Ann. Funct. Anal. 6 (2015), no. 1, 193–209
http://doi.org/10.15352/afa/06-1-14
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

DYNAMICAL SYSTEMS FOR QUASI VARIATIONAL INEQUALITIES

MUHAMMAD ASLAM NOOR*, KHALIDA INAYAT NOOR AND AWAIS GUL KHAN

Communicated by C. P. Niculescu

Abstract. In this paper, using the projection operator, we introduce two new dynamical systems for extended general quasi variational inequalities. These dynamical systems are called extended general implicit projected dynamical system and extended general implicit Wiener-Hopf dynamical system. We prove that these new dynamical systems converge globally exponentially to a unique solution of the extended general quasi variational inequalities under some suitable conditions. Some special cases are also discussed. The ideas and technique of this paper may stimulate further research.

1. Introduction

Quasi variational inequalities were introduced and studied by Bensoussan and Lions [4] in impulse control system. It is well known that the set involved in the quasi variational inequalities depends upon the solution explicitly or implicitly. We remark that if the involved set does not depend upon the solution then quasi variational inequality reduces to the variational inequality, the origin of which can be traced back to Stampacchia [27]. Variational inequalities and quasi variational inequalities provide us a unifying and an efficient framework to study various related and unrelated problems which arise in different branches of pure and applied sciences; see [1, 2, 8, 11, 12, 16, 19, 20, 23, 24, 25, 28] and references therein.

However quasi variational inequalities are more difficult and challenging as compared with variational inequalities. It is still a difficult task to suggest an efficient

Date: Received: Oct. 31, 2013; Accepted: Dec. 10, 2013.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46T99; Secondary 47H05, 49J40.

Key words and phrases. Dynamical systems, Wiener-Hopf equations, global convergence, quasi-variational inequalities.

method for solving quasi variational inequalities. The most common way for solving quasi variational inequalities is to show that the quasi variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation has been used to suggest some projection type methods for solving the quasi variational inequalities; see [17] and references therein. Noor [19] used this fixed point formulation to suggest and investigate the implicit dynamical system for quasi variational inequalities. This dynamical system includes many previously known dynamical systems suggested by Dupuis and Nagurney [7] and Friesz et al. [10] as special cases. In these projected dynamical systems, discontinuity arises and it is due to the discontinuity of the projection operator which appears on the right hand side of the ordinary differential equation. It is well known [15] that the stationary points of the projected dynamical system are solutions of the related variational inequality problem. This is the main reason for the importance of projected dynamical systems. Therefore projected dynamical systems can be used to study financial equilibrium problems, optimization problems, fixed point problems, complementarity problems and all those problems which can be studied in the framework of variational inequalities. Cojocaru et. al. [5] proved that, one can obtain the same results on any finite dimensional Hilbert space, for any closed convex set and a Lipschitz continuous operator. They also observed that if the nonlinear operator is also strictly monotone or strictly pseudomonotone, then there exists a unique stationary point for the projected dynamical system. Xia and Wang [29] have proved that the projected dynamical systems can be used effectively in designing neural network for solving variational inequalities and related optimization problems. Liu and Cao [13] and Liu and Yang [14] have developed the recurrent neural network technique for solving the extended general variational inequalities.

Related to the variational inequalities, one can consider problem of solving the Wiener-Hopf equations. This is mainly due to Shi [26]. He established the equivalence between the Wiener-Hopf equations and the variational inequalities. Noor [18] and Noor [19] introduced and studied a Wiener-Hopf dynamical system for the variational inequalities and implicit Wiener-Hopf dynamical system for quasi variational inequalities respectively. He has shown that the globally asymptotic stability of the Wiener-Hopf dynamical system requires only the pseudomonotonicity.

In recent years, variational inequalities have been generalized and extended in several directions using the novel and innovative techniques. In particular, Noor and Noor [22] and Noor et. al. [24] have introduced and studied some new classes of variational inequalities. These classes are called the extended general quasi variational inequalities. They have suggested some new iterative methods for solving these new classes of quasi variational inequalities. They have also discussed the convergence of these iterative methods and the sensitivity analysis of extended general quasi variational inequalities.

In this paper, we use the fixed point formulation to suggest two types of dynamical systems related to the extended general quasi variational inequality. We use these dynamical systems to study the existence of a solution of the extended general quasi variational inequality. We show that new dynamical systems converge globally exponentially to a unique solution of the extended general quasi variational inequality. Some particular cases are also considered. Results proved in this paper continue to hold for these problems.

2. Formulation and Basic Results

Let $\mathcal{H}$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. Let Ω be any closed and convex set in $\mathcal{H}$. Let $\Omega : x \to \Omega(x)$ be a point-to-set mapping which associates a closed and convex-valued set $\Omega(x)$ of $\mathcal{H}$ with any element x of $\mathcal{H}$.

For given three nonlinear operators $\Upsilon, h_1, h_2 : \mathcal{H} \to \mathcal{H}$, consider a problem of finding $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ and

$$\langle \rho\Upsilon x + h_2(x) - h_1(x), h_1(y) - h_2(x)\rangle \geq 0, \quad \forall y \in \mathcal{H} : h_1(y) \in \Omega(x), \tag{2.1}$$

where $\rho > 0$ is a constant. The inequality of type (2.1) is called the extended general quasi variational inequality. This problem was introduced and studied by Noor and Noor [22]. For the fixed point formulation, existence of a unique solution, equivalence with Wiener-Hopf equation, numerical methods and sensitivity analysis of problem (2.1), see [22, 24].

We now discuss some special cases of problem (2.1).

I. If $\Omega(x) = \Omega$, then problem (2.1) is equivalent to finding $x \in \mathcal{H} : h_2(x) \in \Omega$ and

$$\langle \rho\Upsilon x + h_2(x) - h_1(x), h_1(y) - h_2(x)\rangle \geq 0, \quad \forall y \in \mathcal{H} : h_1(y) \in \Omega, \tag{2.2}$$

which is called the extended general variational inequality. For the formulation, numerical algorithms and recent applications, see [12, 13] and references therein.

II. If $h_2 = h_1$, then problem (2.1) is equivalent to finding $x \in \mathcal{H} : h_1(x) \in \Omega(x)$ and

$$\langle \Upsilon x, h_1(y) - h_1(x)\rangle \geq 0, \quad \forall y \in \mathcal{H} : h_1(y) \in \Omega(x), \tag{2.3}$$

which is known as the general quasi variational inequality, introduced and studied by Noor [17].

