Annals of Functional Analysis
Ann. Funct. Anal. 1 (2010), no. 2, 75–91
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

INFINITE-DIMENSIONAL BICOMPLEX HILBERT SPACES

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Communicated by M. S. Moslehian

ABSTRACT. This paper begins the study of infinite-dimensional modules defined on bicomplex numbers. It generalizes a number of results obtained with finite-dimensional bicomplex modules. The central concept introduced is the one of a bicomplex Hilbert space. Properties of such spaces are obtained through properties of several of their subsets which have the structure of genuine Hilbert spaces. In particular, we derive the Riesz representation theorem for bicomplex continuous linear functionals and a general version of the bicomplex Schwarz inequality. Applications to concepts relevant to quantum mechanics, specifically the bicomplex analogue of the quantum harmonic oscillator, are pointed out.

1. Introduction

The mathematical structure of quantum mechanics consists in Hilbert spaces defined over the field of complex numbers [14]. This structure has been extremely successful in explaining vast amounts of experimental data pertaining largely, but not exclusively, to the world of molecular, atomic and subatomic phenomena.

Bicomplex numbers [2], just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.

Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra

Date: Received: 25 July 2010; Accepted: 23 December 2010.
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2010 *Mathematics Subject Classification.* Primary 16D10; Secondary 30G35, 46C05, 46C50.
Key words and phrases. Bicomplex numbers, bicomplex quantum mechanics, Hilbert spaces, Banach algebras, bicomplex linear algebra.

property. Indeed considerable work has been done over the years on quaternionic quantum mechanics [1]. However, in the past few years it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger's equation for a particle in one dimension was proposed [11] and self-adjoint operators were defined on finite-dimensional bicomplex Hilbert spaces [5, 12].

In this spirit, eigenvalues and eigenkets or eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality [6] over an infinite-dimensional bicomplex module M. The harmonic oscillator is one of the simplest and, at the same time, one of the most important systems of quantum mechanics, involving as it is an infinite-dimensional vector space. However, the module M defined in [6] does not have a property of completeness. Indeed it is made up of finite linear combinations of eigenkets or, in the coordinate basis, of hyperbolic Hermite polynomials. Hence Cauchy sequences of elements of M do not in general converge to an element of M.

In this paper we introduce the mathematical tools necessary to investigate the bicomplex analogue of quantum-mechanical state spaces. This we do by defining the concept of infinite-dimensional bicomplex Hilbert space, closely related to its complex version and already introduced in [5, 12] for finite dimensions. In Section 2, we summarize known algebraic properties of bicomplex numbers and recall the concepts of bicomplex modules and scalar products. Section 3 contains the main results of this paper. Bicomplex Hilbert spaces are introduced and subsets are identified that have the structure of standard complex Hilbert spaces. We derive a general version of the bicomplex Schwarz inequality and a version of Riesz's representation theorem for bicomplex continuous linear functionals. For an arbitrary bicomplex Hilbert space M, the dual space M^* of continuous linear functionals on M can then be identified with M through the bicomplex scalar product $(\cdot, \cdot)$. These are the tools necessary to better justify the concept of self-adjoint operators acting in an infinite-dimensional bicomplex Hilbert space introduced in [6]. Bicomplex Hilbert spaces with countable bases are discussed and results proved on important Hilbert subspaces. Section 4 examines the example of the quantum harmonic oscillator. In the standard case the Hamiltonian eigenfunctions generate the state space $V = L^2(\mathbb{R})$. We construct an infinite-dimensional bicomplex module $M = (\mathbf{e_1}V) \oplus (\mathbf{e_2}V)$, which has the property of completeness and has the eigenfunctions of the bicomplex harmonic oscillator Hamiltonian as a basis.

2. Preliminaries

This section first summarizes a number of known results on the algebra of bicomplex numbers, which will be needed in this paper. Much more details as well as proofs can be found in [2, 10, 11, 12]. Basic definitions related to bicomplex modules and scalar products are also formulated as in [5, 12], but here we make no restrictions to finite dimensions.

2.1. **Bicomplex Numbers.**

2.1.1. *Definition.* The set $\mathbb{T}$ of *bicomplex numbers* is defined as

$$\mathbb{T} := \{w = z_1 + z_2\mathbf{i_2} \mid z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}, \tag{2.1}$$

where $\mathbf{i_1}$ and $\mathbf{i_2}$ are independent imaginary units such that $\mathbf{i_1}^2 = -1 = \mathbf{i_2}^2$. The product of $\mathbf{i_1}$ and $\mathbf{i_2}$ defines a hyperbolic unit $\mathbf{j}$ such that $\mathbf{j}^2 = 1$. The product of all units is commutative and satisfies

$$\mathbf{i_1}\mathbf{i_2} = \mathbf{j}, \qquad \mathbf{i_1}\mathbf{j} = -\mathbf{i_2}, \qquad \mathbf{i_2}\mathbf{j} = -\mathbf{i_1}.$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{T}$ makes up a commutative ring.

Three important subsets of $\mathbb{T}$ can be specified as

$$\begin{aligned} \mathbb{C}(\mathbf{i_k}) &:= \{x + y\mathbf{i_k} \mid x, y \in \mathbb{R}\}, \qquad k = 1, 2; \\ \mathbb{D} &:= \{x + y\mathbf{j} \mid x, y \in \mathbb{R}\}. \end{aligned}$$

Each of the sets $\mathbb{C}(\mathbf{i_k})$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of so-called *hyperbolic numbers*.

2.1.2. *Conjugation and Moduli.* Three kinds of conjugation can be defined on bicomplex numbers. With w specified as in (2.1) and the bar $(\bar{\ })$ denoting complex conjugation in $\mathbb{C}(\mathbf{i_1})$, we define

$$w^{\dagger_1} := \bar{z}_1 + \bar{z}_2\mathbf{i_2}, \qquad w^{\dagger_2} := z_1 - z_2\mathbf{i_2}, \qquad w^{\dagger_3} := \bar{z}_1 - \bar{z}_2\mathbf{i_2}.$$

It is easy to check that each conjugation has the following properties:

$$(s + t)^{\dagger_k} = s^{\dagger_k} + t^{\dagger_k}, \qquad \left(s^{\dagger_k}\right)^{\dagger_k} = s, \qquad (s \cdot t)^{\dagger_k} = s^{\dagger_k} \cdot t^{\dagger_k}.$$

Here $s, t \in \mathbb{T}$ and $k = 1, 2, 3$.

With each kind of conjugation, one can define a specific bicomplex modulus as

$$\begin{aligned} |w|^2_{\mathbf{i_1}} &:= w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \ \in \mathbb{C}(\mathbf{i_1}), \\ |w|^2_{\mathbf{i_2}} &:= w \cdot w^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + 2\,\mathrm{Re}(z_1\bar{z}_2)\mathbf{i_2} \ \in \mathbb{C}(\mathbf{i_2}), \\ |w|^2_{\mathbf{j}} &:= w \cdot w^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) - 2\,\mathrm{Im}(z_1\bar{z}_2)\mathbf{j} \ \in \mathbb{D}. \end{aligned}$$

It can be shown that $|s \cdot t|^2_k = |s|^2_k \cdot |t|^2_k$, where $k = \mathbf{i_1}, \mathbf{i_2}$ or $\mathbf{j}$.

