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ON GENERALIZED BECKNER'S INEQUALITY

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Abstract. In this paper, we present generalizations of Beckner's inequality by using symmetric absolute normalized norms on $\mathbb{R}^2$.

1. Introduction

The aim of this paper is to present generalizations of the following inequality.

Theorem. *Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality*

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for all $u, v \in \mathbb{R}$.

This was shown in 1975 by Beckner [1]. It is known that $\gamma_{p,q}$ in Theorem 1 is the best constant for the inequality, that is, if $\gamma \in [0,1]$ and the inequality

$$\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for all $u, v \in \mathbb{R}$, then we have $\gamma \leq \gamma_{p,q}$. In [9], we constructed an elementary proof of these facts.

To generalize Beckner's inequality, we make use of symmetric absolute normalized norms on $\mathbb{R}^2$. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, normalized if $\|(1,0)\| = \|(0,1)\| = 1$, and symmetric if $\|(x,y)\| = \|(y,x)\|$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by

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AN_2. Bonsall and Duncan [3] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for all $t \in [0,1]$ (cf. [6]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for all $t \in [0,1]$. Note that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

For some other results concerning absolute normalized norms, we refer the readers to [5, 6, 7, 8]. We remark that the norm $\|\cdot\| \in AN_2$ is symmetric if and only if $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. For example, the function ψ_p corresponding to the ℓ_p-norm $\|\cdot\|_p$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t,t\} & \text{if } p = \infty, \end{cases}$$

and satisfies $\psi_p(t) = \psi_p(1-t)$ for all $t \in [0,1]$. Let $\Psi_2^S = \{\psi \in \Psi_2 : \psi(t) = \psi(1-t) \text{ for all } t \in [0,1]\}$.

Using the functions ψ_p and ψ_q, Beckner's inequality can be viewed as follows: Let $1 < p \le q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\frac{\|(u+\gamma_{p,q}v, u-\gamma_{p,q}v)\|_q}{2\psi_q(\frac{1}{2})} \le \frac{\|(u+v,u-v)\|_p}{2\psi_p(\frac{1}{2})}$$

holds for all $u, v \in \mathbb{R}$. Inspired by this observation, in this paper, we consider the following problem.

Problem 1.1. Let $\varphi, \psi \in \Psi_2^S$, and let $\gamma \in [0,1]$. When does the inequality

$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} \le \frac{\|(u+v,u-v)\|_\psi}{2\psi(\frac{1}{2})}$$

hold for all $u, v \in \mathbb{R}$?

For each $\varphi, \psi \in \Psi_2^S$, let $\Gamma(\varphi,\psi)$ be the set of all $\gamma \in [0,1]$ such that

$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} \le \frac{\|(u+v,u-v)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for all $u, v \in \mathbb{R}$. Needless to say, the inequality is trivial if $\gamma = 0$. Thus our main purpose is to clarify the condition that $\Gamma(\varphi,\psi) \neq \{0\}$.

2. Generalizations of Beckner's inequality

The following is an important characterization of absolute norms on $\mathbb{R}^2$. The proof can be found in [2, Proposition IV.1.1] (see, also [7, Lemma 4.1]).

Lemma 2.1. *A norm $\|\cdot\|$ on $\mathbb{R}^2$ is absolute if and only if it is monotone, that is, if $|x_1| \le |x_2|$ and $|y_1| \le |y_2|$ then $\|(x_1,y_1)\| \le \|(x_2,y_2)\|$.*

The following lemma is a key.

Lemma 2.2. *Let $\varphi, \psi \in \Psi_2^S$, and let $\gamma \in [0,1]$. Then the following are equivalent:*

(i) *The inequality*

$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(u+v, u-v)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for all $u, v \in \mathbb{R}$.

(ii) *The inequality*

$$\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(1+u, 1-u)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for all $u \in [0,1]$.