III. If $h_2 = I$, where I the identity operator, then problem (2.1) is equivalent to finding $x \in \Omega(x)$ such that

$$\langle \rho\Upsilon x + x - h_1(x), h_1(y) - x\rangle \geq 0, \quad \forall y \in \mathcal{H} : h_1(y) \in \Omega(x), \tag{2.4}$$

is known as the general quasi variational inequality. This class is quite general and unified one.

IV. If $h_1 = I$, where I the identity operator, then problem (2.1) reduces to finding $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ such that

$$\langle \rho\Upsilon x + h_2(x) - x, y - h_2(x)\rangle \geq 0, \quad \forall y \in \Omega(x), \tag{2.5}$$

is called the general quasi variational inequality.

V. If $h_1 = h_2 = I$, where I the identity operator, then problem (2.1) is equivalent to finding $x \in \Omega(x)$ such that

$$\langle \Upsilon x, y - x\rangle \geq 0, \quad \forall y \in \Omega(x), \tag{2.6}$$

which is known as quasi variational inequality. This problem was introduced and studied by Bensoussan and Lions [4] in the study of impulse control system.
VI. If $\Omega(x) = \Omega$ and $h_2 = h_1$, then problem (2.1) is equivalent to problem of finding $x \in \mathcal{H} : h_1(x) \in \Omega$ such that

$$\langle \Upsilon x, h_1(y) - h_1(x) \rangle \geq 0, \quad \forall y \in \mathcal{H} : h_1(y) \in \Omega. \tag{2.7}$$

This problem is known as general variational inequality. It turned out that odd order and nonsymmetric obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied by problem (2.7), see [12, 14] and the references therein.
VII. If $\Omega(x) = \Omega$ and $h_2 = h_1 = I$, where I is the identity operator, then problem (2.1) is equivalent to problem of finding $x \in \Omega$ such that

$$\langle \Upsilon x, y - x \rangle \geq 0, \quad \forall y \in \Omega, \tag{2.8}$$

which is the original variational inequality. It was introduced and studied by Stampacchia [27]. For the recent applications, numerical algorithms, sensitivity analysis, dynamical systems and formulations of variational inequalities, see [1-29] and the references therein.

We also need the following well-known fundamental results and concepts.

Definition 2.1. A nonlinear operator $\Upsilon : \mathcal{H} \to \mathcal{H}$ is said to be strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle \Upsilon x - \Upsilon y, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.2. A nonlinear operator $\Upsilon : \mathcal{H} \to \mathcal{H}$ is said to be Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|\Upsilon x - \Upsilon y\| \leq \beta \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

From the definitions (2.1) and (2.2), it is clear that $\alpha \leq \beta$.

Remark 2.3. If the nonlinear operator $\Upsilon : \mathcal{H} \to \mathcal{H}$ is strongly monotone with constant $\alpha > 0$ and Υ^{-1} exists, then

$$\left\|\Upsilon^{-1} x - \Upsilon^{-1} y\right\| \leq \frac{1}{\alpha} \|x - y\|, \quad \forall x, y \in \mathcal{H}. \tag{2.9}$$

This shows that inverse operator Υ^{-1} is Lipschitz continuous with constant $\frac{1}{\alpha} > 0$.

Lemma 2.4. [22]. *Let Ω be a closed and convex set in $\mathcal{H}$. Then for a given $z \in \mathcal{H}$, $x \in \Omega$ satisfies*

$$\langle x - z, y - x \rangle \geq 0, \quad \forall y \in \Omega,$$

if and only if

$$x = \Pi_\Omega[z],$$

where Π_Ω is the projection of $\mathcal{H}$ onto the closed and convex set Ω in $\mathcal{H}$.

It is well known that the projection operator Π_Ω is nonexpansive.

Using Lemma 2.4, one can show that problem (2.1) is equivalent to the fixed point problem.

Lemma 2.5. [22]. *Let $\Omega(x)$ be a closed and convex valued set in $\mathcal{H}$. The function $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ is a solution of problem (2.1) if and only if $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ satisfies the relation*

$$h_2(x) = \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x], \tag{2.10}$$

where $\Pi_{\Omega(x)}$ is the implicit projection operator from $\mathcal{H}$ onto the closed and convex valued set $\Omega(x)$ and $\rho > 0$ is a constant.

From Lemma 2.5, it follows that problem (2.1) is equivalent to a fixed point problem (2.10). This equivalent formulation plays a crucial part in developing several iterative methods, see [19, 22, 24].

We would like to mention that the implicit projection operator $\Pi_{\Omega(x)}$ is not non-expansive. However, it satisfies the Lipschitz type continuity. We assume that the implicit projection operator $\Pi_{\Omega(x)}$ satisfies the following condition.

Assumption 2.11.[22]. *The implicit projection operator $\Pi_{\Omega(x)}$ satisfies the condition*

$$\|\Pi_{\Omega(x)}[w] - \Pi_{\Omega(y)}[w]\| \leq \nu\|x-y\|, \quad \text{for all } x, y, w \in \mathcal{H}, \tag{2.11}$$

where $\nu > 0$ is a constant.

Assumption 2.11 has been used to prove the existence of a solution of extended general quasi-variational inequalities as well as analyzing convergence of the iterative methods, see [22, 24].

We now define the residue vector

$$\mathcal{R}(x) = h_2(x) - \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x]. \tag{2.12}$$

It is clear from Lemma 2.5 that problem (2.1) has a solution $x \in \mathcal{H} : h_2(x) \in \Omega(x)$, if and only if $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ is a zero of the equation

$$\mathcal{R}(x) = 0. \tag{2.13}$$

3. Projected Dynamical System

We use the residue vector, defined in (2.12), to consider the following dynamical system

$$\begin{aligned} \frac{dx}{dt} &= -\lambda \mathcal{R}(x) \\ &= \lambda \left\{ \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - h_2(x) \right\}, \quad x(t_0) = x_0 \in \mathcal{H}, \end{aligned} \tag{3.1}$$

associated with problem (2.1), where λ is a constant. The dynamical system (3.1) is called the extended general implicit projected dynamical system. Since right hand side is related to the projection operator, and thus, is discontinuous on the boundary of $\mathcal{H}$. It is clear from the definition that the solution of dynamical system (3.1) belongs to the constraint set $\mathcal{H}$. From this it is clear that the results such as the existence, uniqueness and continuous dependence on the given data can be studied.