In this paper we will often use the Euclidean $\mathbb{R}^4$ norm defined as

$$|w| := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\mathrm{Re}(|w|^2_{\mathbf{j}})}\ .$$

Clearly, this norm maps $\mathbb{T}$ into $\mathbb{R}$. We have $|w| \geq 0$, and $|w| = 0$ if and only if $w = 0$. Moreover [10], for all $s, t \in \mathbb{T}$,

$$|s + t| \leq |s| + |t|, \qquad |s \cdot t| \leq \sqrt{2}\,|s| \cdot |t|.$$

2.1.3. *Idempotent Basis.* Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers $\mathbf{e_1}$ and $\mathbf{e_2}$ defined as

$$\mathbf{e_1} := \frac{1+\mathbf{j}}{2}, \qquad \mathbf{e_2} := \frac{1-\mathbf{j}}{2}.$$

In fact $\mathbf{e_1}$ and $\mathbf{e_2}$ are hyperbolic numbers. They make up the so-called *idempotent basis* of the bicomplex numbers. One easily checks that $(k = 1, 2)$

$$\mathbf{e_1}^2 = \mathbf{e_1}, \quad \mathbf{e_2}^2 = \mathbf{e_2}, \quad \mathbf{e_1} + \mathbf{e_2} = 1, \quad \mathbf{e_k}^{\dagger_3} = \mathbf{e_k}, \quad \mathbf{e_1}\mathbf{e_2} = 0. \tag{2.2}$$

Any bicomplex number w can be written uniquely as

$$w = z_1 + z_2\mathbf{i_2} = z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}, \tag{2.3}$$

where

$$z_{\widehat{1}} = z_1 - z_2\mathbf{i_1} \quad \text{and} \quad z_{\widehat{2}} = z_1 + z_2\mathbf{i_1}$$

both belong to $\mathbb{C}(\mathbf{i_1})$. Note that

$$|w| = \frac{1}{\sqrt{2}}\sqrt{|z_{\widehat{1}}|^2 + |z_{\widehat{2}}|^2}\,. \tag{2.4}$$

The caret notation ($\widehat{1}$ and $\widehat{2}$) will be used systematically in connection with idempotent decompositions, with the purpose of easily distinguishing different types of indices. As a consequence of (2.2) and (2.3), one can check that if $\sqrt[n]{z_{\widehat{1}}}$ is an nth root of $z_{\widehat{1}}$ and $\sqrt[n]{z_{\widehat{2}}}$ is an nth root of $z_{\widehat{2}}$, then $\sqrt[n]{z_{\widehat{1}}}\,\mathbf{e_1} + \sqrt[n]{z_{\widehat{2}}}\,\mathbf{e_2}$ is an nth root of w.

The uniqueness of the idempotent decomposition allows the introduction of two projection operators as

$$P_1 : w \in \mathbb{T} \mapsto z_{\widehat{1}} \in \mathbb{C}(\mathbf{i_1}),$$
$$P_2 : w \in \mathbb{T} \mapsto z_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1}).$$

The P_k $(k = 1, 2)$ satisfy

$$[P_k]^2 = P_k, \qquad P_1\mathbf{e_1} + P_2\mathbf{e_2} = \mathbf{Id},$$

and, for $s, t \in \mathbb{T}$,

$$P_k(s + t) = P_k(s) + P_k(t), \qquad P_k(s \cdot t) = P_k(s) \cdot P_k(t).$$

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2}) \cdot (z'_{\widehat{1}}\mathbf{e_1} + z'_{\widehat{2}}\mathbf{e_2}) = z_{\widehat{1}}z'_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}z'_{\widehat{2}}\mathbf{e_2}.$$

Since 1 is uniquely decomposed as $\mathbf{e_1} + \mathbf{e_2}$, we can see that $w \cdot w' = 1$ if and only if $z_{\widehat{1}}z'_{\widehat{1}} = 1 = z_{\widehat{2}}z'_{\widehat{2}}$. Thus w has an inverse if and only if $z_{\widehat{1}} \neq 0 \neq z_{\widehat{2}}$, and the inverse w^{-1} is then equal to $(z_{\widehat{1}})^{-1}\mathbf{e_1} + (z_{\widehat{2}})^{-1}\mathbf{e_2}$. A nonzero w that does not have an inverse has the property that either $z_{\widehat{1}} = 0$ or $z_{\widehat{2}} = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called *null cone* $\mathcal{NC}$. That terminology comes from the fact that when w is written as in (2.1), zero divisors are such that $z_1^2 + z_2^2 = 0$.

Any hyperbolic number can be written in the idempotent basis as $x_{\hat{1}}\mathbf{e_1} + x_{\hat{2}}\mathbf{e_2}$, with $x_{\hat{1}}$ and $x_{\hat{2}}$ in $\mathbb{R}$. We define the set $\mathbb{D}^+$ of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{x_{\hat{1}}\mathbf{e_1} + x_{\hat{2}}\mathbf{e_2} \mid x_{\hat{1}}, x_{\hat{2}} \geq 0\}.$$

Since $w^{\dagger_3} = \bar{z}_{\hat{1}}\mathbf{e_1} + \bar{z}_{\hat{2}}\mathbf{e_2}$, it is clear that $w \cdot w^{\dagger_3} \in \mathbb{D}^+$ for any w in $\mathbb{T}$.

2.2. **$\mathbb{T}$-Modules.** The set of bicomplex numbers is a commutative ring. Just like vector spaces are defined over fields, modules are defined over rings [3]. A module M defined over the ring $\mathbb{T}$ of bicomplex numbers is called a $\mathbb{T}$-*module* [5, 12].

Definition 2.1. Let M be a $\mathbb{T}$-module. For $k = 1, 2$, we define V_k as the set of all elements of the form $\mathbf{e_k}|\psi\rangle$, with $|\psi\rangle \in M$. Succinctly, $V_1 := \mathbf{e_1}M$ and $V_2 := \mathbf{e_2}M$.

For $k = 1, 2$, addition and multiplication by a $\mathbb{C}(\mathbf{i_1})$ scalar are closed in V_k. Therefore, V_k is a vector space over $\mathbb{C}(\mathbf{i_1})$. Any element $|v_k\rangle \in V_k$ satisfies $|v_k\rangle = \mathbf{e_k}|v_k\rangle$.

For arbitrary $\mathbb{T}$-modules, vector spaces V_1 and V_2 bear no structural similarities. For more specific modules, however, they may share structure. It was shown in [5] that if M is a finite-dimensional free $\mathbb{T}$-module, then V_1 and V_2 have the same dimension. Other similar instances will be examined in Section 3.

Proposition 2.2. *Let M be a $\mathbb{T}$-module and let $|\psi\rangle \in M$. There exist unique vectors $|v_1\rangle \in V_1$ and $|v_2\rangle \in V_2$ such that $|\psi\rangle = |v_1\rangle + |v_2\rangle$.*

Proof. Let $|\psi\rangle \in M$. We can always write

$$|\psi\rangle = \mathbf{e_1}|\psi\rangle + \mathbf{e_2}|\psi\rangle = |v_1\rangle + |v_2\rangle,$$

where $|v_k\rangle := \mathbf{e_k}|\psi\rangle \in V_k$, for $k = 1, 2$. Suppose that $|\psi\rangle = |v_1'\rangle + |v_2'\rangle$, with $|v_k'\rangle \in V_k$. Then

$$|v_1\rangle + |v_2\rangle = |v_1'\rangle + |v_2'\rangle.$$

Multiplying both sides with $\mathbf{e_k}$ and making use of (2.2), we obtain

$$|v_k\rangle = \mathbf{e_k}|v_k\rangle = \mathbf{e_k}|v_k'\rangle = |v_k'\rangle$$

for $k = 1, 2$. $\square$

Henceforth we will write $|\psi\rangle_{\mathbf{k}} = \mathbf{e_k}|\psi\rangle$, keeping in mind that $\mathbf{e_k}|\psi\rangle_{\mathbf{k}} = |\psi\rangle_{\mathbf{k}}$. Proposition 2.2 immediately leads to the following result.

