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ON AN ITERATION PROCESS FOR COMMON FIXED POINTS OF NONSELF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove some weak and strong convergence results for a generalized three-step-three-mappings iteration scheme using a more satisfactory definition of nonself mappings. Our results approximate common fixed points of three nonself total asymptotically nonexpansive mappings in a uniformly convex Banach space.

1. Introduction and preliminaries

Let E be a real normed space, K a nonempty subset of E and $T : K \to K$ a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and assume that $F(T) \neq \emptyset$. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$. Goebel and Kirk [4] proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

Definition 1.1. [1]Let K be a nonempty closed subset of a real normed linear space E. A mapping $T : K \to K$ is called *total asymptotically nonexpansive* if there exist nonnegative real sequences $\{\mu_n\}, \{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and

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a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + l_n, \quad n \in \mathbb{N} \tag{1.1}$$

for all $x,\ y \in K$.

Remark 1.2. If $\phi(\lambda) = \lambda$, then (1.1) reduces to

$$\|T^n x - T^n y\| \le (1 + \mu_n) \|x - y\| + l_n, \quad n \in \mathbb{N}.$$

If $\phi(\lambda) = \lambda$, $l_n = 0$ for all $n \in \mathbb{N}$, then a total asymptotically nonexpansive mapping becomes an asymptotically nonexpansive mapping. If we put $\mu_n = 0$ and $l_n = 0$ in (1.1)for all $n \in \mathbb{N}$, then we have the class of nonexpansive mappings.

The notion of total asymptotically nonexpansive mappings was introduced by Alber et al. [1]. They proved some strong and weak convergence results for this kind of mappings. Mukhamedov and Saburov [8] studied strong convergence of an explicit iteration process involving totally asymptotically I-nonexpansive mappings in Banach spaces. Chidume and Ofoedu [2] studied families of nonself total asymptotically nonexpansive mappings.

A subset K of E is said to be a retract if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Let $P : E \to K$ be a nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called nonself asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n \|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$. Khan and Hussain [6] studied the convergence for nonself asymptotically nonexpansive mappings in Banach spaces. Chidume et al. [3] established demiclosedness principle and gave strong and weak convergence results for such mappings in uniformly convex Banach spaces.

As a generalization of asymptotically nonexpansive nonself mappings, we introduce the following class of nonself total asymptotically nonexpansive mappings .

Definition 1.3. [2]Let K be a nonempty closed and convex subset of a real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself map $T : K \to E$ is said to be total asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}, \{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + l_n, \quad n \in \mathbb{N} \tag{1.2}$$

for all $x,\ y \in K$.

Remark 1.4. If $\phi(\lambda) = \lambda$, then (1.2) reduces to

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le (1 + \mu_n) \|x - y\| + l_n, \quad n \in \mathbb{N}.$$

A nonself total asymptotically nonexpansive mapping becomes

- a nonself asymptotically nonexpansive mapping if $\phi(\lambda) = \lambda$, $l_n = 0$ for all $n \in \mathbb{N}$
- a nonself nonexpansive mapping if $\mu_n = 0$ and $l_n = 0$ for all $n \in \mathbb{N}$.

As a matter of fact, if T is a self-mapping, then P is an identity mapping. In addition, if $T : K \to E$ is asymptotically nonexpansive and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive. Indeed, for all $x, y \in K$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} \|(PT)^n x - (PT)^n y\| &= \left\|PT(PT)^{n-1}x - PT(PT)^{n-1}y\right\| \\ &\leq \left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \\ &\leq \|x-y\| + \mu_n\phi(\|x-y\|) + l_n. \end{aligned}$$

Thus a more satisfactory definition of a nonself total asymptotically nonexpansive mapping is as follows.

Definition 1.5. Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E into K. A nonself mapping $T : K \to E$ is called total asymptotically nonexpansive with respect to P if there exist nonnegative real sequences $\{\mu_n\}, \{l_n\}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|(PT)^n x - (PT)^n y\| \leq \|x-y\| + \mu_n\phi(\|x-y\|) + l_n, \quad n \in \mathbb{N}. \tag{1.3}$$

Proposition 1.6. *Let K be a nonempty subset of E which is also a nonexpansive retract of a real normed linear space E and let $\{T_i\}_{i=1}^3 : K \to E$ be three nonself total asymptotically nonexpansive mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}, n \in \mathbb{N}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that*

$$\|(PT_i)^n x - (PT_i)^n y\| \leq \|x-y\| + \mu_n\phi(\|x-y\|) + l_n, \quad n \in \mathbb{N} \tag{1.4}$$

for all $x, y \in K$ and each $i = 1, 2, 3$.