Definition 3.1. [18]. The dynamical system (3.1) is said to converge to the solution set Ω^* of problem (2.1) if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} \operatorname{dist}\left(x\left(t\right), \Omega^*\right) = 0,$$

where

$$\operatorname{dist}\left(x, \Omega^*\right) = \inf_{y\in\Omega^*} \left\|x - y\right\|.$$

Clearly, if the set Ω^* has a unique point x^*, then we have

$$\lim_{t\to\infty} x\left(t\right) = x^*.$$

The stability of the dynamical system at x^* in the Lyapunov sense, confirms that the dynamical system is also globally asymptotically stable at x^*.

Definition 3.2. [18]. The dynamical system is said to be globally exponentially stable with degree η_1 at x^* if, irrespective of the initial point, the trajectory of the system $x\left(t\right)$ satisfies

$$\left\|x\left(t\right) - x^*\right\| \leq \mu_1 \left\|x\left(t_0\right) - x^*\right\| e^{-\eta_1\left(t-t_0\right)}, \quad \forall t \geq t_0,$$

where $\mu_1 > 0$ and $\eta_1 > 0$ are positive constants independent of the initial point. It is clear that globally exponential stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 3.3. *(Gronwall's Lemma [18]). Let x and y be real valued non-negative continuous functions with domain $\{t : t \geq t_0\}$ and let $\alpha\left(t\right) = \alpha_0\left|t - t_0\right|$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$x\left(t\right) \leq \alpha\left(t\right) + \int_{t_0}^{t} x\left(s\right) y\left(s\right) ds,$$

then

$$x\left(t\right) \leq \alpha\left(t\right) \cdot e^{\int_{t_0}^{t} y(s)ds}.$$

Theorem 3.4. *Let Υ, h_1 and h_2 be the Lipschitz continuous with constants $\beta > 0$, $\delta > 0$ and $\mu > 0$ respectively. Let the Assumption 2.11 holds. Then for each $x_0 \in \mathcal{H}$, there exists a unique continuous solution $x\left(t\right)$ of the dynamical system (3.1) with $x\left(t_0\right) = x_0$ over $[t_0, \infty)$.*

Proof. Let

$$\mathcal{G}\left(x\right) = \lambda\left\{\Pi_{\Omega(x)}\left[h_1\left(x\right) - \rho\Upsilon x\right] - h_2\left(x\right)\right\}.$$

To prove that $\mathcal{G}(x)$ is Lipschitz continuous for all $x_1 \neq x_2 \in \mathcal{H}$, we have to consider

$$\begin{aligned}
\|\mathcal{G}(x_1) - \mathcal{G}(x_2)\| &= \lambda \| \left\{ \Pi_{\Omega(x_1)} [h_1(x_1) - \rho \Upsilon x_1] - h_2(x_1) \right\} \\
&\quad - \left\{ \Pi_{\Omega(x_2)} [h_1(x_2) - \rho \Upsilon x_2] - h_2(x_2) \right\} \| \\
&\leq \lambda \left\| \Pi_{\Omega(x_1)} [h_1(x_1) - \rho \Upsilon x_1] - \Pi_{\Omega(x_2)} [h_1(x_1) - \rho \Upsilon x_1] \right\| \\
&\quad + \lambda \left\| \Pi_{\Omega(x_2)} [h_1(x_1) - \rho \Upsilon x_1] - \Pi_{\Omega(x_2)} [h_1(x_2) - \rho \Upsilon x_2] \right\| \\
&\quad + \lambda \|h_2(x_1) - h_2(x_2)\| \\
&\leq \lambda \nu \|x_1 - x_2\| + \lambda \rho \|\Upsilon x_1 - \Upsilon x_2\| + \lambda \|h_1(x_1) - h_1(x_2)\| \\
&\quad + \lambda \|h_2(x_1) - h_2(x_2)\| \\
&\leq \lambda (\delta + \mu + \nu + \rho \beta) \|x_1 - x_2\|,
\end{aligned}$$

where we have used Assumption 2.11 and the Lipschitz continuity of the operators Υ, h_1, h_2 with constants $\beta > 0, \delta > 0, \mu > 0$ respectively.

This implies that the operator $\mathcal{G}(x)$ is a Lipschitz continuous in $\mathcal{H}$. Hence for each $x_0 \in \mathcal{H}$, there exists a unique and continuous solution $x(t)$ of the dynamical system (3.1), defined in an interval $t_0 \leq t < \Upsilon_1$ with the initial condition $x(t_0) = x_0$. Let $[t_0, \Upsilon_1)$ be its maximal of existence. Now we have to show that $\Upsilon_1 = \infty$.

Consider,

$$\begin{aligned}
\left\| \frac{dx}{dt} \right\| &= \|\mathcal{G}(x)\| \\
&= \lambda \left\| \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] - h_2(x) \right\| \\
&= \lambda \| \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] - \Pi_{\Omega(x)} [h_1(x)] + \Pi_{\Omega(x)} [h_1(x)] \\
&\quad - \Pi_{\Omega(x^*)} [h_1(x^*)] + \Pi_{\Omega(x^*)} [h_1(x^*)] - h_2(x) \| \\
&\leq \lambda \left\| \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] - \Pi_{\Omega(x)} [h_1(x)] \right\| \\
&\quad + \lambda \left\| \Pi_{\Omega(x)} [h_1(x)] - \Pi_{\Omega(x^*)} [h_1(x)] \right\| \\
&\quad + \lambda \left\| \Pi_{\Omega(x^*)} [h_1(x)] - \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| \\
&\quad + \lambda \left\| \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| + \lambda \|h_2(x)\| \\
&\leq \lambda \|h_1(x) - \rho \Upsilon x - h_1(x)\| + \lambda \nu \|x - x^*\| + \lambda \|h_1(x) - h_1(x^*)\| \\
&\quad + \lambda \left\| \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| + \lambda \|h_2(x)\| \\
&\leq \lambda \rho \|\Upsilon x\| + \lambda (\nu + \delta) \|x - x^*\| + \lambda \left\| \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| + \lambda \|h_2(x)\| \\
&\leq \lambda \rho \beta \|x\| + \lambda (\nu + \delta) (\|x\| + \|x^*\|) \\
&\quad + \lambda \left\| \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| + \lambda \mu \|x\| \\
&= \lambda (\rho \beta + \nu + \delta + \mu) \|x\| + \\
&\quad \lambda \left\{ (\nu + \delta) \|x^*\| + \left\| \Pi_{\Omega(x^*)} [h_1(x^*)] \right\| \right\}, \qquad (3.2)
\end{aligned}$$

where we have used Assumption 2.11, and Lipschitz continuity of operators Υ, h_1 and h_2 with constants $\beta > 0, \delta > 0$ and $\mu > 0$ respectively.