Theorem 2.3. *The $\mathbb{T}$-module M can be viewed as a vector space M' over $\mathbb{C}(\mathbf{i_1})$, and $M' = V_1 \oplus V_2$.*

From a set-theoretical point of view, M and M' are identical. In this sense we can say, perhaps improperly, that the **module** M can be decomposed into the direct sum of two vector spaces over $\mathbb{C}(\mathbf{i_1})$, i.e. $M = V_1 \oplus V_2$.

2.3. **Bicomplex Scalar Product.** The norm of a vector is an important concept in vector space theory. We will now generalize it to $\mathbb{T}$-modules, making use of the association established in Theorem 2.3.

Definition 2.4. Let M be a $\mathbb{T}$-module and let M' be the associated vector space. We say that $\|\cdot\| : M \longrightarrow \mathbb{R}$ is a **$\mathbb{T}$-norm** on M if the following holds:

1. $\|\cdot\| : M' \longrightarrow \mathbb{R}$ is a norm;
2. $\big\|w \cdot |\psi\rangle\big\| \leq \sqrt{2}|w| \cdot \big\||\psi\rangle\big\|$, $\forall w \in \mathbb{T}$, $\forall |\psi\rangle \in M$.

A $\mathbb{T}$-module with a **$\mathbb{T}$-norm** is called a **normed $\mathbb{T}$-module**.

In vector space theory, a norm can be induced by a scalar product. Having in mind the use of such norms, we recall the definition of a **bicomplex scalar product** introduced in [12] (the physicists' ordering convention being used).

Definition 2.5. Let M be a $\mathbb{T}$-module. Suppose that with each pair $|\psi\rangle$ and $|\phi\rangle$ in M, taken in this order, we associate a bicomplex number $(|\psi\rangle, |\phi\rangle)$. We say that the association defines a bicomplex scalar (or inner) product if it satisfies the following conditions:

1. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle)$, $\forall |\psi\rangle, |\phi\rangle, |\chi\rangle \in M$;
2. $(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle)$, $\forall \alpha \in \mathbb{T}$, $\forall |\psi\rangle, |\phi\rangle \in M$;
3. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^{\dagger_3}$, $\forall |\psi\rangle, |\phi\rangle \in M$;
4. $(|\psi\rangle, |\psi\rangle) = 0 \;\Leftrightarrow\; |\psi\rangle = 0$, $\forall |\psi\rangle \in M$.

Property 3 implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. Definition 2.5 is intended to be very general. In this paper we shall be more restrictive, by requiring the bicomplex scalar product $(\cdot, \cdot)$ to be *hyperbolic positive*, that is,

$$(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+, \ \forall |\psi\rangle \in M.$$

From Definition 2.5 it is easy to see that the following projection of a bicomplex scalar product:

$$(\cdot, \cdot)_{\widehat{k}} := P_k((\cdot, \cdot)) : M \times M \longrightarrow \mathbb{C}(\mathbf{i_1})$$

is a **standard scalar product** on V_k, for $k = 1, 2$.

Theorem 2.6. *Let $|\psi\rangle, |\phi\rangle \in M$, then*

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e_1}(|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}})_{\widehat{1}} + \mathbf{e_2}(|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}})_{\widehat{2}} \tag{2.5}$$

Proof. Since $|\psi\rangle = |\psi\rangle_{\mathbf{1}} + |\psi\rangle_{\mathbf{2}}$ and $|\phi\rangle = |\phi\rangle_{\mathbf{1}} + |\phi\rangle_{\mathbf{2}}$, we have

$$\begin{aligned}
(|\psi\rangle, |\phi\rangle) &= (|\psi\rangle_{\mathbf{1}} + |\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{1}} + |\phi\rangle_{\mathbf{2}}) \\
&= \mathbf{e_1}\left(|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}}\right) + \mathbf{e_2}\left(|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}}\right) \\
&= \mathbf{e_1}\left\{\mathbf{e_1}P_1((|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}})) + \mathbf{e_2}P_2((|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}}))\right\} \\
&\qquad + \mathbf{e_2}\left\{\mathbf{e_1}P_1((|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}})) + \mathbf{e_2}P_2((|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}}))\right\} \\
&= \mathbf{e_1}P_1((|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}})) + \mathbf{e_2}P_2((|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}})) \\
&= \mathbf{e_1}\left(|\psi\rangle_{\mathbf{1}}, |\phi\rangle_{\mathbf{1}}\right)_{\widehat{1}} + \mathbf{e_2}\left(|\psi\rangle_{\mathbf{2}}, |\phi\rangle_{\mathbf{2}}\right)_{\widehat{2}}.
\end{aligned}$$

□

We point out that a bicomplex scalar product is **completely characterized** by the two standard scalar products $(\cdot,\cdot)_{\widehat{k}}$ on V_k. In fact, if $(\cdot,\cdot)_{\widehat{k}}$ is an arbitrary scalar product on V_k, for $k=1,2$, then $(\cdot,\cdot)$ defined as in (2.5) is a bicomplex scalar product on M.

3. Bicomplex Hilbert Spaces

3.1. **General Results.** In this section we define the notion of a bicomplex Hilbert space and prove the analog of the Riesz representation theorem.

Definition 3.1. Let M be a $\mathbb{T}$-module and let $(\cdot,\cdot)$ be a bicomplex scalar product defined on M. The space $\{M,(\cdot,\cdot)\}$ is called a $\mathbb{T}$-inner product space, or bicomplex pre-Hilbert space. When no confusion arises, $\{M,(\cdot,\cdot)\}$ will simply be denoted by M.

Theorem 3.2. *Let M be a bicomplex pre-Hilbert space. Then $(V_k,(\cdot,\cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) pre-Hilbert space for $k=1,2$.*

Proof. Since $(\cdot,\cdot)_{\widehat{k}}$ is a standard scalar product when M' is restricted to the vector space V_k, then $(V_k,(\cdot,\cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) pre-Hilbert space. □

If V_1 and V_2 are complete, then $M'=V_1\oplus V_2$ is a direct sum of two Hilbert spaces. It is easy to see that M' is also a Hilbert space, when the following natural scalar product is defined over the direct sum [4]:

$$(|\psi\rangle_{\mathbf{1}}\oplus|\psi\rangle_{\mathbf{2}},|\phi\rangle_{\mathbf{1}}\oplus|\phi\rangle_{\mathbf{2}})=(|\psi\rangle_{\mathbf{1}},|\phi\rangle_{\mathbf{1}})_{\widehat{1}}+(|\psi\rangle_{\mathbf{2}},|\phi\rangle_{\mathbf{2}})_{\widehat{2}}\,.$$