Proof. Since $T_i : K \to E$ is a nonself total asymptotically nonexpansive mappings for each $i = 1, 2, 3$, there exist nonnegative real sequences $\{\mu_{in}\}, \{l_{in}\}, n \geq 1$ with $\mu_{in}, l_{in} \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_i(0) = 0$ such that for all $x, y \in K$,

$$\|(PT_i)^n x - (PT_i)^n y\| \leq \|x-y\| + \mu_{in}\phi_i(\|x-y\|) + l_{in}, \ \ n \in \mathbb{N}.$$

Setting

$$\begin{aligned} \mu_n &= \max\{\mu_{1n}, \mu_{2n}, \mu_{3n}\}, \quad l_n = \max\{l_{1n}, l_{2n}, l_{3n}\}, \\ \phi(a) &= \max\{\phi_1(a), \phi_2(a), \phi_3(a)\} \text{ for } a \geq 0, \end{aligned}$$

we obtain nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \in \mathbb{N}$ with $\mu_n, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\begin{aligned} \|(PT_i)^n x - (PT_i)^n y\| &\leq \|x-y\| + \mu_{in}\phi_i(\|x-y\|) + l_{in} \\ &\leq \|x-y\| + \mu_n\phi(\|x-y\|) + l_n, \quad n \in \mathbb{N}, \end{aligned}$$

for all $x, y \in K$, and each $i = 1, 2, 3$. $\square$

In [5], strong convergence results for a modified three step iterative process in the setting of Banach spaces were proved. In 2011, Rashwan and Altwqi [9] approximated common fixed points of three nonself asymptotically nonexpansive mappings in a real uniformly convex Banach space using the following iteration scheme:

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E which is also a nonexpansive retract of E. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three nonself asymptotically nonexpansive mappings with sequences $\mu_n, l_n \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Then for a given $x_1 \in K$,

$$\left\{ \begin{array}{ccc} x_{n+1} & = & P\left((1-\alpha_n)y_n + \alpha_n T_1 \left(PT_1\right)^{n-1} y_n\right) \\ y_n & = & P\left((1-\beta_n)z_n + \beta_n T_2 \left(PT_2\right)^{n-1} z_n\right) \\ z_n & = & P\left((1-\gamma_n)x_n + \gamma_n T_3 \left(PT_3\right)^{n-1} x_n\right), n \in \mathbb{N}, \end{array} \right. \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$.

If $\gamma_n = 0$, then (1.5) reduces to the iteration scheme defined by Thianwan [11] as follows:

$$\left\{ \begin{array}{lcl} x_{n+1} & = & P\left((1-\alpha_n)y_n + \alpha_n T_1 \left(PT_1\right)^{n-1} y_n\right) \\ y_n & = & P\left((1-\beta_n)x_n + \beta_n T_2 \left(PT_2\right)^{n-1} x_n\right), n \in \mathbb{N}, \end{array} \right.$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

If $\gamma_n = 0, \beta_n = 0$, then (1.5) reduces to the iteration scheme defined by Chidume et al. [3] as follows:

$$x_{n+1} = P\left((1-\alpha_n)x_n + \alpha_n T \left(PT\right)^{n-1} x_n\right), n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In wake of the Definition (1.5) , we study the following iteration scheme. Let K be a nonempty closed convex subset of a real normed linear space E with retraction P. Let $T_i : K \to E$ $(i = 1, 2, 3)$ be three nonself total asymptotically nonexpansive mappings with respect to P and sequences $\mu_n, l_n \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Then for a given $x_1 \in K$,

$$\left\{ \begin{array}{lcl} x_{n+1} & = & (1-\alpha_n)y_n + \alpha_n \left(PT_1\right)^n y_n \\ y_n & = & (1-\beta_n)z_n + \beta_n \left(PT_2\right)^n z_n \\ z_n & = & (1-\gamma_n)x_n + \gamma_n \left(PT_3\right)^n x_n, n \in \mathbb{N}, \end{array} \right. \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Following the method of Rashwan and Altwqi [9], it is not difficult to see that our process is able to compute common fixed points at a rate better than (1.5). It is noteworthy that our iteration process is better than [9] because we use a better definition of nonself mappings.