Now, integrating (3.2) over the interval $[t_0, t]$, we have

$$\begin{aligned}\|x(t)\| - \|x(t_0)\| &\leq k_1 \int_{t_0}^{t} 1ds + k_2 \int_{t_0}^{t} \|x(s)\| ds \\ &\leq k_1 (t - t_0) + k_2 \int_{t_0}^{t} \|x(s)\| ds,\end{aligned}$$

from, which by using Lemma 3.3, we have

$$\begin{aligned}\|x(t)\| &\leq \{\|x_0\| + k_1 (t - t_0)\} + k_2 \int_{t_0}^{t} \|x(s)\| ds \\ &\leq \{\|x_0\| + k_1 (t - t_0)\} e^{k_2(t-t_0)}, \quad t \in [t_0, \Upsilon_1),\end{aligned}$$

where

$$\begin{aligned}k_1 &= \lambda \left\{(\nu + \delta) \|x^*\| + \left\|\Pi_{\Omega(x^*)} [h_1 (x^*)]\right\|\right\} \\ k_2 &= \lambda \{\rho\beta + \nu + \delta + \mu\}.\end{aligned}$$

This shows that the solution is bounded on $[t_0, \infty)$. □

We now show that the trajectory of the solution of the dynamical system (3.1) converges globally exponentially to the unique solution of problem (2.1).

Theorem 3.5. *Let the operators Υ, h_1 and $h_2 : \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous with constants $\beta > 0$, $\delta > 0$ and $\mu > 0$ respectively. Let Assumption 2.11 hold. If the operator $h_2 : \mathcal{H} \to \mathcal{H}$ is strongly antimonotone with constant $\eta > 0$, then the dynamical system (3.1) converges globally exponentially to the unique solution of problem (2.1).*

Proof. Since the operators Υ, h_1 and h_2 are Lipschitz continuous, it follows from the Theorem 3.4 that the dynamical system (3.1) has a unique solution $x(t)$ over $[t_0, \Upsilon_1)$ for any fixed $x_0 \in \mathcal{H}$.

Let $x(t) = x(t, t_0; x_0)$ be a solution of problem (3.1). For a given $x^* \in \mathcal{H} : h_2(x^*) \in \Omega(x^*)$, satisfying problem (2.1), consider the following Lyapunov function:

$$\mathcal{L}(x) = \frac{1}{2} \|x(t) - x^*\|^2, \quad x \in \mathcal{H}. \tag{3.3}$$

Thus

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \left\langle x(t) - x^*, \frac{dx}{dt} \right\rangle \\
&= \langle x - x^*, -\lambda \mathcal{R}(x) \rangle \\
&= -\lambda \left\langle x - x^*, h_2(x) - \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] \right\rangle \\
&= -\lambda \left\langle x - x^*, (h_2(x) - h_2(x^*)) + \left(h_2(x^*) - \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x]\right) \right\rangle \\
&= -\lambda \langle h_2(x) - h_2(x^*), x - x^* \rangle \\
&\quad + \lambda \left\langle \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - h_2(x^*), x - x^* \right\rangle \\
&\leq -\lambda \eta \|x - x^*\|^2 + \lambda \left\| \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - h_2(x^*) \right\| \|x - x^*\|, \qquad (3.4)
\end{aligned}
$$

where we have used the strongly antimonotonicity of the operator h_2 with constant $\eta > 0$ and Cauchy Schwarz inequality.

Since $x^* \in \mathcal{H} : h_2(x^*) \in \Omega(x^*)$ is the solution of problem (2.1), therefore using Lemma 2.5,we have

$$
h_2(x^*) = \Pi_{\Omega(x^*)}[h_1(x^*) - \rho \Upsilon x^*]. \qquad (3.5)
$$

Using (3.5), Assumption 2.11 and Lipschitz continuity of operators Υ and h_1 with constants $\beta > 0$ and $\delta > 0$ respectively, we have

$$
\begin{aligned}
&\left\| \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - h_2(x^*) \right\| \\
&\qquad = \left\| \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - \Pi_{\Omega(x^*)}[h_1(x^*) - \rho \Upsilon x^*] \right\| \\
&\qquad \leq \left\| \Pi_{\Omega(x)}[h_1(x) - \rho \Upsilon x] - \Pi_{\Omega(x^*)}[h_1(x) - \rho \Upsilon x] \right\| \\
&\qquad\quad + \left\| \Pi_{\Omega(x^*)}[h_1(x) - \rho \Upsilon x] - \Pi_{\Omega(x^*)}[h_1(x^*) - \rho \Upsilon x^*] \right\| \\
&\qquad \leq \nu \|x - x^*\| + \|h_1(x) - h_1(x^*)\| + \rho \|\Upsilon x - \Upsilon x^*\| \\
&\qquad \leq (\nu + \delta + \rho\beta) \|x - x^*\|. \qquad (3.6)
\end{aligned}
$$

Combining (3.4) and (3.6), we have

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &\leq -\lambda \eta \|x - x^*\|^2 + \lambda (\nu + \delta + \rho\beta) \|x - x^*\|^2 \\
&= \lambda (\nu + \delta - \eta + \rho\beta) \|x - x^*\|^2 \\
&= k_1 \|x - x^*\|^2,
\end{aligned}
$$

where

$$
k_1 = \lambda (\nu + \delta - \eta + \rho\beta) > 0.
$$

Thus for $k_1 = -k_2$, where k_2 is a positive constant, we have

$$
\frac{d\mathcal{L}}{dt} \leq -k_2 \|x - x^*\|^2,
$$

which implies that

$$
\|x(t) - x^*\| \leq \|x(t_0) - x^*\| e^{-k_2(t - t_0)}.
$$

This shows that the trajectory of the solution of the dynamical system (3.1) converges globally exponentially to the unique solution of problem (2.1). □

For suitable and appropriate choice of the operators and spaces one can obtain the results of Noor [18, 19] and others as special cases from our results.