Proof. It is enough to show that (ii) $\Rightarrow$ (i). Suppose that (ii) holds. We first take an arbitrary $u > 1$. Then $|1 \pm \gamma u| \leq |u \pm \gamma|$, which and Lemma 2.1 imply that

$$\begin{aligned}
\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(\frac{1}{2})} &\leq \frac{\|(u+\gamma, u-\gamma)\|_\varphi}{2\varphi(\frac{1}{2})} \\
&= \frac{u\,\|(1+\gamma u^{-1}, 1-\gamma u^{-1})\|_\varphi}{2\varphi(\frac{1}{2})} \\
&\leq \frac{u\,\|(1+u^{-1}, 1-u^{-1})\|_\psi}{2\psi(\frac{1}{2})} \\
&= \frac{\|(1+u, 1-u)\|_\psi}{2\psi(\frac{1}{2})}.
\end{aligned}$$

Next, let $u \leq 0$. Since $\varphi, \psi \in \Psi_2^S$, the assumption and above inequality show that

$$\begin{aligned}
\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(\frac{1}{2})} &= \frac{\|(1-\gamma u, 1+\gamma u)\|_\varphi}{2\varphi(\frac{1}{2})} \\
&\leq \frac{\|(1-u, 1+u)\|_\psi}{2\psi(\frac{1}{2})} \\
&= \frac{\|(1+u, 1-u)\|_\psi}{2\psi(\frac{1}{2})}.
\end{aligned}$$

Thus the inequality

$$\frac{\|(1+\gamma u, 1-\gamma u)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(1+u, 1-u)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for all $u \in \mathbb{R}$.

Finally, take arbitrary $u, v \in \mathbb{R}$. If $u = 0$, we have

$$\frac{\|(\gamma v, -\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} = \gamma|v| \leq |v| = \frac{\|(v, -v)\|_\psi}{2\psi(\frac{1}{2})}.$$

So we assume that $u \neq 0$. Then

$$\begin{aligned}\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} &= \frac{|u|\,\|(1+\gamma u^{-1}v, 1-\gamma u^{-1}v)\|_\varphi}{2\varphi(\frac{1}{2})} \\ &\leq \frac{|u|\,\|(1+u^{-1}v, 1-u^{-1}v)\|_\psi}{2\psi(\frac{1}{2})} \\ &= \frac{\|(u+v, u-v)\|_\psi}{2\psi(\frac{1}{2})}.\end{aligned}$$

This completes the proof. □

We remark that (ii) in the preceding lemma is equivalent to the following condition:

$$\frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} \leq \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}$$

for all $u \in [0,1]$. Thus it follows that

$$\Gamma(\varphi,\psi) = \left\{\gamma \in [0,1] : \frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} \leq \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})} \text{ for all } u \in [0,1]\right\}.$$

for all $\varphi,\psi \in \Psi_2^S$. Moreover, since the function

$$[0,1] \ni \gamma \to \frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})}$$

is continuous and convex for each fixed $u \in [0,1]$, the set $\Gamma(\varphi,\psi)$ is closed and convex. This means that $\Gamma(\varphi,\psi)$ is a subinterval of $[0,1]$. Let $\gamma_{\varphi,\psi} = \max \Gamma(\varphi,\psi)$. We note that $\gamma_{\varphi,\psi}$ is the best constant for the inequality.

In what follows, we study the condition for $\gamma_{\varphi,\psi} > 0$. The following is the simplest result in this direction.

Proposition 2.3. *Let $\varphi,\psi \in \Psi_2^S$. Suppose that $\varphi(t) = \varphi(1/2)$ on $[\delta, 1-\delta]$ for some $0 \leq \delta < 1/2$. Then $\gamma_{\varphi,\psi} > 0$.*

Proof. Let $\gamma = 1 - 2\delta > 0$. Then we have

$$\frac{1}{2} \geq \frac{1-\gamma u}{2} \geq \frac{1-\gamma}{2} = \delta$$

for all $u \in [0,1]$, which implies that

$$\frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} = \frac{\varphi(\frac{1}{2})}{\psi(\frac{1-u}{2})} \leq \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

Thus $\gamma \in \Gamma(\varphi,\psi)$, and so $\gamma_{\varphi,\psi} \geq \gamma > 0$. □

For each $\psi \in \Psi_2^S$, remark that $\psi'_L(1/2) \leq 0$, and that $\psi'_L(1/2) = 0$ if and only if ψ is differentiable at $1/2$, where ψ'_L denotes the left derivative of ψ. Let $\varphi,\psi \in \Psi_2^S$. We consider the following four cases:

(I) $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) = 0$.
(II) $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) < 0$.
(III) $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) = 0$.

(IV) $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) < 0$.

We first present the following theorem concerning cases (II), (III) and (IV).