In this paper, we study convergence of the iterative scheme (1.6) involving three total asymptotically nonexpansive mappings on a nonempty closed convex subset of a uniformly convex Banach space.

In the sequel, we need the following useful known concepts and lemmas in order to prove our main results.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = 1, \quad \|y\| = 1, \quad \varepsilon = \|x-y\|\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A Banach space E is said to satisfy *Opial's condition* [12] if, for each sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{1.7}$$

for all $y \in E$ with $y \neq x$. It is well known that (see [7]) inequality (1.7) is equivalent to

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|.$$

A family $\{T_i : i = 1,2,3\}$ of three nonself mappings of K with the set of common fixed points $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ is said to satisfy condition (B) on K if there is a nondecreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that for all $x \in K$,

$$\max_{1\leq i\leq 3}\{\|x - T_i x\|\} \geq f(d(x,\mathcal{F})).$$

Definition 1.7. Let K be a closed subset of a real Banach space E and let $T : K \to K$ be a mapping. *T is said to be demicompact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$.*

Definition 1.8. Let K be a closed subset of a real Banach space E and let $T : K \to K$ be a mapping. *T is said to be semiclosed (demiclosed) at zero, if for each bounded sequence $\{x_n\}$ in K, the conditions x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.*

Lemma 1.9. [10]*Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *In particular, if $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging strongly to zero, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.10. [13]*Let $p > 1$ and $D > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, with $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda) y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_D$ and $0 \leq \lambda \leq 1$, where B_D is the closed ball with center zero and radius D, $w_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

2. Main results

In this section, we prove some convergence results for the iteration scheme (1.6) for three nonself total asymptotically nonexpansive mappings in Banach spaces. Throughout this paper, we denote by $\mathcal{F} := \bigcap_{i=1}^{3} F(T_i)$, the set of common fixed points of T_1, T_2, T_3 and assume that $\mathcal{F}$ is singleton, In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let K be a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Suppose that $\{x_n\}$ is defined by (1.6) and $\mathcal{F}$ is singleton. Then the sequence $\{x_n\}$ is bounded and for each $q \in \mathcal{F}$, $\lim_{n\to\infty} \|x_n - q\|$ exists.*

Proof. Let $q \in \mathcal{F}$. Since T_3 is a nonself total asymptotically nonexpansive mapping, we have from (1.6) that

$$\begin{aligned}
\|z_n - q\| &\leq (1-\gamma_n)\|x_n - q\| + \gamma_n \|(PT_3)^n x_n - q\| \\
&\leq (1-\gamma_n)\|x_n - q\| + \gamma_n [\|x_n - q\| + \mu_n \phi(\|x_n - q\|) + l_n] \\
&= \|x_n - q\| + \gamma_n \mu_n \phi(\|x_n - q\|) + \gamma_n l_n. \qquad (2.1)
\end{aligned}$$

Since $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$. Moreover, by the hypothesis, $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. In either case, we have

$$\phi(\lambda) \leq \phi(M) + M^*\lambda \qquad (2.2)$$

for some $M, M^* > 0$. Thus from (2.1) and (2.2), we get

$$\begin{aligned}
\|z_n - q\| &\leq \|x_n - q\| + \gamma_n \mu_n [\phi(M) + M^* \|x_n - q\|] + \gamma_n l_n \\
&= (1 + M^* \gamma_n \mu_n)\|x_n - q\| + \phi(M)\gamma_n \mu_n + \gamma_n l_n \\
&\leq (1 + M^* \phi(M) \mu_n)\|x_n - q\| + M^* \phi(M)(\mu_n + l_n) \\
&= (1 + R_1 \mu_n)\|x_n - q\| + R_1(\mu_n + l_n) \qquad (2.3)
\end{aligned}$$

where $R_1 = M^*\phi(M) > 0$.