4. Wiener-Hopf Equation Dynamical System

We now consider problem related to the extended general quasi variational inequality which is called the Wiener-Hopf equation. To be more precise, let $Q_{\Omega(x)} = I - h_1 h_2^{-1} \Pi_{\Omega(x)}$, where I is the identity operator and $\Pi_{\Omega(x)}$ is the projection of $\mathcal{H}$ onto the closed and convex-valued set $\Omega(x)$.

For given non-linear operators $\Upsilon, h_1, h_2 : \mathcal{H} \to \mathcal{H}$, consider problem of finding $z \in \mathcal{H}$ such that

$$\Upsilon h_2^{-1} \Pi_{\Omega(x)} [z] + \rho^{-1} Q_{\Omega(x)} [z] = 0, \tag{4.1}$$

where $\rho > 0$ is a constant. The equation (4.1) is known as the extended general implicit Wiener-Hopf equation.

We now discuss some special cases of problem (4.1).

I. If $\Omega(x) = \Omega$, then problem (4.1) is equivalent to finding $z \in \mathcal{H}$ such that

$$\Upsilon h_2^{-1} \Pi_{\Omega} [z] + \rho^{-1} Q_{\Omega} [z] = 0. \tag{4.2}$$

The problem (4.2) is known as extended general Wiener-Hopf equation, see [21].

II. If $h_2 = h_1$, then problem (4.1) is equivalent to finding $z \in \mathcal{H}$ such that

$$\Upsilon h_1^{-1} \Pi_{\Omega(x)} [z] + \rho^{-1} Q_{\Omega(x)} [z] = 0. \tag{4.3}$$

This problem is called general implicit Wiener–Hopf equation which is equivalent to problem (2.3).

III. If $\Omega(x) = \Omega$ and $h_2 = h_1 = I$, where I is the identity operator, then problem (4.1) is equivalent to finding $z \in \mathcal{H}$ such that

$$\Upsilon \Pi_{\Omega} [z] + \rho^{-1} Q_{\Omega} [z] = 0. \tag{4.4}$$

Problem (4.4) is the original Wiener–Hopf equation, which was introduced by Shi [26]. This problem is equivalent to problem (2.7).

For suitable and appropriate choice of operators and spaces, one can obtain several known and new classes of the Wiener–Hopf equations.

It is known [24] that problem (2.1) and (4.1) are equivalent. For the sake of completeness, we recall this result without proof.

Lemma 4.1. *The problem (2.1) has a solution $x \in \mathcal{H} : h_2(x) \in \Omega(x)$, if and only if, problem (4.1) has a solution $z \in \mathcal{H}$, where*

$$h_2(x) = \Pi_{\Omega(x)} [z], \tag{4.5}$$

$$z = h_1(x) - \rho \Upsilon x, \tag{4.6}$$

where $\rho > 0$ is a constant.

Using Lemma 4.1, the implicit Wiener–Hopf equation (4.1) can be written as

$$h_1(x) - \rho \Upsilon x - h_1 h_2^{-1} \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] + \rho \Upsilon h_2^{-1} \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] = 0, \tag{4.7}$$

from, which we have

$$h_1(x) = \rho \Upsilon x + h_1 h_2^{-1} \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x] - \rho \Upsilon h_2^{-1} \Pi_{\Omega(x)} [h_1(x) - \rho \Upsilon x]. \tag{4.8}$$

Thus it is clear from Lemma 4.1 that $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ is a solution of problem (2.1), if and only if, $x \in \mathcal{H} : h_2(x) \in \Omega(x)$ satisfies the equation (4.7).

Using (4.7) and for a constant λ, we suggest a new dynamical system:

$$\frac{dx}{dt} = \lambda \left\{ h_1 h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] - \rho \Upsilon h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] \right.$$

$$\left. + \rho \Upsilon x - h_1(x) \right\}, \quad x(t_0) = x_0 \in \Omega(x). \tag{4.9}$$

This problem is called the extended general implicit Wiener–Hopf dynamical system. Here the right-hand side is associated with the implicit projection and hence is discontinuous on the boundary of a closed and convex-valued set $\Omega(x)$. It is clear from the definition that the solution to the dynamical system (4.9) belongs to the constraint set $\Omega(x)$. This implies that the results, such as existence, uniqueness and continuous dependence on the data of the solution of (4.9) can be studied.

We now study the main properties of the proposed Wiener–Hopf dynamical system and analyze the global stability of the systems. First of all, we discuss the existence and uniqueness of the dynamical system (4.9) and this is the main motivation of our next result.

Theorem 4.2. *Let the non-linear operators Υ, h_1 and h_2^{-1} be the Lipschitz continuous with constants $\beta > 0, \delta > 0$ and $\theta > 0$ respectively. Let Assumption 2.11 hold. Then for each $x_0 \in \mathcal{H}$, there exists a unique continuous solution $x(t)$ of the dynamical system (4.9) with $x(t_0) = x_0$ over $[t_0, \infty)$.*

Proof. Let

$$\begin{aligned} \mathcal{G}(x) &= \frac{dx}{dt} = \lambda \left\{ h_1 h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] \right. \\ &\quad \left. - \rho \Upsilon h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] + \rho \Upsilon x - h_1(x) \right\}. \end{aligned}$$

To prove that $\mathcal{G}(x)$ is Lipschitz continuous for all $x \neq y \in \mathcal{H}$, we consider

$$\begin{aligned} \|\mathcal{G}(x) - \mathcal{G}(y)\| &= \lambda \| \left\{ h_1 h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] \right. \\ &\quad \left. - \rho \Upsilon h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] + \rho \Upsilon x - h_1(x) \right\} \\ &\quad - \left\{ h_1 h_2^{-1} \Pi_{\Omega(y)} \left[h_1(y) - \rho \Upsilon y \right] \right. \\ &\quad \left. - \rho \Upsilon h_2^{-1} \Pi_{\Omega(y)} \left[h_1(y) - \rho \Upsilon y \right] + \rho \Upsilon y - h_1(y) \right\} \| \end{aligned}$$

$$\begin{aligned} &\leq \lambda \| h_1 h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] - h_1 h_2^{-1} \Pi_{\Omega(y)} \left[h_1(y) - \rho \Upsilon y \right] \| \\ &\quad + \lambda \rho \left\| \Upsilon h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] \right. \\ &\quad \left. - \Upsilon h_2^{-1} \Pi_{\Omega(y)} \left[h_1(y) - \rho \Upsilon y \right] \right\| \\ &\quad + \lambda \rho \left\| \Upsilon x - \Upsilon y \right\| + \lambda \left\| h_1(x) - h_1(y) \right\| \\ &\leq \lambda (\delta + \rho \beta) \left\{ \left\| h_2^{-1} \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] \right. \right. \\ &\quad \left. \left. - h_2^{-1} \Pi_{\Omega(y)} \left[h_1(y) - \rho \Upsilon y \right] \right\| + \|x - y\| \right\}, \end{aligned} \tag{4.10}$$