Theorem 2.4. *Let $\varphi, \psi \in \Psi_2^S$. Then the following hold:*

(i) *If $\varphi'_L(1/2) = 0$ and $\psi'_L(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*
(ii) *If $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*
(iii) *If $\varphi'_L(1/2) < 0$ and $\psi'_L(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi'_L(1/2) < 0$ then

$$\gamma_{\varphi,\psi} \le \frac{\varphi(\frac{1}{2})\psi'_L(\frac{1}{2})}{\psi(\frac{1}{2})\varphi'_L(\frac{1}{2})}.$$

Proof. (i) We first remark that

$$\psi(t) \ge \psi\left(\frac{1}{2}\right) - \psi'_L\left(\frac{1}{2}\right)\left(\frac{1}{2} - t\right)$$

for all $t \in [0, 1/2]$. Since $\varphi'(1/2) = 0$, there exists $t_0 \in [0, 1/2)$ such that

$$\frac{\varphi(\frac{1}{2}) - \varphi(t)}{\frac{1}{2} - t} \ge \frac{\psi'_L(\frac{1}{2})}{2}$$

for all $t \in [t_0, 1/2)$, that is,

$$\varphi(t) \le \varphi\left(\frac{1}{2}\right) - \frac{\psi'_L(\frac{1}{2})}{2}\left(\frac{1}{2} - t\right)$$

for all $t \in [t_0, 1/2]$. Putting $\gamma = 1 - 2t_0 > 0$, we have

$$\frac{1}{2} \ge \frac{1 - \gamma u}{2} \ge \frac{1 - \gamma}{2} = t_0$$

for $u \in [0, 1]$, and hence, by an easy calculation, it follows that

$$\frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} \le \frac{\varphi(\frac{1}{2}) - \psi'_L(\frac{1}{2})\frac{\gamma u}{4}}{\psi(\frac{1}{2}) - \psi'_L(\frac{1}{2})\frac{u}{2}} \le \frac{\varphi(\frac{1}{2}) - \psi'_L(\frac{1}{2})\frac{u}{4}}{\psi(\frac{1}{2}) - \psi'_L(\frac{1}{2})\frac{u}{2}} \le \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

This means that $\gamma \in \Gamma(\varphi, \psi)$. Thus we obtain $\gamma_{\varphi_\psi} \ge \gamma > 0$.

(ii) and (iii): Assume that $\varphi'_L(1/2) < 0$. Put

$$k_0 = \frac{\varphi(\frac{1}{2})\psi'_L(\frac{1}{2})}{\psi(\frac{1}{2})\varphi'_L(\frac{1}{2})}.$$

We first show the inequality $\gamma_{\varphi,\psi} \le k_0$. Suppose that $k_0 < 1$, and that $k_0 < \gamma \le 1$. Since

$$\frac{\gamma\psi(\frac{1}{2})\varphi'_L(\frac{1}{2})}{\varphi(\frac{1}{2})} < \psi'_L\left(\frac{1}{2}\right),$$

there exists $t_0 \in [0, 1/2)$ such that $t \in [t_0, 1/2)$ implies

$$\frac{\psi(\frac{1}{2}) - \psi(t)}{\frac{1}{2} - t} > \frac{\gamma\psi(\frac{1}{2})\varphi'_L(\frac{1}{2})}{\varphi(\frac{1}{2})},$$

that is,

$$\psi(t) < \psi\left(\frac{1}{2}\right) - \frac{\gamma\psi(\frac{1}{2})\varphi'_L(\frac{1}{2})}{\varphi(\frac{1}{2})}\left(\frac{1}{2} - t\right)$$

for all $t \in [t_0, 1/2)$. On the other hand, since

$$\varphi(t) \geq \varphi\left(\frac{1}{2}\right) - \varphi'_L\left(\frac{1}{2}\right)\left(\frac{1}{2} - t\right)$$

for all $t \in [0, 1/2]$, putting $u_0 = 1 - 2t_0$, we have

$$\begin{aligned}\psi\left(\frac{1-u_0}{2}\right) = \psi(t_0) &< \frac{\psi(\frac{1}{2})}{\varphi(\frac{1}{2})}\left(\varphi\left(\frac{1}{2}\right) - \gamma\varphi'_L\left(\frac{1}{2}\right)\left(\frac{1}{2} - t_0\right)\right) \\ &= \frac{\psi(\frac{1}{2})}{\varphi(\frac{1}{2})}\left(\varphi\left(\frac{1}{2}\right) - \frac{\gamma u_0}{2}\varphi'_L\left(\frac{1}{2}\right)\right),\end{aligned}$$

and

$$\varphi\left(\frac{1-\gamma u_0}{2}\right) \geq \varphi\left(\frac{1}{2}\right) - \frac{\gamma u_0}{2}\varphi'_L\left(\frac{1}{2}\right).$$