Next,

$$\begin{aligned}
\|y_n - q\| &\leq (1-\beta_n)\|z_n - q\| + \beta_n \|(PT_2)^n z_n - q\| \\
&\leq (1-\beta_n)\|z_n - q\| + \beta_n [\|z_n - q\| + \mu_n \phi(\|z_n - q\|) + l_n] \\
&\leq \|z_n - q\| + \beta_n \mu_n [\phi(M) + M^* \|z_n - q\|] + \beta_n l_n \\
&= (1 + M^* \beta_n \mu_n)\|z_n - q\| + \phi(M)\beta_n \mu_n + \beta_n l_n \\
&\leq (1 + M^* \phi(M) \mu_n)\|z_n - q\| + M^* \phi(M)(\mu_n + l_n) \\
&= (1 + R_1 \mu_n)\|z_n - q\| + R_1(\mu_n + l_n). \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4), we get

$$\begin{aligned}
\|y_n - q\| &\leq (1+R_1\mu_n)\|z_n - q\| + R_1(\mu_n + l_n) \\
&\leq (1+R_1\mu_n)[(1+R_1\mu_n)\|x_n - q\| + R_1(\mu_n + l_n)] + R_1(\mu_n + l_n) \\
&= \left[1 + \left(2R_1 + R_1^2\mu_n\right)\mu_n\right]\|x_n - q\| + \left(2R_1 + R_1^2\mu_n\right)(\mu_n + l_n) \\
&= (1+R_2\mu_n)\|x_n - q\| + R_2(\mu_n + l_n) \qquad (2.5)
\end{aligned}$$

where $R_2 = 2R_1 + R_1^2\mu_n > 0$.

Thus

$$\begin{aligned}
\|x_{n+1} - q\| &\leq (1-\alpha_n)\|y_n - q\| + \alpha_n\|(PT_1)^n y_n - q\| \\
&\leq (1-\alpha_n)\|y_n - q\| + \alpha_n[\|y_n - q\| + \mu_n\phi(\|y_n - q\|) + l_n] \\
&\leq \|y_n - q\| + \alpha_n\mu_n[\phi(M) + M^*\|y_n - q\|] + \alpha_n l_n \\
&= (1+M^*\alpha_n\mu_n)\|y_n - q\| + \phi(M)\alpha_n\mu_n + \alpha_n l_n \\
&\leq (1+M^*\phi(M)\mu_n)\|y_n - q\| + M^*\phi(M)(\mu_n + l_n) \\
&= (1+R_1\mu_n)\|y_n - q\| + R_1(\mu_n + l_n). \qquad (2.6)
\end{aligned}$$

From (2.5) and (2.6), we are able to write

$$\begin{aligned}
\|x_{n+1} - q\| &\leq (1+R_1\mu_n)\|y_n - q\| + R_1(\mu_n + l_n) \\
&\leq (1+R_1\mu_n)[(1+R_2\mu_n)\|x_n - q\| + R_2(\mu_n + l_n)] \\
&\quad + R_1(\mu_n + l_n) \\
&= (1+R_1\mu_n)(1+R_2\mu_n)\|x_n - q\| \\
&\quad + (1+R_1\mu_n)R_2(\mu_n + l_n) + R_1(\mu_n + l_n) \\
&= [1+\mu_n(R_1 + R_2 + R_1R_2\mu_n)]\|x_n - q\| \\
&\quad + (R_1 + R_2 + R_1R_2\mu_n)(\mu_n + l_n) \\
&= (1+R_3\mu_n)\|x_n - q\| + R_3(\mu_n + l_n) \qquad (2.7)
\end{aligned}$$

where $R_3 = R_1 + R_2 + R_1R_2\mu_n > 0$.

Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty} l_n < \infty$, by Lemma 1.9, we have $\lim_{n\to\infty}\|x_n - q\|$ exists. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three continuous nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Suppose that $\mathcal{F}$ is singleton. Then the sequence $\{x_n\}$ defined by (1.6) converges strongly to a common fixed point of T_1, T_2, T_3 if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F}) = \inf_{q\in\mathcal{F}}\|x_n - q\|, n \in \mathbb{N}$.*

Proof. The necessity is obvious. Let us assume that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and prove that the sequence $\{x_n\}$ converges to a common fixed point of T_1, T_2, T_3. We first show that $\{x_n\}$ is a Cauchy sequence in E. Let $b_n = R_3\mu_n, c_n = R_3(\mu_n + l_n)$. The inequality (2.7) combined with the fact $1 + t \leq \exp(t)$ for all

$t > 0$ yields for all $m \in \mathbb{N}$

$$\begin{aligned}
\|x_{n+m} - q\| &\leq \exp(b_{n+m-1})(\|x_{n+m-1} - q\| + c_{n+m-1}) \\
&\vdots \\
&\leq \exp\left(\sum_{i=n}^{n+m-1} b_i\right)\|x_n - q\| + \left(\sum_{i=n}^{n+m-1} c_i\right)\exp\left(\sum_{i=n}^{n+m-1} b_i\right) \\
&\leq \exp\left(\sum_{i=n}^{\infty} b_i\right)\|x_n - q\| + \left(\sum_{i=n}^{\infty} c_i\right)\exp\left(\sum_{i=n}^{\infty} b_i\right).
\end{aligned}$$

