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ON THE TRANSCENDENTAL RADIUS OF THE VOLTERRA INTEGRATION OPERATOR

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Abstract. The transcendental radius of the Volterra integration operator V acting in the space $L^2(0;1)$ is calculated. The latter is compared with the norm of the self-commutator of V.

1. Introduction and preliminaries

Let A be a linear bounded operator, acting in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. According to [7] there exists a unique complex number c belonging to the closure of the numerical range $W(A)$ such that

$$m(A) = \inf_{\lambda\in\mathbb{C}} \|A - \lambda I\| = \|A - cI\|.$$

Fujii and Prasanna [1] called $m(A)$ transcendental radius of A. Prasanna proved [6] that

$$m^2(A) = \sup_{\|x\|=1} \left\{ \|Ax\|^2 - |\langle Ax, x\rangle|^2 \right\}. \tag{1.1}$$

In [5] a more general problem is considered and is proved that

$$m_T^2(A) = \|T - \lambda_0 A\|^2 = \inf_{\lambda\in\mathbb{C}} \|T - \lambda A\|^2 = \sup_{\|x\|=1} \left\{ \|Tx\|^2 - \frac{|\langle Tx, Ax\rangle|^2}{\|Ax\|^2} \right\}. \tag{1.2}$$

The number λ_0 is unique if the approximate point spectrum of A does not contain 0 and is characterized by the following conditions. There exists a sequence of unit elements $\{x_n\}$ such that

$$\langle (T - \lambda_0 A)x_n, Ax_n\rangle \to 0, \|(T - \lambda_0 A)x_n\| \to \|T - \lambda_0 A\|. \tag{1.3}$$

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In [3] is proved that

$$\lambda_0 = \lim_{n\to\infty} \frac{\langle Tx_n, Ax_n\rangle}{\|Ax_n\|^2}, \tag{1.4}$$

where $\{x_n\}$ is a sequence of unit vectors, approximating the supremum in (1.2). It is easy to see that (1.4) implies both conditions in (1.3). Indeed, denoting

$$\lambda_n = \frac{\langle Tx_n, Ax_n\rangle}{\|Ax_n\|^2}$$

we have

$$\langle (T - \lambda_0 A)\, x_n, Ax_n\rangle = (\lambda_n - \lambda_0) \cdot \|Ax_n\|^2 \to 0.$$

For the second equality

$$\|T - \lambda_0 A\|^2 = \lim_{n\to\infty} \left\{ \|Tx_n\|^2 - |\lambda_n|^2 \cdot \|Ax_n\|^2 \right\}.$$

On the other hand

$$\|(T - \lambda_0 A)\, x_n\|^2 = \|Tx_n\|^2 - 2\,\|Ax_n\|^2 \cdot \operatorname{Re} \lambda_n \overline{\lambda}_0 + |\lambda_0|^2 \cdot \|Ax_n\|^2 .$$

2. Main results

Consider the Volterra integration operator in $L^2\,(0\,;1)$ defined by the formula

$$(Vf)\,(x) = \int_0^x f\,(t)\,dt.$$

Easy calculations show that

$$(VV^* f)\,(x) = \int_0^x tf\,(t)dt + x\int_x^1 f\,(t)dt,$$

$$(V^* V f)\,(x) = \int_0^1 f\,(t)dt - x\int_0^x f\,(t)dt - \int_x^1 tf\,(t)dt.$$

Now we search

$$\inf_{\lambda\in\mathbb{C}} \|V - \lambda I\|.$$

The equality $V\overline{f} - \overline{c}\overline{f} = \overline{Vf - cf}$ implies $\|V - \overline{c}I\| = \|V - cI\|$ and finally, $c \in \mathbb{R}$. Recall ([4], Problem 165) that $W\,(V)$ is bounded by the curve

$$t \mapsto \frac{1-\cos t}{t^2} \pm i\frac{t - \sin t}{t^2},\ 0 \le t \le 2\pi$$

and $\|V\| = \frac{2}{\pi}$, therefore $0 \leqslant c \leqslant \frac{1}{2}$, $m^2\,(V) \leqslant \frac{4}{\pi^2}$.

The operator $S = (V^* - \lambda I)\,(V - \lambda I)$ is defined by the formula

$$(Sf)\,(x) = (1-\lambda)\int_0^1 f\,(t)\,dt - x\int_0^x f\,(t)dt - \int_x^1 tf\,(t)dt + \lambda^2 f\,(x)\,. \tag{2.1}$$

As $V^*V-\lambda\left(V^*+V\right)$ is a self-adjoint compact operator, its norm coincides with the eigenvalue having the greatest absolute value. For the positive operator $S=V^*V-\lambda\left(V^*+V\right)+\lambda^2 I$ one has $\|S\|=\max\left\{eig\left(S\right)\right\}$, where $\left\{eig\left(S\right)\right\}$ is the set of eigenvalues of S.