These imply that

$$\frac{\varphi(\frac{1-\gamma u_0}{2})}{\psi(\frac{1-u_0}{2})} > \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})},$$

which shows $\gamma \notin \Gamma(\varphi, \psi)$. Thus we obtain $\gamma_{\varphi,\psi} \leq k_0$, which also proves (ii).

In the case of (iii), we have $k_0 > 0$. Take an arbitrary γ satisfying $0 < \gamma < \min\{k_0, 1\}$. Since

$$\varphi'_L\left(\frac{1}{2}\right) > \frac{\varphi(\frac{1}{2})\psi'_L(\frac{1}{2})}{\gamma\psi(\frac{1}{2})},$$

there exists $t_0 \in [0, 1/2)$ such that $t \in [t_0, 1/2)$ implies

$$\frac{\varphi(\frac{1}{2}) - \varphi(t)}{\frac{1}{2} - t} \geq \frac{\varphi(\frac{1}{2})\psi'_L(\frac{1}{2})}{\gamma\psi(\frac{1}{2})},$$

that is,

$$\varphi(t) \leq \varphi\left(\frac{1}{2}\right) - \frac{\varphi(\frac{1}{2})\psi'_L(\frac{1}{2})}{\gamma\psi(\frac{1}{2})}\left(\frac{1}{2} - t\right)$$

for all $t \in [t_0, 1/2)$. Moreover, we note that

$$\psi(t) \geq \psi\left(\frac{1}{2}\right) - \psi'_L\left(\frac{1}{2}\right)\left(\frac{1}{2} - t\right)$$

for all $t \in [0, 1/2]$. Now, putting $\gamma_0 = \min\{1 - 2t_0, \gamma\} > 0$, we have

$$\frac{1}{2} \geq \frac{1 - \gamma_0 u}{2} \geq \frac{1 - \gamma_0}{2} \geq t_0$$

for all $u \in [0,1]$, which implies that

$$\begin{aligned}\varphi\left(\frac{1-\gamma_0 u}{2}\right) &\le \varphi\left(\frac{1}{2}\right) - \frac{\gamma_0 u \varphi(\frac{1}{2})\psi_L'(\frac{1}{2})}{2\gamma\psi(\frac{1}{2})} \\ &= \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}\left(\psi\left(\frac{1}{2}\right) - \frac{\gamma_0 u}{2\gamma}\psi_L'\left(\frac{1}{2}\right)\right) \\ &\le \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}\left(\psi\left(\frac{1}{2}\right) - \frac{u}{2}\psi_L'\left(\frac{1}{2}\right)\right).\end{aligned}$$

Then, it follows from

$$\psi\left(\frac{1-u}{2}\right) \ge \psi\left(\frac{1}{2}\right) - \frac{u}{2}\psi_L'\left(\frac{1}{2}\right)$$

that

$$\frac{\varphi(\frac{1-\gamma_0 u}{2})}{\psi(\frac{1-u}{2})} \le \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

for all $u \in [0,1]$. This shows $\gamma_0 \in \Gamma(\varphi,\psi)$, and so we have $\gamma_{\varphi,\psi} \ge \gamma_0 > 0$. $\square$

The following is an application of Theorem 2.4.

Example 2.5. For each $\alpha \in (1/2, 1)$, let ψ_α be an element of Ψ_2^S defined by

$$\psi_\alpha(t) = \begin{cases} 1 + 2(\alpha-1)t & \text{if } t \in [0,1/2], \\ 2\alpha - 1 + 2(1-\alpha)t & \text{if } t \in [1/2,1]. \end{cases}$$

Suppose that $\alpha, \beta \in (1/2, 1)$, and that $\alpha \le \beta$. Then

$$k_0 = \frac{\psi_\alpha(\frac{1}{2})(\psi_\beta)_L'(\frac{1}{2})}{\psi_\beta(\frac{1}{2})(\psi_\alpha)_L'(\frac{1}{2})} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} \le 1.$$