where we have used the Lipschitz continuity of Υ, h_1 with constants $\beta > 0, \delta > 0$, respectively. Now,

$$\begin{aligned}
&\left\|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-h_2^{-1}\Pi_{\Omega(y)}\left[h_1(y)-\rho\Upsilon y\right]\right\| \\
&\leq \theta\left\|\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(y)}\left[h_1(y)-\rho\Upsilon y\right]\right\| \\
&\leq \theta\left\|\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(y)}\left[h_1(x)-\rho\Upsilon x\right]\right\| \\
&\quad +\theta\left\|\Pi_{\Omega(y)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(y)}\left[h_1(y)-\rho\Upsilon y\right]\right\| \\
&\leq \theta\nu\|x-y\|+\theta\left\|h_1(x)-h_1(y)-\rho(\Upsilon x-\Upsilon y)\right\| \\
&\leq \theta\nu\|x-y\|+\theta\left\|h_1(x)-h_1(y)\right\|+\theta\rho\|\Upsilon x-\Upsilon y\| \\
&\leq \theta(\nu+\delta+\rho\beta)\|x-y\|, \qquad (4.11)
\end{aligned}$$

where we have used Assumption (2.11) and Lipschitz continuity of the operators Υ,h_1 and h_2^{-1} with constants $\beta>0$,$\delta>0$ and $\theta>0$, respectively.

Combining (4.10) and (4.11), we have

$$\begin{aligned}
\|\mathcal{G}(x)-\mathcal{G}(y)\| &\leq \lambda(\delta+\rho\beta)\{\theta(\nu+\delta+\rho\beta)\|x-y\|+\|x-y\|\} \\
&= \lambda(\delta+\rho\beta)\{\theta(\nu+\delta+\rho\beta)+1\}\|x-y\|.
\end{aligned}$$

This implies that the operator $\mathcal{G}(x)$ is Lipschitz continuous in $\mathcal{H}$. So for each $x_0\in\mathcal{H}$, there exists a unique and continuous solution $x(t)$ of the dynamical system (4.9), defined in an interval $t_0\leq t<\Upsilon_1$ with the initial condition $x(t_0)=x_0$. Let $[t_0,\Upsilon_1)$ be its maximal of existence. Now we have to show that $\Upsilon_1=\infty$.

Consider,

$$\begin{aligned}
\|\mathcal{G}(x)\| &= \left\|\frac{dx}{dt}\right\| \\
&= \lambda\left\|h_1h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]\right. \\
&\quad \left.-\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]+\rho\Upsilon x-h_1(x)\right\| \\
&\leq \lambda\left\|h_1h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-h_1(x)\right\| \\
&\quad +\lambda\rho\left\|\Upsilon h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Upsilon x\right\| \\
&\leq \lambda(\delta+\rho\beta)\left\|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-x\right\|, \qquad (4.12)
\end{aligned}$$

where we have used the Lipschitz continuity of Υ and h_1^{-1} with constants $\beta>0$ and $\delta>0$, respectively. Now,

$$\begin{aligned}
&\left\|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-x\right\| \\
&= \|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)\right]+h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)\right] \\
&\quad -h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)\right]+h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)\right]-x\| \\
&\leq \left\|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)\right]\right\| \\
&\quad +\left\|h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)\right]-h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x)\right]\right\| \\
&\quad +\left\|h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x)\right]-h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)\right]\right\| \\
&\quad +\left\|h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)\right]\right\|+\|x\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \theta \left\| \Pi_{\Omega(x)} \left[h_1(x) - \rho \Upsilon x \right] - \Pi_{\Omega(x)} \left[h_1(x) \right] \right\| \\
&\quad + \theta \left\| \Pi_{\Omega(x)} \left[h_1(x) \right] - \Pi_{\Omega(x^*)} \left[h_1(x) \right] \right\| \\
&\quad + \theta \left\| \Pi_{\Omega(x^*)} \left[h_1(x) \right] - \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \\
&\quad + \theta \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| + \|x\| \\
&\leq \theta \left\{ \| h_1(x) - \rho \Upsilon x - h_1(x) \| + \nu \| x - x^* \| + \| h_1(x) - h_1(x^*) \| \right. \\
&\quad \left. + \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \right\} + \|x\| \\
&\leq \theta \{ \{ \rho\beta \|x\| + \nu \|x - x^*\| + \delta \|x - x^*\| \\
&\quad + \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \} + \|x\| \\
&\leq \{ \theta (\rho\beta + \nu + \delta) + 1 \} \|x\| \\
&\quad + \theta \left\{ (\nu + \delta) \|x^*\| + \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \right\}, \qquad (4.13)
\end{aligned}
$$

where we have used the Υ, h_1 and h_2^{-1} be the Lipschitz continuous with constants $\beta > 0, \delta > 0$ and $\theta > 0$ respectively and Assumption 2.11.

Combining (4.12) and (4.13), we have

$$
\begin{aligned}
\|\mathcal{G}(x)\| &= \left\| \frac{dx}{dt} \right\| \leq \lambda (\delta + \rho\beta) \{ \theta (\rho\beta + \nu + \delta) + 1 \} \|x\| \\
&\quad + \theta\lambda (\delta + \rho\beta) \left\{ (\nu + \delta) \|x^*\| + \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \right\}. \qquad (4.14)
\end{aligned}
$$

Integrating (4.14) over the interval $[t_0, t]$, we have

$$
\begin{aligned}
\|x(t)\| - \|x(t_0)\| &\leq k_1 \int_{t_0}^{t} ds + k_2 \int_{t_0}^{t} \|x(s)\| \, ds \\
&= k_1 (t - t_0) + k_2 \int_{t_0}^{t} \|x(s)\| \, ds,
\end{aligned}
$$

from which by using the Lemma 3.3, we have

$$
\begin{aligned}
\|x(t)\| &\leq \{ \|x(t_0)\| + k_1 (t - t_0) \} + k_2 \int_{t_0}^{t} \|x(s)\| \, ds \\
&\leq \{ \|x(t_0)\| + k_1 (t - t_0) \} e^{k_2 (t - t_0)}, \quad t \in [t_0, \Upsilon_1),
\end{aligned}
$$

where

$$
\begin{aligned}
k_1 &= \theta\lambda (\delta + \rho\beta) \left\{ (\nu + \delta) \|x^*\| + \left\| \Pi_{\Omega(x^*)} \left[h_1(x^*) \right] \right\| \right\}, \\
k_2 &= \lambda (\delta + \rho\beta) \{ \theta(\rho\beta + \nu + \delta) + 1 \}.
\end{aligned}
$$

This shows that the solution is bounded on $[t_0, \infty)$. □

We now show that the trajectory of the solution of the dynamical system (4.9) converges globally exponentially to the unique solution of problem (2.1).