Calculating the second derivative of (2.1), we get for the eigenfunctions of S the second order differential equation

$$\left(\mu-\lambda^2\right)f''\left(x\right)+f\left(x\right)=0.$$

It is easy to see that the eigenfunction satisfies the condition $f'\left(0\right)=0$, so

$$f\left(x\right)=\cos\frac{x}{\sqrt{\mu-\lambda^2}}.$$

Putting $f\left(t\right)=\cos\alpha t$ into (2.1), we get

$$\left(Sf\right)\left(x\right)=-\frac{\lambda}{\alpha}\sin\alpha-\frac{1}{\alpha^2}\cos\alpha+\frac{1}{\alpha^2}\cos\alpha x+\lambda^2\cos\alpha x,$$

meaning that f is an eigenfunction corresponding to the eigenvalue

$$\mu=\frac{1}{\alpha^2}+\lambda^2, \tag{2.2}$$

if and only if α satisfies

$$\cot\alpha+\alpha\lambda=0. \tag{2.3}$$

For each $\lambda>0$ equation (2.3) has one and only one solution α in each interval $\left(k\pi+\pi/2\,;\,\left(k+1\right)\pi\right), k\in\mathbb{Z}^+$, therefore from (2.2) the greatest eigenvalue of S corresponds to the interval $\left(\pi/2\,;\,\pi\right)$. Then, we have

$$\mu=\frac{1}{\alpha^2}+\frac{\cot^2\alpha}{\alpha^2}=\frac{1}{\alpha^2\sin^2\alpha}.$$

The smallest value of $\|S\|$ corresponds to the greatest value of $\alpha\sin\alpha$, which occurs if $\alpha\in\left(\pi/2\,;\,\pi\right)$ satisfies

$$\tan\alpha+\alpha=0. \tag{2.4}$$

From (2.3) and (2.4) we get $\lambda=1/\alpha^2$. So we arrive to the following result.

Proposition 2.1. *The transcendental radius of the Volterra integration operator is equal to* $\sqrt{1/\alpha^2+1/\alpha^4}$, *where* $\alpha\in\left(\pi/2\,;\,\pi\right)$ *satisfies* (2.4).

The approximate solution of the transcendental equation (2.4), given by MatLab is $\alpha\approx 2.028757838110434$; $\lambda\approx 0.242962685095034$ and

$$\min_{\lambda}\|V-\lambda I\|^2\approx 0.301993551443623.$$

Putting the function $f\left(x\right)=2\left(\frac{\alpha}{2\alpha+\sin 2\alpha}\right)^{1/2}\cos\alpha x$ into (1.4) and (1.1), we get the same values of λ and of the minimal norm.

From the general theory of the Sturm–Liouville operator theory the following result may be deduced.

Corollary 2.2. *The sequence of function* $\left\{\cos\alpha_k x\right\}_{k=1}^{\infty}$, *where* $\left\{\alpha_k\right\}$ *are the positive roots of the equation* $\cot\alpha+\alpha\lambda=0$ *form an orthogonal basis of* $L^2\left(0;\,1\right)$.

Now we intend to show an application of the result above to a problem in operator theory.

In [7] it is shown that the norm of inner derivation $D_T(A) = AT - TA$ is defined by the following formula

$$\sup_{\|A\|=1} \|AT - TA\| = 2m(T).$$

In [2] is proved that for any operator A the self-commutator $C(A) = AA^* - A^*A$ satisfies the following inequality

$$\|AA^* - A^*A\| \leq \|A\|^2. \tag{2.5}$$

For some operators the inequality turns to be the equality.

Example 2.3. Let S be the operator of the simple unilateral shift. Then $S^*S - SS^*$ is the operator of orthogonal projection on the first element of the basis, shifted by S, so $\|S^*S - SS^*\| = 1$ and $\|S\| = 1$, hence $\|S^*S - SS^*\| = \|S\|^2$.

As $C(A - \lambda I) = C(A)$ for any $\lambda \in \mathbb{C}$, inequality (2.5) may be sharpened

$$\|AA^* - A^*A\| \leqslant m^2(A).$$

For the Volterra operator we have

$$((V^*V - VV^*) f)(x) = \int_0^1 f(t)dt - x\int_0^1 f(t)dt - \int_0^1 tf(t)dt.$$

The self-commutator $C(V)$ is a two dimensional self-adjoint operator with unit eigenfunctions $e_1 = \sqrt{2+\sqrt{3}}\left(\left(3-\sqrt{3}\right)x - 1\right)$, $e_2 = \frac{1}{\sqrt{2+\sqrt{3}}}\left(\left(3+\sqrt{3}\right)x - 1\right)$ corresponding to the eigenvalues $\left\{\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}\right\}$, so

$$\|C(V)\| = \frac{\sqrt{3}}{6} \approx 0.288675134\cdots.$$

Easily can be proved that the operator

$$(Bf)(x) = \frac{1}{\sqrt{2\sqrt{3}}}\left(\left(3+\sqrt{3}\right)x - 1\right)\int_0^1 \left(\left(3-\sqrt{3}\right)t - 1\right) f(t)\,dt$$

has the same self-commutator - $C(V) = C(B)$.

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