On the other hand, for each $u \in [0,1]$, we have

$$\begin{aligned}\psi_\alpha\left(\frac{1-k_0 u}{2}\right) &= 1 + (\alpha-1)(1-k_0 u) \\ &= \alpha - (\alpha-1)k_0 u \\ &= \alpha - \frac{\alpha(1-\beta)}{\beta}u \\ &= \frac{\alpha}{\beta}(\beta - (1-\beta)u) \\ &= \frac{\alpha}{\beta}(1 + (\beta-1)(1-u)) \\ &= \frac{\alpha}{\beta}\psi_\beta\left(\frac{1-u}{2}\right).\end{aligned}$$

Thus $k_0 \in \Gamma(\psi_\alpha, \psi_\beta)$, which and Theorem 2.4 imply that

$$\gamma_{\psi_\alpha,\psi_\beta} = k_0 = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

Theorem 2.4 clarifies the relationship between $\gamma_{\varphi,\psi}$ and cases (II), (III) or (IV). However, we have no information about (I). So we next consider several special subcases of (I).

Let $\varphi, \psi \in \Psi_2^S$. Suppose that the second derivatives φ'' and ψ'' are continuous on $(\delta, 1-\delta)$ for some $0 \leq \delta < 1/2$. Then we remark that $\varphi''(1/2) \geq 0$ and $\psi''(1/2) \geq 0$ by convexity. This allows us to consider the following four subcases of (I):

(I-a) $\varphi''(1/2) = 0$ and $\psi''(1/2) = 0$.
(I-b) $\varphi''(1/2) = 0$ and $\psi''(1/2) > 0$.
(I-c) $\varphi''(1/2) > 0$ and $\psi''(1/2) = 0$.
(I-d) $\varphi''(1/2) > 0$ and $\psi''(1/2) > 0$.

Here we do not consider the case (I-a) because of its complexity. For cases (I-b), (I-c) and (I-d), we have the following result.

Theorem 2.6. *Let $\varphi, \psi \in \Psi_2^S$. Suppose that the second derivatives φ'' and ψ'' are continuous on $(\delta, 1-\delta)$ for some $0 \leq \delta < 1/2$. Then the following hold:*

(i) *If $\varphi''(1/2) = 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*
(ii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*
(iii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi''(1/2) > 0$ then

$$\gamma_{\varphi,\psi} \leq \sqrt{\frac{\varphi(\frac{1}{2})\psi''(\frac{1}{2})}{\psi(\frac{1}{2})\varphi''(\frac{1}{2})}}.$$

Proof. (i) For each $\gamma \in (0,1]$, define the function $f_\gamma : [0, 1-2\delta) \to \mathbb{R}$ by the formula

$$f_\gamma(u) = \frac{\psi(\frac{1-u}{2})}{\psi(\frac{1}{2})} - \frac{\varphi(\frac{1-\gamma u}{2})}{\varphi(\frac{1}{2})}.$$

Then, the first and second derivative of f_γ are as follows:

$$f'_\gamma(u) = \frac{1}{2}\left(\gamma\frac{\varphi'(\frac{1-\gamma u}{2})}{\varphi(\frac{1}{2})} - \frac{\psi'(\frac{1-u}{2})}{\psi(\frac{1}{2})}\right),$$

$$f''_\gamma(u) = \frac{1}{4}\left(\frac{\psi''(\frac{1-u}{2})}{\psi(\frac{1}{2})} - \gamma^2\frac{\varphi''(\frac{1-\gamma u}{2})}{\varphi(\frac{1}{2})}\right).$$

So we have $f'_\gamma(0) = 0$ and

$$f''_\gamma(0) = \frac{1}{4}\left(\frac{\psi''(\frac{1}{2})}{\psi(\frac{1}{2})} - \gamma^2\frac{\varphi''(\frac{1}{2})}{\varphi(\frac{1}{2})}\right) = \frac{\psi''(\frac{1}{2})}{4\psi(\frac{1}{2})} > 0.$$