Thus for all $m \in \mathbb{N}$ and $q \in \mathcal{F}$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\
&\leq \left(1 + \exp\left(\sum_{i=n}^{\infty} b_i\right)\right)\|x_n - q\| + \exp\left(\sum_{i=n}^{\infty} b_i\right)\sum_{i=n}^{\infty} c_i \\
&\leq A\left(\|x_n - q\| + \sum_{i=n}^{\infty} c_i\right) \\
&\leq A\left(d(x_n, \mathcal{F}) + \sum_{i=n}^{\infty} c_i\right), \qquad (2.8)
\end{aligned}$$

where $0 < A - 1 = \exp\left(\sum_{i=n}^{\infty} b_i\right) < \infty$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\sum_{i=1}^{\infty} c_i < \infty$, given $\varepsilon > 0$ there exists an integer $n_0 > 0$ such that for all $n > n_0$, we have $d(x_n, \mathcal{F}) < \frac{\varepsilon}{2A}$ and $\sum_{i=n}^{\infty} c_i < \frac{\varepsilon}{2A}$. Thus for all integers $n > n_0$ and $m \in \mathbb{N}$, (2.8) gives

$$\|x_{n+m} - x_n\| \leq \varepsilon.$$

This means that $\{x_n\}$ is a Cauchy sequence in E. Completeness of E now guarantees the existence of $x^* \in E$ such that $x_n \to x^*$.

Next, we show that x^* is a common fixed point of T_1, T_2, T_3. Suppose that $x^* \notin \mathcal{F}$. Since $\mathcal{F}$ is closed subset of E, one has $d(x^*, \mathcal{F}) > 0$. However, for all $q \in \mathcal{F}$, we have

$$\|x^* - q\| \leq \|x_n - x^*\| + \|x_n - q\|.$$

This implies that

$$d(x^*, \mathcal{F}) \leq \|x_n - x^*\| + d(x_n, \mathcal{F}).$$

Since $x_n \to x^*, d(x^*, \mathcal{F}) = 0$ as $n \to \infty$.This contradicts $d(x^*, \mathcal{F}) > 0$. Hence, x^* is a common fixed point of T_1, T_2, T_3 as required. □

Lemma 2.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Suppose that $\mathcal{F}$ is singleton and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$

(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$

(iii) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.
Then $\lim_{n\to\infty} \|(PT_i)^n x_n - x_n\| = 0$ *for all* $i = 1, 2, 3$.

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q\|$ exists. Assume that, for $q \in \mathcal{F}$, $\lim_{n\to\infty} \|x_n - q\| = r$. If $r = 0$, the conclusion is obvious. Suppose $r > 0$. Since $\{x_n - q\}$ is bounded, $\{z_n - q\}$ and $\{y_n - q\}$ are also bounded. As T_i are nonself total asymptotically nonexpansive mappings, we can prove that the sequences $\{(PT_1)^n y_n - q\}, \{(PT_2)^n z_n - q\}, \{(PT_3)^n x_n - q\}$ are all bounded. Using (1.6) and Lemma 1.10, we have for some constant $D_1 > 0$ that

$$\begin{aligned}
\|z_n - q\|^2 &= \|(1-\gamma_n)(x_n - q) + \gamma_n((PT_3)^n x_n - q)\|^2 \\
&\leq (1-\gamma_n)\|x_n - q\|^2 + \gamma_n \|(PT_3)^n x_n - q\|^2 \\
&\quad -\gamma_n(1-\gamma_n) g_1(\|x_n - (PT_3)^n x_n\|) \\
&\leq \|x_n - q\|^2 + D_1(\mu_n + l_n) \\
&\quad -\gamma_n(1-\gamma_n) g_1(\|x_n - (PT_3)^n x_n\|). \qquad (2.9)
\end{aligned}$$

