,B;X)\|_F^2 - \|\mathbb{H}_v(A,B;X)\|_F^2}{\|\mathbb{A}_\tau(A,B;X)\|_F^2 - \|\mathbb{H}_\tau(A,B;X)\|_F^2} \le \left(\frac{1-v}{1-\tau}\right)^2, \tag{4.1}$$

where

$$\mathbb{A}_v(A,B;X) = vAX + (1-v)XB,$$
$$\mathbb{H}_v(A,B;X) = \left[vX^{-1}A^{-1} + (1-v)B^{-1}X^{-1}\right]^{-1}.$$

Proof. Since A and B are positive definite, it follows from the spectral theorem that there exists unitary matrices $U, V \in M_n(\mathbb{C})$ such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where $\Lambda_1 = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n)$, $\Lambda_2 = \mathrm{diag}(\nu_1, \nu_2, \cdots, \nu_n)$ and $\mu_i, \nu_i \ge 0$, $i = 1, 2, \cdots, n$.

Let $Y = U^*XV = [y_{ij}]$, then

$$\begin{aligned} vAX + (1-v)XB &= U(v\Lambda_1 Y + (1-v)Y\Lambda_2)V^* \\ &= U\left[(v\mu_i + (1-v)\nu_j)y_{ij}\right]V^*, \\ \left[vX^{-1}A^{-1} + (1-v)B^{-1}X^{-1}\right]^{-1} &= U\left(vY^{-1}\Lambda_1^{-1} + (1-v)\Lambda_2^{-1}Y^{-1}\right)V^* \\ &= U\left[\left(v\mu_i^{-1} + (1-v)\nu_j^{-1}\right)^{-1} y_{ij}\right]V^*. \end{aligned}$$

Now by using the first inequality in (2.1) with $\lambda = 2$ and the unitary invariance of the Hilbert-Schmidt norm, we have

$$\begin{aligned} &\|vAX + (1-v)XB\|_F^2 - \left\|\left[vX^{-1}A^{-1} + (1-v)B^{-1}X^{-1}\right]^{-1}\right\|_F^2 \\ &= \sum_{i,j=1}^n (v\lambda_i + (1-v)\nu_j)^2|y_{ij}|^2 - \sum_{i,j=1}^n \left(v\lambda_i^{-1} + (1-v)\nu_j^{-1}\right)^{-2}|y_{ij}|^2 \\ &= \sum_{i,j=1}^n \left[(v\lambda_i + (1-v)\nu_j)^2 - \left(v\lambda_i^{-1} + (1-v)\nu_j^{-1}\right)^{-2}\right]|y_{ij}|^2 \\ &\ge \left(\frac{v}{\tau}\right)^2 \sum_{i,j=1}^n \left[(\tau\lambda_i + (1-\tau)\nu_j)^2 - \left(\tau\lambda_i^{-1} + (1-\tau)\nu_j^{-1}\right)^{-2}\right]|y_{ij}|^2 \\ &= \left(\frac{v}{\tau}\right)^2 \left[\sum_{i,j=1}^n (\tau\lambda_i + (1-\tau)\nu_j)^2|y_{ij}|^2 - \sum_{i,j=1}^n \left(\tau\lambda_i^{-1} + (1-\tau)\nu_j^{-1}\right)^{-2}|y_{ij}|^2\right] \\ &= \left(\frac{v}{\tau}\right)^2 \left[\|\tau AX + (1-\tau)XB\|_F^2 - \left\|\left(\tau X^{-1}A^{-1} + (1-\tau)B^{-1}X^{-1}\right)^{-1}\right\|_F^2\right], \end{aligned}$$

which proves the first inequality in (4.1).

The proof of the second inequality in (4.1) can be completed by a similar argument. □

As direct consequences of Theorem 4.1, we have the next two corollarys.