From this scalar product, we can define a **norm** on the vector space M':

$$\begin{aligned}\big\||\phi\rangle\big\| &:= \frac{1}{\sqrt{2}}\sqrt{(|\phi\rangle_{\mathbf{1}},|\phi\rangle_{\mathbf{1}})_{\widehat{1}}+(|\phi\rangle_{\mathbf{2}},|\phi\rangle_{\mathbf{2}})_{\widehat{2}}}\\ &= \frac{1}{\sqrt{2}}\sqrt{\big||\phi\rangle_{\mathbf{1}}\big|_1^2+\big||\phi\rangle_{\mathbf{2}}\big|_2^2}\,.\end{aligned}\tag{3.1}$$

Here we wrote

$$\big||\phi\rangle_{\mathbf{k}}\big|_k=\sqrt{(|\phi\rangle_{\mathbf{k}},|\phi\rangle_{\mathbf{k}})_{\widehat{k}}}\,,\tag{3.2}$$

where $|\cdot|_k$ is the natural scalar-product-induced norm on V_k. The $1/\sqrt{2}$ factor in (3.1) is introduced so as to relate in a simple manner the norm with the bicomplex scalar product. Indeed we have

$$\big\||\phi\rangle\big\|=\frac{1}{\sqrt{2}}\sqrt{(|\phi\rangle_{\mathbf{1}},|\phi\rangle_{\mathbf{1}})_{\widehat{1}}+(|\phi\rangle_{\mathbf{2}},|\phi\rangle_{\mathbf{2}})_{\widehat{2}}}=\big|\sqrt{(|\phi\rangle,|\phi\rangle)}\big|,$$

which is easily seen through (2.5), (2.4) and the remark on roots made after that last equation.

It is easy to check that $\|\cdot\|$ is a $\mathbb{T}$**-norm** on M and that the $\mathbb{T}$-module M is **complete** with respect to the following metric on M:

$$d(|\phi\rangle,|\psi\rangle)=\big\||\phi\rangle-|\psi\rangle\big\|.$$

Thus M is a **complete** $\mathbb{T}$**-module**.

Let us summarize what we found by means of a definition and a theorem.

Definition 3.3. A bicomplex Hilbert space is a $\mathbb{T}$-inner product space M which is complete with respect to the induced $\mathbb{T}$-norm (3.1).

Theorem 3.4. *Let $\{M, (\cdot,\cdot)\}$ be a bicomplex pre-Hilbert space, and let the induced space V_k be complete with respect to the inner product $(\cdot,\cdot)_{\widehat{k}}$ for $k = 1, 2$. Then $\{M, (\cdot,\cdot)\}$ is a bicomplex Hilbert space.*

The converse of this theorem is also true, as shown by the following two results.

Theorem 3.5. *Let M be a bicomplex Hilbert space. The normed vector space $(V_k, ||\cdot||)$ is closed in M for $k = 1, 2$.*

Proof. Let $\{|\psi_n\rangle\} \in V_k$ be a sequence converging to a ket $|\psi\rangle \in M$. Using Property 2 of Definition 2.4, with $w = \mathbf{e_k}$, we see that $|\psi_n\rangle_{\mathbf{k}} \to |\psi\rangle_{\mathbf{k}}$ when $n \to \infty$. Thus $|\psi\rangle_{\mathbf{k}} = |\psi\rangle$ and the sequence $\{|\psi_n\rangle\}$ converges in V_k. Hence $(V_k, ||\cdot||)$ is closed in M. □

Corollary 3.6. *Let M be a bicomplex Hilbert space. Then $(V_k, (\cdot,\cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) Hilbert space for $k = 1, 2$.*

Proof. By Theorem 3.2, $(V_k, (\cdot,\cdot)_{\widehat{k}})$ is a normed space over $\mathbb{C}(\mathbf{i_1})$. Using the $\mathbb{T}$-norm introduced in (3.1) and Theorem 3.5, it is easy to see that $(V_k, |\cdot|_k)$ is complete since

$$\big||\phi\rangle_{\mathbf{k}}\big|_k = \sqrt{2}\big\||\phi\rangle_{\mathbf{k}}\big\|, \quad \forall |\phi\rangle_{\mathbf{k}} \in V_k.$$

Hence V_k is a complex (in $\mathbb{C}(\mathbf{i_1})$) Hilbert space. □

As a direct application of this result, we obtain the following **bicomplex Riesz representation theorem**.

Theorem 3.7 (Riesz)**.** *Let $\{M, (\cdot,\cdot)\}$ be a bicomplex Hilbert space and let $f : M \to \mathbb{T}$ be a continuous linear functional on M. Then there is a unique $|\psi\rangle \in M$ such that $\forall |\phi\rangle \in M$, $f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$.*

Proof. From the functional f on M we can define a functional $P_k(f)$ on V_k ($k = 1, 2$) as

$$P_k(f)(|\phi\rangle_{\mathbf{k}}) := P_k(f(|\phi\rangle_{\mathbf{k}})).$$

It is clear that $P_k(f)$ is linear and continuous if f is.

We now apply Riesz's theorem [9] to $P_k(f)$ and find a unique $|\psi_k\rangle \in V_k$ such that $\forall |\phi\rangle_{\mathbf{k}} \in V_k$,

$$P_k(f)(|\phi\rangle_{\mathbf{k}}) = (|\psi_k\rangle, |\phi\rangle_{\mathbf{k}})_{\widehat{k}}\,.$$

Letting $|\psi\rangle := |\psi_1\rangle + |\psi_2\rangle$ and making use of Theorem 2.6, we get for every $|\phi\rangle$ in M:

$$\begin{aligned}(|\psi\rangle, |\phi\rangle) &= \mathbf{e_1}(|\psi_1\rangle, |\phi\rangle_{\mathbf{1}})_{\widehat{1}} + \mathbf{e_2}(|\psi_2\rangle, |\phi\rangle_{\mathbf{2}})_{\widehat{2}} \\ &= \mathbf{e_1}P_1(f)(|\phi\rangle_{\mathbf{1}}) + \mathbf{e_2}P_2(f)(|\phi\rangle_{\mathbf{2}}) \\ &= \mathbf{e_1}P_1(f(|\phi\rangle_{\mathbf{1}})) + \mathbf{e_2}P_2(f(|\phi\rangle_{\mathbf{2}})).\end{aligned}$$

On the other hand,

$$\begin{aligned} f(|\phi\rangle) &= \mathbf{e_1}P_1(f(|\phi\rangle)) + \mathbf{e_2}P_2(f(|\phi\rangle)) \\ &= \mathbf{e_1}P_1(f(\mathbf{e_1}|\phi\rangle_\mathbf{1} + \mathbf{e_2}|\phi\rangle_\mathbf{2})) + \mathbf{e_2}P_2(f(\mathbf{e_1}|\phi\rangle_\mathbf{1} + \mathbf{e_2}|\phi\rangle_\mathbf{2})) \\ &= \mathbf{e_1}P_1(\mathbf{e_1}f(|\phi\rangle_\mathbf{1}) + \mathbf{e_2}f(|\phi\rangle_\mathbf{2})) + \mathbf{e_2}P_2(\mathbf{e_1}f(|\phi\rangle_\mathbf{1}) + \mathbf{e_2}f(|\phi\rangle_\mathbf{2})) \\ &= \mathbf{e_1}P_1(f(|\phi\rangle_\mathbf{1})) + \mathbf{e_2}P_2(f(|\phi\rangle_\mathbf{2})). \end{aligned}$$

Hence $f(|\phi\rangle) = (|\psi\rangle, |\phi\rangle)$. If $|\psi'\rangle$ works also, then $(|\psi\rangle, |\phi\rangle) = (|\psi'\rangle, |\phi\rangle)$ for any $|\phi\rangle$ in M, and therefore $(|\psi\rangle - |\psi'\rangle, |\phi\rangle) = 0$. Letting $|\phi\rangle = |\psi\rangle - |\psi'\rangle$, we conclude that $|\psi\rangle - |\psi'\rangle = 0$, that is, $|\psi\rangle$ is unique. □

Theorem 3.7 means that for an arbitrary bicomplex Hilbert space M, the dual space M^* of continuous linear functionals on M can be identified with M through the bicomplex inner product $(\cdot, \cdot)$.