From these facts, the function f_γ is positive on the interval $[0, u_0]$ for some $u_0 \in (0,1]$. Let $\gamma_0 = \gamma u_0 > 0$. Take an arbitrary $u \in [0,1]$ and put $v = u_0 u$. Then

$$0 \leq v \leq \min\{u_0, u\},$$

and so

$$\frac{1-u}{2} \leq \frac{1-v}{2} \leq \frac{1}{2},$$

which implies that

$$\psi\left(\frac{1-u}{2}\right) \geq \psi\left(\frac{1-v}{2}\right).$$

Hence we have

$$\begin{aligned} f_{\gamma_0}(u) &= \frac{\psi(\frac{1-u}{2})}{\psi(\frac{1}{2})} - \frac{\varphi(\frac{1-\gamma_0 u}{2})}{\varphi(\frac{1}{2})} \\ &\geq \frac{\psi(\frac{1-v}{2})}{\psi(\frac{1}{2})} - \frac{\varphi(\frac{1-\gamma v}{2})}{\varphi(\frac{1}{2})} \\ &= f_\gamma(v) \geq 0. \end{aligned}$$

This shows that $\gamma_0 \in \Gamma(\varphi, \psi)$. Thus $\gamma_{\varphi,\psi} \geq \gamma_0 > 0$.

We next suppose that $\varphi''(1/2) > 0$. Put

$$k_0 = \sqrt{\frac{\varphi(\frac{1}{2})\psi''(\frac{1}{2})}{\psi(\frac{1}{2})\varphi''(\frac{1}{2})}}.$$

(ii) As in the proof of (i), $f''_\gamma(0) < 0$ if $\gamma > k_0$. Then it follows that $f_\gamma(u_0) < 0$ for some $u_0 \in (0, 1-2\delta)$. This means that $\gamma \notin \Gamma(\varphi, \psi)$, which shows that $\gamma_{\varphi,\psi} \leq k_0$. This proves (ii).

(iii) In this case, we obtain $k_0 > 0$. Moreover, for each γ with $0 < \gamma < \min\{1, k_0\}$, we have $f''_\gamma(0) > 0$. Hence the function f_γ is positive on some non-trivial interval $[0, u_0]$. Finally, we obtain $\gamma_{\varphi,\psi} > 0$ by an argument similar to that in the first paragraph. □

Remark 2.7. We remark that

$$\sqrt{\frac{\psi_q(\frac{1}{2})\psi''_p(\frac{1}{2})}{\psi_p(\frac{1}{2})\psi''_q(\frac{1}{2})}} = \sqrt{\frac{p-1}{q-1}} = \gamma_{p,q},$$

where $\gamma_{p,q}$ is the best constant for Beckner's inequality. This gives another aspect of the value $\gamma_{p,q}$.

We next consider the duality of the Beckner type inequality. Then we need the following lemma.

Lemma 2.8. *Suppose that $\varphi, \psi \in \Psi_2^S$. For each $\gamma \in [0,1]$, let*

$$A_\gamma = \begin{pmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1+\gamma \end{pmatrix}.$$

Then $\gamma \in \Gamma(\varphi, \psi)$ if and only if

$$\left\| A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi) \right\| \leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

Proof. Let $\gamma \in \Gamma(\varphi, \psi)$. Then, by Lemma 2.2,

$$\frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(u+v, u-v)\|_\psi}{2\psi(\frac{1}{2})}$$

for all $u, v \in \mathbb{R}$. Take arbitrary $u, v \in \mathbb{R}$, and put

$$u_1 = u + v \quad \text{and} \quad v_1 = u - v,$$

respectively. Applying the inequality for u_1 and v_1, we obtain

$$\frac{\|((1+\gamma)u + (1-\gamma)v, (1-\gamma)u + (1+\gamma)v)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(2u, 2v)\|_\psi}{2\psi(\frac{1}{2})},$$

that is,

$$\|A_\gamma(u, v)\|_\varphi \leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}\|(u, v)\|_\psi.$$

Thus we have

$$\big\|A : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

Conversely, suppose that

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

Let $u, v \in \mathbb{R}$. Putting

$$u_1 = \frac{u+v}{2} \quad \text{and} \quad v_1 = \frac{u-v}{2},$$

we have

$$\begin{aligned}
\|(u + \gamma v, u - \gamma v)\|_\varphi &= \|((1+\gamma)u_1 + (1-\gamma)v_1, (1-\gamma)u_1 + (1+\gamma)v_1)\|_\varphi \\
&= \|A_\gamma(u_1, v_1)\|_\varphi \\
&\leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}\|(u_1, v_1)\|_\psi \\
&= \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}\|(u + v, u - v)\|_\psi.
\end{aligned}$$