It follows from (1.6) and (2.9) that for some constant $D_2 > 0$,

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1-\beta_n)(z_n - q) + \beta_n((PT_2)^n z_n - q)\|^2 \\
&\leq (1-\beta_n)\|z_n - q\|^2 + \beta_n \|(PT_2)^n z_n - q\|^2 \\
&\quad -\beta_n(1-\beta_n) g_2(\|z_n - (PT_2)^n z_n\|) \\
&\leq \|x_n - q\|^2 + D_2(\mu_n + l_n) - \gamma_n(1-\gamma_n) g_1(\|x_n - (PT_3)^n x_n\|) \\
&\quad -\beta_n(1-\beta_n) g_2(\|z_n - (PT_2)^n z_n\|). \qquad (2.10)
\end{aligned}$$

Similarly, from (1.6), (2.9) and (2.10), there is a constant $D_3 > 0$ such that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(y_n - q) + \alpha_n((PT_1)^n y_n - q)\|^2 \\
&\leq (1-\alpha_n)\|y_n - q\|^2 + \alpha_n \|(PT_1)^n y_n - q\|^2 \\
&\quad -\alpha_n(1-\alpha_n) g_3(\|y_n - (PT_1)^n y_n\|) \\
&\leq \|x_n - q\|^2 + D_3(\mu_n + l_n) - \gamma_n(1-\gamma_n) g_1(\|x_n - (PT_3)^n x_n\|) \\
&\quad -\beta_n(1-\beta_n) g_2(\|z_n - (PT_2)^n z_n\|) \\
&\quad -\alpha_n(1-\alpha_n) g_3(\|y_n - (PT_1)^n y_n\|). \qquad (2.11)
\end{aligned}$$

From (2.11), we can write

$$\gamma_n(1-\gamma_n) g_1(\|x_n - (PT_3)^n x_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + D_3(\mu_n + l_n), \quad (2.12)$$

$$\beta_n(1-\beta_n) g_2(\|z_n - (PT_2)^n z_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + D_3(\mu_n + l_n), \quad (2.13)$$

$$\alpha_n(1-\alpha_n) g_3(\|y_n - (PT_1)^n y_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + D_3(\mu_n + l_n). \quad (2.14)$$

Since $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$, $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$ and $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$, there exist $n_0 \in \mathbb{N}$ and

$m_1, m_1^*, m_2, m_2^*, m_3, m_3^* \in (0,1)$ such that $0 < m_1 < \alpha_n < m_1^*$, $0 < m_2 < \beta_n < m_2^*$ and $0 < m_3 < \gamma_n < m_3^*$ for all $n \geq n_0$. This implies by (2.12) that

$$m_3 (1 - m_3^*) g_1 (\|x_n - (PT_3)^n x_n\|) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + D_3 (\mu_n + l_n) \tag{2.15}$$

for all $n \geq n_0$. It follows from (2.15) that $m \geq n_0$

$$\begin{aligned} \sum_{n=n_0}^{m} g_1 (\|x_n - (PT_3)^n x_n\|) &\leq \frac{1}{m_3 (1 - m_3^*)} \left(\sum_{n=n_0}^{m} \left(\|x_n - q\|^2 - \|x_{n+1} - q\|^2\right) \right. \\ &\quad \left. + D_3 \sum_{n=n_0}^{m} (\mu_n + l_n) \right) \\ &\leq \frac{1}{m_3 (1 - m_3^*)} \left(\|x_{n_0} - q\|^2 + D_3 \sum_{n=n_0}^{m} (\mu_n + l_n) \right). \end{aligned}$$

Then $\sum_{n=n_0}^{\infty} g_1 (\|x_n - (PT_3)^n x_n\|) < \infty$ and therefore

$$\lim_{n\to\infty} g_1 (\|x_n - (PT_3)^n x_n\|) = 0.$$

Since g_1 is strictly increasing and continuous with $g_1(0) = 0$, we have

$$\lim_{n\to\infty} \|x_n - (PT_3)^n x_n\| = 0. \tag{2.16}$$

In a similar way, (2.13) and (2.14) imply

$$\lim_{n\to\infty} \|z_n - (PT_2)^n z_n\| = 0, \tag{2.17}$$

$$\lim_{n\to\infty} \|y_n - (PT_1)^n y_n\| = 0. \tag{2.18}$$

It follows from (1.6) that

$$\|z_n - x_n\| \leq \gamma_n \|x_n - (PT_3)^n x_n\|.$$

Thus by (2.16),

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{2.19}$$

Again, from (1.6)

$$\|y_n - z_n\| \leq \beta_n \|z_n - (PT_2)^n z_n\|,$$

so that(2.17) yields $\lim_{n\to\infty} \|y_n - z_n\| = 0$. From (2.19) and $\lim_{n\to\infty} \|y_n - z_n\| = 0$, we have

$$\lim_{n\to\infty} \|y_n - x_n\| \leq \lim_{n\to\infty} \|y_n - z_n\| + \lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{2.20}$$