We close this section by proving a general version of Schwarz's inequality in a bicomplex Hilbert space.

Theorem 3.8 (Bicomplex Schwarz inequality)**.** *Let $|\psi\rangle, |\phi\rangle \in M$. Then*

$$|(|\psi\rangle, |\phi\rangle)| \leq \sqrt{2}\big\||\psi\rangle\big\| \; \big\||\phi\rangle\big\|.$$

Proof. From the complex (in $\mathbb{C}(\mathbf{i_1})$) Schwarz inequality we have

$$|(|\psi\rangle_\mathbf{k}, |\phi\rangle_\mathbf{k})_{\widehat{k}}|^2 \leq \big||\psi\rangle_\mathbf{k}\big|_k^2 \cdot \big||\phi\rangle_\mathbf{k}\big|_k^2, \quad \forall |\psi\rangle_\mathbf{k}, |\phi\rangle_\mathbf{k} \in V_k.$$

Therefore, if $|\psi\rangle, |\phi\rangle \in M$, we obtain from (2.4) and (3.2)

$$\begin{aligned} |(|\psi\rangle, |\phi\rangle)| &= |\mathbf{e_1}(|\psi\rangle_\mathbf{1}, |\phi\rangle_\mathbf{1})_{\widehat{1}} + \mathbf{e_2}(|\psi\rangle_\mathbf{2}, |\phi\rangle_\mathbf{2})_{\widehat{2}}| \\ &= \frac{1}{\sqrt{2}}\sqrt{|(|\psi\rangle_\mathbf{1}, |\phi\rangle_\mathbf{1})_{\widehat{1}}|^2 + |(|\psi\rangle_\mathbf{2}, |\phi\rangle_\mathbf{2})_{\widehat{2}}|^2} \\ &\leq \frac{1}{\sqrt{2}}\sqrt{\big||\psi\rangle_\mathbf{1}\big|_1^2 \cdot \big||\phi\rangle_\mathbf{1}\big|_1^2 + \big||\psi\rangle_\mathbf{2}\big|_2^2 \cdot \big||\phi\rangle_\mathbf{2}\big|_2^2} \\ &\leq \sqrt{2}\big\||\psi\rangle\big\| \; \big\||\phi\rangle\big\|. \end{aligned}$$

□

This result is a direct generalization of the bicomplex Schwarz inequality obtained in [12] and proved here for an arbitrary bicomplex Hilbert space M without any extra condition on the bicomplex scalar product.

3.2. **Countable $\mathbb{T}$-Modules.** In this section we investigate more specific $\mathbb{T}$-modules, namely those that have a countable basis. Such modules have another important Hilbert subspace.

3.2.1. *Schauder $\mathbb{T}$-Basis.*

Definition 3.9. Let M be a normed $\mathbb{T}$-module. We say that M has a **Schauder $\mathbb{T}$-basis** if there exists a countable set $\{|m_1\rangle \dots |m_l\rangle \dots\}$ of elements of M such that every element $|\psi\rangle \in M$ admits a unique decomposition as the sum of a convergent series $|\psi\rangle = \sum_{l=1}^{\infty} w_l|m_l\rangle$, $w_l \in \mathbb{T}$.

If $\{|m_l\rangle\}$ is a Schauder $\mathbb{T}$-basis in M, it follows that $\sum_{l=1}^{\infty} w_l|m_l\rangle = 0$ if and only if $w_l = 0$, $\forall l \in \mathbb{N}$. Moreover, if a normed $\mathbb{T}$-module M has a **Schauder $\mathbb{T}$-basis**, then the vector space M' is automatically a normed vector space with the following classical Schauder basis:

$$\{|m_1\rangle_{\mathbf{1}}, |m_1\rangle_{\mathbf{2}} \dots |m_l\rangle_{\mathbf{1}}, |m_l\rangle_{\mathbf{2}} \dots\}.$$

The normed space M' with a Schauder basis is necessarily of infinite dimension since it contains subspaces of an arbitrary finite dimension. For more details on Schauder bases see [7].

A normed $\mathbb{T}$-module with a Schauder $\mathbb{T}$-basis is called a **countable $\mathbb{T}$-module**. For the rest of this section we will only consider bicomplex Hilbert spaces constructed from countable $\mathbb{T}$-modules. We now show that in this context, it is always possible to construct an orthonormal Schauder $\mathbb{T}$-basis in M.

Theorem 3.10 (Orthonormalization)**.** *Let M be a bicomplex Hilbert space and let $\{|s_l\rangle\}$ be an arbitrary Schauder $\mathbb{T}$-basis of M. Then $\{|s_l\rangle\}$ can always be orthonormalized.*

Proof. Using Theorem 2.3 and Corollary 3.6, we can write $M = V_1 \oplus V_2$, where V_1 and V_2 are the associated Hilbert spaces. Hence $\{|s_l\rangle_{\mathbf{k}}\}$ is a Schauder basis of V_k $(k = 1, 2)$. Let $\{|s'_l\rangle_{\mathbf{k}}\}$ be the orthonormal basis constructed from $\{|s_l\rangle_{\mathbf{k}}\}$ in V_k [7, p. 59]. For all $l \in \mathbb{N}$ we have that

$$(|s'_l\rangle_{\mathbf{1}} + |s'_l\rangle_{\mathbf{2}}, |s'_l\rangle_{\mathbf{1}} + |s'_l\rangle_{\mathbf{2}}) = \mathbf{e_1}\big||s'_l\rangle_{\mathbf{1}}\big|_1^2 + \mathbf{e_2}\big||s'_l\rangle_{\mathbf{2}}\big|_2^2 = 1$$

and

$$\big(|s'_l\rangle_{\mathbf{1}} + |s'_l\rangle_{\mathbf{2}}, |s'_p\rangle_{\mathbf{1}} + |s'_p\rangle_{\mathbf{2}}\big) = 0$$

if $l \neq p$. From this we conclude that the set $\{|s'_l\rangle_{\mathbf{1}} + |s'_l\rangle_{\mathbf{2}}\}$ is an orthonormal basis in M. $\square$