Corollary 4.2. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive definite. If v is a real number with $0 \leq v \leq 1$, then*

$$\begin{aligned} \|vAX + (1-v)XB\|_F^2 &\geq \left\|A^v X B^{1-v}\right\|_F^2 \\ &\geq \left\|\left[vX^{-1}A^{-1} + (1-v)B^{-1}X^{-1}\right]^{-1}\right\|_F^2. \end{aligned} \tag{4.2}$$

Proof. The inequalities (4.2) follow from the proof of Theorem 4.1 by the inequalities (1.1). □

Corollary 4.3. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive definite. If v is a real number with $0 \leq v \leq 1/2$, then*

$$\begin{aligned} &4v^2\left[\left\|\frac{AX+XB}{2}\right\|_F^2 - \left\|\left(\frac{X^{-1}A^{-1}+B^{-1}X^{-1}}{2}\right)^{-1}\right\|_F^2\right] \\ &\leq \|vAX + (1-v)XB\|_F^2 - \left\|\left[vX^{-1}A^{-1} + (1-v)B^{-1}X^{-1}\right]^{-1}\right\|_F^2 \\ &\leq 4(1-v)^2\left[\left\|\frac{AX+XB}{2}\right\|_F^2 - \left\|\left(\frac{X^{-1}A^{-1}+B^{-1}X^{-1}}{2}\right)^{-1}\right\|_F^2\right]. \end{aligned} \tag{4.3}$$

Note that (4.2) can be regarded as the arithmetic-geometric-harmonic mean inequalities for the Hilbert-Schmidt norm. The inequalities (4.3) present the Hilbert-Schmidt norm version of (2.6) which contain the refinement and reverse of (4.2).

5. Determinant inequalities

In this section, the singular values of A are denoted by $s_j(A)$, $j = 1, 2, \cdots, n$ and we adhere to the convention that singular values are sorted in non-increasing order. $\det(A)$ denotes the determinant of A.

Obviously, by (1.2), for positive definite matrices $A, B \in M_n(\mathbb{C})$ and $0 \leq v \leq 1$,

$$A!_v B \leq A\nabla_v B.$$

So we have the following proposition.

Proposition 5.1. *Let A, $B \in M_n(\mathbb{C})$ be positive definite matrices, $0 \leq v \leq 1$ and $\lambda \geq 1$. Then*

$$\det\left(A!_v B\right)^\lambda \leq \det\left(A\nabla_v B\right)^\lambda. \tag{5.1}$$

Proof. Putting the positive definite matrix $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, by arithmetic-harmonic mean inequality, then we have

$$(v + (1-v)s_i(T))^{\lambda} \geq \left(v + (1-v)s_i(T)^{-1}\right)^{-\lambda}$$

for all $i = 1, 2, \cdots, n$.

$$\begin{aligned}\det(v + (1-v)T)^{\lambda} &= \prod_{i=1}^{n}(v + (1-v)s_i(T))^{\lambda} \\ &\geq \prod_{i=1}^{n}\left(v + (1-v)s_i(T)^{-1}\right)^{-\lambda} \\ &= \det(v + (1-v)T^{-1})^{-\lambda}.\end{aligned}$$

Multiplying the both sides by $\left(\det A^{1/2}\right)^{\lambda}$, we deduce the result by the multiplicativity of the determinant. □

Next, we will improve the inequality (5.1), the following two lemmas should be mentioned.

Lemma 5.2. (Minkowski's product inequality [8, p.560]) *Let $a = [a_i], b = [b_i], i = 1, 2, \cdots, n$ such that a_i and b_i are positive real numbers. Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n}(a_i + b_i)\right)^{\frac{1}{n}}.$$

Equality holds if and only if a=b.

Lemma 5.3. *Let a and b be positive real numbers with $a > b$. If $\lambda \geq 1$, then*

$$a^{\lambda} - b^{\lambda} \geq (a-b)^{\lambda}.$$

Theorem 5.4. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If v, τ and λ are real numbers with $0 < v \leq \tau < 1$ and $\lambda \geq 1$, then*

$$\left(\frac{v}{\tau}\right)^{\lambda} \det\left(A\nabla_{\tau}B - A!_{\tau}B\right)^{\frac{\lambda}{n}} \leq \det\left(A\nabla_{v}B\right)^{\frac{\lambda}{n}} - \det\left(A!_{v}B\right)^{\frac{\lambda}{n}}. \tag{5.2}$$