Theorem 4.3. *Let the operators Υ, h_1 and h_2^{-1} be Lipschitz continuous with constants $\beta > 0, \delta > 0$ and $\theta > 0$ respectively. Let the operator h_1 be strongly monotone with constants $\sigma > 0$. Let Assumption 2.11 hold. Then the dynamical* *system* (4.9) *converges globally exponentially to the unique solution of problem* (2.1).

Proof. Since the operators Υ, h_1 and h_2^{-1} satisfies all the conditions of Theorem 4.2, therefore it follows from Theorem 4.2 that the dynamical system (4.9) has a unique solution $x(t)$ over $[t_0, \Upsilon_1)$ for any fixed point $x_0 \in \mathcal{H}$.

Let $x(t) = x(t, t_0; x_0)$ be a solution of (4.9). For a given $x^* \in \mathcal{H} : h_2(x^*) \in \Omega(x)$, satisfying problem (2.1), consider the following Lyapunov function:

$$\mathcal{L}(x) = \frac{1}{2}\|x(t) - x^*\|^2, \quad x \in \mathcal{H}.$$

Thus by using (4.9), we have

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \left\langle x(t) - x^*, \frac{dx}{dt} \right\rangle \\
&= \lambda\langle x - x^*, h_1 h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] \\
&\quad -\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] + \rho\Upsilon x - h_1(x)\rangle \\
&= -\lambda\langle x - x^*, h_1(x) - h_1 h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] \\
&\quad +\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] - \rho\Upsilon x\rangle \\
&= -\lambda\langle x - x^*, h_1(x) - h_1(x^*) + h_1(x^*) - h_1 h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] \\
&\quad +\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] - \rho\Upsilon x\rangle \\
&= -\lambda\langle x - x^*, h_1(x) - h_1(x^*)\rangle + \lambda\langle h_1 h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] \\
&\quad -\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] + \rho\Upsilon x - h_1(x^*), x - x^*\rangle \\
&\leq -\lambda\sigma\|x - x^*\|^2 + \lambda\|x - x^*\|\,\|h_1 h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] \\
&\quad -\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}[h_1(x) - \rho\Upsilon x] + \rho\Upsilon x - h_1(x^*)\|, \qquad (4.15)
\end{aligned}$$

where we have used the strongly monotonicity of the operator h_1 with constant $\sigma > 0$.

Since $x^* \in \mathcal{H} : h_2(x^*) \in \Omega(x)$ is the solution of problem (2.1), therefore using (4.8), we have

$$
\begin{aligned}
& \left\|\left\{h_1 h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]+\rho\Upsilon x\right\}-h_1(x^*)\right\| \\
= \;& \|\left\{h_1 h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]\right. \\
& \left.-\rho\Upsilon h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]+\rho\Upsilon x\right\}-\left\{h_1 h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]\right. \\
& \left.-\rho\Upsilon h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]+\rho\Upsilon x^*\right\}\| \\
\leq \;& \left\|h_1 h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-h_1 h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]\right\| \\
& +\rho\left\|\Upsilon h_2^{-1}\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Upsilon h_2^{-1}\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]\right\| \\
& +\rho\left\|\Upsilon x-\Upsilon x^*\right\| \\
\leq \;& \theta(\delta+\rho\beta)\left\|\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]\right\|+\rho\beta\|x-x^*\| \\
\leq \;& \theta(\delta+\rho\beta)\{\left\|\Pi_{\Omega(x)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(x^*)}\left[h_1(x)-\rho\Upsilon x\right]\right\| \\
& +\left\|\Pi_{\Omega(x^*)}\left[h_1(x)-\rho\Upsilon x\right]-\Pi_{\Omega(x^*)}\left[h_1(x^*)-\rho\Upsilon x^*\right]\right\|\}+\rho\beta\|x-x^*\| \\
\leq \;& \theta(\delta+\rho\beta)\{\nu\|x-x^*\| \\
& +\|(h_1(x)-h_1(x^*))-\rho(\Upsilon x-\Upsilon x^*)\|\}+\rho\beta\|x-x^*\| \\
\leq \;& \theta(\delta+\rho\beta)\{\nu\|x-x^*\|+\|h_1(x)-h_1(x^*)\| \\
& +\rho\|\Upsilon x-\Upsilon x^*\|\}+\rho\beta\|x-x^*\| \\
\leq \;& \theta(\delta+\rho\beta)(\nu+\delta+\rho\beta)\|x-x^*\|+\rho\beta\|x-x^*\| \\
= \;& \{\theta(\delta+\rho\beta)(\nu+\delta+\rho\beta)+\rho\beta\}\|x-x^*\|,
\end{aligned} \tag{4.16}
$$

where we have used Assumption 2.11 and the Lipschitz continuity of the operators Υ, h_1 and h_2^{-1} with constants $\beta>0, \delta>0$ and $\theta>0$ respectively.