It is interesting to note that the normalizability of kets requires that the scalar product belongs to $\mathbb{D}^+$. To see this, let us write $(|m_1\rangle, |m_1\rangle) = a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2}$ with $a_{\widehat{1}}, a_{\widehat{2}} \in \mathbb{R}$, and let

$$|m'_1\rangle = (z_{\widehat{1}}\mathbf{e_1} + z_{\widehat{2}}\mathbf{e_2})|m_1\rangle,$$

with $z_{\widehat{1}}, z_{\widehat{2}} \in \mathbb{C}(\mathbf{i_1})$ and $z_{\widehat{1}} \neq 0 \neq z_{\widehat{2}}$. We get

$$\begin{aligned}(|m'_1\rangle, |m'_1\rangle) &= (|z_{\widehat{1}}|^2\mathbf{e_1} + |z_{\widehat{2}}|^2\mathbf{e_2})\,(|m_1\rangle, |m_1\rangle)\\ &= (|z_{\widehat{1}}|^2\mathbf{e_1} + |z_{\widehat{2}}|^2\mathbf{e_2})(a_{\widehat{1}}\mathbf{e_1} + a_{\widehat{2}}\mathbf{e_2})\\ &= c_{\widehat{1}}a_{\widehat{1}}\mathbf{e_1} + c_{\widehat{2}}a_{\widehat{2}}\mathbf{e_2},\end{aligned}$$

with $c_{\widehat{k}} = |z_{\widehat{k}}|^2 \in \mathbb{R}^+$. The normalization condition of $|m'_1\rangle$ becomes

$$c_{\widehat{1}}a_{\widehat{1}}\mathbf{e_1} + c_{\widehat{2}}a_{\widehat{2}}\mathbf{e_2} = 1,$$

or $c_{\widehat{1}}a_{\widehat{1}} = 1 = c_{\widehat{2}}a_{\widehat{2}}$. This is possible only if $a_{\widehat{1}} > 0$ and $a_{\widehat{2}} > 0$. In other words, $(|m_1\rangle, |m_1\rangle) \in \mathbb{D}^+$.

3.2.2. *Projection in a Specific Vector Space.* Let $\{|m_1\rangle \ldots |m_l\rangle \ldots\}$ be a Schauder $\mathbb{T}$-basis associated with the bicomplex Hilbert space $\{M, (\cdot,\cdot)\}$. That is, any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{l=1}^{\infty} w_l |m_l\rangle, \tag{3.3}$$

with $w_l \in \mathbb{T}$. As was shown in [12] for the finite-dimensional case, an important subset V of M is the set of all kets for which all w_l in (3.3) belong to $\mathbb{C}(\mathbf{i_1})$. It is obvious that V is a non-empty normed vector space over complex numbers with Schauder basis $\{|m_1\rangle \ldots |m_l\rangle \ldots\}$. Let us write $w_l = \mathbf{e_1} z_{l\hat{1}} + \mathbf{e_2} z_{l\hat{2}}$, so that

$$|\psi\rangle = \sum_{l=1}^{\infty} w_l |m_l\rangle = \sum_{l=1}^{\infty} \mathbf{e_1} z_{l\hat{1}} |m_l\rangle + \sum_{l=1}^{\infty} \mathbf{e_2} z_{l\hat{2}} |m_l\rangle.$$

From Theorem 3.5 we know that the two series on the right-hand side separately converge to elements of V_1 and V_2. However, it is not obvious that $\sum_{l=1}^{\infty} z_{l\hat{k}} |m_l\rangle$ $(k = 1, 2)$ converges since $\mathbf{e_k}$ is not invertible. To prove this result, we need the following theorem.

Theorem 3.11. *Let $\{|\psi_n\rangle\}$ be an orthonormal sequence in the bicomplex Hilbert space M and let $\{\alpha_n\}$ be a sequence of bicomplex numbers. Then the series $\sum_{n=1}^{\infty} \alpha_n |\psi_n\rangle$ converges in M if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges in $\mathbb{R}$.*

Proof. The series $\sum_{n=1}^{\infty} \alpha_n |\psi_n\rangle$ converges if and only if for $k = 1, 2$, the series

$$\sum_{n=1}^{\infty} \mathbf{e_k} \alpha_n |\psi_n\rangle = \sum_{n=1}^{\infty} P_k(\alpha_n) |\psi_n\rangle_{\mathbf{k}}$$

converges. However, in the Hilbert space V_k, it is well-known [7, p. 59] that $\sum_{n=1}^{\infty} P_k(\alpha_n) |\psi_n\rangle_{\mathbf{k}}$ converges if and only if $\sum_{n=1}^{\infty} |P_k(\alpha_n)|^2$ converges in $\mathbb{R}$. Since

$$|\alpha_n|^2 = \frac{|P_1(\alpha_n)|^2 + |P_2(\alpha_n)|^2}{2},$$

we find that the series $\sum_{n=1}^{\infty} \alpha_n |\psi_n\rangle$ converges in M if and only if $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges in $\mathbb{R}$. $\square$

From Theorem 3.11 we see that if $\{|m_1\rangle \ldots |m_l\rangle \ldots\}$ is an **orthonormal** Schauder $\mathbb{T}$-basis and

$$\sum_{l=1}^{\infty} (\mathbf{e_1} z_{l\hat{1}} + \mathbf{e_2} z_{l\hat{2}}) |m_l\rangle$$

converges in M, then the series

$$\sum_{l=1}^{\infty} |\mathbf{e_1} z_{l\hat{1}} + \mathbf{e_2} z_{l\hat{2}}|^2$$

converges in $\mathbb{R}$. In particular, $\sum_{l=1}^{\infty} |z_{l\widehat{k}}|^2$ also converges. Hence $\sum_{l=1}^{\infty} z_{l\widehat{k}}|m_l\rangle$ converges and this allows to define projectors P_1 and P_2 from M to V as

$$P_k(|\psi\rangle) := \sum_{l=1}^{\infty} z_{l\widehat{k}}|m_l\rangle, \qquad k = 1, 2.$$

Therefore, any $|\psi\rangle \in M$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e_1} P_1(|\psi\rangle) + \mathbf{e_2} P_2(|\psi\rangle). \tag{3.4}$$

As in the finite-dimensional case [12], one can easily show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

$$P_k(s|\psi\rangle + t|\phi\rangle) = P_k(s)P_k(|\psi\rangle) + P_k(t)P_k(|\phi\rangle).$$

It will be useful to rewrite (3.4) as

$$|\psi\rangle = \mathbf{e_1}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}|\psi\rangle_{\widehat{2}} = |\psi\rangle_{\mathbf{1}} + |\psi\rangle_{\mathbf{2}},$$

where $|\psi\rangle_{\widehat{k}} := P_k(|\psi\rangle)$. Note that the scalar product in Theorem 2.6 can be reinterpreted as

$$(|\psi\rangle, |\phi\rangle) = \mathbf{e_1}(|\psi\rangle_{\widehat{1}}, |\phi\rangle_{\widehat{1}})_{\widehat{1}} + \mathbf{e_2}(|\psi\rangle_{\widehat{2}}, |\phi\rangle_{\widehat{2}})_{\widehat{2}}.$$

The $\mathbb{T}$-norm defined in (3.1) can be written as

$$\big|\big||\phi\rangle\big|\big| = \frac{1}{\sqrt{2}}\sqrt{\big||\phi\rangle_{\widehat{1}}\big|_1^2 + \big||\phi\rangle_{\widehat{2}}\big|_2^2}, \tag{3.5}$$

where

$$\big||\phi\rangle_{\widehat{k}}\big|_k^2 = \big(|\phi\rangle_{\widehat{k}}, |\phi\rangle_{\widehat{k}}\big)_{\widehat{k}} = (|\phi\rangle_{\mathbf{k}}, |\phi\rangle_{\mathbf{k}})_{\widehat{k}} = \big||\phi\rangle_{\mathbf{k}}\big|_k^2. \tag{3.6}$$

Just to avoid confusion, we recall that a bold index $\mathbf{k}$ on a ket (like in $|\psi\rangle_{\mathbf{k}}$) means that the ket belongs to V_k. Kets $|\psi\rangle_{\widehat{1}}$ and $|\psi\rangle_{\widehat{2}}$, on the other hand, belong to V. Indices $\widehat{1}$ and $\widehat{2}$ on a bicomplex number denote idempotent projection. The double bar denotes the $\mathbb{T}$-norm in M while the single bar denotes standard norms in V_k or V.