Proof. By the first inequality of (2.1) and taking the positive definite matrix $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$,

$$\left(\frac{v}{\tau}\right)^{\lambda} \leq \frac{(v + (1-v)s_i(T))^{\lambda} - \left(v + (1-v)s_i(T)^{-1}\right)^{-\lambda}}{(\tau + (1-\tau)s_i(T))^{\lambda} - \left(\tau + (1-\tau)s_i(T)^{-1}\right)^{-\lambda}}$$

holds for all $s_i(T) \neq 1$ $(i = 1, 2, \cdots, n)$. Since the determinant of a positive definite matrix is the product of its singular values. Thus

$$\begin{aligned}
&\det(vI + (1-v)T)^{\frac{\lambda}{n}} \\
&\quad = \left(\det(vI + (1-v)T)^{\lambda}\right)^{\frac{1}{n}} \\
&\quad = \left(\prod_{i=1}^{n}(vI + (1-v)s_i(T))^{\lambda}\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} A_v(1, s_i(T))^{\lambda}\right)^{\frac{1}{n}} \\
&\quad \geq \left(\prod_{i=1}^{n}\left[\left(\frac{v}{\tau}\right)^{\lambda}\left(A_\tau(1, s_i(T))^{\lambda} - H_\tau(1, s_i(T))^{\lambda}\right) + H_v(1, s_i(T))^{\lambda}\right]\right)^{\frac{1}{n}} \\
&\quad \geq \left(\frac{v}{\tau}\right)^{\lambda}\prod_{i=1}^{n}\left[A_\tau(1, s_i(T))^{\lambda} - H_\tau(1, s_i(T))^{\lambda}\right]^{\frac{1}{n}} + \prod_{i=1}^{n}\left[H_v(1, s_i(T))^{\lambda}\right]^{\frac{1}{n}} \\
&\quad \geq \left(\frac{v}{\tau}\right)^{\lambda}\prod_{i=1}^{n}\left[(A_\tau(1, s_i(T)) - H_\tau(1, s_i(T)))^{\lambda}\right]^{\frac{1}{n}} + \prod_{i=1}^{n}\left[H_v(1, s_i(T))^{\lambda}\right]^{\frac{1}{n}} \\
&\quad = \left(\frac{v}{\tau}\right)^{\lambda}\det\left[(I\nabla_\tau T - I!_\tau T)^{\lambda}\right]^{\frac{1}{n}} + \left[\det(I!_v T)^{\lambda}\right]^{\frac{1}{n}}.
\end{aligned}$$

The second inequality is obtained by Minkowski's product inequality and the last inequality is obtained by Lemma 5.3. The inequality (5.2) follows from multiplying the both sides by $\left(\det A^{1/2}\right)^{\lambda/n}$. □

Remark 5.5. If $\lambda = 1$ in the inequality (5.2), then

$$\frac{v}{\tau}\det(A\nabla_\tau B - A!_\tau B)^{\frac{1}{n}} \leq \det(A\nabla_v B)^{\frac{1}{n}} - \det(A!_v B)^{\frac{1}{n}}.$$

If $\lambda = 1$ and $\tau = 1/2$ in the inequality (5.2), then

$$2v\det(A\nabla B - A!B)^{\frac{1}{n}} \leq \det(A\nabla_v B)^{\frac{1}{n}} - \det(A!_v B)^{\frac{1}{n}}.$$

Corollary 5.6. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If v and τ are two real numbers with $0 < v \leq \tau < 1$, then*

$$\det A!_v B + \left(\frac{v}{\tau}\right)^{n}\det(A\nabla_\tau B - A!_\tau B) \leq \det A\nabla_v B. \tag{5.3}$$

Proof. The proof is similar to that of Theorem 5.4. □

Corollary 5.7. *Let $A, B \in M_n(\mathbb{C})$ be positive definite. If v is a real number with $0 \leq v \leq 1/2$, then*

$$\det A!_v B + (2v)^{n}\det(A\nabla B - A!B) \leq \det A\nabla_v B. \tag{5.4}$$

Note that the inequality (5.4) is the determinant version of the inequality (1.7). The inequalities (5.2) and (5.3) can be treated as two generalizations of (5.4).

Acknowledgement. The authors wish to thank the editor and anonymous referees for the careful reading of the manuscript and useful comments.

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