Then it follows that $\gamma \in \Gamma(\varphi, \psi)$. □

Let $\psi \in \Psi_2$, and let $\|\cdot\|_\psi^*$ be the dual norm of $\|\cdot\|_\psi$. Then, as in [4], one has $\|\cdot\|_\psi^* \in AN_2$, and the function $\psi^* \in \Psi_2$ corresponding to $\|\cdot\|_\psi^*$ is given by

$$\psi^*(s) = \sup_{0 \leq t \leq 1} \frac{(1-s)(1-t) + st}{\psi(t)}$$

for all $s \in [0, 1]$. We remark that $\psi \in \Psi_2^S$ implies $\psi^* \in \Psi_2^S$.

The following is our purpose.

Theorem 2.9. *Let $\varphi, \psi \in \Psi_2^S$. Then $\gamma_{\varphi,\psi} = \gamma_{\psi^*,\varphi^*}$.*

Proof. Since $\min_{0 \leq t \leq 1} \varphi(t) = \varphi(1/2)$, it follows that

$$\varphi^*\left(\frac{1}{2}\right) = \sup_{0 \leq t \leq 1} \frac{(1-t)/2 + t/2}{\varphi(t)} = \frac{1}{2\varphi(\frac{1}{2})}.$$

We similarly have $\psi^*(1/2) = 1/2\psi(1/2)$, which implies that

$$\frac{\psi^*(\frac{1}{2})}{\varphi^*(\frac{1}{2})} = \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}.$$

Now, take an arbitrary $\gamma \in [0,1]$ and define a matrix A_γ as in Lemma 2.8. We remark that $A_\gamma^* = A_\gamma$, where A_γ^* is the adjoint operator of A_γ. Hence Lemma 2.8 assures that $\gamma \in \Gamma(\varphi,\psi)$ if and only if

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_\psi) \to (\mathbb{R}^2, \|\cdot\|_\varphi)\big\| \leq \frac{2\varphi(\frac{1}{2})}{\psi(\frac{1}{2})}$$

if and only if

$$\big\|A_\gamma : (\mathbb{R}^2, \|\cdot\|_{\varphi^*}) \to (\mathbb{R}^2, \|\cdot\|_{\psi^*})\big\| \leq \frac{2\psi^*(\frac{1}{2})}{\varphi^*(\frac{1}{2})}$$

if and only if $\gamma \in \Gamma(\psi^*, \varphi^*)$. Thus we have $\Gamma(\varphi,\psi) = \Gamma(\psi^*,\varphi^*)$. The proof is complete. □

Finally, we extend generalized Beckner's inequality to normed spaces.

Theorem 2.10. *Let X be a normed space. Suppose that $\varphi, \psi \in \Psi_2^S$, and that $\gamma \in \Gamma(\varphi,\psi)$. Then the inequality*

$$\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(x+y, x-y)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for all $x, y \in X$.

Proof. Take arbitrary $x, y \in X$, and put

$$z = x+y \quad \text{and} \quad w = x-y,$$

respectively. We also put

$$u = \frac{\|z\|+\|w\|}{2} \quad \text{and} \quad v = \frac{\|z\|-\|w\|}{2}.$$

Then we have

$$\begin{aligned}
\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(\frac{1}{2})} &= \frac{\|(\frac{1+\gamma}{2}z + \frac{1-\gamma}{2}w, \frac{1-\gamma}{2}z + \frac{1+\gamma}{2}w)\|_\varphi}{2\varphi(\frac{1}{2})} \\
&\leq \frac{\|(\frac{1+\gamma}{2}\|z\| + \frac{1-\gamma}{2}\|w\|, \frac{1-\gamma}{2}\|z\| + \frac{1+\gamma}{2}\|w\|)\|_\varphi}{2\varphi(\frac{1}{2})} \\
&= \frac{\|(u+\gamma v, u-\gamma v)\|_\varphi}{2\varphi(\frac{1}{2})} \\
&\leq \frac{\|(u+v, u-v)\|_\varphi}{2\psi(\frac{1}{2})} \\
&= \frac{\|(x+y, x-y)\|_\psi}{2\psi(\frac{1}{2})}.
\end{aligned}$$

This completes the proof. □

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