In Corollary 3.6 we showed that $(V_k, (\cdot,\cdot)_{\widehat{k}})$ is a Hilbert space for $k = 1, 2$. The same is true with V.

Theorem 3.12. *Let M be a bicomplex Hilbert space with orthonormal Schauder $\mathbb{T}$-basis $\{|m_l\rangle\}$. Then $(V, (\cdot,\cdot)_{\widehat{k}})$ is a complex (in $\mathbb{C}(\mathbf{i_1})$) Hilbert space for $k = 1, 2$.*

Proof. It is easy to see that $(V, (\cdot,\cdot)_{\widehat{k}})$ is a normed space over $\mathbb{C}(\mathbf{i_1})$ for $k = 1, 2$. Without lost of generality, let us consider the case $k = 1$. Let $\{|\psi_n\rangle\} \in V$ be a Cauchy sequence with respect to norm $|\cdot|_1$ specified in (3.6). We can see that $\{|\psi_n\rangle_{\mathbf{1}}\}$ is also a Cauchy sequence. By Corollary 3.6, $\{|\psi_n\rangle_{\mathbf{1}}\}$ converges to a ket $|\psi\rangle_{\mathbf{1}} = \mathbf{e_1}\sum_{l=1}^{\infty} w_l|m_l\rangle \in V_1$ with respect to the norm $|\cdot|_1$, where $\sum_{l=1}^{\infty} w_l|m_l\rangle \in M$. However,

$$\mathbf{e_1}\sum_{l=1}^{\infty} w_l|m_l\rangle = \sum_{l=1}^{\infty} \mathbf{e_1} w_l|m_l\rangle = \sum_{l=1}^{\infty} \mathbf{e_1} P_1(w_l)|m_l\rangle.$$

Hence, from the discussion after Theorem 3.11, we have that

$$|\psi\rangle_{\mathbf{1}} = \mathbf{e_1}\sum_{l=1}^{\infty} P_1(w_l)|m_l\rangle,$$

where $\sum_{l=1}^{\infty} P_1(w_l)|m_l\rangle \in V$. Thus $|\psi_n\rangle_{\mathbf{1}} = \mathbf{e_1}|\psi_n\rangle \to \mathbf{e_1}\sum_{l=1}^{\infty} P_1(w_l)|m_l\rangle$ whenever $n \to \infty$. Hence by (3.6) we get

$$|\psi_n\rangle \to \sum_{l=1}^{\infty} P_1(w_l)|m_l\rangle \in V$$

whenever $n \to \infty$. □

Theorem 3.13. *Let M be a bicomplex Hilbert space with $\{|m_l\rangle\}$ an orthonormal Schauder $\mathbb{T}$-basis of M. If the norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent on V, then the normed vector space $(V, ||\cdot||)$ is closed in M.*

Proof. Let $\{|\psi_n\rangle\} \in V$ be a converging sequence to a ket $|\psi\rangle \in M$. Thus,

$$|\psi_n\rangle = \sum_{l=1}^{\infty} z_{nl}|m_l\rangle = \mathbf{e_1}|\psi_n\rangle_{\widehat{1}} + \mathbf{e_2}|\psi_n\rangle_{\widehat{2}}$$

and

$$|\psi\rangle = \sum_{l=1}^{\infty} w_l|m_l\rangle = \mathbf{e_1}|\psi\rangle_{\widehat{1}} + \mathbf{e_2}|\psi\rangle_{\widehat{2}}$$

where $z_{nl} \in \mathbb{C}(\mathbf{i_1})$ and $w_l \in \mathbb{T}$, $\forall n, l \in \mathbb{N}$. Since $\{|\psi_n\rangle\} \in V$, we have that $|\psi_n\rangle_{\widehat{1}} = |\psi_n\rangle_{\widehat{2}}$, $\forall n \in \mathbb{N}$. Using (3.5) for the $\mathbb{T}$-norm, we find that

$$\big||\psi_n\rangle_{\widehat{k}} - |\psi\rangle_{\widehat{k}}\big|_k \to 0$$

whenever $n \to \infty$, for $k = 1, 2$. Hence,

$$|\psi\rangle_{\widehat{1}} = |\psi\rangle_{\widehat{2}}$$

since the norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent on V. □

Since all norms on finite-dimensional vector spaces are equivalent, the normed vector space $(V, ||\cdot||)$ is always closed in M whenever M is finite-dimensional. In the infinite-dimensional case, it is possible to find a simple condition to obtain the closure.

Definition 3.14. Let $\{|m_l\rangle\}$ be an orthonormal Schauder $\mathbb{T}$-basis of M and let V be the associated vector space. We say that a scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V if, $\forall |\psi\rangle, |\phi\rangle \in V$, we have $(|\psi\rangle, |\phi\rangle) \in \mathbb{C}(\mathbf{i_1})$.

We note that the property of being $\mathbb{C}(\mathbf{i_1})$-closed is basis-dependent. That is, a scalar product may be $\mathbb{C}(\mathbf{i_1})$-closed under V defined through a basis $\{|m_l\rangle\}$, but not under V' defined through a basis $\{|s_l\rangle\}$.

Corollary 3.15. *Let M be a bicomplex Hilbert space M with $\{|m_l\rangle\}$ an orthonormal Schauder $\mathbb{T}$-basis of M. If the scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V then the inner space $(V, ||\cdot||)$ is closed in M.*

Proof. Equation (2.5) is true whether the bicomplex scalar product is $\mathbb{C}(\mathbf{i_1})$-closed under V or not. When it is $\mathbb{C}(\mathbf{i_1})$-closed, we have for $k = 1, 2$

$$(|\psi\rangle, |\phi\rangle)_{\widehat{k}} = P_k((|\psi\rangle, |\phi\rangle)) = (|\psi\rangle, |\phi\rangle), \qquad \forall |\psi\rangle, |\phi\rangle \in V.$$

Hence, $|\cdot|_1 = |\cdot|_2$ and by Theorem 3.13 the inner space $(V, ||\cdot||)$ is closed in M. $\square$

4. The Harmonic Oscillator

Complex Hilbert spaces are fundamental tools of quantum mechanics. We should therefore expect that bicomplex Hilbert spaces should be relevant to any attempted generalization of quantum mechanics to bicomplex numbers. Steps towards such a generalization were made in [6, 11, 12]. In [6], in particular, the problem of the bicomplex quantum harmonic oscillator was investigated in detail.

To summarize the main results obtained, let us first recall the function space introduced in [6]. Let n be a nonnegative integer and let α be a positive real number. Consider the following function of a real variable x:

$$f_{n,\alpha}(x) := x^n \exp\left\{-\alpha x^2\right\}.$$

Let S be the set of all finite linear combinations of functions $f_{n,\alpha}(x)$, with complex coefficients. Furthermore, let a bicomplex function $u(x)$ be defined as

$$u(x) = \mathbf{e_1} u_{\widehat{1}}(x) + \mathbf{e_2} u_{\widehat{2}}(x), \tag{4.1}$$

where $u_{\widehat{1}}$ and $u_{\widehat{2}}$ are both in S. The set of all functions $u(x)$ is a $\mathbb{T}$-module, denoted by M_S.