A combined effect of (2.17) and (2.19) yields

$$\begin{aligned} \|x_n - (PT_2)^n x_n\| &\leq \|x_n - z_n\| + \|z_n - (PT_2)^n z_n\| + \|(PT_2)^n z_n - (PT_2)^n x_n\| \\ &\leq 2\|x_n - z_n\| + \|z_n - (PT_2)^n z_n\| + \mu_n \phi (\|z_n - x_n\|) + l_n. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \|x_n - (PT_2)^n x_n\| = 0. \tag{2.21}$$

Also from

$$\begin{aligned}\|x_n - (PT_1)^n x_n\| &\leq \|x_n - y_n\| + \|y_n - (PT_1)^n y_n\| + \|(PT_1)^n y_n - (PT_1)^n x_n\| \\ &\leq 2\|x_n - y_n\| + \|y_n - (PT_1)^n y_n\| + \mu_n \phi(\|y_n - x_n\|) + l_n,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|x_n - (PT_1)^n x_n\| = 0. \tag{2.22}$$

This completes the proof. □

Lemma 2.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three continuous nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$. Suppose that $\mathcal{F}$ is singleton and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$
(iii) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.
Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for* $i = 1, 2, 3$.

Proof. Note that

$$\begin{aligned}\|x_n - (PT_1)^n y_n\| &\leq \|x_n - (PT_1)^n x_n\| + \|(PT_1)^n x_n - (PT_1)^n y_n\| \\ &\leq \|x_n - (PT_1)^n x_n\| + \|x_n - y_n\| + \mu_n \phi(\|y_n - x_n\|) + l_n.\end{aligned}$$

Since T_1 is total asymptotically nonexpansive mapping, (2.20) and (2.22) give

$$\lim_{n\to\infty} \|x_n - (PT_1)^n y_n\| = 0. \tag{2.23}$$

Using (1.6), (2.20) and (2.23), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|(1-\alpha_n)(y_n - x_n) + \alpha_n((PT_1)^n y_n - x_n)\| \\ &\leq (1-\alpha_n)\|y_n - x_n\| + \alpha_n \|(PT_1)^n y_n - x_n\|,\end{aligned}$$

so that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.24}$$

Lemma 2.3 and (2.24) imply

$$\begin{aligned}\left\|x_n - (PT_i)^{n-1} x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - (PT_i)^{n-1} x_{n-1}\right\| \\ &\quad + \left\|(PT_i)^{n-1} x_{n-1} - (PT_i)^{n-1} x_n\right\| \\ &\leq 2\|x_n - x_{n-1}\| + \left\|x_{n-1} - (PT_i)^{n-1} x_{n-1}\right\| \\ &\quad + \mu_{n-1}\phi(\|x_n - x_{n-1}\|) + l_{n-1}.\end{aligned}$$

This gives

$$\lim_{n\to\infty} \left\|x_n - (PT_i)^{n-1} x_n\right\| = 0. \tag{2.25}$$

Since each T_i is continuous and P is nonexpansive retraction, it follows from (2.25) that

$$\lim_{n\to\infty} \|(PT_i)^n x_n - T_i x_n\| = \lim_{n\to\infty} \left\|PT_i (PT_i)^{n-1} x_n - PT_i x_n\right\| = 0 \tag{2.26}$$

for $i = 1, 2, 3$.

Hence, by Lemma 2.3 and (2.26), we have

$$\|x_n - T_i x_n\| \leq \|x_n - (PT_i)^n x_n\| + \|(PT_i)^n x_n - T_i x_n\|.$$

Consequently,

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{2.27}$$

□

Theorem 2.5. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three continuous nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and that one of T_1, T_2, T_3 is demicompact. Suppose that $\{x_n\}$ is defined by (1.6), $\mathcal{F}$ is singleton and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$

(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$

(iii) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^3$.