Combining (4.15) and (4.16), we have

$$
\begin{aligned}
\frac{d\mathcal{L}}{dt} &\leq -\lambda\sigma\|x-x^*\|^2+\lambda\{\rho\beta+\theta(\delta+\rho\beta)(\nu+\delta+\rho\beta)\}\|x-x^*\|^2 \\
&= \lambda\{-\sigma+\rho\beta+\theta(\delta+\rho\beta)(\nu+\delta+\rho\beta)\}\|x-x^*\|^2 \\
&= \lambda k_1\|x-x^*\|^2,
\end{aligned}
$$

where

$$k_1=-\sigma+\rho\beta+\theta(\delta+\rho\beta)(\nu+\delta+\rho\beta)$$

Let $k_1=-k_2$, where $k_2>0$, then

$$\frac{d\mathcal{L}}{dt}\leq -\lambda k_2\|x-x^*\|^2,$$

which implies that

$$\|x(t)-x^*\|\leq\|x(t_0)-x^*\|e^{-\lambda k_2(t-t_0)},$$

This shows that the trajectory of the solution of the dynamical system (4.9) converges globally exponentially to the unique solution of problem (2.1). □

5. Conclusion

In this paper, we have introduced two new dynamical systems associated with the extended general quasi variational inequalities. These dynamical systems have been used to investigate the unique existence of the solutions of the quasi variational inequalities. Several special cases are discussed. Results obtained in this paper can be extended for quasi split feasibility problems, see [23], which is another direction for future research. The ideas and the technique of this paper may inspire the interested readers to discover novel and innovative applications of quasi variational inequalities in pure and applied sciences. This research is supported by HEC NRPU project No: 20-1966/R&D 11-2553.

Acknowledgement: Authors are grateful to Dr. S. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research facilities.

References

1. Q.H. Ansari, J. Balooee and J.-C. Yao, *Extended general nonlinear quasi-variational inequalities and projection dynamical systems,* Taiwanese J. Math. **17** (2013), no. 4, 1321-1352.
2. A.S. Antipin, M. Jacimovic and N. Mijailovic, *A second-order continuous method for solving quasi-variational inequalities,* Comput. Math. Math. Phy. **51** (2013), no. 11, 1856–1863.
3. A. Baiocchi and A. Capelo, *Variational and Quasi-Variational Inequalities*, J. Willey and Sons, New York, 1984.
4. A. Bensoussan and J.L. Lions, *Applications Des Inequations Variationnelles En Control Eten Stochastique*, Dunod, Paris, 1978.
5. M.-G. Cojocaru, P. Daniele and A. Nagurney, *Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces with applications,* J. Optim. Theory Appl. **127** (2005), no. 3, 549–563.
6. J. Dong, D. Zhang and A. Nagurney, *A projected dynamical systems model of general financial equilibrium with stability analysis,* Math. Comput. Modell. **24** (1996), no. 2, 35–44.
7. P. Dupuis and A. Nagurney, *Dynamical systems and variational inequalities,* Ann. Oper. Res. **44** (1993), no. 1, 7–42.
8. F. Facchinei, C. Kanzow and S. Sagratella, *Solving quasi-variational inequalities via their KKT conditions,* Math. Program. Ser. A. (2013), doi:10.1007/s10107-013-0637-0.
9. T.L. Friesz, D. Bernstein, N.J. Mehta, R.L. Tobin and S. Ganjalizadeh, *Day-to-day dynamic network disequilibria and idealized traveler information systems,* Oper. Res. **42** (1994), no. 6, 1120–1136.
10. T.L. Friesz, D. Bernstein and R. Stough, *Dynamic systems, variational inequalities and control theoretic models for predicting time-varying urban network flows,* Trans. Sci. **30** (1996), no. 1, 14–31.
11. A.S. Kravchuk and P.J. Neittaanmki, *Variational and Quasi-variational Inequalities in Mechanics*, Springer,Dordrecht, Holland, 2007.
12. F. Lenzen, F. Becker, J. Lellmann, S. Petra and C. Schnörr, *A class of quasi-variational inequalities for adaptive image denoising and decomposition,* Comput. Optim. Appl. **54** (2013), no. 2, 371–398.
13. Q. Liu and J. Cao, *A recurrent neural network based on projection operator for extended general variational inequalities. Systems*, IEEE. Trans. Syst. Man Cybern. Part B Cybern. **40** (2010), no. 3, 928–938.
14. Q. Liu and Y. Yang, *Global exponential system of projection neural networks for system of generalized variational inequalities and related nonlinear minimax problems,* Neurocomputing. **73** (2010), no. 10, 2069–2076.
15. A. Nagurney and A.D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic, Boston, 1996.
16. Z.D. Mitrovic, *Some existence results on a class of inclusions.* Ann. Funct. Anal. **2** (2011), no. 2, 51–58.
17. M.A. Noor, *Quasi variational inequalities,* Appl. Math. Lett. **1** (1988), no. 4, 367–370.

18. M.A. Noor, *A Wiener–Hopf dynamical system for variational inequalities,* New Zealand J. Math. **31** (2002), 173–182.
19. M.A. Noor, *Implicit dynamical systems and quasi variational inequalities,* Appl. Math. Comput. **134** (2003), no. 1, 69–81.
20. M.A. Noor, *Existence results for quasi variational inequalities,* Banach J. Math. Anal. **1 (2007)**, no. 2, 186–194.
21. M.A. Noor, *Auxiliary principle technique for extended general variational inequalities,* Banach J. Math. Anal. **2** (2008), no. 1, 33–39.
22. M.A. Noor and K.I. Noor, *Sensitivity analysis of some quasi variational inequalities,* J. Adv. Math. Studies. **6** (2013), no. 1, 43–52.
23. M.A. Noor and K.I. Noor, *Some new classes of quasi split feasibility problems,* Appl. Math. Inform. Sci. **7** (2013), no. 4, 1547–1552.
24. M.A. Noor, K.I. Noor and A.G. Khan, *Some iterative schemes for solving extended general quasi variational inequalities,* Appl. Math. Inform. Sci. **7** (2013), no. 3, 917–925.
25. Y. Shehu, *Iterative methods for fixed points and equilibrium problems,* Ann. Funct. Anal. **1** (2010), no. 2, 121–132.
26. P. Shi, *Equivalence of variational inequalities with Wiener–Hopf equations,* Proc. Amer. Math. Soc. **111** (1991), no. 2, 339–346.
27. G. Stampacchia, *Formes bilineaires coercivites sur les ensembles convexes,* C. R. Acad. Sci. Paris **258** (1964), 4413–4416.
28. Z.-M. Wang, M.K. Kang and Y.J. Cho, *Convergence theorems based on the shrinking projection method for hemi-relatively nonexpansive mappings, variational inequalities and equilibrium problems,* Banach J. Math. Anal. **6** (2012), no. 1, 11–34.
29. Y. Xia and J. Wang, *On the stability of globally projected dynamical systems,* J. Optim. Theory Appl. **106** (2000), no. 1, 129–150.

Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

E-mail address: noormaslam@gmail.com
E-mail address: khalidan@gmail.com
E-mail address: awaisgulkhan@gmail.com