Let $u(x)$ and $v(x)$ both belong to M_S. We define a mapping (u, v) of this pair of functions into $\mathbb{D}^+$ as follows:

$$(u, v) := \int_{-\infty}^{\infty} u^{\dagger_3}(x) v(x) \mathrm{d}x = \int_{-\infty}^{\infty} \left[\mathbf{e_1} \bar{u}_{\widehat{1}}(x) v_{\widehat{1}}(x) + \mathbf{e_2} \bar{u}_{\widehat{2}}(x) v_{\widehat{2}}(x)\right] \mathrm{d}x. \tag{4.2}$$

It is not hard to see that (4.2) is always finite and satisfies all the properties of a bicomplex scalar product.

Let $\xi = \mathbf{e_1}\xi_{\widehat{1}} + \mathbf{e_2}\xi_{\widehat{2}}$ be in $\mathbb{D}^+$ and let us define two operators X and P that act on elements of M_S as follows:

$$X\{u(x)\} := x u(x), \qquad P\{u(x)\} := -\mathbf{i_1}\hbar\xi \frac{\mathrm{d}u(x)}{\mathrm{d}x}.$$

It is not difficult to show that

$$[X, P] = \mathbf{i_1}\hbar\xi I.$$

Note that the action of X and P on elements of M_S always yields elements of M_S. That is, X and P are defined on all of M_S.

Let m and ω be two positive real numbers. We define the bicomplex harmonic oscillator Hamiltonian as follows:

$$H := \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2.$$

The problem of the bicomplex quantum harmonic oscillator consists in finding the eigenvalues and eigenfunctions of H.

That problem was solved in [6]. The results can be summarized as follows. Let θ_k $(k = \widehat{1}, \widehat{2})$ be defined as

$$\theta_k := \sqrt{\frac{m\omega}{\hbar\xi_k}}\, x.$$

Bicomplex harmonic oscillator eigenfunctions can then be written as (the most general eigenfunction would have different l indices in the two terms):

$$\begin{aligned}\phi_l(x) &= \mathbf{e_1}\phi_{l\widehat{1}} + \mathbf{e_2}\phi_{l\widehat{2}} \\ &= \mathbf{e_1}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{1}}}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta_{\widehat{1}}^2/2} H_l(\theta_{\widehat{1}}) + \mathbf{e_2}\left[\sqrt{\frac{m\omega}{\pi\hbar\xi_{\widehat{2}}}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta_{\widehat{2}}^2/2} H_l(\theta_{\widehat{2}}),\end{aligned} \tag{4.3}$$

where H_l are Hermite polynomials [8]. Equation (4.3) can be written more succinctly as

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}}\frac{1}{2^l l!}\right]^{1/2} \mathrm{e}^{-\theta^2/2} H_l(\theta),$$

where

$$\theta := \mathbf{e_1}\theta_{\widehat{1}} + \mathbf{e_2}\theta_{\widehat{2}} \quad \text{and} \quad H_l(\theta) := \mathbf{e_1}H_l(\theta_{\widehat{1}}) + \mathbf{e_2}H_l(\theta_{\widehat{2}}).$$

Finally, it was shown in [6] that $\tilde{M}$, the collection of all finite linear combinations of bicomplex functions $\phi_l(x)$, with bicomplex coefficients, is a $\mathbb{T}$-module. Specifically,

$$\tilde{M} := \left\{ \sum_l w_l \phi_l(x) \mid w_l \in \mathbb{T} \right\}. \tag{4.4}$$

Since $\tilde{M}$ only involves finite linear combinations of the functions ϕ_l, it is not complete, as was pointed out in [6]. With the methods developed in this paper, however, we can extend $\tilde{M}$ to a complete module, in fact to a bicomplex Hilbert space.

Just like in Definition 2.1, we can define two vector spaces $\tilde{V}_1$ and $\tilde{V}_2$ as $\tilde{V}_1 = \mathbf{e_1}\tilde{M}$ and $\tilde{V}_2 = \mathbf{e_2}\tilde{M}$. From (4.3) and (4.4), it is clear that $\tilde{V}_1$ contains all functions $\mathbf{e_1}\phi_{l\widehat{1}}$ and $\tilde{V}_2$ contains all $\mathbf{e_2}\phi_{l\widehat{2}}$. Now the functions $\phi_{l\widehat{1}}$ and $\phi_{l\widehat{2}}$ are normalized eigenfunctions of the usual quantum harmonic oscillator (with $\hbar$ replaced by $\hbar\xi_{\widehat{1}}$ or $\hbar\xi_{\widehat{2}}$). It is well-known [13] that, as a Schauder basis, these eigenfunctions generate $L^2(\mathbb{R})$.

Let $u(x)$ be defined as in (4.1), except that $u_{\widehat{1}}(x)$ and $u_{\widehat{2}}(x)$ are both taken as $L^2(\mathbb{R})$ functions. Clearly, the set of all $u(x)$ makes up a $\mathbb{T}$-module, which we shall denote by M. With the scalar product (4.2), M becomes a bicomplex pre-Hilbert space. Since $L^2(\mathbb{R})$ is complete we get from Theorem 3.4:

Corollary 4.1. *M is a bicomplex Hilbert space.*

The following theorem shows that the ϕ_l make up a Schauder $\mathbb{T}$-basis of M.

Theorem 4.2. *Any bicomplex function u in M can be expanded uniquely as $u = \sum_{l=0}^{\infty} w_l\phi_l$, where w_l is a bicomplex number and ϕ_l is given in (4.3).*

Proof. Let $u = \mathbf{e_1} u_{\widehat{1}} + \mathbf{e_2} u_{\widehat{2}}$. Since $u_{\widehat{1}}$ and $u_{\widehat{2}}$ both belong to $L^2(\mathbb{R})$ and since the functions $\phi_{l\widehat{1}}$ and $\phi_{l\widehat{2}}$ are Schauder bases of $L^2(\mathbb{R})$, one can write

$$u_{\widehat{1}} = \sum_{l=0}^{\infty} c_{l\widehat{1}} \phi_{l\widehat{1}}, \qquad u_{\widehat{2}} = \sum_{l=0}^{\infty} c_{l\widehat{2}} \phi_{l\widehat{2}},$$

where $c_{l\widehat{1}}$ and $c_{l\widehat{2}}$ belong to $\mathbb{C}(\mathbf{i_1})$. Letting $w_l = \mathbf{e_1} c_{l\widehat{1}} + \mathbf{e_2} c_{l\widehat{2}}$, the desired expansion follows. It must be unique since otherwise, either $u_{\widehat{1}}$ or $u_{\widehat{2}}$ would have two different expansions. $\square$

5. Conclusion

We have derived a number of new results on infinite-dimensional bicomplex modules and Hilbert spaces, including a generalization of the Riesz representation theorem for bicomplex continuous linear functionals and a general version of the bicomplex Schwarz inequality. The perspective of further investigating the extent to which quantum mechanics generalizes to bicomplex numbers motivates us in developing additional mathematical tools related to infinite-dimensional bicomplex Hilbert spaces and operators acting on them. We believe that results like the Riesz-Fischer theorem and the spectral theorem can also be extended to infinite-dimensional Hilbert spaces.

Acknowledgments

DR is grateful to the Natural Sciences and Engineering Research Council of Canada for financial support. RGL would like to thank the Québec FQRNT Fund for the award of a postgraduate scholarship.

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