Proof. Without any loss of generality, we may assume that T_1 is demicompact. From the facts that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ that converges strongly to some $x^* \in K$ as $k \to \infty$. By Lemma 2.4, $T_1 x_{n_k} \to x^*, T_2 x_{n_k} \to x^*, T_3 x_{n_k} \to x^*$. Continuity of T_i gives $T_i x_{n_k} \to T_i x^*$ for all $i = 1, 2, 3$. Now using (2.26), we have

$$\|(PT_i)^{n_k} x_{n_k} - T_i x^*\| \leq \|(PT_i)^{n_k} x_{n_k} - T_i x_{n_k}\| + \|T_i x_{n_k} - T_i x^*\|.$$

This implies

$$\lim_{n\to\infty} \|(PT_i)^{n_k} x_{n_k} - T_i x^*\| = 0 \text{ for all } i = 1, 2, 3. \tag{2.28}$$

Observe that

$$\|x^* - T_1 x^*\| \leq \|x^* - x_{n_k}\| + \|x_{n_k} - (PT_1)^{n_k} x_{n_k}\| + \|(PT_1)^{n_k} x_{n_k} - T_1 x^*\|.$$

Using (2.28) and Lemma 2.3, we get $T_1 x^* = x^*$ and so $x^* \in F(T_1)$.

To prove $x^* \in F(T_2)$, note that

$$\|x^* - T_2 x^*\| \leq \|x^* - x_{n_k}\| + \|x_{n_k} - (PT_2)^{n_k} x_{n_k}\| + \|(PT_2)^{n_k} x_{n_k} - T_2 x^*\|.$$

We can similarly say that $x^* \in F(T_3)$. Therefore, $x^* \in \mathcal{F}$. According to Lemma 2.1, the limit $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Hence $\lim_{n\to\infty} x_{n_k} = x^* \in \mathcal{F}$. This completes the proof. □

Corollary 2.6. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three continuous nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$ and T_1, T_2, T_3 are satisfying the condition (B). Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Suppose that $\{x_n\}$ is defined by (1.6), $\mathcal{F}$ is singleton and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$
(iii) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.
Then the sequence $\{x_n\}$ *converges strongly to a common fixed points of* $\{T_i\}_{i=1}^3$.

Proof. By Lemma 2.4, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2, 3$. Since T_1, T_2, T_3 satisfy the condition (B), $f(d(x_n, \mathcal{F})) \le \max_{1\le i\le 3} \{\|x_n - T_i x_n\|\}$. Thus $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. The rest of the proof follows as in proof of Theorem 2.2. □

Theorem 2.7. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E satisfying Opial's condition which is also a nonexpansive retract of E and $T_1, T_2, T_3 : K \to E$ be three continuous nonself total asymptotically nonexpansive mappings defined by (1.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} l_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \le M^*\lambda$ for all $\lambda \ge M$. Suppose that $\{x_n\}$ is defined by (1.6), $\mathcal{F}$ is singleton and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ which satisfy the following conditions:*
(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$
(iii) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.
If the mappings $I - T_i$ $(i = 1, 2, 3)$ are semiclosed at zero, then the sequence $\{x_n\}$ converges weakly to a common fixed points of $\{T_i\}_{i=1}^3$.

Proof. Let $q_1 \in \mathcal{F}$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q_1\|$ exists and $\{x_n\}$ is bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_1 \in K$. Using Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0, \tag{2.29}$$

for $i = 1, 2, 3$. Since the mappings $I - T_i$ $(i = 1, 2, 3)$ are semiclosed at zero, therefore, we have $T_i q_1 = q_1$. That is, $q_1 \in \mathcal{F}$.

Finally, let us prove that $\{x_n\}$ converges weakly to q_1. Suppose on contrary that there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_2 \in K$ and $q_1 \ne q_2$. Then by the same method as given above, we can also prove that $q_2 \in \mathcal{F}$. Since $q_1, q_2 \in \mathcal{F}$, according to Lemma 2.1 $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q_1\| = r_1, \quad \lim_{n\to\infty} \|x_n - q_2\| = r_2, \tag{2.30}$$

where $r_1, r_2 \ge 0$. Because of the Opial's condition of E, we obtain

$$\begin{aligned} r_1 &= \limsup_{n_k\to\infty} \|x_{n_k} - q_1\| < \limsup_{n_k\to\infty} \|x_{n_k} - q_2\| = r_2 \\ \text{and } r_2 &= \limsup_{n_j\to\infty} \|x_{n_j} - q_2\| < \limsup_{n_j\to\infty} \|x_{n_j} - q_1\| = r_1. \end{aligned}$$

This is a contradiction. Hence $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to q_1. This completes the proof. □

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